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HOMOLOGICAL MIRROR SYMMETRY WITH HIGHER PRODUCTS

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The homological mirror symmetry conjecture formulated by M. Kontsevich in [5] claims that derived categories of Fukaya's symplectic A_∞-categogy $F(M)$ of a Calabi-Yau manifold M and of coherent sheaves on a mirror dual Calabi-Yau manifold X are equivalent. In particular, this means that one can identify the associative product on Ext-groups between coherent sheaves on X with the corresponding product in the Floer cohomology of Lagrangians submanifolds in M (defined by Fukaya in [1]). The drawback of this conjecture is that one has an A_∞-category on the symplectic side of the story and the usual category on the complex side, so one has to make the usual category out of $F(M)$. In this note we fix this problem by constructing an A_∞-category on the complex side and formulate a more general conjecture involving A_∞-categories on both sides.

Let X be a compact complex manifold equipped with a hermitian metric. Inspired by Merkulov's paper [6] we define an A_∞-category $\mathcal{D}^b_\infty(X)$ which is a refined version of the derived category of X. The objects of $\mathcal{D}^b_\infty(X)$ are bounded complexes of holomorphic vector bundles on X equipped with hermitian metrics. The morphisms from $E^\bullet$ to $F^\bullet$ are elements of $\mathrm{Ext}(E^\bullet, F^\bullet) = \oplus_q H^q(X, (E^\bullet)^* \otimes F^\bullet)$ which can be thought of as harmonic $(0,q)$-forms with values in $(E^\bullet)^* \otimes F^\bullet$. The A_∞-structure has $m_1 = 0$ while m_2 is the usual composition of Ext's. In particular, forgetting higher products we obtain the usual derived category of X. The higher products measure in some sense to which extent the product of harmonic forms with values in hermitian bundles fails to be harmonic. More precisely, we use the construction of A_∞-structure on a subcomplex homotopy equivalent to a dg-algebra (see [3], [4], [6]). This construction gives an A_∞-structure on the algebra $\mathrm{Ext}(E, E)$ where E is a hermitian holomorphic vector bundle (or complex of such bundles). It follows from the results of the homological perturbation theory (see [4]) that up to homotopy equivalence our A_∞-structures do not depend on choices of metrics.

Now the more natural formulation the homological mirror conjecture is that for mirror dual Calabi-Yau manifolds there exists an A_∞-functor from $F(M)$ to $\mathcal{D}^b_\infty(X)$ which is a homotopy equivalence on morphisms.

The only non-trivial case in which the original Kontsevich's conjecture is verified is that of elliptic curves considered in [9]. In the present paper we show that generic triple products on the subcategory of line bundles on elliptic curves defined using complex structure and using symplectic structure are canonically homotopic to each other. This gives a partial verification for our generalized version of the homological mirror conjecture in the case of elliptic curves.

1. A_∞-CATEGORIES

1.1. Recall that a ($\mathbb{Z}$-graded) A_∞-algebra is a $\mathbb{Z}$-graded vector space A equipped with linear maps $m_k : A^{\otimes k} \to A$ for $k \geq 1$ of degree $2-k$ satisfying the following A_∞-constraints

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{l(\tilde{a}_1+...+\tilde{a}_j)+j(l-1)+(k-1)l} m_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n) = 0$$

where $\tilde{a}_i = \deg(a_i) \mod (2)$. A more conceptual way to spell this axiom is to say that m_k are components of a coderivation ∂ on the bar-construction of A such that $\partial^2 = 0$ (see [10]). For a pair of A_∞-algebras A and B there is a natural notion of a A_∞-morphism from A to B. Namely, such a morphism consists of the data $(f_n, n \geq 1)$ where $f_n : A^{\otimes n} \to B$ is a linear of degree $1-n$ such that

$$\sum_{1 \leq k_1 < k_2 < ... < k_i = n} (-1)^i m_i(f_{k_1}(a_1, \ldots, a_{k_1}), f_{k_2-k_1}(a_{k_1+1}, \ldots, a_{k_2}), \ldots, f_{n-k_{i-1}}(a_{k_{i-1}+1}, \ldots, a_n)) =$$

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{l(\tilde{a}_1+...+\tilde{a}_j+n)+j(l-1)} f_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n)$$

Again this is equivalent to having a morphism between the corresponding bar-constructions in the category of differential coalgebras.

One can compose A_∞-morphisms in the natural way. The identity A_∞-morphism consists of $f_1 = \mathrm{id}$, $f_n = 0$ for $n \geq 2$. If $(f_n) : A \to B$ is an A_∞-morphism such that f_1 is an isomorphism of underlying abelian groups then f_1^{-1} extends to an A_∞-morphism $B \to A$ which is inverse to (f_n). In the case B and A have the same underlying spaces and $f_1 = \mathrm{id}$ we will call sometimes the data $(f_n, n \geq 2)$ a homotopy between two A_∞-structures (m_n) and (m'_n) on the same space. For example, if we have $m_1 = m'_1 = 0$ and $m_2(a,b) = m'_2(a,b) = ab$ then f_2 gives a homotopy between m_3 and m'_3 in the following sense:

$$m'_3(a_1,a_2,a_3) - m_3(a_1,a_2,a_3) = a_1 f_2(a_2,a_3) + f_2(a_1,a_2)a_3 - f_2(a_1a_2,a_3) + f_2(a_1,a_2a_3).$$

1.2. The definition of A_∞-category is very similar to that of an A_∞-algebras (see [1]). Namely, an A_∞-category $\mathcal{C}$ consists of a class of objects $\mathrm{Ob}\,\mathcal{C}$, for every pair of objects E_1 and E_2 graded spaces of morphisms $\mathrm{Hom}(E_1, E_2)$ equipped with a differential m_1, and compositions

$$m_k : \mathrm{Hom}(E_1, E_2) \otimes \mathrm{Hom}(E_2, E_3) \otimes \ldots \otimes \mathrm{Hom}^*(E_k, E_{k+1}) \to \mathrm{Hom}(E_1, E_{k+1})$$

of degree $2-k$ for all $k \geq 2$. The associativity constraint is that these compositions define a structure of A_∞-algebra on $\oplus_{ij} \mathrm{Hom}(E_i, E_j)$ for every collection $E_1, \ldots, E_n \in \mathrm{Ob}\,\mathcal{C}$.

An A_∞-functor (see [2]) $\phi : \mathcal{C} \to \mathcal{C}'$ between A_∞-categories consists of the map $\phi : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}'$ and of the linear maps

$$f_k : \mathrm{Hom}_{\mathcal{C}}(E_1, E_2) \otimes \mathrm{Hom}_{\mathcal{C}}(E_2, E_3) \otimes \ldots \otimes \mathrm{Hom}_{\mathcal{C}}(E_k, E_{k+1}) \to \mathrm{Hom}_{\mathcal{C}'}(\phi(E_1), \phi(E_{k+1}))$$

of degree $1-k$ for $k \geq 1$, which define A_∞-morphisms $\oplus_{ij} \mathrm{Hom}_{\mathcal{C}}(E_i, E_j) \to \oplus_{ij} \mathrm{Hom}_{\mathcal{C}'}(\phi(E_i), \phi(E_j))$.

1.3. Recall that if (A, d) is a dg-algebra, $Q : A \to A$ is an odd operator then the subcomplex $B = (1 - dQ - Qd)(A) \subset A$ has a canonical A_∞-structure defined as follows (see [6]). The differential m_1 is just the restriction of d to B. Then one defines inductively linear maps $\lambda_n : A^{\otimes n} \to A$ for $n \geq 2$ by setting $\lambda_2(a_1, a_2) = a_1 a_2$,

$$\lambda_n(a_1, \ldots, a_n) = (-1)^{n-1}(Q\lambda_{n-1}(a_1, \ldots, a_{n-1})a_n - (-1)^{n\tilde{a}_1} a_1 (Q\lambda_{n-1}(a_2, \ldots, a_n)) - \sum_{\substack{k+l=n,\\ k,l\geq 2}} (-1)^{k+(l-1)(\tilde{a}_1+\ldots+\tilde{a}_k)} (Q\lambda_k(a_1, \ldots, a_k))(Q\lambda_l(a_{k+1}, \ldots, a_n)).$$

Now the products m_n for $n \geq 2$ are defined by the formula

$$m_n(b_1, \ldots, b_n) = \mathrm{pr}(\lambda_n(b_1, \ldots, b_n))$$

where $\mathrm{pr} = 1 - Qd - dQ$. For example, $m_2(b_1, b_2) = \mathrm{pr}(b_1 b_2)$,

$$m_3(b_1, b_2, b_3) = \mathrm{pr}(Q(b_1 b_2)b_3 - (-1)^{\tilde{b}_1} b_1 Q(b_2 b_3)).$$

Assume in addition that we have $\mathrm{pr}\,|_B = \mathrm{id}$, $\mathrm{pr} \circ Q = Q^2 = 0$. Then we can apply the homological perturbation theory (see [4], 4.2) to conclude that the natural embedding $B \to A$ and the projection $\mathrm{pr} : A \to B$ extend to A_∞-morphisms of A_∞-algebras (where A has trivial m_k with $k \geq 3$). In the situation below we will also have $d|_B = 0$, so that B is isomorphic to the cohomology $H(A)$ of (A, d). Thus, in this particular case the perturbation theory implies that A_∞-structures on $H(A)$ obtained by different operators Q are homotopic. Note that the higher products on $H(A)$ thus defined coincide with the Massey products when the latter are well-defined and univalued.

We apply this machinery in the following situation: $(A,d) = (C^\infty(X, \Omega^{0,\bullet}(E^* \otimes E), \overline{\partial}))$ is the Dolbeault complex computing $\mathrm{Ext}(E,E)$, where E is a holomorphic vector bundle on a compact complex manifold X. Now assume that X and E are equipped with hermitian metrics. Then there is an induced metric on $E^* \otimes E$ which defines a conjugate-linear $*$-operator

$$\overline{*} : \Omega^{p,q}(E^* \otimes E) \to \Omega^{n-p,n-q}(E^* \otimes E).$$

The induced metric on the Dolbealt complex is given by

$$(\alpha, \beta) = \int_X \alpha \wedge \overline{*}\beta.$$

The conjugate operator to $\overline{\partial}$ with respect to this metric is

$$\overline{\partial}^* = -\overline{*}\overline{\partial}\overline{*}$$

(see [11]). Now we set $Q = \overline{\partial}^* G$ where G is the Green operator corresponding to the $\overline{\partial}$-laplacian $\Delta = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$. Then by definition

$$\mathrm{pr} = 1 - \overline{\partial}Q - Q\overline{\partial} = 1 - \Delta G - G\Delta$$

is the projector on the space $\mathcal{H}^{0,\bullet}(E^* \otimes E)$ of harmonic forms of type $(0,\bullet)$. Thus, the above general construction gives an A_∞-structure on $\mathcal{H}^{0,\bullet}(E^* \otimes E) \simeq \mathrm{Ext}(E,E)$ extending the standard algebra structure on it (and having $m_1 = 0$).

Similarly, fixing a hermitian metric on X we can define an A_∞-category $\mathcal{V}_\infty(X)$ as follows. The objects of $\mathcal{V}_\infty(X)$ are hermitian vector bundles on X. Morhisms from E to F in $\mathcal{C}(X)$ are elements of $\mathrm{Ext}(E,F)$ (identified with the space of harmonic forms $\mathcal{H}^{0,\bullet}(E^* \otimes F)$). The composition

$$m_k : \mathrm{Ext}(E_0,E_1) \otimes \ldots \otimes \mathrm{Ext}(E_{k-1},E_k) \to \mathrm{Ext}(E_0,E_k) \tag{1.3.1}$$

is defined as follows. Consider the hermitian holomorphic vector bundle $E = \oplus_{i=0}^k E_i$. Then the left hand side of (1.3.1) is the direct summand of $\mathrm{Ext}(E,E)^{\otimes k}$ while the right hand side is the direct summand of $\mathrm{Ext}(E,E)$. So we just define (1.3.1) as the corresponding component of the map

$$m_k : \mathrm{Ext}(E,E)^{\otimes k} \to \mathrm{Ext}(E,E)$$

coming from the A_∞-structure on $\mathrm{Ext}(E,E)$.

1.4. The definition of the category $\mathcal{D}^b_\infty(X)$ for a hermitian complex manifold X follows the same pattern. One just has to observe that there is an analogue of the theory of harmonic forms for a complex of hermitian holomorphic vector bundles $(E^\bullet, \delta)$ (the differential δ is holomorphic). Namely, we consider the complex of forms $\Omega^{0,\bullet}(E^\bullet) =$

$\oplus_{p,q}\Omega^{0,p}(E^q)$ with the total differential $\overline{\partial}+\delta$ and total grading $p+q$. Then the operator $\overline{\partial}+\delta$ is elliptic, so we can introduce the corresponding Laplace operator $\Delta_{\overline{\partial}+\delta}$, Green operator $G_{\overline{\partial}+\delta}$, etc. In particular, we have a notion of harmonic forms with values in $E^\bullet$. They don't have to be bihomogeneous so the space of harmonic forms has just one grading coming from the total grading on $\Omega^{0,\bullet}(E^\bullet)$. Now we can apply the same procedure as before to define the A_∞-structure on the space of harmonic forms which can be identified with $\mathrm{Ext}(E^\bullet, E^\bullet)$ (=morphisms in the derived category $\mathcal{D}^b(X)$). Now $\mathcal{D}^b_\infty(X)$ has as objects complexes of hermitian holomorphic vector bundles with morphisms between $E^\bullet$ and $F^\bullet$ being the space of $\mathrm{Ext}(E^\bullet, F^\bullet)$. The A_∞-structure is defined as above.

1.5. One can generalize slightly the above construction as follows. For a holomorphic vector bundle E on X let us denote $H^{p,q}(E) = H^q(X, \Omega^p_{hol}\otimes E)$. Then for an n-tuple $E_1, \dots, E_n$ of hermitian bundles on X (a hermitian metric on X is fixed) we can define products

$$m_n : H^{p_1,q_1}(E_1)\otimes\ldots\otimes H^{p_n,q_n}(E_n) \to H^{p_1+\ldots+p_n, q_1+\ldots+q_n-n+2}(E_1\otimes\ldots\otimes E_n).$$

Indeed, this product is obtained by applying the general construction to the Dolbeault complex of forms of all types with values in the algebra $\mathcal{A} = \oplus_{i\le j}\mathcal{A}_{ij}$, where $\mathcal{A}_{ij} = E_i\otimes E_{i+1}\ldots\otimes E_j$, equipped with the operator $Q = \overline{\partial}^* G$. The products (1.3.1) can be recovered from this more general construction. The further possible generalization is to let the product m_k depend on a collection of hermitian metrics h_{ij}, $1\le i<j\le k$ on X, by considering the operator Q on the Dolbeault complex with values in $\mathcal{A}$ such that $Q = \oplus Q_{ij}$, where Q_{ij} is defined using the metric h_{ij} (note that only Q_{ij} with $i<j$ are used in the definition of m_k). These generalized products still satisfy an analogue of the A_∞-constraint.

1.6. The higher products defined above are compatible with the Serre duality in the way described below. Recall that the Serre duality is a perfect pairing between $H^{p,q}(E)$ and $H^{d-p,d-q}(E^*)$. where d is the complex dimension of X. If X and E are equipped with hermitian metrics then this pairing is induced by the pairing

$$\langle\alpha,\beta\rangle = \int \alpha\wedge\beta$$

between the corresponding Dolbeault complexes. More generally, for any bilinear holomorphic pairing $b : E\otimes_{\mathcal{O}} F\to\mathcal{O}$ we have the induced pairing $\langle\cdot,\cdot\rangle_b$ between $H^{p,q}(E)$ and $H^{d-p,d-q}(F)$. Now assume

that we have $n+1$ hermitian holomorphic bundles $E_1, \dots, E_{n+1}$ and a polylinear holomorphic map

$$f : E_1 \otimes_{\mathcal{O}} \dots \otimes_{\mathcal{O}} E_{n+1} \to \mathcal{O}.$$

Theorem 1.1. *One has*

$$\langle m_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle_f = (-1)^{n(\tilde{\alpha}_1+1)} \langle \alpha_1, m_n(\alpha_2, \dots, \alpha_{n+1}) \rangle_f$$

where $\alpha_i \in H^{p_i,q_i}(E_i)$, $i = 1, \dots, n+1$, $p_1 + \dots + p_{n+1} = d$, $q_1 + \dots + q_{n+1} = d + n - 2$.

Lemma 1.2. *Let E be a hermitian holomorphic bundle. Then for any $\alpha \in \Omega^{p,q+1}(E)$, $\beta \in \Omega^{d-p,d-q}(E^*)$ one has*

$$\int_X Q\alpha \wedge \beta = (-1)^{\tilde{\alpha}} \int_X \alpha \wedge Q\beta$$

where $Q = \overline{\partial}^* G$.

Proof. This follows from self-adjointness of G and from the formula

$$\int \overline{\partial}^* \alpha \wedge \beta = (-1)^{p+q+1} \int \alpha \wedge \overline{\partial}^* \beta$$

which in turn follows from the equalities $\overline{\partial}^* = -\overline{*}\overline{\partial}\overline{*}$ and $\beta = (-1)^{p+q}\overline{*}(\overline{*}\beta)$. □

Proof of theorem 1.1. First of all Lemma 1.2 implies that for a harmonic form β one has

$$\int \mathrm{pr}(\alpha) \wedge \beta = \int \alpha \wedge \beta.$$

Thus, we have to prove the identity

$$\int \lambda_n(\alpha_1, \dots, \alpha_n) \wedge \alpha_{n+1} = (-1)^{n(\tilde{\alpha}_1+1)} \int \alpha_1 \wedge \lambda_n(\alpha_2, \dots, \alpha_{n+1}). \tag{1.6.1}$$

Following [6] let us rewrite the inductive definition of λ_n as follows:

$$\lambda_n(x_1, \dots, x_n) = \sum_{\substack{k+l=n,\\ k,l \geq 1}} (-1)^{k+(l-1)(\tilde{x}_1+\dots+\tilde{x}_k)} Q\lambda_k(x_1, \dots, x_k) Q\lambda_l(x_{k+1}, \dots, n)$$

where we denote formally $\lambda_1 = -Q^{-1}$ so that $Q\lambda_1(x) = -x$. Hence, the LHS of (1.6.1) is equal to the integral of

$$\sum_{\substack{k+l=n,\\ k,l\geq 1}} (-1)^{k+(l-1)(\tilde{x}_1+\ldots+\tilde{x}_k)} Q\lambda_k(\alpha_1,\ldots,\alpha_k)Q\lambda_l(\alpha_{k+1},\ldots,\alpha_n)\alpha_{n+1} =$$

$$(-1)^{1+n\tilde{\alpha}_1}\alpha_1 Q\lambda_{n-1}(\alpha_2,\ldots,\alpha_n)\alpha_{n+1} +$$

$$(-1)^n \sum_{\substack{k+l=n+1,\\ k,l\geq 2}} (-1)^{l(\tilde{\alpha}_1+\ldots+\tilde{\alpha}_k)} Q\lambda_k(\alpha_1,\ldots,\alpha_k)\lambda_l(\alpha_{k+1},\ldots,\alpha_{n+1}) -$$

$$(-1)^n \sum_{\substack{k+l+m=n+1,\\ k,m\geq 2; l\geq 1}} (-1)^{l+(m-1)(\tilde{\alpha}_{k+1}+\ldots+\tilde{\alpha}_{k+l})+(l+m)(\tilde{\alpha}_1+\ldots+\tilde{\alpha}_k)} Q\lambda_k(\alpha_1,\ldots,\alpha_k) \times$$

$$Q\lambda_l(\alpha_{k+1},\ldots,\alpha_{k+l})Q\lambda_m(\alpha_{k+l+1},\ldots,\alpha_{n+1}).$$

Similarly, we can rewrite the RHS of (1.6.1) as the integral of

$$(-1)^{1+n(\tilde{\alpha}_1)}\alpha_1 Q\lambda_{n-1}(\alpha_2,\ldots,\alpha_n)\alpha_{n+1} +$$

$$(-1)^n \sum_{\substack{k+l=n+1,\\ k,l\geq 2}} (-1)^{k+(l-1)(\tilde{\alpha}_1+\ldots+\tilde{\alpha}_k)} \lambda_k(\alpha_1,\ldots,\alpha_k)Q\lambda_l(\alpha_{k+1},\ldots,\alpha_{n+1}) -$$

$$(-1)^n \sum_{\substack{k+l+m=n+1,\\ k,m\geq 2; l\geq 1}} (-1)^{l+(m-1)(\tilde{\alpha}_{k+1}+\ldots+\tilde{\alpha}_{k+l})+(l+m)(\tilde{\alpha}_1+\ldots+\tilde{\alpha}_k)} Q\lambda_k(\alpha_1,\ldots,\alpha_k) \times$$

$$Q\lambda_l(\alpha_{k+1},\ldots,\alpha_{k+l})Q\lambda_m(\alpha_{k+l+1},\ldots,\alpha_{n+1}).$$

It remains to apply Lemma 1.2 to conclude that

$$\int Q\lambda_k(\alpha_1,\ldots,\alpha_k)\lambda_l(\alpha_{k+1},\ldots,\alpha_{n+1}) = (-1)^{k+\tilde{\alpha}_1+\ldots+\tilde{\alpha}_k}\int \lambda_k(\alpha_1,\ldots,\alpha_k)Q\lambda_l(\alpha_{k+1},\ldots,\alpha_{n+1}).$$

□

2. Case of elliptic curve

In this section we'll study the category $\mathcal{V}_\infty(X)$ in the case when $X = E_\tau = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ is a complex elliptic curve. All holomorphic vector bundles on elliptic curves are obtained from line bundles using the operations of push-forward under isogeny, tensoring with local systems, and direct sums. Therefore, it is plausible that the entire A_∞-category $\mathcal{V}_\infty(E_\tau)$ can be reconstructed from its full A_∞-subcategory consisting of line bundles (see the argument of [9] dealing with the case of m_2). More precisely, we choose canonical representatives in each isomorphism class of holomorphic line bundles on E_τ and canonical hermitian metrics on them and consider the corresponding subcategory in $\mathcal{V}_\infty(E_\tau)$. We show that generic triple products in this subcategory are homotopic to the triple products between the corresponding objects in the Fukaya category of the symplectic torus. More precisely we consider *transversal*

products, i.e. such that the corresponding configuration of geodesic circles in $\mathbb{R}^2/\mathbb{Z}^2$ is transversal.

2.1. Let $p : \mathbb{C} \to E_\tau$ be the projection. We denote by L the holomorphic line bundle of degree 1 on E_τ equipped with a trivialization of p^*L such that the theta-function

$$\theta(x,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i\tau n^2 + 2\pi i n x)$$

is a pull-back of a holomorphic section of L. For every $u \in \mathbb{C}$ let us denote by $L(k,u)$ the holomorphic line bundle $L^{k-1} \otimes t_u^* L$ on E_τ. Up to an isomorphism $L(k,u)$ depends only on u modulo the lattice $\mathbb{Z} + \mathbb{Z}\tau$. More precisely, we have $L(k,u) = L(k,u+1)$ and a natural isomorphism

$$L(k,u) \tilde{\to} L(k,u+\tau) : f(x) \mapsto \exp(-2\pi x) f(x).$$

We fix a hermitian metric on $L(k,u)$ which under the natural trivialization of $\pi^*L(k,u)$ has form

$$\langle f, g\rangle = \int_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} f(x)\overline{g(x)} \exp(-2\pi t(kx_2^2 + 2x_2u_2))dx_1dx_2$$

where $t = \mathrm{Im}(\tau)$, $x = x_1 + \tau x_2$, $u = u_1 + \tau u_2$. The natural isomorphisms $L(k,u) \simeq L(k, u+m+n\tau)$ are compatible with metrics. Also we have an isomorphism of hermitian line bundles

$$L(k,u) \otimes L(k',u') \simeq L(k+k', u+u').$$

Note that every C^∞-section $f(x)$ of $L(k,u)$ can be written in the form

$$f(x) = \sum_{n\in\mathbb{Z}} \varphi_n(x_2) \exp(2\pi i n x_1).$$

If $k \neq 0$ then $|k|$ functions $(\varphi_n, n = 0, \ldots, |k|)$ determine the rest. Moreover, the functions $\varphi_n(x_2)\exp(\pi i(k\tau x_2^2 + 2ux_2))$ belong to the Schwarz space of functions on $\mathbb{R}$ with derivatives of all orders rapidly decreasing at infinity . If $g(x) = \sum_{n\in\mathbb{Z}} \psi_n(x_2)\exp(2\pi i n x_1)$ is another C^∞-section of $L(k,u)$ then we have

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}/k\mathbb{Z}} \int_{\mathbb{R}} \varphi_n(s)\overline{\psi_n(s)} \exp(-2\pi t(ks^2 + 2u_2 s))ds. \tag{2.1.1}$$

On the other hand, the line bundle $L(0,u)$ is trivial as C^∞-bundle. Indeed, C^∞-sections of $L(0,u)$ are functions $f(x)$ on $\mathbb{C}$ such that $f(x+1) = f(x)$, $f(x+\tau) = \exp(-2\pi i u) f(x)$. In other words, the function $\exp(2\pi i u x_2) f(x)$, where $x = x_1 + \tau x_2$, is doubly periodic with respect

to $\mathbb{Z}+\mathbb{Z}\tau$. In particular, any C^∞-section of $L(0,u)$ can be written as a Fourier series

$$f(x)=\sum_{m,n} c_{m,n}\exp(2\pi i(mx_1+(n-u)x_2))$$

where $\varphi_{u,m,n}(x)=\exp(2\pi i(mx_1+(n-u)x_2))$ form an orthonormal basis with respect to the hermitian metric on $L(0,u)$. Note that $\overline{\partial}\varphi_{u,m,n}(x)=\frac{\pi}{t}(m\tau-n+u)\varphi_{u,m,n}(x)$. Hence, if $c_{m,n}$ are the Fourier coefficients of $f\in C^\infty(L(0,u))$ then $\frac{\pi}{t}(m\tau-n+u)c_{m,n}$ are the Fourier coefficients of $\overline{\partial}f$.

2.2. If $k>0$ then the basis of global sections of $L(k,u)$ is given by k theta-functions with characteristics

$$\theta_{a/k}(kx+u,k\tau);a\in\mathbb{Z}/k\mathbb{Z}$$

where for $r\in\mathbb{Q}/\mathbb{Z}$ we denote

$$\theta_r(x,\tau)=\sum_{n\in\mathbb{Z}}\exp(\pi i\tau(n+r)^2+2\pi i(n+r)x)=\exp(\pi i\tau r^2+2\pi irx)\theta(x+r\tau,\tau).$$

Using (2.1.1) it is easy to see that this basis is orthogonal with respect to the hermitian metric and

$$||\theta_{a/k}(kx+u,k\tau)||^2=\frac{1}{\sqrt{2tk}}\exp(\frac{2\pi tu_2^2}{k}).$$

Similarly, for $k>0$ the basis of harmonic $(0,1)$-forms with values in $L(-k,-u)$ is given by

$$\overline{\theta_{a/k}(kx+u,k\tau)}\exp(-2\pi t(kx_2^2+2x_2u_2))d\overline{x},a\in\mathbb{Z}/k\mathbb{Z}$$

Note that $\theta_{b/k}(kx+u,k\tau)\overline{\theta_{a/k}(kx+v,k\tau)}\exp(-2\pi t(kx_2^2+2x_2v_2))$ is a section of $L(0,u-v)$. The corresponding Fourier coefficients are

$$c_{m,n}=\int_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau}\theta_{b/k}(kx+u,k\tau)\overline{\theta_{a/k}(kx+v,k\tau)}\times$$
$$\exp(-2\pi t(kx_2^2+2x_2v_2)-2\pi i(mx_1+(n-u+v)x_2))dx_1dx_2=$$
$$\langle\theta_{b/k}(kx+u,k\tau)\exp(-2\pi i(mx_1+(n-u+v)x_2)),\theta_{a/k}(kx+v,k\tau)\rangle,$$

where we take the hermitian product of sections of $L(k,v)$. Using the formula (2.1.1) we find that $c_{m,n}=0$ unless $m\equiv b-a \mod(k)$ in which case

$$c_{m,n}=\exp(\frac{\pi i}{k}((m+a)^2\tau-a^2\overline{\tau}+2(m+a)u-2a\overline{v}))\times$$
$$\int_{\mathbb{R}}\exp(-2\pi tks^2+2\pi is((m+a)\tau-a\overline{\tau}-n+u-\overline{v}))ds=$$
$$\frac{1}{\sqrt{2tk}}\exp(-\frac{\pi}{2tk}(|\gamma|^2+2\overline{\gamma}u-2\gamma\overline{v}+(u-\overline{v})^2)+\frac{\pi in}{k}(m+2a)).$$

where $\gamma = m\tau - n$.

2.3. We start by computing the triple product

$$m_3 : H^1(L(-k,0)) \otimes H^0(L(k,u)) \otimes H^0(L(l,v) \to H^0(L(l,u+v))$$

where $k, l > 0$, $u \notin \mathbb{Z} + \mathbb{Z}\tau$. By definition this product is given by the formula

$$m_3(\alpha, \beta_1, \beta_2) = \mathrm{pr}(Q(\alpha\beta_1)\beta_2 + \alpha Q(\beta_1\beta_2))$$

where α, β_1, β_2 are harmonic representatives of the corresponding classes, pr is the harmonic projector. Note that $\beta_1\beta_2$ is a holomorphic section of $L(k+l, u+v)$, hence $Q(\beta_1\beta_2) = 0$. On the other hand, since $u \notin \mathbb{Z} + \mathbb{Z}\tau$ we have $H^*(L(0,u)) = 0$, hence, $Q(\alpha\beta_1) = \overline{\partial}^{-1}(\alpha\beta_1)$, so $m_3(\alpha, \beta_1, \beta_2) = \mathrm{pr}(\overline{\partial}^{-1}(\alpha\beta_1)\beta_2)$. We can compute this using explicit bases of theta functions:

$$m_3(\overline{\theta_{a/k}(kx, k\tau)} \exp(-2\pi t k x_2^2) d\overline{x}, \theta_{b/k}(kx+u, k\tau), \theta_{c/l}(lx+v, l\tau)) = \sqrt{2tl} \times$$

$$\sum_{d=0}^{l-1} \langle F(x)\theta_{c/l}(lx+v, l\tau), \theta_{d/l}(lx+u+v, l\tau)\rangle \exp(-\frac{2\pi t(u_2+v_2)^2}{l})\theta_{d/l}(lx+u+v, l\tau)$$

where $F(x) \in C^\infty(L(0,u))$ is determined by

$$\overline{\partial} F(x) = \overline{\theta_{a/k}(kx, k\tau)} \exp(-2\pi t k x_2^2)\theta_{b/k}(kx+u, k\tau).$$

Using the computation of the Fourier coefficients of the right hand side we find that

$$F(x) = \sum_{\substack{m,n, \\ m \equiv b-a \mod (k)}} a_{m,n}\varphi_{u,m,n}(x)$$

where

$$a_{m,n} = \frac{t}{\pi\sqrt{2tk}(m\tau - n + u)} \exp(-\frac{\pi}{2tk}(|m\tau - n|^2 + 2(m\overline{\tau} - n)u + u^2) + \frac{\pi i}{k}(mn + 2na)).$$

We have

$$\langle F(x)\theta_{c/l}(lx+v, l\tau), \theta_{d/l}(lx+u+v, l\tau)\rangle =$$

$$\int_{\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau} F(x)\theta_{c/l}(lx+v, l\tau)\overline{\theta_{d/l}(lx+u+v, l\tau)} \exp(-2\pi t(lx_2^2 + 2x_2(u_2+v_2)))dx_1 dx_2 =$$

$$\sum_{\substack{m,n, \\ m \equiv b-a \mod (k)}} a_{m,n}\overline{b_{m,n}}$$

where $b_{m,n}$ are the Fourier coefficients of $\theta_{d/l}(lx+u+v, l\tau)\overline{\theta_{c/l}(lx+v, l\tau)} \exp(-2\pi t(lx_2^2 + 2x_2v_2))$ (this is a C^∞-section of $L(0,u)$). We have $b_{m,n} = 0$ unless $m \equiv d - c \mod (l)$ in which case

$$b_{m,n} = \frac{1}{\sqrt{2tl}} \exp(-\frac{\pi}{2tl}(|m\tau - n|^2 + 2(m\overline{\tau} - n)(u+v) - 2(m\tau - n)\overline{v} + (u+v-\overline{v})^2) + \frac{\pi i n}{l}(m+2c)).$$

Denoting $w = u + v$ we obtain

$$\langle F(x)\theta_{c/l}(lx+v,l\tau),\theta_{d/l}(lx+u+v,l\tau)\rangle = \frac{1}{2\pi\sqrt{kl}} \times$$

$$\sum \frac{\exp(-\frac{\pi(k+l)}{2tkl}Q(\gamma,u)+\frac{2\pi i}{l}((u+n)w_2-mw_1)+\frac{2\pi t}{l}w_2^2+\pi in(\frac{m+2c}{l}-\frac{m+2a}{k}))}{\gamma+u}$$

where the sum is taken over $\gamma = m\tau + n \in \mathbb{Z}+\mathbb{Z}\tau$ such that $m \equiv b-a \mod (k)$ and $m \equiv d-c \mod (l)$, and

$$Q(\gamma,u) = |\gamma|^2 + 2\overline{\gamma}u + u^2$$

Thus, we get the following formula for the triple product:

$$m_3(\alpha,\beta_1,\beta_2) = \sum_{d\in\mathbb{Z}/d\mathbb{Z}} G^d_{a,b,c}(u,w)\theta_{d/l}(lx+w,l\tau)$$

where $\alpha = \frac{\pi\sqrt{2k}}{\sqrt{t}}\overline{\theta_{a/k}(kx,k\tau)}\exp(-2\pi tkx_2^2)d\overline{x} \in H^1(L(-k,0))$, $\beta_1 = \theta_{b/k}(kx+u,k\tau) \in H^0(L(k,u))$, $\beta_2 = \theta_{c/l}(lx+v,l\tau)) \in H^0(L(l,v))$,

$$G^d_{a,b,c}(u,w) =$$

$$\sum_{\substack{\gamma=m\tau+n,\\ m\equiv b-a(k),m\equiv d-c(l)}} \frac{\exp(-\frac{\pi(k+l)}{2tkl}Q(\gamma,u)+\frac{2\pi i}{l}((u+n)w_2-mw_1)+\pi in(\frac{m+2c}{l}-\frac{m+2a}{k})}{\gamma+u}$$

The function $G^d_{a,b,c}(u,w)$ is meromorphic in u with poles of order 1 at the lattice points and satisfies

$$G^d_{a,b,c}(u+1,w) = \exp(2\pi i(\frac{b}{k}-\frac{c}{l}))G^d_{a,b,c}(u,w) \tag{2.3.1}$$

$$G^d_{a,b,c}(u+\tau,w) = \exp(-\frac{\pi i(k+l)}{kl}(2u+\tau)+2\pi i\frac{w}{l})G^d_{a,b+1,c-1}(u,w) \tag{2.3.2}$$

2.4. Now let us compare the above triple product with the corresponding product in the Fukaya category. According to the dictionary from [9] the corresponding four special lagrangians in the symplectic torus $\mathbb{R}^2/\mathbb{Z}^2$ are $\Lambda_1 = (t,0)$, $\Lambda_2 = (t,-kt)$, $\Lambda_3 = (t,-u_2)$, $\Lambda_4 = (t,lt-w_2)$. The lagrangians Λ_1 and Λ_2 are equipped with trivial connections, while Λ_3 (resp. Λ_4) comes with the connection $-2\pi i u_1 dx$ (resp. $-2\pi i w_1 dx$). The morphisms α, β_1 and β_2 in the derived category of E_τ correspond to the intersection points $(\frac{a}{k},0) \in \Lambda_1\cap\Lambda_2$, $(\frac{u_2+b}{k},-u_2) \in \Lambda_2\cap\Lambda_3$, and $(\frac{w_2-u_2+c}{l},-u_2) \in \Lambda_3\cap\Lambda_4$ respectively, while $\theta_{d/l}(lx+w,l\tau)$ corresponds to the point $(\frac{w_2+d}{l},0) \in \Lambda_4\cap\Lambda_1$. Up to some factors which appear in

the definition of the functor in [9] the corresponding Fukaya product is given by

$$m_3'(\alpha,\beta_1,\beta_2)=\sum_{d\in\mathbb{Z}/d\mathbb{Z}} G^d_{a,b,c}(u,w)\theta_{d/l}(lx+w,l\tau)$$

where

$$F^d_{a,b,c}(u,w)=$$

$$-2\pi i\cdot\sum_{\substack{m\equiv b-a(k),\\ m\equiv d-c(l)}}\frac{\exp(\frac{\pi i(k+l)}{kl}(\tau m^2+2mu)-2\pi i\frac{mw}{l}+2\pi i(m\tau+u)(n_0-\frac{d}{l}+\frac{a}{k}))}{1-\exp(2\pi i(m\tau+u))}$$

where n_0 is the minimal integer n such that $n\geq\frac{w_2+d}{l}-\frac{a}{k}$. It is easy to see that the function $F^d_{a,b,c}(u,w)$ is meromorphic in u with poles of order 1 at the lattice points and satisfies (2.3.1) and (2.3.2). Notice that the residues of $F^d_{a,b,c}(u,w)$ and of $G^d_{a,b,c}(u,w)$ at $u=0$ are equal to 1. Therefore, for fixed w, a, and d the functions $H_{b,c}(u)=G^d_{a,b,c}(u,w)-F^d_{a,b,c}(u,w)$ are holomorphic in u and satisfy

$$H_{b,c}(u+1)=\exp(2\pi i(\frac{b}{k}-\frac{c}{l}))H_{b,c}(u) \tag{2.4.1}$$

$$H_{b,c}(u+\tau)=\exp(-\frac{\pi i(k+l)}{kl}(2u+\tau)+2\pi i\frac{w}{l})G_{b+1,c-1}(u) \tag{2.4.2}$$

Let r be the greatest common divisor of k and l. Iterating these equations kl/r times we obtain

$$H_{b,c}(u+\frac{kl}{r})=H_{b,c}(u) \tag{2.4.3}$$

$$H_{b,c}(u+\frac{kl}{r}\tau)=\exp(-\pi i\frac{(k+l)kl}{r^2}\tau-2\pi i\frac{(k+l)u-kw}{r})H_{b,c}(u) \tag{2.4.4}$$

The space of holomorphic functions in u satisfying (2.4.3) and (2.4.4) has the basis $e(s)=\theta_{s/N}(\frac{(k+l)u-kw}{r},\frac{(k+l)kl}{r^2}\tau)$ where $s\in\mathbb{Z}/N\mathbb{Z}$, $N=\frac{(k+l)kl}{r^2}$. Note that the action of the operator $f(u)\mapsto f(u+1)$ on this basis is diagonal, hence, the space of holomorphic functions satisfying (2.4.1) and (2.4.4) has as basis $e((b/k-c/l+p)\frac{kl}{r})$, where $p\in\mathbb{Z}/\frac{k+l}{n}\mathbb{Z}$. Thus, we should have

$$H_{b,c}(u)=\sum_p f_{b,c,p}e((b/k-c/l+p)\frac{kl}{r})$$

where p runs through $\mathbb{Z}/\frac{k+l}{n}\mathbb{Z}$. Note that in this formula we have to fix representatives for the residue classes b and c. A more convenient

point of view is the following. Consider the set T of triples of integers (b, c, p) modulo the equivalence relation generated by

$$(b, c, p) \equiv (b, c, p + \frac{k+l}{r}),$$
$$(b, c, p) \equiv (b + k, c, p - 1),$$
$$(b, c, p) \equiv (b, c + l, p + 1)$$

Forgetting p we get the projection $\phi_1 : T \to \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$. On the other hand, we have a well-defined map

$$\phi_2 : T \to \mathbb{Z}/N\mathbb{Z} : (b, c, p) \mapsto (\frac{b}{k} - \frac{c}{l} + p)\frac{kl}{r}$$

Now the above formula can be rewritten as

$$H_{b,c}(u) = \sum_{\sigma \in \phi_1^{-1}(b,c)} f_\sigma e(\phi_2(\sigma)).$$

Now taking into account the relation (2.4.2) we derive the relation $f_{b,c,p} = f_{b+1,c-1,p}$. Consider the action of $\mathbb{Z}$ on T generated by $(b, c, p) \mapsto (b + 1, c - 1, p)$. It is easy to see that the orbits of this action coincide with the fibers of the map

$$\phi_3 : T \to \mathbb{Z}/(k + l)\mathbb{Z} : (b, c, p) \mapsto b + c + kp.$$

Therefore, the coefficients f_σ depend only on $\phi_3(\sigma)$. Summarizing (and recalling the dependence on a, d and w) we can write

$$G^d_{a,b,c}(u, w) - F^d_{a,b,c}(u, w) = \sum_{\sigma \in \phi_1^{-1}(b,c)} f^d_{a,\phi_3(\sigma)}(w)\theta_{\phi_2(\sigma)/N}(\frac{(k + l)u - kw}{r}, \frac{(k + l)kl}{r^2}\tau). \quad (2.4.5)$$

2.5. Recall that the addition formula for theta-functions (see [7], prop. 6.4) gives the following identity

$$\theta_{b/k}(kx + u, k\tau)\theta_{c/l}(lx + v, l\tau) =$$
$$\sum_{p \in \mathbb{Z}/\frac{k+l}{r}\mathbb{Z}} \theta_{(\frac{b}{k} - \frac{c}{l} + p)\frac{r}{k+l}}(\frac{(k + l)u - kw}{r}, \frac{(k + l)kl}{r^2}\tau)\theta_{\frac{b+c+kp}{k+l}}((k + l)x + w, (k + l)\tau)$$

where $w = u + v$. In this formula one has to fix representatives of the residue classes b and c. Using the notation above we can rewrite it as follows

$$\theta_{b/k}(kx + u, k\tau)\theta_{c/l}(lx + v, l\tau) =$$
$$\sum_{\sigma \in \phi_1^{-1}(b,c)} \theta_{\frac{\phi_2(\sigma)}{N}}(\frac{(k + l)u - kw}{r}, \frac{(k + l)kl}{r^2}\tau)\theta_{\frac{\phi_3(\sigma)}{k+l}}((k + l)x + w, (k + l)\tau)$$

Comparing this with (2.4.5) we conclude that we can write the difference between the two types of triple product of the elements $\alpha \in H^1(L(-k,0))$, $\beta_1 \in H^0(L(k,u))$ and $\beta_2 \in H^0(L(l,v))$ in the form

$$m_3(\alpha,\beta_1,\beta_2) - m_3'(\alpha,\beta_1,\beta_2) = n_2(\alpha,\beta_1\beta_2)$$

where the linear map $n_2 : H^1(L(-k,0)) \otimes H^0(L(k+l,w)) \to H^0(L(l,w))$ is defined by the formula

$$n_2(\alpha, \theta_{\frac{q}{k+l}}((k+l)x+w,(k+l)\tau)) = \sum_{d\in\mathbb{Z}/l\mathbb{Z}} f_{a,q}^d \theta_{d/l}(lx+w,l\tau). \tag{2.5.1}$$

where $q \in \mathbb{Z}/(k+l)\mathbb{Z}$.

Note that we have

$$m_3(\beta_2,\beta_1,\alpha) = -\operatorname{pr}(\beta_2(\overline{\partial}^{-1}(\beta_1\alpha)) = -\operatorname{pr}(\overline{\partial}^{-1}(\alpha\beta_1)\beta_2) = -m_3(\alpha,\beta_1,\beta_2).$$

Similar formula holds for m_3' so we get a homotopy between $m_3(\beta_2,\beta_1,\alpha)$ and $m_3'(\beta_2,\beta_1,\alpha)$ by setting $n_2(\beta,\alpha) = -n_2(\alpha,\beta)$.

2.6. We claim that using A_∞-axioms one can express all transversal triple products between line bundles on E_τ in terms of the ones we computed in 2.3 and in terms of the triple products of the form

$$H^0(L(d,u_1)) \otimes H^1(L(-d,u_2)) \otimes H^0(L(d,u_3)) \to H^0(L(d,u_1+u_2+u_3))$$

where $u_1+u_2 \notin \mathbb{Z}+\mathbb{Z}\tau$, $u_2+u_3 \notin \mathbb{Z}+\mathbb{Z}\tau$. The latter triple products in both categories coincide as the univalued well-defined Massey products (see [8]).

First consider the triple product $m_3(r,s,t)$ where $r \in H^1(L(-d_1,u_1))$, $s \in H^0(L(d_2,u_2))$, $t \in H^0(L(d_3,u_3))$, where $d_i > 0$ and $d_2+d_3 > d_1$. If $d_2 = d_1$ and $u_1+u_2 \notin \mathbb{Z}+\mathbb{Z}\tau$ this product reduces to the one considered in 2.3 by a translation of argument. Otherwise, there are two possibilities:

a) $d_2 > d_1$. In this case we can write s as a linear combination of products $s's''$ where $s' \in H^0(L(d_1,u_2'))$, $s'' \in H^0(L(d_2-d_1,u_2''))$, $u_2'+u_2'' = u_2$. Moreover, we can assume that u_1+u_2' doesn't belong to the lattice $\mathbb{Z}+\mathbb{Z}\tau$. Then applying the A_∞-constraint to the quadruple r,s',s'',t we obtain that $m_3(r,s's'',t)$ is a linear combination of $m_3(r,s',s'')t$ and $m_3(r,s',s''t)$ (note that $rs'=0$ by assumption while $m_3(s',s'',t)=0$ as an element of $H^{-1}(L(d_2+d_3,u_2+u_3))$).

b) $d_2 < d_1$. In this case we write t as a linear combination of products $t't''$ where $t' \in H^0(L(d_1-d_2,u_3'))$, $t'' \in H^0(L(d_2+d_3-d_1,u_3''))$, $u_3 = u_3'+u_3''$. Moreover, we can assume that $u_1+u_2+u_3'$ doesn't belong to the lattice $\mathbb{Z}+\mathbb{Z}\tau$. This implies that $m_3(r,s,t')=0$. Hence, applying the A_∞-constraint to the quadruple r,s,t',t'' we get $m_3(r,s,t't'') = \pm m_3(rs,t',t'') \pm m_3(r,st',t'')$ (note that $m_3(s,t',t'')=0$).

One deals similarly with transversal products of the type

$$H^0(L(d_1,u_1))\otimes H^1(L(d_2,u_2))\otimes H^0(L(-d_3,u_3))\to H^0(L(d_1+d_2-d_3,u_1+u_2+u_3))$$

where $d_i>0$.

Now let us consider $m_3(r,s,t)$ where $r\in H^0(L(d_1,u_1))$, $s\in H^1(L(-d_2,u_2))$, $t\in H^0(L(d_3,u_3))$, $d_1+d_3>d_2$, $d_i>0$. If $d_1=d_3=1$ then we necessarily have $d_2=1$, hence this is a univalued Massey product. Otherwise, we can use recursion in $max(d_1,d_3)$. Indeed, if $d_1>1$ then we can write r as a linear combination of $r'r''$ where $r'\in H^0(L(1,u_1'))$, $r''\in H^0(L(d_1-1,u_2''))$. Now

$$m_3(r'r'',s,t)=\pm r'm_3(r'',s,t)\pm m_3(r',r''s,t)\pm m_3(r',r'',st)\pm m_3(r',r'',s)t$$

where the products $m_3(r',r'',st)$ and $m_3(r',r'',s)$ are of the type treated above. Similarly, if $d_3>1$ then we can write t as linear combination of $t't''$ to get a recursion formula for $m_3(r,s,t)$.

Finally, using the compatibility of m_3 with the Serre duality and the obvious cyclic symmetry of m_3' we can reduce all non-zero transversal higher products m_3 and m_3' to the ones considered above. For example, the product

$$H^1(L_1)\otimes H^1(L_2)\otimes H^0(L_3)\to H^1(L_1\otimes L_2\otimes L_3)$$

by Serre duality reduces to

$$H^1(L_2)\otimes H^0(L_3)\otimes H^0((L_1\otimes L_2\otimes L_3)^{-1})\to H^0(L_1^{-1}).$$

Using these formulas it is easy to see that the homotopy constructed in 2.5 induces homotopy in all the other cases of transversal triple products between the line bundles.

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