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Classical Lebesgue Integration Theorems for the Riemann Integral

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Abstract

In this paper, we use the concept of gauges to provide easy proofs (proofs that are not difficult and do not use any measure or Lebesgue integration theory except for the notion of measure zero) of integration theorems for the Riemann integral that are closely related to classical Lebesgue integration theorems.

An undergraduate mathematics student will most likely be rigorously exposed to the Riemann integral, working directly with its definition and proving various results about the integral. However, if the student goes on to take higher courses in mathematics, he or she will undoubtedly be exposed to the Lebesgue integral, which compared to the Riemann integral, is very different and lot harder. Because of this, students sometimes have a difficult time adjusting to the Lebesgue integral and hence, some authors even claim that, "the Lebesgue integral is quite frankly too difficult to be taught in an undergraduate class."

To ease this transition from the Riemann integral to the Lebesgue integral, some authors have called for the teaching of what is known as the Kurzweil-Henstock integral (or sometimes called the Gauge integral) for the following two reasons: the integral only slightly differs in definition when compared to the Riemann integral and in some cases the even generalizes the Lebesgue integral. Therefore, a student can be exposed to a few of the standard integration theorems for Lebesgue integral, but stay within a comfortable framework.

However, the proofs of these integration theorems for the Kurzweil-Henstock integral are quite difficult and might not be understood by students in advanced calculus or introduction to analysis courses. One way to avoid this is to use the

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concept of gauges to prove classical integration theorems in modified forms for the Riemann integral, and hence, with the aid of gauges, we will prove the following three theorems for the Riemann integral:

1) Riemann integrability and a.e. equality implies equal integrals
2) the dominated convergence theorem
3) the monotone convergence theorem

For an elementary discussion of the Kurzweil-Henstock integral, one should consult [1, p. 274-311], and for a more detailed discussion of the Kurzweil-Henstock integral, one should consult [3]. For a detailed discussion of Lebesgue integration, one should consult any Real Analysis text, for example [2].

Before we start our proofs, we need to state a few definitions and theorems for the the reader who has never seen them before and we will give a definition of the Riemann integral for the sake of clarifying our notation.

Definition 1 *A sequence of functions $\{f_n(x)\}$ is said to be **uniformly bounded** over a set S if there exists an $M > 0$ such that for any $x \in S$ and $n \in \mathbb{Z}^+$, $|f_n(x)| < M$.*

Definition 2 *A **partition** Q of an interval $[a, b]$ is a collection of non-overlapping closed intervals $\{[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]\}$ who's union is $[a, b]$. If for each $i = 1, 2, 3, ..., n$, a point t_i has been chosen from each interval $[x_{i-1}, x_i]$, then the collection of points $\{t_i\}$ and intervals Q together is called a **tagged partition**.*

Definition 3 *A **gauge** on an interval $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. If $\delta(t)$ is a gauge on $[a, b]$, then a tagged partition Q is said to be $\delta(t)$ - fine if for each $i = 1, 2, 3, ..., n$, $t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$.*

The following theorem shows the existence of a $\delta(t)$ - fine partition Q for any gauge $\delta(t)$. For a proof one should consult [1, p.146].

Theorem 1 *If $\delta(t)$ is a gauge on $[a, b]$, then there exists a $\delta(t)$ - fine partition of $[a, b]$.*

Definition 4 *If Q is a tagged partition, then the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ is the number*

$$S(f, Q) = \sum f(t_i)(x_i - x_{i-1}).$$

*The number $max\{(x_1 - x_0), \cdots, (x_{n-1} - x_n)\}$ is said to be the **norm** of Q and is denoted by $||Q||$. A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** over $[a, b]$ if there exists an $L \in \mathbb{R}$ such that for any $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that for any tagged partition Q with $||Q|| < \delta_\epsilon$,*

$$|S(f, Q) - L| < \epsilon,$$

*and L is called the **Riemann integral** of $f(x)$ over $[a, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be **Kurzweil-Henstock** integrable (or "Gauge integrable" or KH integrable for short) over $[a, b]$ if there exists an $L \in \mathbb{R}$ such that for any $\epsilon > 0$,*

there exists a gauge $\delta_\epsilon(t)$ on $[a,b]$ such that for any $\delta_\epsilon(t)$ - fine tagged partition Q of $[a,b]$,

$$|S(f,Q) - L| < \epsilon.$$

*and L is said to be the **Kurzweil-Henstock integral** (or the KH integral) of $f(x)$ over $[a,b]$. The class of all Riemann integrable functions over $[a,b]$ is denoted by $R[a,b]$ and the class of all KH integrable functions over $[a,b]$ is denoted by $R^*[a,b]$.*

Two important facts should be noted at this point. First, it can easily shown that if $f(x) \in R[a,b]$ then $f(x) \in R^*[a,b]$ and the Riemann integral of $f(x)$ over $[a,b]$ is the same as the KH integral of $f(x)$ over $[a,b]$. Second, the KH integral can be defined over subsets of $\mathbb{R}$ that are more general then closed and bounded intervals, although we will not define how this will be done. For such a definition and a proof of the above statement, see [1, p. 274-300].

Theorem 2 *If $f(x) \in R[a,b]$ then $f(x)$ is bounded over $[a,b]$.*

For a proof of this one can consult any introduction to analysis text, for example [1, p. 200]. It should be noted that this is not true for functions in $R^*[a,b]$. For examples of unbounded KH functions, one should again consult [3].

Theorem 3 (Lebesgue's Criteria) A function $f(x)$ is in $R[a,b]$ if and only if the set of discontinuities in $[a,b]$ has measure zero.

Although many different books prove this criteria, in the spirit of our topic, one can consult [1, p.347] for an easy proof that utilizes gauges. It is extremely important to note that this does not characterize KH integrable functions. In fact, it can be shown that the set of all KH integrable functions is a proper subset of the set of all Lebesgue integrable functions (when dealing with integration over the real line.) For a detailed discussion of KH integrable functions, can again consult [3].

Finally, we introduce a very important concept known as measure zero.

Definition 5 *A set $G \subseteq \mathbb{R}$ is said to have **measure zero** if for any $\epsilon > 0$, there exists a collection of intervals $I = \{(a_n, b_n) : n \in \mathbb{Z}^+\}$ such that $G \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \epsilon$. If $X \subseteq \mathbb{R}$ and a property holds on all of X except for possibly a subset $S \subseteq X$ of measure zero, then the property is said to hold **almost everywhere** (or a.e. for short) on X.*

Now we are in a position to prove our first theorem, that two functions that are in R$[a,b]$ and equivalent a.e. on $[a,b]$ have the same integral over $[a,b]$ (this of course is the most elementary and fundamental of all integration theorems for the Lebesgue integral.) In general, however, if one function that is in R$[a,b]$ is equivalent to another function a.e. on $[a,b]$, then the second function might not even be integrable over the interval, as the easy example where $f(x) = 0$ for $x \in [0,1]$ and $g(x) = 0$ for $x \in [0,1] \cap \mathbb{Q}$ and $g(x) = 1$ for $x \in [0,1] \backslash \mathbb{Q}$ shows. Note that this partly gives an intuitive justification for the definition of almost everywhere in the sense that if we define two functions to be "almost equal to each other," then their respective integrals (if they exist) should be the same.

Lemma 1 *Let $f(x)$ and $g(x)$ be in $R[a,b]$ where $f(x) = g(x)$ a.e. on $[a,b]$, then $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.*

PROOF: We will prove that $\int_a^b F(x)\,dx = 0$ where F(x) = f(x) - g(x) for all $x \in [a,b]$. $F(x) \in R[a,b]$ so that $F(x)$ is bounded on $[a,b]$ and for any $\epsilon > 0$, there exists an L and a $\delta_\epsilon > 0$ such that for any tagged partition Q of $[a,b]$ with $||Q|| < \delta_\epsilon$, $|S(F,Q) - L| < \epsilon$. Let G be the set of all $x \in [a,b]$ such that $F(x) \neq 0$. By our hypothesis, G has measure zero so that there exists a collection of intervals $I = \{[a_n, b_n] : n \in \mathbb{Z}^+\}$ such that $G \subseteq \bigcup_{n=1}^{\infty}[a_n, b_n]$, $[a_n, b_n] \subseteq [a,b]$ for each $n \in \mathbb{Z}^+$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \frac{\epsilon}{2M}$ where $|F(x)| < M$ on $[a,b]$.

Now define a gauge $\delta(t)$ on $[a,b]$ where if $t \in G$ then $\delta(t) = min\{(b_i - a_i), \frac{\delta_\epsilon}{4}\}$ where i is the first $i \in \mathbb{Z}^+$ such that $t \in [a_i, b_i]$ where $[a_i, b_i] \in I$ and $\delta(t) = \frac{\delta_\epsilon}{4}$ otherwise. Therefore, for any $\delta(t)$ - fine tagged partition Q of $[a,b]$, the only nonzero contributions to $S(F,Q)$ are from those partition intervals of Q that have partition points t that are in G so that

$$|S(F(x),Q)| = |\sum F(t_i)(x_{i+1} - x_i)| \leq \sum |F(t_i)|(x_{i+1} - x_i)| \leq$$

$$M\sum(x_i - x_{i-1}) \leq \frac{2M\epsilon}{2M} = \epsilon$$

so that since $||Q|| < \delta_\epsilon$ because Q is $\delta(t)$ - fine, by definition $L = 0$. QED

Now we prove a version of the dominated convergence theorem that is suitable for the Riemann Integral. One should note that the essential content of the theorem is not lost when we pass it from the Lebesgue integral to the Riemann integral: that we can interchange limits when a convergent sequence of functions is "dominated" in some sense (and when the limit function is Riemann integrable in the Riemann integral case.)

Theorem 4 (Dominated Convergence Theorem) Let $\{f_n(x)\}$ be a uniformly bounded sequence of functions in $R[a,b]$ that converges to a function $f(x) \in R[a,b]$ on $[a,b]$, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

.

PROOF:

Let $F_n(x) = f_n(x) - f(x)$ for all $x \in [a,b]$. Since $\{f_n(x)\}$ is uniformly bounded on $[a,b]$ and $f(x)$ is bounded on $[a,b]$, $\{F_n(x)\}$ is clearly uniformly bounded on $[a,b]$ so let $|F_n(x)| < M$ for some $M > 0$ and all $n \in \mathbb{Z}^+$.

For any $\epsilon > 0$, let $S = \{t : |F_n(t)| < \epsilon\}$ for any $n \in \mathbb{Z}^+$ and define a gauge $\delta(t)$ on $[a,b]$ where $\delta(t) = \frac{\epsilon}{4M(b-a)}$ if $t \in S$ and $\delta(t) = \frac{1}{n(t)}$ otherwise, where

$n(t)$ is the last integer n such that $|F_n(t)| \geq \epsilon$. Now for any $N \in \mathbb{Z}^+$ such that $\frac{1}{N} < \frac{\epsilon}{4M(b-a)}$ and for any Riemann sum with a $\delta(t)$ - fine tagged partition Q of $[a,b]$, divide the tags into two sets $\{t_i'\}$ and $\{t_i''\}$ (with corresponding partition points x_i' and x_i'') where the first set consists of all tags of Q that are in S and the second consists of all tags of Q that are not in S, so that for any integer $n > N$,

$$|S(f_n, Q) - S(f, Q)| = |S(F_n, Q)| \leq |\sum F_n(t_i{}')(x_i' - x_{i-1}')| +$$

$$|\sum F_n(t_i'')(x_i'' - x_{i-1}'')| < \frac{\epsilon}{2}$$

since the definition of our gauge implies that the first and second terms are both $< \frac{\epsilon}{4}$.

Finally, since $f(x) \in R[a,b]$, there exists an $L \in \mathbb{R}$ and a $\gamma > 0$ such that for any Q of $[a,b]$ with $||Q|| < \gamma$, $|S(f,Q) - L| < \frac{\epsilon}{2}$. Now let $\delta'(t) = \min\{\delta(t), \frac{\gamma}{4}\}$. For any integer $n > N$ and $\delta'(t)$ fine tagged partition Q of $[a,b]$, the previous paragraph implies that

$$|S(f_n, Q) - L| \leq |S(f_n, Q) - S(f, Q)| + |S(f, Q) - L| < \epsilon,$$

but $||Q|| < \gamma$ and $f(x) \in R[a,b]$ implies that $f(x) \in R^*[a,b]$ with equal integral so that we obtain our desired result. QED

It should be noted that we actually proved a slightly stronger result, which we state in the following.

Corollary 1 *Let $\{f_n(x)\}$ be a uniformly bounded sequence of functions in $R^*[a,b]$ that converge to a bounded function $f(x) \in R^*[a,b]$ on $[a,b]$, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Finally, we prove our last desired result: the monotone convergence theorem. It turns out that this will follow trivially from the dominated convergence theorem.

Theorem 5 (Monotone Convergence Theorem) Let $\{f_n(x)\}$ be a monotone sequence of functions in $R[a,b]$ (monotone on $[a,b]$) such that $f_n(x)$ converges to a function $f(x) \in R[a,b]$ on $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

PROOF: We claim that $\{f_n(x)\}$ is uniformly bounded on $[a,b]$, which will immediately prove our desired result. $f(x)$ and $f_1(x)$ are both in $R[a,b]$, so that there exists an $M > 0$ and $M' > 0$ such that $|f(x)| < M$ and $|f_1(x)| < M'$ for

any $x \in [a,b]$. Let $L = \max\{M+1, M'\}$ We claim that for all $n \in \mathbb{Z}^+, |f_n(x)| \leq L$.

Now assume $f_1(x) \geq f(x)$ for all $x \in [a,b]$. If $f_1(x) \leq f(x)$ for all $x \in [a,b]$ then the following can be easily modified to fit this case. Now choose any $x' \in [a,b]$. If $f_n(x')$ converges to $f(x')$ then clearly $\{f_n(x')\}$ must be a decreasing sequence so that for all $n \in \mathbb{Z}^+, f_n(x) \leq f_1(x')$. However, choosing n large enough so that $|f_n(x') - f(x')| \leq 1$ gives us that $|f_n(x')| \leq 1 + |f(x')|$ so that for all $n \in \mathbb{Z}^+, |f_n(x)| \leq L$ by the definition of L. QED

The reader might have noticed that above proof only relied on the monotonicity of $\{f_n(x)\}$ on $[a,b]$ and the boundedness of each $f_n(x)$ and $f(x)$ on $[a,b]$. Therefore, we again proved a slightly stronger result that is stated below. It should be noted that results that generalize corollary 1 and corollary 2 exist. For a discussion of these, one should consult [3, p.115-135].

Corollary 2 *Let $\{f_n(x)\}$ be a monotone and bounded sequence of functions in $R^*[a,b]$ (monotone on $[a,b]$) such that $f_n(x)$ converges to a function $f(x) \in R^*[a,b]$ on $[a,b]$,,then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

References

[1] Bartle, R.G. and Sherbert D.R. Introduction to Real Analysis, John Wiley and Sons, Inc., 1986.

[2] Bruckner, A.M., Bruckner, J.B., and Thomas, B.S. Real Analysis, Prentice Hall, Inc., 1981.

[3] Bartle, R.G Modern Theory of Integration, American Mathematical Society, 2001.