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Quadrisecants Give New Lower Bounds for the Ropelength of a Knot

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Abstract. Using the existence of a special quadrisecant line, we show the ropelength of any nontrivial knot is at least 15.66. This improves the previously known lower bound of 12. Numerical experiments have found a trefoil with ropelength less than 16.34, so our new bounds are quite sharp.

1. Introduction

The ropelength problem asks to minimize the length of a knotted curve subject to maintaining an embedded tube of fixed radius around the curve; this is a mathematical model of tying the knot tight in rope of fixed thickness.

More technically, the thickness $\tau(K)$ of a space curve K is defined [**GM**] to be twice the infimal radius $r(a, b, c)$ of circles through any three distinct points of K. It is known [**CKS**] that $\tau(K) = 0$ unless K is $C^{1,1}$, meaning that its tangent direction is a Lipschitz function of arclength. When K is C^1, we can define normal tubes around K, and then indeed $\tau(K)$ is the supremal diameter of such a tube that remains embedded. We note that in the existing literature thickness is sometimes defined to be the radius rather than diameter of this thick tube.

We define ropelength to be the (scale-invariant) quotient of length over thickness. Because this is semi-continuous even in the C^0 topology on closed curves, it is not hard to show [**CKS**] that any (tame) knot or link type has a ropelength minimizer.

Cantarella, Kusner and Sullivan gave certain lower bounds for the ropelength of links; these are sharp in certain simple cases where each component of the link is planar [**CKS**]. However, these examples are still the only known ropelength minimizers. Recent work [**CFKSW**] describes a much more complicated tight (ropelength-critical) configuration B_0 of the Borromean rings. (Although the somewhat different Gehring notion of thickness is used there, B_0 should still be tight, and presumably minimizing, for the ordinary ropelength we consider here.) Each component of B_0 is still planar, and it seems significantly more difficult to describe explicitly the shape of any tight (ropelength-critical) knot.

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Numerical experiments [**Pie, Sul, Raw**] suggest that the minimum ropelength for a trefoil is slightly less than 16.34, and that there is another tight trefoil with different symmetry and ropelength about 18.7. For comparison, numerical simulations of the tight figure-eight knot show ropelength just over 21. The best lower bound in [**CKS**] was 10.726; this was improved by Diao [**Dia**], who showed that any knot has ropelength more than 12 (meaning that "no knot can be tied in one foot of one-inch rope").

Here, we use the idea of quadrisecants, lines that intersect a knot in four distinct places, to get better lower bounds for ropelength. Almost 70 years ago, Pannwitz [**Pan33**] showed, using polygonal knots, that a generic representative of any nontrivial knot type must have a quadrisecant. Kuperberg [**Kup**] extended this result to all knots by showing that generic knots have essential (or topologically nontrivial) quadrisecants. We will define this precisely below, as essential quadrisecants are exactly what we need for our improved ropelength bounds. (Note that a curve arbitrarily close to a round circle, with ropelength thus near π, can have a nonessential quadrisecant.)

By comparing the orderings of the four points along the knot and along the quadrisecant, we distinguish three types of quadrisecants. For each of these types we use geometric arguments to obtain a lower bound for the ropelength of the knot having a quadrisecant of that type. The worst of these three bounds is 13.936.

In her doctoral dissertation [**Den**], Denne shows that any ($C^{1,1}$) knot has an essential quadrisecant of alternating type. This result, combined with our Theorem 9.4, shows that any nontrivial knot has ropelength at least 15.66.

Nontrivial links also necessarily have quadrisecants. We briefly consider ropelength bounds obtained for links with different types of quadrisecants. These provide another interpretation of the argument showing that the tight Hopf link has ropelength 4π, as in the Gehring link problem. But we have not found any way to improve the known ropelength estimates for other links.

2. Definitions and lemmas

Definition. A *knot* is an oriented simple closed curve K in $\mathbb{R}^3$. Any two points a and b on a knot K divide it into two complementary subarcs γ_{ab} and γ_{ba}. Here γ_{ab} is the arc from a to b following the given orientation on K. If $x \in \gamma_{ab}$, we will sometimes write $\gamma_{axb} = \gamma_{ab}$ to emphasize the order of points along K. We will use ℓ_{ab} to denote the length of γ_{ab}; by comparison, $|a-b|$ denotes the distance from a to b in space, the length of the segment $\overline{ab}$.

Definition. An *n-secant line* for a knot K is an oriented line in $\mathbb{R}^3$ whose intersection with K has at least n components. An *n-secant* is an ordered n-tuple of points in K (no two of which lie in a common straight subarc of K) which lie in order along an n-secant line. We will use *secant*, *trisecant* and *quadrisecant* to mean 2-secant, 3-secant and 4-secant, respectively. The *midsegment* of a quadrisecant *abcd* is the segment $\overline{bc}$.

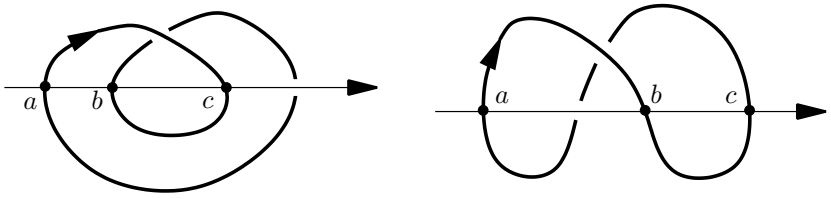


FIGURE 1. These trisecants *abc* are reversed (left) and direct (right) because the cyclic order of the points along K is *acb* and *abc*, respectively. Flipping the orientation of the knot or the trisecant would change its type.

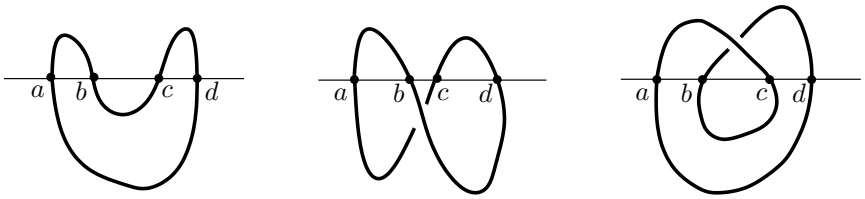


FIGURE 2. Here we see quadrisecants *abcd* on each of three knots. From left to right, these are simple, flipped and alternating, because the dihedral order of the points along K is *abcd*, *abdc* and *acbd*, respectively.

The orientation of a trisecant either agrees or disagrees with that of the knot. In detail, the three points of a trisecant *abc* occur in that linear order along the trisecant line, but may occur in either cyclic order along the oriented knot. (Cyclic orders are cosets of the cyclic group C_3 in the symmetric group S_3.) These could be labeled by their lexicographically least elements (*abc* and *acb*), but we choose to call them *direct* and *reversed* trisecants, respectively, as in Figure 1. Changing the orientation of either the knot or the trisecant changes its class. Note that *abc* is direct if and only if $b \in \gamma_{ac}$.

Similarly, the four points of a quadrisecant *abcd* occur in that order along the quadrisecant line, but may occur in any order along the knot K. Of course, the order along K is only a cyclic order, and if we ignore the orientation on K it is really just a dihedral order, meaning one of the three cosets of the dihedral group D_4 in S_4. Picking the lexicographically least element in each, we could label these cosets *abcd*, *abdc* and *acbd*. We will call the corresponding classes of quadrisecants *simple*, *flipped* and *alternating*, respectively, as in Figure 2. Note that this definition ignores the orientation of K, and switching the orientation of the quadrisecant does not change its type.

When discussing a quadrisecant *abcd*, we will usually choose to orient K so that $b \in \gamma_{ad}$. That means the cyclic order of points along K will be *abcd*, *acbd* or *abdc*, depending on the type of the quadrisecant.

In some sense, alternating quadrisecants are the most interesting. These have also been called NSNS quadrisecants, because if we view $\overrightarrow{cd}$ and $\overrightarrow{ab}$ as the North

and South ends of the midsegment $\overline{bc}$, then when *abcd* is an alternating quadrisecant, K visits these ends alternately NSNS as it goes through the points *acbd*. It was noted [**CKKS**] that the midsegment of an alternating quadrisecant for K is automatically in the *second hull* of K. Denne shows [**Den**] that any nontrivial knot has an alternating quadrisecant. Budney *et al.* [**BCSS**] have shown that the finite-type (Vassiliev) knot invariant of type 2 can be computed by counting alternating quadrisecants with appropriate multiplicity.

3. Knots with unit thickness

Because the ropelength problem is scale invariant, we find it most convenient to rescale any knot K to have thickness (at least) 1. This implies that K is a $C^{1,1}$ curve with curvature bounded above by 2.

For any point $a \in \mathbb{R}^3$, let $B(a)$ denote the open unit ball centered at a. Our first lemma, about the local structure of a thick knot, is by now standard. (Compare [**Dia**, Lem. 4] and [**CKS**, Lem. 5].)

LEMMA 3.1. *Let K be a knot of unit thickness. If $a \in K$, then $B(a)$ contains a single unknotted arc of K; this arc has length at most π and is transverse to the nested spheres centered at a. If ab is a secant of K with $|a-b| < 1$, then the ball of diameter $\overline{ab}$ intersects K in single unknotted arc (either γ_{ab} or γ_{ba}) whose length is at most* $\arcsin|a-b|$.

PROOF. If there were an arc of K tangent at some point c to one of the spheres around a within $B(a)$, then triples near (a, c, c) would have radius less than $\frac{1}{2}$. For the second statement, if K had a third intersection point c with the sphere of diameter $\overline{ab}$, then we would have $r(a, b, c) < \frac{1}{2}$. The length bounds come from Schur's lemma. □

An immediate corollary is:

COROLLARY 3.2. *If K has unit thickness, $a, b \in K$ and $p \in \gamma_{ab}$ with $a, b \notin B(p)$ then the complementary arc γ_{ba} lies outside $B(p)$.* □

The following lemma should be compared to [**Dia**, Lem. 5] and [**CKS**, Lem. 4], but here we give a slightly stronger version with a new proof.

LEMMA 3.3. *Let K be a knot of unit thickness. If $a \in K$, then the radial projection of $K \setminus \{a\}$ to the unit sphere $\partial B(a)$ is 1-Lipschitz, i.e., it does not increase length.*

PROOF. Consider what this projection does infinitesimally near a point $b \in K$. Let $d = |a - b|$ and let θ be the angle at b between K and the segment $\overline{ab}$. The projection stretches by a factor $1/d$ near b, but does not see the radial part of the tangent vector to K. Thus the local Lipschitz constant on K is $(\sin\theta)/d$. Now consider the circle through a and tangent to K at b. Plane geometry (see Figure 3) shows that its radius is $r = d/(2\sin\theta)$, but it is a limit of circles through three points of K, so by the three-point characterization of thickness, r is at least $\frac{1}{2}$; that is, the Lipschitz constant $1/2r$ is at most 1. □

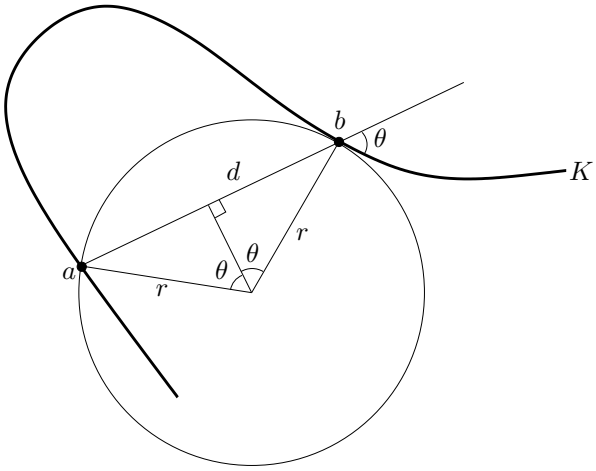


FIGURE 3. In the proof of Lemma 3.3, the circle tangent to K at b and passing through $a \in K$ has radius $r = d/(2\sin\theta)$ where $d = |a - b|$ and θ is the angle at b between K and $\overrightarrow{ab}$.

COROLLARY 3.4. *Suppose K has unit thickness, and $p, a, b \in K$ with $p \notin \gamma_{ab}$. Let $\angle apb$ be the angle between the vectors $a - p$ and $b - p$. Then $\ell_{ab} \geq \angle apb$. In particular, if apb is a reversed trisecant in K, then $\ell_{ab} \geq \pi$.*

PROOF. By the lemma, ℓ_{ab} is at least the length of the projected curve on $B(p)$, which in turn is at least the spherical distance $\angle apb$ between its endpoints. For a trisecant apb we have $\angle apb = \pi$, and $p \notin \gamma_{ab}$ exactly when the trisecant is reversed. □

Observe that a quadrisecant $abcd$ includes four trisecants: abc, abd, acd and bcd. Simple, flipped and alternating quadrisecants have different numbers of reversed trisecants. We can apply Corollary 3.4 to these trisecants to get simple lower bounds on ropelength for any curve with a quadrisecant.

THEOREM 3.5. *The ropelength of a knot with a simple, flipped or alternating quadrisecant is at least π, 2π or 3π, respectively.*

PROOF. Rescale K to have unit thickness, so its ropelength equals its length ℓ. Let $abcd$ be the quadrisecant, and orient K in the usual way, so that $b \in \gamma_{ad}$. In the case of a simple quadrisecant, the trisecant dba is reversed, so $\ell \geq \ell_{da} \geq \pi$, using Corollary 3.4. In the case of a flipped quadrisecant, the trisecants cba and bcd are reversed, so $\ell \geq \ell_{ca} + \ell_{bd} \geq 2\pi$. In the case of an alternating quadrisecant, the trisecants abc, bcd and dca are reversed, so $\ell \geq \ell_{ac} + \ell_{bd} + \ell_{da} \geq 3\pi$. □

Of course, any closed curve has ropelength at least π, independent of whether it is knotted or has any quadrisecants, because its total curvature is at least 2π. But curves arbitrarily close (in the C^1 or even C^∞ sense) to a round circle can have simple quadrisecants, so at least the first bound in the theorem above is sharp.

Although Kuperberg has shown that any nontrivial (tame) knot has a quadrisecant, and Denne shows that in fact it has an alternating quadrisecant, the bounds from Theorem 3.5 are not as good as the previously known bounds of [**Dia**] or even [**CKS**]. To improve our bounds, in Section 5 we will consider Kuperberg's notion of an essential quadrisecant.

4. Length bounds in terms of segment lengths

Given a thick knot K with quadrisecant *abcd*, we can bound its length in terms of the distances along the quadrisecant line. Throughout this paper, we will abbreviate these three distances as $r := |a-b|$, $s := |b-c|$ and $t := |c-d|$. We start with some lower bounds for r, s and t for quadrisecants of certain types.

LEMMA 4.1. *If abcd is a flipped quadrisecant for a knot of unit thickness, then the midsegment has length $s \geq 1$. Furthermore if $r \geq 1$ then the whole arc γ_{ca} lies outside $B(b)$; similarly if $t \geq 1$ then γ_{bd} lies outside $B(c)$.*

PROOF. Orient the knot in the usual way. If $s = |b-c| < 1$, then by Lemma 3.1 either $\ell_{cab} < \pi/2$ or $\ell_{bdc} < \pi/2$. But since *cba* and *bcd* are reversed trisecants, we have $\ell_{ca} \geq \pi$ and $\ell_{bd} \geq \pi$. This is a contradiction because $\ell_{cab} = \ell_{ca} + \ell_{ab}$ and $\ell_{bdc} = \ell_{bd} + \ell_{dc}$. The second statement follows immediately from Corollary 3.2. □

LEMMA 4.2. *If abcd is an alternating quadrisecant for a knot of unit thickness, then $r \geq 1$ and $t \geq 1$. With the usual orientation, the entire arc γ_{da} thus lies outside of $B(b) \cup B(c)$. If $s \geq 1$ as well, then γ_{ac} lies outside of $B(b)$ and γ_{bd} lies outside of $B(c)$.*

PROOF. If $r = |a-b| < 1$, then by Lemma 3.1 either $\ell_{acb} < \pi/2$ or $\ell_{bda} < \pi/2$. Similarly, if $t = |c-d| < 1$, then either $\ell_{cbd} < \pi/2$ or $\ell_{dac} < \pi/2$. But as in the proof of Theorem 3.5, we have $\ell_{ac} \geq \pi$ and $\ell_{bd} \geq \pi$, contradicting any choice of the inequalities above. Thus we have $r, t \geq 1$. Because a and d are outside $B(b)$ and $B(c)$, the remaining statements follow from Corollary 3.2. □

As suggested by the discussion above, we will often find ourselves in the situation where we have an arc of a knot known to stay outside a unit ball $B(x)$. We can compute exactly the minimum length of such an arc in terms of the following functions.

DEFINITION. For $r \geq 1$, let $f(r) := \sqrt{r^2-1} + \arcsin(1/r)$. For $r, s \geq 1$ and $\theta \in [0, \pi]$, the minimum length function is defined by

$$m(r,s,\theta) := \begin{cases} \sqrt{r^2+s^2-2rs\cos\theta} & \text{if } \theta \leq \arccos(1/r) + \arccos(1/s) \\ f(r) + f(s) + (\theta - \pi) & \text{if } \theta \geq \arccos(1/r) + \arccos(1/s) \end{cases}.$$

The function $f(r)$ will arise again in other situations. The function m was defined exactly to make the following bound sharp:

LEMMA 4.3. *Any arc γ from a to b staying outside $B(p)$ has length at least $m(|a-p|, |b-p|, \angle apb)$.*

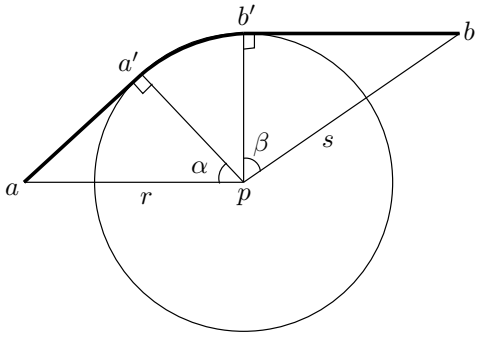


FIGURE 4. If points a, b are at distances $r, s \geq 1$ (respectively) from p, then the shortest curve from a to b avoiding $B(p)$ is planar. Either it is a straight segment or (in the case illustrated) it includes an arc of $\partial B(p)$. In either case, its length is $m(r, s, \angle apb)$.

PROOF. Let $r := |a - p|$ and $s := |b - p|$ be the distances to p (with $r, s \geq 1$) and let $\theta := \angle apb$ be the angle between $a - p$ and $b - p$. The shortest path from a to b staying outside $B := B(p)$ either is the straight segment or is the C^1 join of a straight segment from a to ∂B, a great-circle arc in ∂B, and a straight segment from ∂B to b. In either case, we see that the path lies in the plane through a, p, b (shown in Figure 4). In this plane, draw the lines from a and b tangent to ∂B. Let $\alpha := \angle apa'$ and $\beta := \angle bpb'$, where a' and b' are the points of tangency. Then $\cos\alpha = 1/r$ and $\cos\beta = 1/s$. Clearly if $\alpha + \beta \geq \theta$ then the shortest path is the straight segment from a to b, with length $\sqrt{r^2 + s^2 - 2rs\cos\theta}$. If $\alpha + \beta \leq \theta$ then the shortest path consists of the C^1 join described above, with length

$$\sqrt{r^2 - 1} + \big(\theta - (\alpha + \beta)\big) + \sqrt{s^2 - 1} = f(r) + f(s) + (\theta - \pi).$$

□

An important special case is when $\theta = \pi$. Here we are always in the case $\alpha + \beta \leq \theta$, so we get:

COROLLARY 4.4. *If a and b lie at distances r and s along opposite rays from p (so that $\angle apb = \pi$) then the length of any arc from a to b avoiding $B(p)$ is at least*

$$f(r) + f(s) = \sqrt{r^2 - 1} + \arcsin(1/r) + \sqrt{s^2 - 1} + \arcsin(1/s).$$

□

We note that the special case of this formula when $r = s$ also appears in recent papers [**EGK, DGR**] investigating the geometric dilation of planar graphs. Gromov had given a lower bound for the distortion of a closed curve (see [**KS**]); in [**EGK, DGR**] sharper bounds in terms of the diameter and width of the curve are derived using this minimum-length arc avoiding a ball. (Although the bounds are stated there only for plane curves they apply equally well to space curves.)

LEMMA 4.5. *Let $abcd$ be an alternating quadrisecant for a knot of unit thickness (oriented in the usual way). Let $r := |a - b|$, $s := |b - c|$ and $t := |c - d|$ be the*

lengths of the segments along abcd. Then $\ell_{ad} \geq f(r) + s + f(t)$. *The same holds if abcd is a simple quadrisecant as long as* $r, t \geq 1$.

PROOF. In either case (and as we already noted in Lemma 4.2 for the alternating case) we find, using Corollary 3.2, that γ_{da} lies outside of $B(b) \cup B(c)$. As in the proof of Lemma 4.3, the shortest arc from d to a outside these balls will be the C^1 join of various pieces: these alternate between straight segments in space and great-circle arcs in the boundaries of the balls. Here, the straight segment in the middle always has length exactly $s := |b - c|$. As in Corollary 4.4, the overall length is then at least $f(r) + s + f(t)$ as desired. □

5. Essential secants

We have seen that the existence of a quadrisecant for K is not enough to get good lower bounds on ropelength, because some quadrisecants do not capture the knottedness of K. Kuperberg [**Kup**] introduced the notion of *essential* secants and quadrisecants (which he called "topologically nontrivial"). We will see below that these give us much better ropelength bounds.

We extend Kuperberg's definition to say when an arc γ_{ab} of a knot K is essential, capturing part of the knottedness of K. Generically, the knot K together with the segment $S = \overline{ab}$ forms a knotted Θ-graph in space (that is, a graph with three edges connecting the same two vertices). To adapt Kuperberg's definition, we consider such knotted Θ-graphs.

DEFINITION. Suppose α, β and γ are three disjoint simple arcs from p to q, forming a knotted Θ-graph. Then we say that the ordered pair (α, β) is *inessential* if there is a disk D bounded by the knot $\alpha \cup \beta$ having no interior intersections with the knot $\alpha \cup \gamma$. (We allow self-intersections of D, and interior intersections with β; the latter are certainly necessary if $\alpha \cup \beta$ is knotted.)

An equivalent definition, illustrated in Figure 5, is as follows: Let $X := \mathbb{R}^3 \setminus (\alpha \cup \gamma)$, and consider a parallel curve δ to $\alpha \cup \beta$ in X. Here by *parallel* we mean that $\alpha \cup \beta$ and δ cobound an annulus embedded in X. We choose the parallel to be homologically trivial in X. (Since the homology of the knot complement X is $\mathbb{Z}$, this simply means we take δ to have linking number zero with $\alpha \cup \gamma$. This determines δ uniquely up to homotopy.) Let $h(\alpha, \beta) \in \pi_1(X)$ denote the (free) homotopy class of δ. Then (α, β) is *inessential* if $h(\alpha, \beta)$ is trivial.

We say that (α, β) is *essential* if it is not inessential, meaning that $h(\alpha, \beta)$ is nontrivial. Now let λ be a meridian loop (linking $\alpha \cup \gamma$) in the knot complement X. If the commutator $[\lambda, h(\alpha, \beta)]$ is nontrivial then we say (α, β) is *strongly essential*.

This notion is clearly a topological invariant of the (ambient isotopy) class of the knotted Θ-graph. For an introduction to the theory of knotted graphs, see [**Kau**]; note that since the vertices of the Θ-graph have degree three, in our situation there is no distinction between what Kauffman calls topological and rigid vertices. The three arcs approaching one vertex can be braided arbitrarily without affecting the topological type of the knotted Θ-graph.

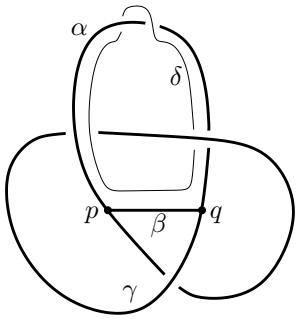


FIGURE 5. In this knotted Θ-graph $\alpha \cup \beta \cup \gamma$, the ordered pair (α, β) is essential. To see this, we find the parallel $\delta = h(\alpha, \beta)$ to $\alpha \cup \beta$ which has linking number zero with $\alpha \cup \gamma$, and note that it is homotopically nontrivial in the knot complement $\mathbb{R}^3 \setminus (\alpha \cup \gamma)$. In this illustration, β is the straight segment $\overline{pq}$, so we equally say that the arc α of the knot $\alpha \cup \gamma$ is essential.

LEMMA 5.1. *In a knotted Θ-graph $\alpha \cup \beta \cup \gamma$, the pair (α, β) is strongly essential if and only if (γ, β) is.*

PROOF. Here we should be more careful about basepoints and orientations for the homotopy classes $h(\alpha, \beta)$ and $h(\gamma, \beta)$. So fix a basepoint near p and orient both these loops to follow β outwards and then return backwards along α or γ. Note that the product $h^{-1}(\alpha, \beta)h(\gamma, \beta)$ is homotopic to a parallel of the knot $\alpha \cup \gamma$. Since a torus has abelian fundamental group, this parallel commutes with the meridian λ. It follows that $[\lambda, h(\alpha, \beta)] = [\lambda, h(\gamma, \beta)]$. □

This commutator $[\lambda, h(\alpha, \beta)]$ that comes up in the definition of strongly essential will later be referred to as the *loop l_β along β*; it can be represented by a curve which follows a parallel β' of β, then loops around $\alpha \cup \gamma$ along a meridian near q, then follows β'^{-1}, then loops backwards along a meridian near p. (See Figure 6.)

We apply the definition of essential to arcs of a knot as follows.

DEFINITION. If K is a knot and $a, b \in K$, let $S = \overline{ab}$. Assuming S has no interior intersections with K, we say that γ_{ab} is *(strongly) essential* for K if (γ_{ab}, S) is (strongly) essential in the knotted Θ-graph $K \cup S$. If S does intersect K, then we say γ_{ab} is *(strongly) essential* if for any $\varepsilon > 0$ there exists some ε-perturbation of S (in the C^1 sense, with endpoints fixed) to a curve S' such that $K \cup S'$ is an embedded Θ-graph in which (γ_{ab}, S') is (strongly) essential.

Note that Lemma 5.1 shows that γ_{ab} is strongly essential if and only if γ_{ba} is. The second part of the definition ensures that the set of essential secants is closed in the set of all secants of K.

In [**CKKS**] it was shown that if K is an unknot, then any arc γ_{ab} is inessential. In our context, this follows immediately, because the homology and homotopy groups of $X := \mathbb{R}^3 \setminus K$ are equal for an unknot, so any curve δ having linking

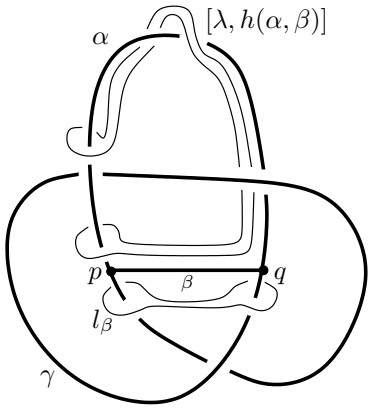


FIGURE 6. If λ is a meridian curve linking $\alpha \cup \gamma$, then the commutator $[\lambda, h(\alpha, \beta)]$ is homotopic to the loop l_β along β.

number zero with K is homotopically trivial in X. We can use Dehn's lemma to prove a converse statement:

THEOREM 5.2. *If $a, b \in K$ and both γ_{ab} and γ_{ba} are inessential, then K is unknotted.*

PROOF. Let S be the secant $\overline{ab}$, perturbed if necessary to avoid interior intersections with K. We know that $\gamma_{ab} \cup S$ and $\gamma_{ba} \cup S$ bound disks whose interiors are disjoint from K. Glue these two disks together along S to form a disk D spanning K. This disk may have self intersections, but these occur away from K, which is the boundary of D. By Dehn's lemma, we can replace D by an embedded disk, so K is unknotted. □

DEFINITION. A secant ab of K is *essential* if both subarcs γ_{ab} and γ_{ba} are essential. A secant ab is *strongly essential* if γ_{ab} (or, equivalently, γ_{ba}) is strongly essential.

To call a quadrisecant $abcd$ essential, we could follow Kuperberg and require that the secants ab, bc and cd all be essential. But instead, depending on the order type of the quadrisecant, we require this only of those secants whose length could not already be bounded as in Theorem 3.5, namely those secants whose endpoints are consecutive along the knot. That is, for simple quadrisecants, all three secants must be essential; for flipped quadrisecants the end secants ab and cd must be essential; for alternating quadrisecants, the middle secant bc must be essential. Figure 7 shows a knot with essential and inessential quadrisecants.

More formally, we can give the following definition for any n-secant.

DEFINITION. An n-secant $a_1 a_2 \cdots a_n$ is *essential* if, for each secant $a_i a_{i+1}$ where one of the arcs $\gamma_{a_i a_{i+1}}$ and $\gamma_{a_{i+1} a_i}$ includes no other a_j, we have $a_i a_{i+1}$ essential.

Kuperberg introduced the notion of essential secants and showed:

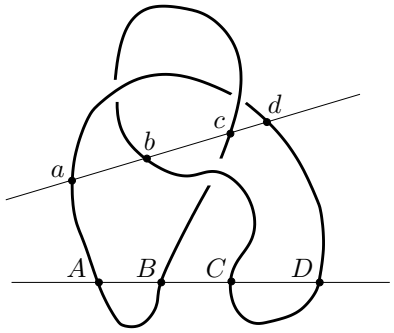


FIGURE 7. This trefoil knot has two quadrisecants. Quadrisecant $abcd$ is alternating and essential (meaning that bc is essential, although here in fact also ab and cd are essential). Quadrisecant $ABCD$ is simple and inessential, since AB and CD are inessential (although BC is essential).

THEOREM 5.3. *If K is a nontrivial knot parameterized by a generic polynomial, then K has an essential quadrisecant.* □

Since he did not distinguish the different order types, Kuperberg actually obtained a quadrisecant $abcd$ where all three segments ab, bc and cd are essential. This is more than we will need here. Thus we have given a weaker definition of essential for flipped and alternating quadrisecants, so these will be easier to produce.

Kuperberg used the fact that a limit of essential quadrisecants must still be a quadrisecant in order to show that every knot has a quadrisecant. Since we want an essential quadrisecant for every knot, we need to show that being essential is preserved in such limits. This will be a corollary of a more general theorem below, which says that by showing that knotted graphs that are nearby in a C^1 sense are necessarily ambient isotopic.

6. Convergence of knotted graphs

To know that essential secants stay essential in a limit, we need to know something about convergence of knotted Θ-graphs. Because the proof is hardly more difficult, we state a more general result: any two knotted graphs of finite total curvature which are sufficiently close in an appropriate norm must be ambient isotopic.

Milnor [**Mil50**] defined the total curvature of an arbitrary curve as the supremal total curvature of inscribed polygons. Curves of finite total curvature are exactly those rectifiable curves whose unit tangent vector has bounded variation, so they have curvature well-defined as a Radon measure. (Compare the discussion in [**CFKSW**].) Milnor showed any knotted curve of finite total curvature has an ambient-isotopic inscribed polygon; in extending this result to curves in Hadamard spaces, Alexander and Bishop [**AB98**] filled a slight gap in Milnor's argument. Here we use a similar argument to show the same holds for knotted graphs (with topological, or nonrigid, vertices).

A knotted graph of finite total curvature is a graph (of fixed finite combinatorial type) embedded in space such that each arc has finite total curvature. Any graph Γ of finite total curvature has well-defined one-sided tangent vectors everywhere; these are equal and opposite except at countably many *corners* of Γ (and of course at the vertices of the graph).

We will prove that if Γ is a knotted graph of finite total curvature, then any graph which is sufficiently C^1-close to Γ is isotopic to Γ. Here, convergence in C^1 (for graphs of a fixed combinatorial type) means convergence in sup norm (for some choice of parametrizations) both of the curves themselves and of their one-sided tangent vectors.

LEMMA 6.1. *Let B be a ball centered at p, and suppose Γ and Γ' each consist of n labeled arcs starting at p and proceeding out to ∂B transverse to the nested spheres around p. Then Γ and Γ' are ambient isotopic. In fact, any isotopy of ∂B which takes the n points of Γ to those of Γ' can be extended to an isotopy of the whole ball.*

PROOF. It suffices to show that Γ is isotopic to a configuration with n straight radial segments to the same n points on ∂B. For then the same is true of Γ', and any two straight configurations are clearly isotopic (for instance by extending radially any given boundary isotopy).

To straighten Γ, we comb the n strands inwards from ∂B, which we identify with the unit sphere. For $0 < \lambda \leq 1$, let $\lambda p_i(\lambda)$ be the n points of intersection of Γ with the sphere $\lambda \partial B$. (Thus the p_i are points on the unit sphere.) Start with f_1 being the identity map, and define a family of maps f_λ from the unit sphere to itself, continuous in λ, such that f_λ takes $p_i(1)$ to $p_i(\lambda)$ for each i. The combing isotopy, at each time t, maps each concentric sphere $\lambda \partial B$ to itself, using the map $f_{\lambda+(1-\lambda)t} \circ f_\lambda^{-1}$. Thus at time t on the sphere of size λ, we see the same picture that was initially on the sphere of size $\lambda + (1-\lambda)t$; it follows that at time 1 the strands are all straight. □

Note that if two (or more) strands of Γ leave p with a common tangent line, then they might twist (or braid) around each other infinitely often. But outside any sphere $\lambda \partial B$ they can twist only finitely many times, so the isotopy constructed in the lemma has no trouble combing this out.

THEOREM 6.2. *Suppose Γ is a knotted graph of finite total curvature. Then any Γ' which is sufficiently close to Γ in the C^1 sense is ambient isotopic to Γ.*

PROOF. We begin by selecting a finite number of points p_j on Γ (including all its vertices) such that these points divide Γ into arcs α_k each of total curvature less than $\pi/17$. (Note that any corner in Γ of turning angle at least $\pi/17$ must also be included among the p_j.)

Let δ be the minimum distance between any two arcs α_k which are not incident to a common p_j. (In particular, no two of the p_j are at distance less than δ.) Consider disjoint open balls B_j of radius $\delta/17$ centered at the p_j. Each arc α_k leaving p_j proceeds monotonically outwards to the boundary of B_j (since its curvature is

too small to double back), and B_j contains no other arcs (since δ was chosen small enough).

Note that $\Gamma \setminus \bigcup B_j$ is a compact union of disjoint arcs $\beta_k \subset \alpha_k$. Let $\varepsilon \leq 2\delta/17$ be the minimum distance between any two of these arcs β_k. For each β_k, consider a tube T_k foliated by parallel disks of radius $\varepsilon/17$ centered at the points of β_k and lying in planes normal to the vector between the ends of β_k. The disks at the end of these tubes lie almost (but not quite) in the boundaries of the balls B_j. But around each p_j we can fit a smooth (not quite round) sphere $\tilde{S}_j$ through these disks and lying close to ∂B_j. This sphere encloses a topological ball $\tilde{B}_j$ which is foliated by homothetic copies of $\tilde{S}_j$ shrinking down around p_j; these spheres are close enough to round that they are still transverse to Γ.

We have thus found a neighborhood N of Γ which is foliated by smooth spheres near the p_j and flat disks near the β_k; the graph Γ is transverse to this foliation F.

Now consider a nearby Γ'. If it is sufficiently close in the C^1 sense, corresponding points will be within $\varepsilon/17^2$, and corresponding one-sided tangent vectors will be within an angle $\pi/17$. Thus Γ' lies within this neighborhood N and it is transverse to the foliation F, except perhaps near the p_j. To find an ambient isotopy from Γ' to Γ, we proceed in two stages.

First we find the points $p'_j \in \Gamma'$ corresponding to $p_j \in \Gamma$, and construct an isotopy I supported in the $\tilde{B}_j$ as follows: the map is linear in time, moving each point on a straight line; p'_j moves to p_j, each boundary point $q \in \tilde{S}_j$ is fixed, and the map is linear on each segment $\overline{p_j q}$. Because p'_j was so close to p_j compared to the size of $\tilde{B}_j$, the resulting $I_1(\Gamma')$ at time 1 is still C^1-close to Γ, and is now transverse to the foliation F.

Second, we construct an isotopy J which is supported in N and which preserves each leaf of the foliation F. On the disks in the tubes around the β_k this is easy: we have to move one given point in the disk to the center, and can choose to do this in a continuous way. On the boundary $\tilde{S}_j$ we define J to match these motions in the disks and fix the rest of the sphere. Finally, we use Lemma 6.1 to fill in the isotopy in the interior of the balls. □

Note that the several constants 17 used in this proof were arbitrary and are far from sharp. For smooth curves K, thickness gives a good notion of how close another curve must be in order to be isotopic. For general graphs Γ of finite total curvature, we have not attempted to find an analogous notion, but have been content to simply prove above that sufficiently close graphs are isotopic.

We expect that the isotopy guaranteed by this theorem can be chosen to be small—that is, close to the identity—in the sense that the C^1 distance between any curve and its image is no more than a constant times that between the original graphs Γ and Γ'. This might follow from the next lemma:

LEMMA 6.3. *Suppose an ambient isotopy is given for $t \in [0,1]$ by $I_t(p) = p + tf(p)$, where $f : \mathbb{R}^3 \to \mathbb{R}^3$ has one-sided directional derivatives $\partial_v f$ everywhere, is λ-Lipschitz and is bounded in norm by δ. Then for any graph Γ, the image*

$\Gamma' := I_1(\Gamma)$ is C^1-close to Γ: corresponding points are within distance δ and the one-sided tangent vectors differ in angle by at most λ.

PROOF. Clearly each point $p \in \Gamma$ is within distance δ of $p' := I_1(p)$. If v is a one-sided unit tangent vector to Γ at p, then its image, a one-sided tangent vector to Γ' at p', is $v + \partial_v f$, where $|\partial_v f| < \lambda$ by the Lipschitz condition. This differs in angle from v by at most $\arctan \lambda < \lambda$. □

It is unclear, however, whether the construction we gave above in the proof of Theorem 6.2 has this property near the points p_i. Luckily, below we will only need the simplest special case of this result:

PROPOSITION 6.4. *Suppose that K is a knot of thickness $\tau > 0$ and K' is a knot of finite total curvature which is C^1-close to K in the sense that for some $\varepsilon < 1/12$, corresponding points are within distance $\varepsilon\tau$ and corresponding one-sided tangent vectors are within angle ε. Then there is an ambient isotopy I taking K' to K, which satisfies the hypotheses of Lemma 6.3 with $\lambda := 8\varepsilon$ and $\delta := 1.01\varepsilon\tau$.*

PROOF. First note that K is $C^{1,1}$ and scale so that K has thickness $\tau = 1$. Clearly K' lies within the thick tube around K and is transverse to the foliation by normal disks to K. It seems natural to build an isotopy disk-by-disk, but if K is not piecewise C^2 the normal disks might not form a piecewise C^1 foliation, so the resulting isotopy would not necessarily have the one-sided derivatives we need.

Instead, choose a finite number of points p_i along K which are spaced at arclength less than ε. Build a tube around K foliated by disks D_p of radius $2/3$ centered at points $p \in K$; at the points p_i use normal disks but between p_i and p_{i+1} use instead disks rotating at a constant speed. These disks are within angle $\varepsilon < 1/12$ of normal, and their union T includes the tube of radius $1/2$ around K.

Now the knot K' is contained in T and transverse to the D_p. We define the isotopy I to be linear in time, with I_0 the identity and I_1 mapping each D_p to itself. Consider a point $q' \in K'$. We know there is a corresponding point $q \in K$ within distance ε and with tangent vector within angle ε of those at q'. But if $p \in K$ is the point such that $q' \in D_p$, then $|p - q'| < \varepsilon \text{arcsec}(12) < \delta$. By considering also the nearest point $p^* \in K$ to q', trigonometric calculations show that the tangent vector at p is within angle 1.1ε of that at q and thus within 2.1ε of the one-sided tangent vectors at q'. It follows that K' is almost normal to D_p, within angle 3.1ε.

We now build the map $I_1(x) = x + f(x)$ on each D_p by fixing the boundary, moving q' to p, and mapping each line segment $q'p'$ with $p' \in \partial D_p$ linearly to the line segment pp'. That is, $f(x) = g(x)(p - q')$ where $g(x)$ is the scalar function which is 1 at q' and 0 on ∂D_p and linear on each segment from q' to the boundary. Thus g has Lipschitz constant at most $1/(\frac{2}{3} - \varepsilon) < 1.98$ and on this fixed D_p, the Lipschitz constant of f is thus at most $1.98\delta < 2\varepsilon$.

To find the overall Lipschitz constant for f, next consider its values $f = p - q'$ along the curve K'. This function along K' is 6ε-Lipschitz, because both p and q' move almost normal to the disks D_p (within angle ε and 3.1ε, respectively), and the disks rotate at most at unit speed while $|p - q'| < \varepsilon$. Writing any variation

direction as a sum of components within D_p and along K', we find that f has Lipschitz constant at most 8ε at points along K'. At other points in the tube, normal variation might move somewhat more quickly through the disks D_p, but the coefficient g drops off towards zero and more than compensates for this. □

We have now developed all the technical machinery we need to prove that limits of essential arcs remain essential.

COROLLARY 6.5. *If $C^{1,1}$ knots K_i have essential arcs $\gamma_{a_ib_i}$, and if the K_i converge (in C^1) to some limit K with $a_i \to a$ and $b_i \to b$, then the arc γ_{ab} is essential for K.*

PROOF. First assume that $S := \overline{ab}$ has no interior intersections with K. Then $K \cup S$ is a knotted Θ-graph. If any $S_i := \overline{a_ib_i}$ has an interior intersection with K_i, perturb it slightly as in the definition of essential. Now $K_i \cup S_i$ approach $K \cup S$ in the C^1 sense, so by Theorem 6.2 they are eventually isotopic to $K \cup S$. Since $\gamma_{a_ib_i}$ are essential, the same is true of γ_{ab} in the isotopic Θ-graph.

If S does have interior intersections with K, we proceed in a similar way. The (perturbed if necessary) Θ-graphs $K_i \cup S_i$ have $\gamma_{a_ib_i}$ as essential arcs. By Proposition 6.4, the K_i are (eventually) isotopic to K with small isotopies I_i. In particular, if we let S_i' be the image of S_i under the time-one map of I_i then by Lemma 6.3, the S_i' approach S in the C^1 sense. Since $K_i \cup S_i$ is isotopic to its image $K \cup S_i'$, the arc γ_{ab} is essential in $K \cup S_i'$. Our definition of essential allows exactly such perturbations of $K \cup S$ in this case. □

COROLLARY 6.6. *Every nontrivial $C^{1,1}$ knot has an essential quadrisecant.*

PROOF. Any $C^{1,1}$ knot K is a limit of polynomial knots, which can be taken to be generic in the sense of [**Kup**]. Then by Theorem 5.3 they have essential quadrisecants. Some subsequence of these quadrisecants converges to a quadrisecant for K, which is essential by Corollary 6.5. □

7. Arcs becoming essential

We showed in Corollary 6.6 that every nontrivial $C^{1,1}$ knot has an essential quadrisecant. Our aim is to find the least length of an essential arc γ_{pq} for a knot K, and use this to get better lower bounds on the ropelength of knots. This leads us to consider what happens when arcs change from inessential to essential.

THEOREM 7.1. *Suppose γ_{ac} is in the boundary of the set of essential arcs for a knot K. (That is, γ_{ac} is essential, but there are inessential arcs of K with endpoints arbitrarily close to a and c.) Then K must intersect the interior of segment $\overline{ac}$, and in fact there is some essential trisecant abc.*

PROOF. If K did not intersect the interior of segment $\overline{ac}$ (in a component separate from a and c) then all nearby secant segments would be ambient isotopic, and thus would all be essential. Therefore a and c are the first and third points of some trisecant abc. We first assume that b is the only point of K in the interior of $\overline{ac}$ and will return to the more general case at the end of the proof. We will show

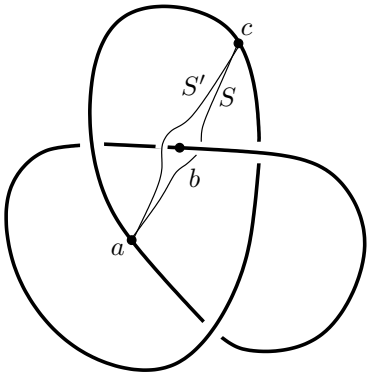


FIGURE 8. Secant ac is essential but some nearby secants are not. By Theorem 7.1 there must be an essential trisecant abc, because there are perturbations S and S' of $\overline{ac}$ which are essential and inessential, respectively.

that ab is a strongly essential secant; by symmetry the same is true of bc, and thus abc is an essential trisecant, as desired.

Let S and S' be two perturbations of $\overline{ac}$, with $K \cup S$ forming Θ-graph in which (γ_{ac}, S) is essential, and $K \cup S'$ forming a Θ-graph in which (γ_{ac}, S') is inessential. (The first exists because γ_{ac} is essential, and the second because it is near inessential arcs.) These must differ merely by going to the two different sides of K near b. (See Figure 8.) Now, by the definition of essential, the homotopy classes $h(\gamma_{ac}, S)$ and $h(\gamma_{ac}, S')$ must differ: the first of these is nontrivial in $\pi_1(\mathbb{R}^3 \setminus K)$ and the second is trivial.

The loops $h(\gamma_{ac}, S)$ and $h(\gamma_{ac}, S')$ each have linking number zero with K. Thus they differ not only by the meridian loop around K near b (changing S to S') but also by a meridian of K somewhere along the arc γ_{ac}, say near a. This means that the nontrivial homotopy class $h^{-1}(\gamma_{ac}, S)h(\gamma_{ac}, S')$ can in fact be represented by a loop $l_{\overline{ab}}$ along $\overline{ab}$. (See Figure 9.) Thus by definition, ab is strongly essential, as desired.

We now consider the case when K intersects the interior of $\overline{ac}$ in more than one place. Write this n-secant as $ab_1b_2 \dots b_{n-2}c$, assuming for the moment that K only intersects $\overline{ac}$ at these discrete points. We wish to show one of the trisecants ab_ic is essential.

From the definition of essential, we know there is a C^1 perturbation of $\overline{ac}$ to a path S such that (γ_{ac}, S) is essential. But again, since some nearby secants are inessential, we can also find an inessential perturbation S'. Now S and S' differ by loops around some subset of the b_i. (See Figure 10.) We can start with S' and splice these loops in one-by-one, ending with S. At some stage (adding a loop around a particular b_i) the path first becomes essential. The argument given above then applies to show that ab_i (and by symmetry also b_ic) is strongly essential.

Finally, it is possible that some component of $K \cap \overline{ac}$ is an interval around b_i. In this case S and S' could differ not just by a single loop around this segment but

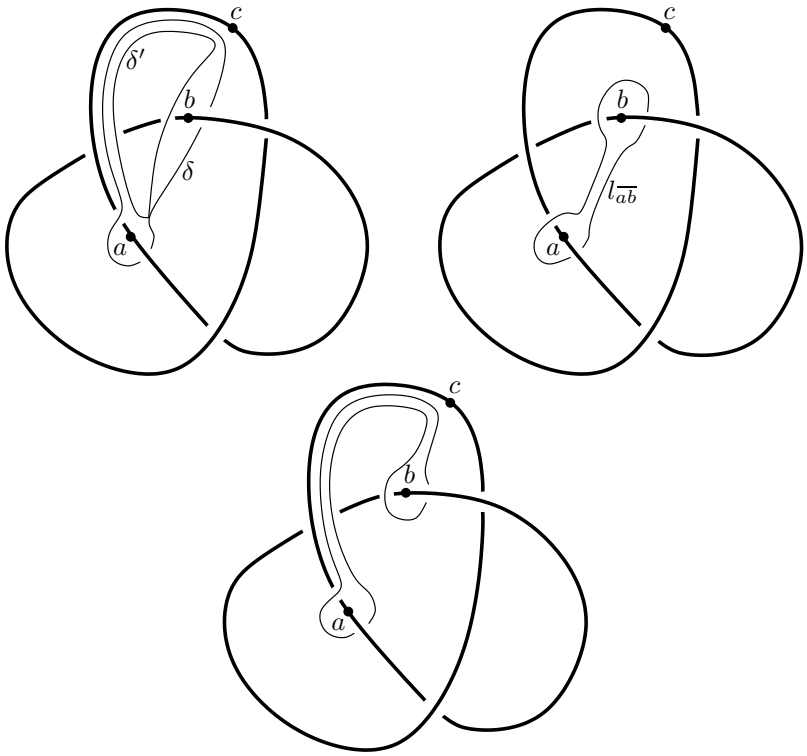


FIGURE 9. At the top left, we see the loops $\delta = h(\gamma_{ac}, S)$ and $\delta' = h(\gamma_{ac}, S')$, where S and S' are the essential and inessential perturbations of $\overline{ac}$ as in Figure 8. The product $\delta^{-1}\delta'$ is homotopic to the loop $l_{\overline{ab}}$ (top right) along $\overline{ab}$. At the bottom we see an intermediate stage of the homotopy.

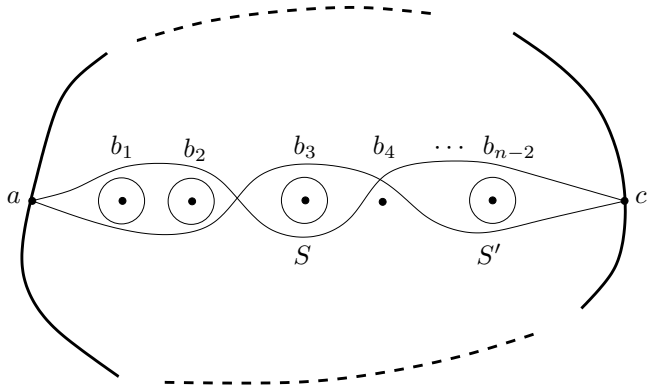


FIGURE 10. In case $\overline{ac}$ intersects K at many points b_i, it still has essential and inessential perturbations S and S', and these differ by loops about some subset of the b_i.

by some finite number of such loops. This, however, does not affect the rest of our argument. □

8. Minimum arclength for essential subarcs of a knot

We will improve our previous ropelength bounds by getting bounds on the length of an essential arc. A first bound is very easy:

LEMMA 8.1. *If secant ab is essential in a knot of unit thickness then $|a-b| \geq 1$, and if arc γ_{ab} is essential then $\ell_{ab} \geq \pi$.*

PROOF. If $|a-b| < 1$ then by Lemma 3.1 the ball B of diameter $\overline{ab}$ contains a single unknotted arc (say γ_{ab}) of K. Now for any perturbation S of $\overline{ab}$ which is disjoint from γ_{ab}, we can span $\gamma_{ab} \cup S$ by an embedded disk within B, whose interior is then disjoint from K. This means that γ_{ab} (and thus ab) is inessential.

Knowing that sufficiently short arcs starting at any given point a are inessential, consider now the shortest arc γ_{aq} which is essential. From Theorem 7.1 there must be a trisecant apq with both secants ap and pq essential, implying by the first part that a and q are outside $B(p)$. Since ap is essential, by the definition of q we have $p \notin \gamma_{aq}$, meaning that apq is reversed. From Corollary 3.4 we get $\ell_{ab} \geq \ell_{aq} \geq \pi$. □

Intuitively, we expect an essential arc γ_{ab} of a knot to "wrap at least halfway around" some point on the complementary arc γ_{ba}. Although when $|a-b| = 2$ we can have $\ell_{ab} = \pi$, when $|a-b| < 2$ we expect a better lower bound for ℓ_{ab}. Even though in fact an essential γ_{ab} might instead "wrap around" some point on itself, we can still derive the desired bound.

LEMMA 8.2. *If γ_{ab} is an essential arc in a unit-thickness knot and $|a-b| < 2$, then $\ell_{ab} \geq 2\pi - 2\arcsin(|a-b|/2)$.*

PROOF. Note that $|a-b| \in [1,2]$, so $2\pi - 2\arcsin(|a-b|/2) \leq 5\pi/3$. As in the previous proof, let γ_{aq} be the shortest essential arc from a, and find a reversed trisecant apq. We have $b \notin \gamma_{aq}$ and $\ell_{aq} \geq \pi$, so we may assume $\ell_{qb} < 2\pi/3$ or the bound is trivially satisfied.

Since γ_{qp} is essential, $\ell_{qp} \geq \pi > \ell_{qb}$, so $b \in \gamma_{qp}$. If $b \notin B(p)$ then the whole arc γ_{aqb} stays outside $B(p)$. Let Π denote the radial projection to $\partial B(p)$ as in Figure 11. From Lemma 3.3, this projection does not increase length. Because $|\Pi a - \Pi b| \leq |a-b|$, we have $2\pi - 2\arcsin(|\Pi a - \Pi b|/2) \geq 2\pi - 2\arcsin(|a-b|/2)$. It therefore suffices to consider the case $\gamma_{ab} \subset \partial B(p)$. For any two points $x, y \in \partial B(p)$, the spherical distance between them is $2\arcsin(|x-y|/2)$. Thus

$$\begin{aligned}\ell_{ab} = \ell_{aq} + \ell_{qb} &\geq \pi + 2\arcsin(|q-b|/2) \\ &= \pi + 2\arccos(|a-b|/2) = 2\pi - 2\arcsin(|a-b|/2).\end{aligned}$$

So we now assume that $|b-p| < 1$. Let γ_{qy} be the shortest essential arc starting from q, and note $|q-y| \geq 2$. Since $\ell_{qy} \geq \pi > \ell_{qb}$ we have $b \in \gamma_{qy}$. Let $h := |p-y| \leq \ell_{yp}$ and note that $h \in [0,1]$ since $b \in B(p)$. (See Figure 12.) Since $|q-y| \geq 2$, we have $|p-q| \geq 2-h$, so $\ell_{aq} \geq \pi/2 + f(2-h)$ by Corollary 4.4. On the other hand, since $\ell_{bp} \leq \pi/2$ (by Lemma 3.1) and $\ell_{qy} \geq \pi$, we have $\ell_{qb} \geq \pi/2 + \ell_{yp} \geq \pi/2 + h$.

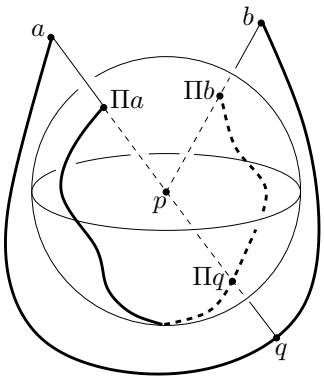


FIGURE 11. In the proof of Lemma 8.2, projecting γ_{ab} to the unit ball around p increases neither its length nor the distance between its endpoints. The projected curve includes antipodal points Πa and Πq, which bounds its length from below.

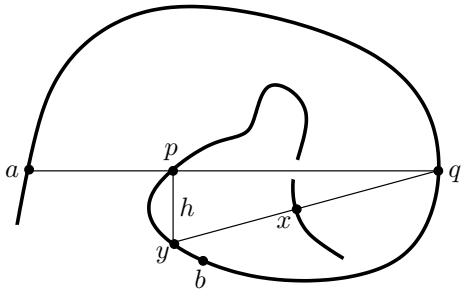


FIGURE 12. In the most intricate case in the proof of Lemma 8.2, we let γ_{aq} be the first essential arc from a, giving an essential trisecant apq. We then let γ_{qy} be the first essential arc from q, giving an essential trisecant qxy. Since $|x-y| \geq 1$ and $|x-q| \geq 1$, setting $h = |p-y|$ we have $|p-q| \geq 2-h$.

Thus $\ell_{ab} \geq \pi + f(2-h) + h$. An elementary calculation shows that the right-hand side is an increasing function of $h \in [0,1]$, minimized at $h = 0$, where its value is $\pi + f(2) = 7\pi/6 + \sqrt{3} > 5\pi/3$. That is, we have as desired

$$\ell_{ab} \geq 5\pi/3 \geq 2\pi - 2\arcsin(|a-b|/2).$$

□

If we define the continuous function

$$g(r) := \begin{cases} 2\pi - 2\arcsin(r/2) & \text{if } 0 \leq r \leq 2, \\ \pi & \text{if } r \geq 2. \end{cases}$$

then we can collect the results of the previous two lemmas as:

COROLLARY 8.3. *If γ_{ab} is an essential arc in a knot K of unit thickness, then $\ell_{ab} \geq g(|a-b|)$.* □

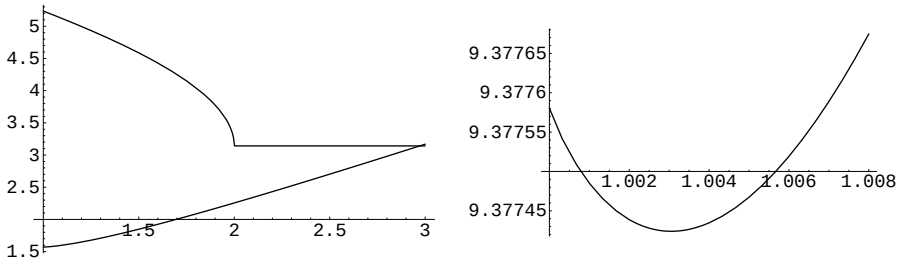


FIGURE 13. Left, a plot of $f(r)$ and $g(r)$ for $r \in [1,3]$, and right, a plot of $2f(r)+g(r)+r$ for $r \in [1, 1.008]$.

9. Main results

We now prove ropelength bounds for knots with different types of quadrisecants. The following lemma will be used repeatedly.

LEMMA 9.1. *Recall that* $f(r) := \sqrt{r^2-1} + \arcsin(1/r)$ *and that* $g(r) := 2\pi - 2\arcsin(r/2)$ *for* $r \le 2$ *while* $g(r) := \pi$ *for* $r \ge 2$. *The following functions of* $r \ge 1$ *have the minima indicated:*

The minimum of $f(r)$ *is* $\pi/2$ *and occurs at* $r = 1$.

The minimum of $f(r)+g(r)$ *is* $7\pi/6 + \sqrt{3} > 5.397$ *and occurs at* $r = 2$.

The minimum of $g(r)+r$ *is* $\pi + 2 > 5.141$ *and also occurs at* $r = 2$.

The minimum of $2f(r)+g(r)+r$ *is just over* 9.3774 *and occurs for* $r \approx 1.00305$.

PROOF. Note that f is increasing, and g is constant for $r \ge 2$. Thus the minima will occur in the range $r \in [1,2]$, where $f' = \frac{1}{r}\sqrt{r^2-1}$ and $g' = -2/\sqrt{4-r^2}$. Elementary calculations then give the results we want, where $r \approx 1.00305$ is a polynomial root expressible in radicals. See also Figure 13. □

THEOREM 9.2. *A knot with an essential simple quadrisecant has ropelength at least* $10\pi/3 + 2\sqrt{3} + 2 > 15.936$.

PROOF. Rescale the knot K to have unit thickness, let *abcd* be the quadrisecant and orient K in the usual way. Then the length of K is $\ell_{ab} + \ell_{bc} + \ell_{cd} + \ell_{da}$. As before, let $r = |a-b|$, $s = |b-c|$ and $t = |c-d|$.

Corollary 8.3 bounds γ_{ab}, γ_{bc} and γ_{cd}. The quadrisecant is essential, so from Lemma 8.1 we have $r, s, t \ge 1$, and Lemma 4.5 may be applied to bound ℓ_{da}. Thus the length of K is at least

$$\begin{aligned} & g(r) + g(s) + g(t) + \big(f(r) + s + f(t)\big) \\ &\quad = \quad \big(g(r) + f(r)\big) + \big(g(s) + s\big) + \big(g(t) + f(t)\big). \end{aligned}$$

Since this is a sum of functions in the individual variables, we can minimize each term separately. These are the functions considered in Lemma 9.1, so the minima are achieved at $r = s = t = 2$. Adding the three values together, we find the ropelength of K is at least $10\pi/3 + 2\sqrt{3} + 2 > 15.936$. □

THEOREM 9.3. *A knot with an essential flipped quadrisecant has ropelength at least* $10\pi/3 + 2\sqrt{3} > 13.936$.

PROOF. Rescale K to have unit thickness, let *abcd* be the quadrisecant. With the usual orientation, the length of K is $\ell_{ab} + \ell_{bd} + \ell_{dc} + \ell_{ca}$. Since the quadrisecant is essential, from Lemma 8.1 and Lemma 4.1 we have $r, s, t \geq 1$. We apply Corollary 8.3 to γ_{ab} and γ_{dc} and Corollary 4.4 to γ_{bd} and γ_{ca}.

Thus the length of K is at least

$$\begin{aligned} & g(r) + \big(f(r) + f(s)\big) + g(t) + \big(f(s) + f(t)\big) \\ = \quad & \big(g(r) + f(r)\big) + 2f(s) + \big(g(t) + f(t)\big). \end{aligned}$$

Again we minimize the terms separately using Lemma 9.1. We find the ropelength of K is at least $10\pi/3 + 2\sqrt{3} > 13.936$. □

THEOREM 9.4. *A knot with an essential alternating quadrisecant has ropelength at least* 15.66.

PROOF. Rescale K to have unit thickness, let *abcd* be the quadrisecant and orient K in the usual way. Then the ropelength of K is $\ell_{ac} + \ell_{cb} + \ell_{bd} + \ell_{da}$. Again, let $r = |a - b|$, $s = |b - c|$ and $t = |c - d|$.

The quadrisecant is essential, so from Lemma 8.1 and Lemma 4.2 we see that $r, s, t \geq 1$. Thus Lemma 4.5 may be applied to γ_{da}. We apply Corollary 4.4 to γ_{ac} and γ_{bd}, and Corollary 8.3 to γ_{cb}.

We find that the length of K is at least

$$\begin{aligned} & \big(f(r) + f(s)\big) + \big(f(s) + f(t)\big) + g(s) + \big(f(r) + s + f(t)\big) \\ = \quad & 2f(r) + \big(2f(s) + g(s) + s\big) + 2f(t). \end{aligned}$$

Again, we can minimize in each variable separately, using Lemma 9.1. Hence the ropelength of K is at least $2\pi + 9.377 > 15.66$. □

THEOREM 9.5. *Any nontrivial knot has ropelength at least* 13.936.

PROOF. Any knot of finite ropelength is $C^{1,1}$, so by Corollary 6.6 it has an essential quadrisecant. This must be either simple, alternating, or flipped, so one of the theorems above applies; we inherit the worst of the three bounds. □

In her doctoral dissertation [**Den**], Denne shows:

THEOREM 9.6. *Any nontrivial* $C^{1,1}$ *knot has an essential alternating quadrisecant.* □

Combining this with Theorem 9.4 gives:

COROLLARY 9.7. *Any nontrivial knot has ropelength at least* 15.66. □

We note that this bound is better even than the conjectured bound of 15.25 from [**CKS**, Conj. 26]. We also note that our bound cannot be sharp, for a curve which is C^1 at b cannot simultaneously achieve the bounds for ℓ_{cb} and ℓ_{bd} when $s \approx 1.003$. Probably a careful analysis based on the tangent directions at b and c could

yield a slightly better bound. However, we note again that numerical simulations have found trefoil knots with ropelength no more than 5% greater than our bound, so there is not much further room for improvement.

10. Quadrisecants and Links

Quadrisecants may be used in a similar fashion to give lower bounds for the ropelength of a nontrivial link. For a link, n-secant lines and n-secants are defined as before. They can be classified in terms of the order in which they intersect the different components of the link. We begin our considerations with a simple lemma bounding the length of any nonsplit component of a link.

LEMMA 10.1. *Suppose L is a link, and A is a component of L not split from the rest of the link. Then for any point $a \in A$, there is a trisecant axa' where $a' \in A$ but x lies on some other component. If L has unit thickness, then A has length at least 2π.*

PROOF. For the given a, the union of all secants aa' is a disk spanning A. Because A is not split from $L \setminus A$, this disk must be cut by $L \setminus A$, at some point x. This gives the desired trisecant. If L has unit thickness, then A stays outside $B(x)$, so as in Corollary 3.4, we have $\ell_{aa'} \geq \pi$ and $\ell_{a'a} \geq \pi$. □

This construction of a trisecant is adapted from Ortel's original solution of the Gehring link problem. (See [**CFKSW**].) The length bound can also be viewed as a special case of [**CKS**, Thm. 10], and immediately implies:

COROLLARY 10.2. *Any nonsplit link with k components must have ropelength at least $2\pi k$.* □

This bound is sharp in the case of the tight Hopf link, where each component has length exactly 2π.

We know that nontrivial links have many trisecants, and want to consider when they have quadrisecants. Pannwitz [**Pan33**] was the first to show the existence of quadrisecants for certain links:

PROPOSITION 10.3. *If A and B are disjoint generic polygonal knots, linked in the sense that neither one is homotopically trivial in the complement of the other, then $A \cup B$ has a quadrisecant line intersecting them in the order $ABAB$.* □

Note that, when A or B is unknotted, Pannwitz's hypothesis is equivalent to having nonzero linking number. The theorem says nothing, for instance, about the Whitehead link.

Kuperberg [**Kup**] extended this result to apply to all nontrivial link types (although with no information about the order type), and for generic links he again guaranteed an essential quadrisecant:

PROPOSITION 10.4. *A generic nontrivial link has an essential quadrisecant.* □

Here, a secant of a link L is automatically essential if its endpoints lie on different components of L. If its endpoints are on the same component K, we

apply our previous definition of inessential secants, but require the disk to avoid all of L, not just the component K.

Combining Kuperberg's result with Corollary 6.5 (which extends easily to links) immediately gives us:

THEOREM 10.5. *Every nontrivial $C^{1,1}$ link has an essential quadrisecant.* □

Depending on the order in which the quadrisecant visits the different components of the link, we can hope to get bounds on the ropelength. Sometimes, as for $ABAB$ quadrisecants, we cannot improve the bound of Corollary 10.2. Here of course, the example of the tight Hopf link (which does have an $ABAB$ quadrisecant) shows that this bound has no room for improvement.

Note that Lemma 8.1 extends immediately to links, since if a and b are on different components of a unit-thickness link we automatically have $|a-b| \geq 1$.

THEOREM 10.6. *Let L be a link of unit thickness with an essential quadrisecant Q. If Q has type $AAAB$, $AABA$, $ABBA$ or $ABCA$, then the length of component A is at least $7\pi/3 + 2\sqrt{3}$, $8\pi/3 + 1 + \sqrt{3}$, $2\pi + 2$ or $2\pi + 2$, respectively.*

PROOF. Suppose $a_1a_2a_3b$ is an essential quadrisecant of type $AAAB$, and set $r := |a_2 - a_1|$, $s := |a_3 - a_2|$. The quadrisecant is essential, so from Lemma 8.1 we have $r, s \geq 1$ and we may apply Corollary 4.4 to $\gamma_{a_1a_3}$ and Corollary 8.3 to $\gamma_{a_1a_2}$ and $\gamma_{a_2a_3}$. Thus the length of A is at least $f(r) + g(r) + f(s) + g(s)$. As before, we can minimize in each variable separately, using Lemma 9.1. Hence the length of A is at least $7\pi/3 + 2\sqrt{3}$.

Now suppose $a_1a_2ba_3$ is an essential quadrisecant of type $AABA$, and set $r := |a_1 - a_2|$, $s := |a_2 - b|$ and $t := |b - a_3|$. Since the quadrisecant is essential, we have $r, s, t \geq 1$ and we may apply Corollary 8.3 to $\gamma_{a_1a_2}$, Corollary 4.4 to $\gamma_{a_2a_3}$ and Lemma 4.5 to $\gamma_{a_1a_3}$. We find that the length of A is at least

$$\begin{aligned} & g(r) + \big(f(s) + f(t)\big) + \big(f(r) + s + f(t)\big) \\ = \quad & \big(f(r) + g(r)\big) + \big(f(s) + s\big) + 2f(t). \end{aligned}$$

Again, minimizing in each variable separately using Lemma 9.1, we find the length of A is at least $8\pi/3 + 1 + \sqrt{3}$.

Finally suppose a_1bca_2 is an essential quadrisecant of type $ABBA$ or type $ABCA$. Because the quadrisecant is essential we must have $|a_1 - b| \geq 1$, $|b - c| \geq 1$ and $|c - a_2| \geq 1$. Just as in the proof of Lemma 4.5, we find that $\ell_{a_1a_2} \geq \pi + 1$ and $\gamma_{a_2a_1} \geq \pi + 1$, showing that the length of A is at least $2\pi + 2$ as desired. □

Unfortunately, we do not know any link classes which would be guaranteed to have one of these types of quadrisecants, so we know no way to apply this theorem.

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