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TORIC VARIETIES - DEGENERATIONS AND FUNDAMENTAL GROUPS

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Abstract. In this paper we calculate fundamental groups of branch curves and Galois covers for four toric varieties. In order to do it we need to study properties and degenerations of these toric varieties and the braid monodromies of the branch curves in $\mathbb{CP}^2$.

We find that the fundamental groups of the branch curves for the first two toric varieites are Artin braid groups $\mathcal{B}_5$ and $\mathcal{B}_6$, and the two others are certain quotients of $\mathcal{B}_4$ and $\mathcal{B}_6$. The quotients of these groups by the normal subgroups generated by the squares of the generators are S_5, S_6, S_4 and S_6 respectively.

In all cases, the fundamental groups of the related Galois covers are trivial.

1. Introduction

Let X be a projective algebraic surface embedded in a projective space $\mathbb{CP}^N$. Take a general linear subspace V in $\mathbb{CP}^N$ of dimension $N-3$. Then the projection centered at V to $\mathbb{CP}^2$ defines a finite map $f : X \to \mathbb{CP}^2$. Let $B \subset \mathbb{CP}^2$ be the branch curve of f. Denote $\pi_1(\mathbb{CP}^2 \setminus B)$ to be *the fundamental group of the branch curve*. The group $\Pi_{(B)}$ is defined to be its quotient by the normal subgroup generated by the squares

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of the generators. Let X_{Gal} be the Galois cover of X with respect to f, and we define $\pi_1(X_{Gal})$ to be its fundamental group. These groups are imporatnt invariants for classification of surfaces, hence we want to compute them.

In this work we attack four toric varieties. The first surface is $X_1 := F_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ the Hirzebruch surface of degree one in $\mathbb{CP}^6$ embedded by the line bundle with the class $s + 3g$, where s is the negative section and g is a general fiber. The second is $X_2 := F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ the Hirzebruch surface of degree zero in $\mathbb{CP}^7$ embedded by $\mathcal{O}(1, 3)$. We generalize the results to the case where X_2 is embedded in $\mathbb{CP}^{2n+1}$ by $\mathcal{O}(1, n)$. The third is $X_3 := F_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(3))$ in $\mathbb{CP}^5$ embedded by the class $s+3g$. The fourth is a singular toric surface X_4 with one A_1 singular point embedded in $\mathbb{CP}^6$. A_1-singularity is an isolated normal singularity of dimension two whose resolution consists of one (-2)-curve (i.e. a nonsingular rational curve on a surface with -2 as its selfintersection number). For the first three cases, we use different triangulations of tetragons from those treated in [21] and [22].

The research was initiated when Moishezon and Teicher defined the generators of a braid group from line arrangement in $\mathbb{CP}^2$, which is the branch curve of a generic projection from a union of projective planes [21], namely degeneration. In order to explain the process of such degeneration, they used schematic figures consisting of triangulations of triangles and tetragons ([17], [20], [21]). Degenerations, braid monodromies and fundamental groups were studied also by Auroux - Donaldson - Katzarkov - Yotov in [10]. This work consults both works and gives a geometric meaning to these schematic figures from the view point of toric geometry ([12], [23]).

Moishezon and Teicher studied the cases when X is the projective plane embedded by $\mathcal{O}(3)$ [21], or when X are Hirzebruch surfaces $F_k(a, b)$ for a, b relatively prime [16]. Later works present the group $\pi_1(\mathbb{CP}^2 \setminus B)$ and $\Pi_{(B)}$ related to $K3$ surfaces [4], $\mathbb{CP}^1 \times T$ where T is a complex torus ([5], [6]), $T \times T$ ([2], [3]), Hirzebruch surfaces $F_k(a, b)$ where a, b are non relatively prime [9].

Main results of this work are

- Theorems 11 and 14: $\pi_1(\mathbb{CP}^2 \setminus B_1) \cong \mathcal{B}_5$ and $\pi_1(\mathbb{CP}^2 \setminus B_2) \cong \mathcal{B}_6$. Moreover, $\Pi_{(B_1)} \cong S_5$ and $\Pi_{(B_2)} \cong S_6$.

- Theorems 18 and 21: $\pi_1(\mathbb{CP}^2\setminus B_3)\cong\mathcal{B}_4/ < \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 >$ and $\pi_1(\mathbb{CP}^2\setminus B_4)\cong\mathcal{B}_6/ < [\Gamma_1,\Gamma_4^2\Gamma_2^2],[\Gamma_3,\Gamma_2^2\Gamma_5^2],[\Gamma_2,\Gamma_1^2\Gamma_4\Gamma_1^{-2}],\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_5\Gamma_4\Gamma_3^2\Gamma_2^2\Gamma_1^2 >$. Moreover, $\Pi_{(B_3)}\cong S_4$ and $\Pi_{(B_4)}\cong S_6$.

We continue and calculate the group $\pi_1(X_{Gal})$ for these cases. The results in this work motivate us to enlarge our research in toric geometry and to compute these fundamental groups related Galois covers of toric varieties. Moishezon and Teicher proved in [20] that for $X=\mathbb{CP}^1\times\mathbb{CP}^1$ the group $\pi_1(X_{Gal})$ is a finite abelian group on $n-2$ generators each of order g.c.d.(a,b) (a and b are the parameters of the embedding). In [5] the treated surface is $X=\mathbb{CP}^1\times T$ and $\pi_1(X_{Gal})=\mathbb{Z}^{10}$. In [6] the same surface was embedded in $\mathbb{CP}^{2n-1}$ and $\pi_1(X_{Gal})=\mathbb{Z}^{4n-2}$. In [7] and [8] the surface $X=T\times T$ is attacked, and $\pi_1(X_{Gal})$ is nilpotent of class 3. In [9] this group was computed for the Hirzebruch surface $F_1(2,2)$ and it is $\mathbb{Z}_2^{10}$. The results concerning the fundamental groups for the Galois covers in this work are summarized as follows:

- The group $\pi_1((X_i)_{Gal})$ is trivial for $i=1,2,3,4$.

We note that the Coxeter quotient, defined as $C=\Pi_{(B)}/ < \Gamma_i=\Gamma_{i'} >$ is also an important invariant. The group C is isomorphic to a symmetric group S_n for Hirzebruch surfaces ([9], [16]) and $\mathbb{CP}^1\times\mathbb{CP}^1$ ([17], [20]). The case of $\mathbb{CP}^1\times T$ and $T\times T$ are the first in which C is a larger group, namely $C\cong\mathbb{Z}_5\rtimes S_6$ [5] and $C\cong K_C\rtimes S_{18}$, where K_C is a central extension of $\mathbb{Z}^{34}$ by $\mathbb{Z}$ [7]. Here the group C is isomorphic to the group $\Pi_{(B)}$, that means it is isomorphic to a symmetric group.

The work is done along the following lines. We degenerate X into a union X_0 of planes. Then, X_0 is composed of $n=\deg(X_0)$ planes. B_0 is the union of the intersection lines (as depicted in Figures 1, 5, 7, 8). The lines are numerated for future use. It is very difficult to deal with $\pi_1(\mathbb{C}^2\setminus B)$ directly, so we consider first $\pi_1(\mathbb{C}^2\setminus B_0)$ and use the regeneration rules from [22] to get a braid monodromy factorization of B from the one of B_0. Then we can use the van Kampen Theorem [26] to get a finite presentation of $\pi_1(\mathbb{C}^2\setminus B)$ with generators $\Gamma_1,\Gamma_{1'},\cdots\Gamma_m,\Gamma_{m'}$ ($2m$ is the degree of B). A presentation of $\pi_1(\mathbb{CP}^2\setminus B)$ is obtained by adding the projective relation $\Gamma_{m'}\Gamma_m\cdots\Gamma_{1'}\Gamma_1=1$. Then we calculate the group $\Pi_{(B)}=\pi_1(\mathbb{CP}^2\setminus B)/ < \Gamma_i^2,\Gamma_{i'}^2 >$

and $\pi_1(X_{Gal}) = ker(\Pi_{(B)} \to S_n)$. The reader might want to check the papers [5], [7] and [24] in order to get the feeling of the type of presentation we are dealing with.

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3. Degeneration of toric surfaces

In their process to calculate the braid monodromy, Moishezon and Teicher studied the projective degeneration of $V_3 = (\mathbb{CP}^2, \mathcal{O}(3))$ [21] and Hirzebruch surfaces [16]. Since $\mathbb{CP}^2$ and the Hirzebruch surfaces are toric surfaces, we shall describe the projective degeneration of toric surfaces in this section.

3.1. **Basic notions.** We outline definitions we need in toric geometry and refer to [12] and [23] for further statements and proofs.

Definition 1. Toric variety *A toric variety is a normal algebraic variety X that contains an algebraic torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an algebraic action $T \times X \to X$ of T on X, that is an extension of the natural action of T on itself.*

Let M be a free $\mathbb{Z}$-module of rank n ($n \geq 1$) and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ the extension of the coefficients to the real numbers. Let $T := \mathrm{Spec}\mathbb{C}[M]$ be an algebraic torus of dimension n. Then M is considered as the character group of T, i.e. $T = \mathrm{Hom}_{\mathrm{gr}}(M, \mathbb{C}^*)$.

We denote an element $m \in M$ by $e(m)$ as a function on T, which is also a rational function on X. Let L be an ample line bundle on X. Then we have

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}e(m), \tag{1}$$

where P is an integral convex polytope in $M_{\mathbb{R}}$ defined as the convex hull $\mathrm{Conv}\{m_0, m_1, \ldots, m_r\}$ of a finite subset $\{m_0, m_1, \ldots, m_r\} \subset M$. Conversely we can construct a pair (X, L) of a polarized toric variety from an integral convex polytope P so that the above isomorphism holds (see [12, Section 3.5] or [23, Section 2.4]). If an affine automorphism φ of M transforms P to P_1, then φ induces an isomorphism of polarized toric varieties (X, L) to (X_1, L_1), where (X_1, L_1) corresponds to P_1.

Example 2. *Let $M = \mathbb{Z}^2$. Then $V_3 = (\mathbb{CP}^2, \mathcal{O}(3))$ corresponds to the integral convex polytope $P_3 := Conv\{(0,0), (3,0), (0,3)\}$.*

Example 3. *The Hirzebruch surface $F_d = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(d))$ of degree d has generators s, g in the Picard group consisting of the negative section $s^2 = -d$ and general fiber $g^2 = 0$. A line bundle L with $[L] = as + bg$ in $Pic(F_d)$ is ample if $a > 0$ and $b > ad$. Then this pair (F_d, L) corresponds to $P_{d(a,b)} := Conv\{(0,0), (b - ad, 0), (b, a), (0, a)\}$.*

Next we consider degenerations of toric surfaces as Moishezon - Teicher defined. We recall the definition from [21].

Definition 4. Projective degeneration *A degeneration of X is a proper surjective morphism with connected fibers $\pi : V \to \mathbb{C}$ from an algebraic variety V such that the restriction $\pi : V \setminus \pi^{-1}(0) \to \mathbb{C} \setminus \{0\}$ is smooth and that $\pi^{-1}(t) \cong X$ for $t \neq 0$.*

When X is projective with an embedding $k : X \hookrightarrow \mathbb{CP}^n$, a degeneration of X $\pi : V \to \mathbb{C}$ is called **a projective degeneration** *of k if there exists a morphism $F : V \to \mathbb{CP}^n \times \mathbb{C}$ such that the restriction $F_t = F\,|_{\pi^{-1}(t)} : \pi^{-1}(t) \to \mathbb{CP}^n \times t$ is an embedding of $\pi^{-1}(t)$ for all $t \in \mathbb{C}$ and that $F_1 = k$ under the identification of $\pi^{-1}(1)$ with X.*

Moishezon and Teicher called the triangulation of P_3 consisting of nine standard triangles as a schematic figure of a union of nine projective planes [21]. In the theory

of toric varieties, however, the lattice points $P_3 \cap M$ correspond to rational functions of degree 3 on $V_3 \cong \mathbb{CP}^2$. Let $m_0 = (0,0), m_1 = (1,0), m_2 = (0,1), \ldots, m_9 = (0,3) \in \mathbb{Z}^2$. Then we may write $e(m_0) = x_0^3, e(m_1) = x_0^2 x_1, e(m_2) = x_0^2 x_2, \ldots, e(m_9) = x_2^3$ with a suitable choice of the homogeneous coordinates of $\mathbb{CP}^2$. The Veronese embedding $V_3 \hookrightarrow \mathbb{CP}^9$ is given by $z_i = e(m_i)$ for $i = 0, 1, \ldots, 9$ with the homogeneous coordinates $[z_0 : z_1 : \cdots : z_9]$ of $\mathbb{CP}^9$. Let $P_1 := \mathrm{Conv}\{(0,0),(1,0),(0,1)\}$, which corresponds to $(\mathbb{CP}^2, \mathcal{O}(1))$. The subset $P_1 \subset P_3$ corresponds to the linear subspace $\{z_3 = \cdots = z_9 = 0\} \subset \mathbb{CP}^9$. Thus a triangulation of P_3 into a union of nine standard triangles means the subvariety of dimension two consisting of the union of nine projective planes in $\mathbb{CP}^9$ and each standard triangle defines a linear subspace of dimension two with corresponding coordinates.

3.2. **Constructing the degeneration of toric surfaces.** In the following we construct a semistable degeneration of toric surfaces according to Hu [13]. Let $M = \mathbb{Z}^2$. Let P be a convex polyhedron in $M_{\mathbb{R}}$ corresponding to a polarized toric surface (X, L). The lattice points $P \cap M$ define the embedding $\varphi_L : X \to \mathbb{P}(\Gamma(X, L))$. Let Γ be a triangulation of P consisting of standard triangles with vertices in $P \cap M$. Let $h : P \cap M \to \mathbb{Z}_{\geq 0}$ be a function on the lattice points in P with values in nonnegative integers. Let $\tilde{M} = M \oplus \mathbb{Z}$ and let $\tilde{P} = \mathrm{Conv}\{(x,0),(x,h(x)); x \in P \cap M\}$ the integral convex polytope in $\tilde{M}_{\mathbb{R}}$. We want to choose h to satisfy the conditions that $(x, h(x))$ for $x \in P \cap M$ are vertices of $\tilde{P}$ and that for each edge in Γ joining x and $y \in P \cap M$ there is an edge joining $(x, h(x))$ and $(y, h(y))$ as a face of $\partial \tilde{P}$. We call that $\tilde{P}$ realizes the triangulation Γ if these conditions are satisfied. Now we assume that $\tilde{P}$ realizes the triangulation Γ. Then $\tilde{P}$ defines a polarized toric 3-fold $(\tilde{X}, \tilde{L})$. From the construction, $\tilde{X}$ has a fibration $p : \tilde{X} \to \mathbb{CP}^1$ satisfying that $p^{-1}(t) \cong X$ with $t \neq 0$ and that $p^{-1}(0)$ is a union of projective planes. Furthermore we see that $p^{-1}(\mathbb{CP}^1 \setminus \{0\}) \cong \mathbb{C} \times X$. Thus the flat family $p : \tilde{X} \to \mathbb{CP}^1$ gives a degeneration of X into a union of projective planes with the configuration diagram Γ. Hu treats only nonsingular toric varieties of any dimension. The difficulty of this construction is to find a triangulation Γ. Here we restrict ourselves to toric surfaces. Then we can find a triangulation for any integral convex polygon P.

Example 5. *Let* $m_0 = (0,0), m_1 = (1,0), m_2 = (0,1), m_3 = (1,1) \in M = \mathbb{Z}^2$. *Let* $P = Conv\{m_0, m_1, m_2, m_3\}$. *Then* P *defines the polarized surface* $(X = \mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(1,1))$. *Let* Γ *be the triangulation of* P *defined by adding the edge connecting* m_1 *and* m_2. *Define as* $h(m_0) = h(m_3) = 0, h(m_1) = h(m_2) = 1$. *Let* $\tilde{M} := M \oplus \mathbb{Z}$. *Set* $m_i = (m_i, 0)$ *in* $\tilde{M}$ *and* $m_1^+ = (m_1, 1), m_2^+ = (m_2, 1)$. *Then the integral convex polytope* $\tilde{P} := Conv\{m_0, \ldots, m_3, m_1^+, m_2^+\}$ *in* $\tilde{M}$ *defines the polarized toric 3-fold* $(\tilde{X}, \tilde{L})$. *By definition* $\tilde{X}$ *has a fibration* $p : \tilde{X} \to \mathbb{CP}^1$. *The global sections of* $\tilde{L}$ *defines an embeddind of* $\tilde{X}$ *as follows: Let* $[z_0 : \cdots : z_5]$ *be the homogeneous coordinates of* $\mathbb{CP}^5$. *The equations* $z_i = e(m_i)$ *for* $i = 0, 1, 2, 3$ *and* $z_4 = e(m_1^+), z_5 = e(m_2^+)$ *define the embedding* $\tilde{X} \to \mathbb{CP}^5$. *The defining equation of* $\tilde{X}$ *is* $z_0 z_3 = z_1 z_2, z_1 z_5 = z_2 z_4$. *The fiber* $p^{-1}(\infty)$ *is given by* $\{z_0 z_3 = z_1 z_2, z_4 = z_5 = 0\}$ *which is isomorphic to* $X = \mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathbb{P}(\Gamma(X, \mathcal{O}(1,1))) \cong \mathbb{CP}^3$, *and the fiber* $p^{-1}(0)$ *is given by* $\{z_0 z_3 = 0, z_1 = z_2 = 0\}$ *which is a union of two projective planes in* $\mathbb{CP}^3$.

Lemma 6. *The line bundle* $\tilde{L}$ *on* $\tilde{X}$ *is very ample.*

Proof. Let $m_1, m_2, m_3 \in P \cap M$ be three vertices of a standard triangle in the triangulation Γ of P. Set $m_i^- = (m_i, 0), m_i^+ = (m_i, h(m_i))$ in $\tilde{M}_{\mathbb{R}}$ for $i = 1, 2, 3$. Denote $Q = \mathrm{Conv}\{m_i^\pm; i = 1, 2, 3\}$ the integral convex polytope with vertices $\{m_i^\pm; i = 1, 2, 3\}$. Then we divide $\tilde{P}$ into a union of triangular prisms like Q. We can divide Q into a union of standard 3-simplices. We may assume $h(m_1) \geq h(m_2) \geq h(m_3)$ by renumbering m_i if necessary. Then we can divide Q into a union of $Q_0 = \mathrm{Conv}\{m_1^+, m_2^+, m_3^+, (m_1, h(m_1) - 1)\}$ and $Q_1 = \mathrm{Conv}\{m_1^-, (m_1, h(m_1) - 1), m_2^\pm, m_2^\pm\}$. Here Q_0 is a standard 3-simplex and Q_1 has the similar shape as Q but less volume than that of Q. Thus we obtain a division of $\tilde{P}$ into a union of standard 3-simplices. This is not always triangulation of $\tilde{P}$, but this gives a covering of $\tilde{P}$ consisting of standard 3-simplices. From the theory of polytopal semigroup ring (see, for instance, [11] and [25]), we see that $\tilde{L}$ is simply generated, hence very ample. □

We claim that $\tilde{X}$ also defines a projective degeneration of (X, L). Denote $\varphi_{\tilde{L}} : \tilde{X} \to \mathbb{P}(\Gamma(\tilde{X}, \tilde{L}))$ the embedding defined by global section of $\tilde{L}$. Let $P \cap M = \{m_0, m_1, \ldots, m_r\}$. Set $m_i^- = (m_i, 0), m_i^+ = (m_i, h(m_i))$ in $\tilde{M}$ for $i = 0, \ldots, r$. Now we define a morphism $\Phi : \tilde{X} \to \mathbb{CP}^r$ as the i-th homogeneous coordinate of $\Phi(x)$

for $x \in \tilde{X}$ is $\lambda e(m_i^+) + \mu e(m_i^-)$, where $e(m_i^\pm)$ are homogeneous coordinates of $\varphi_{\tilde{L}}(x)$ in $\mathbb{P}(\Gamma(X, L))$ and $[\lambda : \mu]$ is the homogeneous coordinate of $p(x)$ in $\mathbb{CP}^1$. Then the restriction of Φ to $p^{-1}(\lambda = 0)$ coincides with φ_L and $\Phi(p^{-1}(\mu = 0))$ is a union of projective planes corresponding to the triangulation Γ of P. Let $V := \tilde{X} \setminus p^{-1}(\lambda = 0)$. Then $F := \Phi \times p : V \to \mathbb{CP}^r \times \mathbb{C}$ is a projective degeneration of φ_L.

Theorem 7. *Let P be an integral convex polyhedron of dimension 2 corresponding to a polarized toric surface (X, L) and let Γ a triangulation of P consisting of standard triangles with vertices in M. Assume that $\tilde{P}$ is an integral convex polytope in $\tilde{M}_{\mathbb{R}}$ realizing the triangulation Γ. Then $\tilde{P}$ defines a polarized toric 3-fold $(\tilde{X}, \tilde{L})$, which gives a projective degeneration of (X, L) to a union of projective planes.*

3.3. **Degeneration of the four toric surfaces.** In this paper we study four degenerations of polarized toric surfaces, each one of them is defined by integral convex polygon P. We choose a triangulation Γ for each P. And define a function $h : P \cap M \to \mathbb{Z}_{\geq 0}$ so that the integral convex polytope $\tilde{P}$ of dimension 3 should realize the triangulation Γ of P.

The first surface is the Hirzebruch surface $X_1 := F_1$ of degree one embedded in $\mathbb{CP}^6$ by the very ample line bundle L_1 whose class is $s + 3g$, where s is the negative section and g is a general fiber. We mentioned this surface as a polarized toric surface in Example 3, which is corresponding to the integral convex polygon $P_{1(1,3)}$ in $M = \mathbb{Z}^2$. Let $m_i = (i, 0)$ for $i = 0, 1, 2, 3$ and $m_j = (j - 3, 1)$ for $j = 4, 5, 6$. Then $P_{1(1,3)} = \mathrm{Conv}\{m_0, m_3, m_4, m_6\}$. Let Γ_1 be the triangulation of $P_{1(1,3)}$ obtained by adding the edges $m_1\bar{m}_4, m_2\bar{m}_4, m_2\bar{m}_5, m_3\bar{m}_5$. This triangulation is slightly different from the one treated in [21]. We define a function $h_1 : P_{1(1,3)} \cap M \to \mathbb{Z}_{\geq 0}$ as $h_1(m_0) = h_1(m_3) = h_1(m_6) = 0, h_1(m_1) = h_1(m_5) = 2, h_1(m_2) = h_1(m_4) = 3$. Then we can define an integral convex polytope $\tilde{P}$ in $\tilde{M} = M \oplus \mathbb{Z}$ realizing the triangulation Γ_1 of $P_{1(1,3)}$. Hence we have a projective degeneration of $\varphi_1 := \varphi_{L_1} : F_1 \hookrightarrow \mathbb{CP}^6$.

The second surface is $X_2 := \mathbb{CP}^1 \times \mathbb{CP}^1$ embedded in $\mathbb{CP}^7$ by $\mathcal{O}(3, 1)$. This embedded toric surface corresponds to the convex polygon $P_{3,1} := \mathrm{Conv}\{(0, 0), (3, 0), (0, 1), (3, 1)\}$ in $M = \mathbb{Z}^2$. Let $m_i = (i, 0)$ for $i = 0, 1, 2, 3$ and $m_j = (j - 4, 1)$ for $j = 4, 5, 6, 7$. Then $P_{3,1} = \mathrm{Conv}\{m_0, m_3, m_4, m_7\}$. Let Γ_2 be the triangulation of $P_{3,1}$ obtained by adding

the edges $m_0\bar{m}_5, m_1\bar{m}_5, m_1\bar{m}_6, m_2\bar{m}_6, m_2\bar{m}_7$. We define a function $h_2 : P_{3,1} \cap M \to \mathbb{Z}_{\geq 0}$ as $h_2(m_3) = h_2(m_4) = 0, h_2(m_0) = h_2(m_7) = 1, h_2(m_5) = 2, h_2(m_1) = h_2(m_2) = h_2(m_6) = 3$. Then we have a projective degeneration of $\varphi_2 := \varphi_{\mathcal{O}(3,1)} : \mathbb{CP}^1 \times \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^7$ corresponding to the triangulation Γ_2.

The third surface is the Hirzebruch surface $X_3 := F_2$ of degree 2 embedded in $\mathbb{CP}^5$ by the ample line bundle L_2 whose class is $s + 3g$. The corresponding polygon is $P_{2(1,3)}$. Let $m_i = (i, 0)$ for $i = 0, 1, 2, 3$ and $m_j = (j - 3, 1)$ for $j = 4, 5$ in $M = \mathbb{Z}^2$. Then $P_{2(1,3)} = \mathrm{Conv}\{m_0, m_3, m_4, m_5\}$ up to affine automorphism of M. Let Γ_3 be the triangulation of $P_{2(1,3)}$ obtained by adding the edges $m_1\bar{m}_4, m_1\bar{m}_5, m_2\bar{m}_5$. We define a function $h_3 : P_{2(1,3)} \cap M \to \mathbb{Z}_{\geq 0}$ as $h_3(m_0) = h_3(m_3) = h_3(m_4) = 0, h_3(m_1) = h_3(m_2) = h_3(m_5) = 1$. Then we have a projective degeneration of $\varphi_3 := \varphi_{L_2} : F_2 \hookrightarrow \mathbb{CP}^5$ corresponding to the triangulation Γ_3.

The last surface is a singular toric surface X_4 embedded in $\mathbb{CP}^6$ corresponding to the polygon $P_4 := \mathrm{Conv}\{(0, 0), (2, 0), (0, 1), (1, 2), (2, 1)\}$. Let $m_i = (i, 0)$ for $i = 0, 1, 2$, $m_j = (j - 3, 1)$ for $j = 3, 4, 5$ and $m_6 = (1, 2)$. Let Γ_4 be the triangulation of P_4 obtained by adding the edges $\{m_i\bar{m}_4; i = 0, 1, 2, 3, 5, 6\}$. We define a function $h_4 : P_4 \cap M \to \mathbb{Z}_{\geq 0}$ as $h_4(m_0) = h_4(m_2) = h_4(m_3) = h_4(m_5) = h_4(m_6) = 0, h_4(m_1) = 1, h_4(m_4) = 2$. Then we have a projective degeneration of $\varphi_4 : X_4 \hookrightarrow \mathbb{CP}^6$ corresponding to the triangulation Γ_4.

Note that degenerations for Hirzebruch surfaces were constructed in [14] and were explained in [10]. The fundamental groups of branch curves for some of those surfaces in $\mathbb{CP}^2$ were computed in [27]. In order to avoid duplications (our toric varieties are of type Hirzebruch), we explain briefly the general setting of degeneration for X_1, X_2 and X_3, give the braid monodromy factorization, the resulting groups, and give sketch of proof (using the Moishezon - Teicher techniques, regeneration Lemmas [19] and van Kampen Theorem [26]). For X_4 we give a detailed proof.

4. The surfaces X_1 and X_2

In this section we show that by Moishezon - Teicher braid monodromy techniques and van Kampen theorem we get the same results of Zariski [27], i.e. the groups

$\pi_1(\mathbb{CP}^2 \setminus B_1)$ and $\pi_1(\mathbb{CP}^2 \setminus B_2)$ are Artin braid groups. This proves that $\Pi_{(B_1)}$ and $\Pi_{(B_2)}$ are symmetric groups.

4.1. **The surface X_1.** Let $X_1 = F_1(3,1)$ be the Hirzebruch surface, as defined in Section 3. The construction of the degeneration of Hirzebruch surfaces of type $F_1(p,q)$ (for $p > q \geq 2$) appears in [14] and [15]. In [10], Auroux - Donaldson - Katzarkov - Yotov dedicate Section 6.2 to the costruction of degeneration of F_1 surfaces and to the presentations of the fundamental groups of branch curves.

The degeneration of X_1 into a union of five planes $(X_1)_0$ is embedded in $\mathbb{CP}^6$. The numeration of lines is fixed according to the numeration of the vertices in Section 3, see Figure 1. Note that each of the points m_2, m_4, m_5 is contained in three distinct

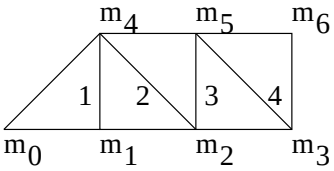

FIGURE 1. Degeneration of X_1

planes, while each of m_1, m_3 is contained in two planes.

Take a generic projection $f_1 : X_1 \to \mathbb{CP}^2$. The union of the intersection lines is the ramification locus in $(X_1)_0$ of $f_1^0 \colon (X_1)_0 \to \mathbb{CP}^2$, denoted by R_0. Let $(B_1)_0 = f_1^0(R_0)$ be the degenerated branch curve. It is a line arrangement, $(B_1)_0 = \bigcup_{j=1}^{4} L_j$.

Denote the singularities of $(B_1)_0$ as $f_1^0(m_i) = m_i, i = 1, \ldots, 5$ (the points m_0, m_6 do not lay on numerated lines, hence they are not singularities of $(B_1)_0$). The points m_1 and m_3 (resp. m_2, m_4, m_5) are called 1-points (resp. 2-points). They were studied in [5], [9], [17] and [22]. Other singularities may be intersection points, preferably called *parasitic intersections* in the works of Moishezon - Teicher ([17], [22]).

The regeneration of $(X_1)_0$ induces a regeneration of $(B_1)_0$ in such a way that each point on the typical fiber, say c, is replaced by two close points c, c'. The regeneration occurs as follows. First we regenerate in a neighbourhood of m_1, m_3 to get conics. Now, in a neighbourhood of m_2, m_4, m_5, the diagonal line regenerates to a conic [19, Regenerations Lemmas]), which is tangent to the line it intersected with [22, Lemma

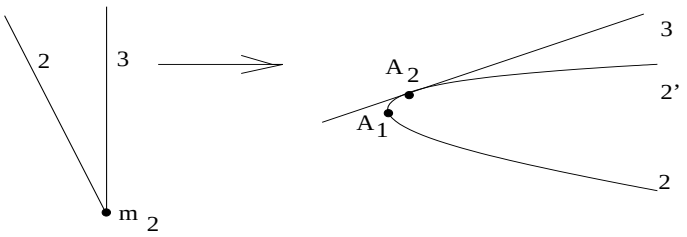

FIGURE 2. Regeneration around the point m_2

1]. See Figure 2 for the regeneration around m_2. When the line regenerates, the tangency regenerates into three cusps, [19, Regeneration Lemmas].

The resulting curve B_1 is of degree 8. Zariski [27] showed that B_1 is dual to rational curve of degree 5. We are interested in the braid monodromy factorization of B_1, and the groups $\pi_1(\mathbb{C}^2 \setminus B_1), \pi_1(\mathbb{CP}^2 \setminus B_1), \Pi_{(B_1)}$ and $\pi_1((X_1)_{Gal})$.

Notation 8. *We denote by Z_{ij} the counterclockwise halftwist of i and j along a path below the real axis. Denote by $Z^2_{i,jj'}$ the product $Z^2_{ij} \cdot Z^2_{ij'}$, and by $Z^2_{ii',jj'}$ the product $Z^2_{ij} \cdot Z^2_{i'j} \cdot Z^2_{ij'} \cdot Z^2_{i'j'}$. Likewise $Z^3_{i,jj'}$ denotes the product $(Z^3_{ij})^{Z_{jj'}} \cdot (Z^3_{ij}) \cdot (Z^3_{ij})^{Z_{jj'}{}^{-1}}$. Conjugation of braids is defined as $a^b = b^{-1}ab$.*

Theorem 9. *The braid monodromy factorization of the curve B_1 is the product of*

$$\varphi_{m_1} = Z_{11'} \tag{2}$$

$$\varphi_{m_2} = Z^3_{2'\ 33'} \cdot Z_{2\ 2'}{}^{Z^2_{2'\ 33'}} \tag{3}$$

$$\varphi_{m_3} = Z_{44'} \tag{4}$$

$$\varphi_{m_4} = Z^3_{11'\ 2} \cdot Z_{2\ 2'}{}^{Z^2_{11'\ 2}} \tag{5}$$

$$\varphi_{m_5} = Z^3_{33'\ 4} \cdot Z_{4\ 4'}{}^{Z^2_{33'\ 4}} \tag{6}$$

and the product of parasitic intersection braids

$$Z^2_{1\ 1',3\ 3'} \cdot Z^2_{1\ 1',4\ 4'} \cdot Z^2_{2\ 2',4\ 4'}. \tag{7}$$

Proof. The monodromies φ_{m_1} and φ_{m_3} (resp. $\varphi_{m_2}, \varphi_{m_4}, \varphi_{m_5}$) are derived from the regenerations around 1-points (resp. 2-points), so the induced monodromies are (2) and (4) (resp. (3), (5), (6)). See for example the braids of φ_{m_2} in Figure 3. The parasitic

intersections were formulated in [18], these are the intersections of the projections of the lines L_1 and L_3, L_1 and L_4, L_2 and L_4. See Figure 4.



FIGURE 3.

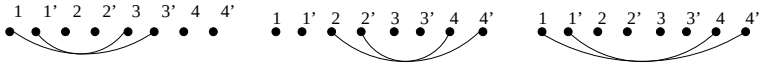

FIGURE 4. Parasitic intersections braids in B_1

Summing the degrees of the braids gives 56. Since the degree of the factorization is 56 [18, Cor. V.2.3], there are no other involved braids. □

Notation 10. *$\Gamma_{ii'}$ stands for Γ_i or $\Gamma_{i'}$. The relation $\langle \Gamma_a, \Gamma_b \rangle = e$ means $\Gamma_a \Gamma_b \Gamma_a = \Gamma_b \Gamma_a \Gamma_b$.*

Theorem 11. *The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by $\{\Gamma_j\}_{j=1,2,3,4}$ subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \textit{for i=1,2,3} \tag{8}$$

$$[\Gamma_1, \Gamma_i] = e \quad \textit{for i=3,4} \tag{9}$$

$$[\Gamma_2, \Gamma_4] = e \tag{10}$$

$$[\Gamma_4, \Gamma_3 \Gamma_2 \Gamma_1^2 \Gamma_2 \Gamma_3] = e. \tag{11}$$

Moreover, the group $\pi_1(\mathbb{CP}^2 \setminus B_1)$ is isomorphic to the Artin braid group $\mathcal{B}_5$ with the relations (8) - (10) and the group $\Pi_{(B_1)}$ is isomorphic to S_5.

Proof. The group $\pi_1(\mathbb{C}^2 \setminus B_1)$ is generated by the elements $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^4$, where Γ_j and $\Gamma_{j'}$ are loops in $\mathbb{C}^2$ around j and j' respectively.

By the van Kampen Theorem, the two branch points braids give the following relations

$$\Gamma_i = \Gamma_{i'} \quad \textit{for } i=1,4. \tag{12}$$

From the local braid monodromies $\varphi_{m_2}, \varphi_{m_4}, \varphi_{m_5}$ (for example Figures 3), we have branch points relations and triple relations

$$\langle \Gamma_{2'}, \Gamma_{33'} \rangle = \langle \Gamma_{2'}, \Gamma_3^{-1}\Gamma_{3'}\Gamma_3 \rangle = e \tag{13}$$

$$\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1} = \Gamma_2 \tag{14}$$

$$\langle \Gamma_{11'}, \Gamma_2 \rangle = \langle \Gamma_1^{-1}\Gamma_{1'}\Gamma_1, \Gamma_2 \rangle = e \tag{15}$$

$$\Gamma_2\Gamma_{1'}\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_2^{-1} = \Gamma_{2'} \tag{16}$$

$$\langle \Gamma_{33'}, \Gamma_4 \rangle = \langle \Gamma_3^{-1}\Gamma_{3'}\Gamma_3, \Gamma_4 \rangle = e \tag{17}$$

$$\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_4\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_4^{-1} = \Gamma_{4'}. \tag{18}$$

The parasitic intersections braids contribute commutative relations

$$[\Gamma_{11'}, \Gamma_{ii'}] = e \quad \textit{for } i=3,4 \tag{19}$$

$$[\Gamma_{22'}, \Gamma_{44'}] = e. \tag{20}$$

Relations (14), (16) and (18) get the forms $\Gamma_2^{-1}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_2 = \Gamma_{3'}, \Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_{2'}$ and $\Gamma_4^{-2}\Gamma_3\Gamma_4^2 = \Gamma_{3'}$ respectively. Relation (11) is derived by these three. Substituing these three relations in the other relations above gives relations (8), (9) and (10).

Adding the projective relation $\Gamma_{4'}\Gamma_4 \cdots \Gamma_{1'}\Gamma_1 = e$, we can omit relation (11) and get $\pi_1(\mathbb{CP}^2 \setminus B_1) \cong \mathcal{B}_5$. Therefore $\Pi_{(B_1)} \cong S_5$. □

Corollary 12. *$\pi_1((X_1)_{Gal})$ is trivial.*

4.2. **The surface X_2.** In [17], Moishezon - Teicher embed the surface $X_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ into a big projective space by the linear system $(\mathcal{O}(i), \mathcal{O}(j))$, where $i \geq 2, j \geq 3$. They used its degeneration to compute the fundamental group of the Galois cover corresponding to the generic projection of the surface onto $\mathbb{CP}^2$.

In this paper the embedding is by the linear system $(\mathcal{O}(1), \mathcal{O}(3))$. The degeneration of X_2 is a union of six planes embedded in $\mathbb{CP}^7$. It is viewed in Figure 5.

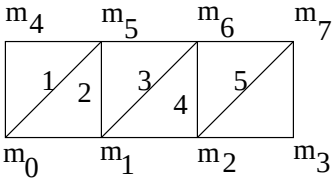

FIGURE 5. Degeneration of X_2

In a similar regeneration (see Sections 4 and 5) we get the branch curve B_2, which has a degree of 10. Moishezon [15] pointed out that the curve B_2 is the projective dual of a rational nodal curve of degree 6.

Theorem 13. *The braid monodromy factorization of the curve B_2 is the product of*

$$\varphi_{m_0} = Z_{11'} \tag{21}$$

$$\varphi_{m_1} = Z^3_{22',\ 3} \cdot Z_{3\ 3'}{}^{Z^2_{22',\ 3}} \tag{22}$$

$$\varphi_{m_2} = Z^3_{44',\ 5} \cdot Z_{5\ 5'}{}^{Z^2_{44',\ 5}} \tag{23}$$

$$\varphi_{m_5} = Z^3_{1',\ 22'} \cdot Z_{1\ 1'}{}^{Z^2_{1',\ 22'}} \tag{24}$$

$$\varphi_{m_6} = Z^3_{3',\ 44'} \cdot Z_{3\ 3'}{}^{Z^2_{3',\ 44'}} \tag{25}$$

$$\varphi_{m_7} = Z_{55'} \tag{26}$$

and the parasitic intersections braids

$$Z^2{}_{1\ 1',3\ 3'} \cdot \prod_{i=1}^{2} Z^{2\ (3)(3')}_{ii',4\ 4'} \cdot \prod_{i=1}^{3} Z^2{}_{ii',5\ 5'}. \tag{27}$$

Proof. By modification of indices in Figure 3 we get the needed factorization. See parasitic intersections braids in Figure 6.

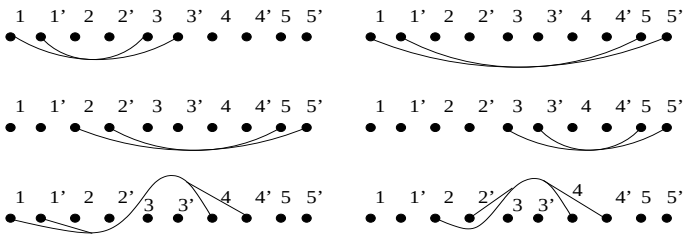

FIGURE 6. Parasitic intersections braids of B_2

□

Theorem 14. *The fundamental group $\pi_1(\mathbb{C}^2\backslash B_2)$ is generated by $\{\Gamma_j\}_{j=1,2,3,4,5}$ subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1}\rangle = e \quad \textit{for i=1,2,3,4} \tag{28}$$

$$[\Gamma_1, \Gamma_i] = e \quad \textit{for i=3,4,5} \tag{29}$$

$$[\Gamma_2, \Gamma_i] = e \quad \textit{for i=4,5} \tag{30}$$

$$[\Gamma_3, \Gamma_5] = e \tag{31}$$

$$\Gamma_2^{-1}\Gamma_1^{-2}\Gamma_2^{-1}\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2 = \Gamma_4^{-1}\Gamma_5^{-2}\Gamma_4^{-1}\Gamma_3\Gamma_4\Gamma_5^2\Gamma_4. \tag{32}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_2)$ is isomorphic to $\mathcal{B}_6$ and the group $\Pi_{(B_2)}$ is isomorphic to S_6.

Proof. This proof is similar to the one of Theorem 11. □

Corollary 15. *$\pi_1((X_2)_{Gal})$ is trivial.*

One can generalize this result as follows. Take $X_2 := \mathbb{CP}^1 \times \mathbb{CP}^1$ embedded in $\mathbb{CP}^{n+1}$ by $\mathcal{O}(n,1)$. This embedded toric surface corresponds to the convex polygon $P_{n,1} := \mathrm{Conv}\{(0,0),(n,0),(0,1),(n,1)\}$. This extends the results for this kind of toric variety, namely for an embedding $(\mathcal{O}(i),\mathcal{O}(j))$, where $i \geq 1, j \geq 3$.

Corollary 16. *In this case, the group $\Pi_{(B_2)}$ is isomorphic to S_{2n} and $\pi_1((X_2)_{Gal})$ is trivial.*

5. The surfaces X_3 and X_4

In this section we show that the groups $\pi_1(\mathbb{CP}^2 \setminus B_3)$ and $\pi_1(\mathbb{CP}^2 \setminus B_4)$ are certain quotients of Artin braid groups. In both cases, the groups $\Pi_{(B_3)}$ and $\Pi_{(B_4)}$ are symmetric groups.

5.1. **The surface X_3.** The degeneration of X_3 is a union of four planes embedded in $\mathbb{CP}^5$, as depicted in Figure 7.

Its branch curve $(B_3)_0$ in $\mathbb{CP}^2$ is a line arrangment. Regenerating it, the diagonal line regenerates to a conic, which is tangent to the lines L_1, L_3. When the lines

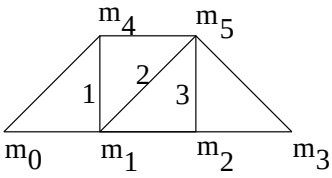

FIGURE 7. Degeneration of X_3

regenerate, each tangency regenerates into three cusps. We obtain the branch curve B_3, whose degree is 6 and which is dual to a rational nodal curve of degree 4.

Theorem 17. *The braid monodromy factorization related to B_3 is the product of*

$$\varphi_{m_1} = Z^3_{11'\ 2} \cdot Z_{2\ 2'}{}^{Z^2_{11'\ 2}} \tag{33}$$

$$\varphi_{m_5} = Z^3_{2'\ 33'} \cdot Z_{2\ 2'}{}^{Z^2_{2'\ 33'}} \tag{34}$$

$$\varphi_{m_2} = Z_{33'} \tag{35}$$

$$\varphi_{m_4} = Z_{11'} \tag{36}$$

and the parasitic intersections braid

$$Z^2_{1\ 1',3\ 3'}. \tag{37}$$

Proof. Similar proof as in Theorem 9. □

Theorem 18. *The fundamental group $\pi_1(\mathbb{C}^2 \setminus B_3)$ is generated by $\Gamma_1, \Gamma_2, \Gamma_3$ subject to the relations*

$$\langle \Gamma_i, \Gamma_{i+1} \rangle = e \quad \textit{for i=1,2} \tag{38}$$

$$[\Gamma_1, \Gamma_3] = e \tag{39}$$

$$\Gamma_1^{-2}\Gamma_2\Gamma_1^2 = \Gamma_3^{-2}\Gamma_2\Gamma_3^2. \tag{40}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_3)$ is isomorphic to $\mathcal{B}_4/< \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 >$, and the group $\Pi_{(B_3)}$ is isomorphic to S_4.

Proof. By the van Kampen Theorem [26]

$$\Gamma_i = \Gamma_{i'} \quad \textit{for } i=1,3 \tag{41}$$

$$\langle \Gamma_{11'}, \Gamma_2 \rangle = \langle \Gamma_1^{-1}\Gamma_{1'}\Gamma_1, \Gamma_2 \rangle = e \tag{42}$$

$$\Gamma_2\Gamma_{1'}\Gamma_1\Gamma_2\Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_2^{-1} = \Gamma_{2'} \tag{43}$$

$$\langle \Gamma_{2'}, \Gamma_{33'} \rangle = \langle \Gamma_{2'}, \Gamma_3^{-1}\Gamma_{3'}\Gamma_3 \rangle = e \tag{44}$$

$$\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1} = \Gamma_2 \tag{45}$$

$$[\Gamma_{11'}, \Gamma_{33'}] = e. \tag{46}$$

From this presentation one can easily derive relations (38) - (40).

The projective relation $\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1 = e$ and (40) get the same form $\Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 = e$. Therefore $\pi_1(\mathbb{CP}^2 \setminus B_3) \cong \mathcal{B}_4 / < \Gamma_2\Gamma_3^2\Gamma_2\Gamma_1^2 >$ and $\Pi_{(B_3)} \cong S_4$. □

Corollary 19. *$\pi_1((X_3)_{Gal})$ is trivial.*

5.2. **The surface X_4.** The degeneration of X_4 is a union of six planes $(X_4)_0$ embedded in $\mathbb{CP}^7$ (Figure 8). It has A_1 singularity as explained in the introduction.

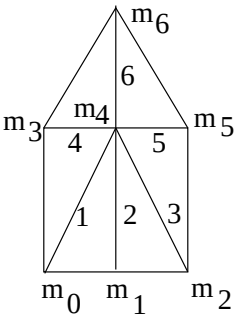

FIGURE 8. Degeneration of X_4

The regeneration of $(X_4)_0$ induces a regeneration on its branch curve $(B_4)_0$. Around the 1-points, the regeneration yields conics. A regeneration around 6-points was studied already, therefore we state briefly that we get a curve B_4 of degree 12 with six branch points, 24 cusps, and 24 nodes.

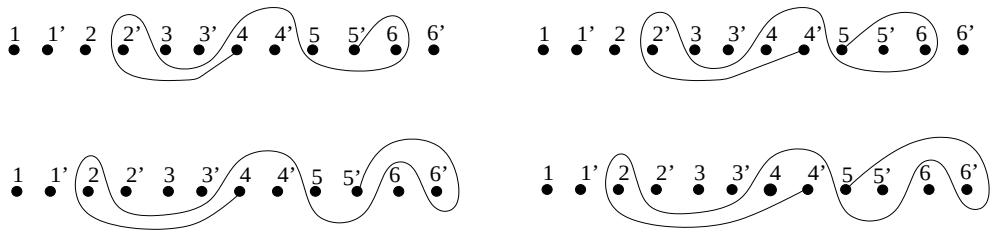

FIGURE 9.

Theorem 20. *The braid monodromy factorization of the curve B_4 is the product of*

$$\varphi_{m_i} = Z_{i+1,i+1'} \quad \textit{for } i=0,\ 1,\ 2,\ 3 \tag{47}$$

$$\varphi_{m_i} = Z_{ii'} \quad \textit{for } i=5,\ 6 \tag{48}$$

$$\begin{aligned}\varphi_{m_4} = {} & Z^3_{1',2\,2'} \cdot \left(Z^2_{1',ii'}\right)^{Z^2_{1',2\,2'}}_{i=3,5,6} \cdot \left(Z^3_{1',4\,4'}\right)^{Z^2_{1',2\,2'}} \cdot \left(Z^2_{1,\ ii'}\right)_{i=3,5,6} \cdot \\ & Z_{1\ 1'}{}^{Z^2_{1',4\,4'}Z^2_{1',2\,2'}} \cdot \left(Z^2_{3',ii'}\right)_{i=4,6} \cdot \left(Z^3_{2\,2',3}\right)^{Z^2_{1',2\,2'}} \cdot \left(Z^3_{3',5\,5'}\right)^{Z^2_{3',4\,4'}} \cdot \\ & \left(Z^2_{3',ii'}\right)^{Z^{-2}_{ii',5\,5'}}_{i=4,6} \cdot Z_{3\ 3'}{}^{Z^2_{2\,2',3}Z^2_{3',5\,5'}Z^2_{3',4\,4'}Z^2_{1',2\,2'}} \cdot H^{Z^2_{1',2\,2'}Z^2_{2\,2',3}}\end{aligned} \tag{49}$$

where

$$\begin{aligned}H = {} & Z^3_{2',4\,4'} \cdot Z^3_{2,4\,4'} \cdot Z^3_{5\,5',6} \cdot \left(Z^3_{5\,5',6'}\right)^{Z^{-2}_{6\,6'}} \cdot Z^2_{2\,6} \cdot \left(Z^2_{2'\,6}\right)^{Z^2_{2',4\,4'}} \cdot \left(Z^2_{2\,6'}\right)^{Z^2_{2,4\,4'}Z^{-2}_{6\,6'}} \cdot \\ & \left(Z^2_{2'\,6'}\right)^{Z^{-2}_{2\,2'}Z^{-2}_{6\,6'}} \cdot h_1 \cdot h_2 \cdot h_3 \cdot h_4\end{aligned}$$

and h_1, h_2, h_3, h_4 are the braids related to the paths in Figure 9.

Proof. The regeneration around the 1-points $m_0, m_1, m_2, m_3, m_5, m_6$ and the 6-point m_4 was studied already, for example in [1] and [22]. The resulting braids (47) and (48) (resp. (49)) are related to the 1-points (resp. 6-point). □

Theorem 21. *The fundamental group $\pi_1(\mathbb{C}^2 \setminus B_4)$ is generated by $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ subject to the relations*

$$\langle \Gamma_1, \Gamma_i \rangle = e \quad \textit{for } i=2,4 \tag{50}$$

$$\langle \Gamma_3, \Gamma_i \rangle = e \quad \textit{for } i=2,5 \tag{51}$$

$$[\Gamma_1, \Gamma_i] = e \quad \textit{for } i=3,5 \tag{52}$$

$$[\Gamma_2, \Gamma_i] = e \quad \textit{for } i=4,5 \tag{53}$$

$$[\Gamma_4, \Gamma_i] = e \quad \textit{for } i=3,5 \tag{54}$$

$$[\Gamma_2, \Gamma_1^2 \Gamma_4 \Gamma_1^{-2}] = e \tag{55}$$

$$[\Gamma_1, \Gamma_4^2 \Gamma_2^2] = [\Gamma_3, \Gamma_2^2 \Gamma_5^2] = e. \tag{56}$$

The group $\pi_1(\mathbb{CP}^2 \setminus B_4)$ is isomorphic to the quotient of $\mathcal{B}_6$ by the subgroup $< [\Gamma_1, \Gamma_4^2\Gamma_2^2], [\Gamma_3, \Gamma_2^2\Gamma_5^2], [\Gamma_2, \Gamma_1^2\Gamma_4\Gamma_1^{-2}], \Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_5\Gamma_4\Gamma_3^2\Gamma_2^2\Gamma_1^2 >$, and $\Pi_{(B_4)}$ is isomorphic to S_6.

Proof. We apply the van Kampen Theorem [26] in order to get the presentation of $\pi_1(\mathbb{C}^2 \setminus B_4)$. The monodromies $\varphi_{m_0}, \varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3}, \varphi_{m_5}, \varphi_{m_6}$ contribute the relations

$$\Gamma_i = \Gamma_{i'} \quad \textit{for } i=1, \ldots, 6. \tag{57}$$

The monodromy φ_4 (except the H factors) contributes branch points relations, commutative relations and triple relations (see (49)):

$$\langle \Gamma_{1'}, \Gamma_{22'} \rangle = \langle \Gamma_{1'}, \Gamma_{2'}\Gamma_2\Gamma_{2'}^{-1} \rangle = e \tag{58}$$

$$\langle \Gamma_{1'}, \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_{44'}\Gamma_{2'}\Gamma_2 \rangle = \langle \Gamma_{1'}, \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_{4'}\Gamma_4\Gamma_{4'}^{-1}\Gamma_{2'}\Gamma_2 \rangle = e \tag{59}$$

$$\langle \Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_2\Gamma_{2'}\Gamma_2\Gamma_{1'}^{-1}\Gamma_2^{-1}\Gamma_{2'}^{-1}, \Gamma_3 \rangle \tag{60}$$

$$= \langle \Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_2^{-1}\Gamma_{2'}\Gamma_2\Gamma_{1'}^{-1}\Gamma_2^{-1}\Gamma_{2'}^{-1}, \Gamma_3 \rangle = \langle \Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}^{-1}\Gamma_2^{-1}\Gamma_{2'}^{-1}, \Gamma_3 \rangle = e$$

$$\langle \Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_4^{-1}\Gamma_{4'}^{-1}, \Gamma_{55'} \rangle = \langle \Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_4^{-1}\Gamma_{4'}^{-1}, \Gamma_{5'}\Gamma_5\Gamma_{5'}^{-1} \rangle = e \tag{61}$$

$$[\Gamma_{1'}, \Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_{ii'}\Gamma_{2'}\Gamma_2] = e \quad \textit{for 3,5,6} \tag{62}$$

$$[\Gamma_1, \Gamma_{ii'}] = e \quad \textit{for 3,5,6} \tag{63}$$

$$[\Gamma_{3'}, \Gamma_{ii'}] = e \quad \textit{for i=4,6} \tag{64}$$

$$[\Gamma_{3'}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_{4'}\Gamma_4\Gamma_{4'}^{-1}\Gamma_{5'}\Gamma_5\Gamma_{4'}\Gamma_4] \tag{65}$$

$$= [\Gamma_{3'}, \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_{4'}\Gamma_4\Gamma_{4'}\Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_{5'}\Gamma_5\Gamma_{4'}\Gamma_4] = e$$

$$[\Gamma_{3'}, \Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_{66'}\Gamma_{5'}\Gamma_5] = e \tag{66}$$

$$\Gamma_{1'}^{\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_4^{-1}\Gamma_{4'}^{-1}} = \Gamma_1 \tag{67}$$

$$\Gamma_3^{\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_{1'}^{-1}\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}} = \Gamma_{3'}^{\Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_5^{-1}\Gamma_{5'}^{-1}}. \tag{68}$$

The H factors contribute the relations

$$\langle \bar{\Gamma}_{22'}, \bar{\Gamma}_4 \rangle = \langle \bar{\Gamma}_{22'}, \bar{\Gamma}_{4'} \rangle = \langle \bar{\Gamma}_{22'}, \bar{\Gamma}_{4'}\bar{\Gamma}_4\bar{\Gamma}_{4'}^{-1} \rangle = e \tag{69}$$

$$\langle \bar{\Gamma}_{55'}, \bar{\Gamma}_6 \rangle = \langle \bar{\Gamma}_{5'}\bar{\Gamma}_5\bar{\Gamma}_{5'}^{-1}, \bar{\Gamma}_6 \rangle = e \tag{70}$$

$$\langle \bar{\Gamma}_{55'}, \bar{\Gamma}_6^{-1}\bar{\Gamma}_{6'}\bar{\Gamma}_6 \rangle = \langle \bar{\Gamma}_{5'}\bar{\Gamma}_5\bar{\Gamma}_{5'}^{-1}, \bar{\Gamma}_6^{-1}\bar{\Gamma}_{6'}\bar{\Gamma}_6 \rangle = e \tag{71}$$

$$[\bar{\Gamma}_2, \bar{\Gamma}_6] = [\bar{\Gamma}_{2'}, \bar{\Gamma}_6^{\bar{\Gamma}_{4'}\bar{\Gamma}_4}] = [\bar{\Gamma}_2, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6\bar{\Gamma}_{4'}\bar{\Gamma}_4}] = [\bar{\Gamma}_{2'}^{\bar{\Gamma}_2}, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6}] = e \tag{72}$$

$$\bar{\Gamma}_4^{\bar{\Gamma}_{2'}^{-1}\bar{\Gamma}_4^{-1}\bar{\Gamma}_{4'}^{-1}} = \bar{\Gamma}_{5'}^{\bar{\Gamma}_6^{-1}} \tag{73}$$

$$\bar{\Gamma}_{4'}^{\bar{\Gamma}_{2'}^{-1}\bar{\Gamma}_4^{-1}\bar{\Gamma}_{4'}^{-1}} = \bar{\Gamma}_5^{\bar{\Gamma}_{5'}^{-1}\bar{\Gamma}_6^{-1}} \tag{74}$$

$$\bar{\Gamma}_4^{\bar{\Gamma}_2^{-1}\bar{\Gamma}_4^{-1}\bar{\Gamma}_{4'}^{-1}} = \bar{\Gamma}_{5'}^{\bar{\Gamma}_6^{-1}\bar{\Gamma}_{6'}^{-1}\bar{\Gamma}_6} \tag{75}$$

$$\bar{\Gamma}_{4'}^{\bar{\Gamma}_2^{-1}\bar{\Gamma}_4^{-1}\bar{\Gamma}_{4'}^{-1}} = \bar{\Gamma}_5^{\bar{\Gamma}_{5'}^{-1}\bar{\Gamma}_6^{-1}\bar{\Gamma}_{6'}^{-1}\bar{\Gamma}_6} \tag{76}$$

where $\bar{\Gamma}_i = \Gamma_i$ for i = 1,3',4,4',5,5',6,6', $\bar{\Gamma}_{1'} = \Gamma_{1'}^{\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}}$, $\bar{\Gamma}_2 = \Gamma_2^{\Gamma_{2'}\Gamma_2\Gamma_3^{-1}\Gamma_{1'}^{-1}\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}}$, $\bar{\Gamma}_{2'} = \Gamma_{2'}^{\Gamma_2\Gamma_3^{-1}\Gamma_{1'}^{-1}\Gamma_3\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}}$, $\bar{\Gamma}_3 = \Gamma_3^{\Gamma_2^{-1}\Gamma_{2'}^{-1}\Gamma_3^{-1}}$.

We can simplify the relations and add other needed ones. First we simplify (69) - (76) by replacing $\bar{\Gamma}_{1'}, \bar{\Gamma}_2, \bar{\Gamma}_{2'}$ and $\bar{\Gamma}_3$ with the above given expressions. Then we replace each $\Gamma_{i'}$ in the given relations with Γ_i. We get the following relations:

(77) $\langle \Gamma_i, \Gamma_{i+1} \rangle = e$ *for i=1,2*

(78) $\langle \Gamma_5, \Gamma_i \rangle = e$ *for i=3,6*

(79) $\langle \Gamma_1, \Gamma_2^{-2}\Gamma_4\Gamma_2^2 \rangle = \langle \Gamma_1, \Gamma_2^{-1}\Gamma_4\Gamma_2 \rangle = e$

(80) $[\Gamma_1, \Gamma_i] = e$ *for i=3,5,6*

(81) $[\Gamma_2, \Gamma_6] = e$

(82) $[\Gamma_3, \Gamma_i] = e$ *for i=4,6*

(83) $[\Gamma_1, \Gamma_4^2\Gamma_2^2] = [\Gamma_3, \Gamma_2^2\Gamma_5^2] = e$

(84) $[\Gamma_1, \Gamma_2^{-2}\Gamma_i\Gamma_2^2] = e$ *for i=3,5,6*

(85) $[\Gamma_3, \Gamma_5^{-2}\Gamma_i\Gamma_5^2] = e$ *for i=4,6*

(86) $[\Gamma_1^{-1}\Gamma_2\Gamma_1, \Gamma_4^{-2}\Gamma_6\Gamma_4^2] = e$

(87) $\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1 = \Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_2.$

Now we rewrite (87) as $\Gamma_5 = \Gamma_6^{-1}\Gamma_4\Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_6$. When replacing Γ_5 with this expression in $\langle \Gamma_5, \Gamma_6 \rangle = e$, we get $\langle \Gamma_4, \Gamma_6 \rangle = e$. Now we rewrite (87) as $\Gamma_1^{-1}\Gamma_2\Gamma_1 = \Gamma_3^{-1}\Gamma_4^{-1}\Gamma_6\Gamma_5\Gamma_6^{-1}\Gamma_4\Gamma_3$, and replacing $\Gamma_1^{-1}\Gamma_2\Gamma_1$ by this expression in (86), gives $[\Gamma_4, \Gamma_5] = e$. And when rewriting it as $\Gamma_6 = \Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_5^{-1}$ and replacing Γ_6 with this expression in $[\Gamma_3, \Gamma_6] = e$, we get $[\Gamma_2, \Gamma_5] = e$. Now, replacing Γ_5 in this relation with $\Gamma_6^{-1}\Gamma_4\Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_6$, gives $[\Gamma_2, \Gamma_4] = e$. These obtained

relations simplify (77) - (87) to

$$\langle \Gamma_1, \Gamma_i \rangle = e \quad \textit{for i=2,4} \tag{88}$$

$$\langle \Gamma_3, \Gamma_i \rangle = e \quad \textit{for i=2,5} \tag{89}$$

$$\langle \Gamma_6, \Gamma_i \rangle = e \quad \textit{for i=4,5} \tag{90}$$

$$[\Gamma_1, \Gamma_i] = e \quad \textit{for i=3,5,6} \tag{91}$$

$$[\Gamma_2, \Gamma_i] = e \quad \textit{for i=4,5,6} \tag{92}$$

$$[\Gamma_3, \Gamma_i] = e \quad \textit{for i=4,6} \tag{93}$$

$$[\Gamma_4, \Gamma_5] = e \tag{94}$$

$$[\Gamma_1, \Gamma_4^2\Gamma_2^2] = [\Gamma_3, \Gamma_2^2\Gamma_5^2] = e \tag{95}$$

$$\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1 = \Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_2. \tag{96}$$

Now we omit the generator Γ_6 from all relations by replacing it with $\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_5^{-1}$. The relations $[\Gamma_2, \Gamma_6] = e, [\Gamma_3, \Gamma_6] = e, \langle \Gamma_4, \Gamma_6 \rangle = e$ and $\langle \Gamma_5, \Gamma_6 \rangle = e$ are transformed to $[\Gamma_1, \Gamma_3] = e, [\Gamma_1, \Gamma_5] = e, \langle \Gamma_1, \Gamma_4 \rangle = e$ and $\langle \Gamma_2, \Gamma_3 \rangle = e$ respectively. These ones exist already in the list of relations. There is one added relation $[\Gamma_2, \Gamma_1^2\Gamma_4\Gamma_1^{-2}] = e$ which is obtained in the same way from $[\Gamma_1, \Gamma_6] = e$.

Therefore $\pi_1(\mathbb{C}^2 \setminus B_4)$ has relations (50) - (56) and therefore is isomorphic to $\mathcal{B}_6/ < [\Gamma_1, \Gamma_4^2\Gamma_2^2], [\Gamma_3, \Gamma_2^2\Gamma_5^2], [\Gamma_2, \Gamma_1^2\Gamma_4\Gamma_1^{-2}] >$.

Now, the projective relation $\Gamma_{6'}\Gamma_6\Gamma_{5'}\Gamma_5\Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1 = e$ gets the form $\Gamma_6^2\Gamma_5^2\Gamma_4^2\Gamma_3^2\Gamma_2^2\Gamma_1^2 = e$. We replace Γ_6 again with the relevant expression and get $\Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_5\Gamma_4\Gamma_3^2\Gamma_2^2\Gamma_1^2 = e$. Therefore

$$\pi_1(\mathbb{CP}^2\setminus B_4) \cong \mathcal{B}_6/ < [\Gamma_1, \Gamma_4^2\Gamma_2^2], [\Gamma_3, \Gamma_2^2\Gamma_5^2], [\Gamma_2, \Gamma_1^2\Gamma_4\Gamma_1^{-2}], \Gamma_5\Gamma_4\Gamma_3\Gamma_2\Gamma_1^2\Gamma_2^{-1}\Gamma_3^{-1}\Gamma_5\Gamma_4\Gamma_3^2\Gamma_2^2\Gamma_1^2 > .$$

Hence, the group $\Pi_{(B_4)}$ is isomorphic to S_6. □

Corollary 22. *$\pi_1((X_4)_{Gal})$ is trivial.*

References

[1] Amram, M., *Galois Covers of Algebraic Surfaces,* Ph.D. Dissertation, 2001.

[2] Amram, M., Teicher, M., *On the degeneration, regeneration and braid monodromy of the surface $T \times T$*, Acta Applicandae Mathematicae, 75(1), 2003, 195-270.

[3] Amram, M., Teicher, M., *The fundamental group of the complement of the branch curve of the surface* $T \times T$, Osaka, Japan, 40(4), 2003, 1-37.

[4] Amram, M., Ciliberto, C., Miranda, R., Teicher, M., *Braid monodromy factorization for a non-prime* $K3$ *surface branch curve*, submitted.

[5] Amram, M., Goldberg, D., Teicher, M., Vishne U., *The fundamental group of a Galois cover of* $\mathbb{CP}^1 \times T$, Algebraic and Geometric Topology Volume 2, 2002, paper no. 20, 403-432.

[6] Amram, M., Goldberg, D., *Higher degree Galois covers of* $\mathbb{CP}^1 \times T$, Algebraic and Geometric Topology 4, 2004, paper no. 37, 841-859.

[7] Amram, M., Teicher, M., Vishne, U., *The Coxeter quotient of the fundamental group of a Galois cover of* $T \times T$, to appear.

[8] Amram, M., Teicher, M., Vishne, U., *The fundamental group of the Galois cover of the surface* $T \times T$, submitted.

[9] Amram, M., Teicher, M., Vishne, U., *Fundamental groups of Galois covers of Hirzebruch surfaces* $F_1(2,2)$, submitted.

[10] Auroux, D., Donaldson, S. K., Katzarkov, L., Yotov, M., *Fundamental groups of complements of plane curves and symplectic invariants*, Topology 43, 2004, 1285-1318.

[11] Bruns, W., Gubeladze, J., Trung, N.V., *Normal polytopes, triangulations, and Koszul algebras* J. reine angew. Math. **485**, 1997, 123-160.

[12] Fulton, W., *Introduction to Toric Varieties, Ann. of Math. Studies* **131** Princeton Univ. Press, 1993.

[13] Hu, S., *Semi-stable degeneration of toric varieties and their hypersurfaces*, 2003, AG/0110091.

[14] Moishezon, B., *Algebraic surfaces and the arithmetic of braids, II*, Cont. Math. 44, 1985, 311-344.

[15] Moishezon, B., *On cuspidal branch curves*, J. Algebraic Geometry **2** 1993.

[16] Moishezon, B., Robb, A., Teicher, M., *On Galois covers of Hirzebruch surfaces*, Math. Ann. **305**, 1996, 493-539.

[17] Moishezon, B., Teicher, M., *Simply connected algebraic surfaces of positive index*, Invent. Math. **89**, 1987, 601-643.

[18] Moishezon, B., Teicher, M., *Braid group technique in complex geometry I, Line arrangements in* $\mathbb{CP}^2$, Contemporary Math. **78**, 1988, 425-555.

[19] Moishezon, B., Teicher, M., *Braid group technique in complex geometry II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lect. Notes in Math. Vol. **1479**, 1990.

[20] Moishezon, B., Teicher, M., *Finite fundamental groups, free over* $\mathbb{Z}/c\mathbb{Z}$*, Galois covers of* $\mathbb{CP}^2$, Math. Ann. **293** 1992, 749-766.

[21] Moishezon, B., Teicher, M., *Braid group technique in complex geometry III: Projective degeneration of V_3,* Contemp. Math. **162**, 1993, 313-332.

[22] Moishezon, B., Teicher, M., *Braid group technique in complex geometry IV: Braid monodromy of the branch curve S_3 of $V_3 \to \mathbb{CP}^2$ and application to $\pi_1(\mathbb{CP}^2 - S_3, *)$,* Contemporary Math. **162**, 1993, 332-358.

[23] T. Oda, *Convex Bodies and Algebraic Geometry,* Springer-Verlag, 1985.

[24] Rowen, L.H., Teicher, M., Vishne, U., *Coxeter Covers of the Symmetric Groups*, J. Group Theory, 2004, to appear.

[25] B. Sturmfels, *Equations defining toric varieties,* Algebraic Geometry-Santa Cruz 1995, Proc. Sympos. Pure Math. **62**, 1997, 437-449.

[26] van Kampen, E.R., *On the fundamental group of an algebraic curve*, Amer. J. Math. **55**, 1933, 255-260.

[27] Zariski, O., *On the Poincare group of rational plane curve*, Amer. J. Math., **58** 1936, 607-619.

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