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THE HEUN EQUATION AND THE CALOGERO-MOSER-SUTHERLAND SYSTEM I: THE BETHE ANSATZ METHOD

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Abstract. We propose and develop the Bethe Ansatz method for the Heun's equation. As an application, we justify the holomorphic perturbation for the 1-particle Inozemtsev model from the trigonometric model.

1. Introduction

Olshanetsky and Perelomov proposed the family of integrable quantum systems, which is called the Calogero-Moser-Sutherland system or the Olshanetsky-Perelomov system ([5]). In early 90's, Ochiai, Oshima and Sekiguchi classified the integrable models of quantum mechanics which are invariant under the action of a Weyl group with some assumption ([4]). For the B_N ($N \geq 3$) case, the generic model coincides with the Inozemtsev's model. Oshima and Sekiguchi found that the Hamiltonian of the 1-particle Inozemtsev model is transposed to the Heun's equation with full parameters ([6]). Here, the Heun's equation is the general second-order differential equation with four regular singular points.

In this article we will investigate the solution of the Heun's equation motivated by the analysis of the Calogero-Moser-Sutherland system.

In the articles ([8, 3]), we justified the regular perturbation for the Calogero-Moser-Sutherland model of type A_N from the trigonometric one. In ([8]), the author obtained results by applying the Bethe Ansats method for the A_N cases. We remark that the Bethe Ansatz for the A_N cases was established by Felder and Varchenko ([1]) by investigating

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the asymptotic of the integral expression of the solution of the KZB equation.

For the A_1 case, the Bethe Ansatz method was found in the textbook of Whittaker and Watson ([10]), although they did not use the word "Bethe Ansatz". Whittaker and Watson wrote them in the chapter "Lamé's equation". The Lamé equation is the special case of the Heun equation ([7]).

In this article we will propose and develop the Bethe Ansatz method for the Heun equation, and justify the regular perturbation for the 1-particle Inozemtsev model from the trigonometric model.

The Bethe Ansatz method replace the problem to find the eigenstates and spectral for the Hamiltonian with the problem to solve the transcendental equations for the finite variables which is called the Bethe Ansatz equation. To apply the Bethe Ansatz method and write down the Bethe Ansatz equation in our case, we heavily utilize the elliptic functions.

It would be impossible to solve the transcendental equation explicitly. But if we consider the trigonometric limit, then we have the algebraic equations instead of the transcendental equations and may solve the equations.

On the other hand, the Hamiltonian of the BC_1 Calogero-Sutherland model appears when we take the trigonometric limit, and the eigenstates for the BC_1 Calogero-Sutherland model are described by the Jacobi polynomial.

We will connect the Jacobi polynomial with the solution of the Bethe Ansatz equation for the trigonometric case, and will have the holomorphy of the perturbation. The holomorphy of the perturbation for the 1-particle Inozemtsev model from the trigonometric model follows from the holomorphy of the solution of the Bethe Ansatz equation with respect to the parameter of the periods of the elliptic functions.

This article is organized as follows. In section 2, we clarify the relationship between the Heun equation and the 1-particle Inozemtsev

system following ([6]). We note that the Hamiltonian of the 1-particle Inozemtsev model is given in terms of doubly periodic functions.

In section 3, we introduce the Bethe Ansatz method for the 1-particle Inozemtsev system, or equivalently the Heun equation. In section 3.1, we discuss the invariant subspace of the doubly periodic functions. In sections 3.2 and 3.3, we introduce the Bethe Ansatz equation and discuss the completeness for a given eigenvalue E. In this step, the relationship between the completeness and the eigenvalue of the invariant subspace will be discussed. In section 3.4, we rewrite the Bethe Ansatz equation in terms of the theta function in order to take the trigonometric limit.

In section 4, we take the trigonometric limit and solve the trigonometric Bethe Ansatz equation.

In section 5, we justify the holomorphic perturbation for the 1-particle Inozemtsev model from the trigonometric model. The key observation is that the holomorphy of the perturbation follows from the holomorphy of the solution of the Bethe Ansatz equation.

In section 6, we give comments.

In the appendix, we note definitions and formulae of the elliptic functions.

2. The Heun equation and the Hamiltonian of the 1-particle Inozemtsev system

The Hamiltonian of the 1-particle Inozemtsev model is given as follows:

$$H := -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i), \tag{2.1}$$

where $\wp(x)$ is the Weierstrass' $\wp$-function with periods $(2\omega_1, 2\omega_3)$, $\omega_0 := 0$, $\omega_2 := -(\omega_1+\omega_3)$ and l_i $(i = 0, 1, 2, 3)$ are coupling constants. In this article, we will consider the eigenvalues and the eigenfunctions of the Hamiltonian H. Let $f(x)$ be an eigenfunction of H with eigenvalue

E, i.e.,

(2.2)
$$(H-E)f(x)=\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i)-E\right)f(x)=0.$$

We will make a correspondence between the equation (2.1) and the Heun equation as was done in ([6]). We set $z=\wp(x)$ and $e_i=\wp(\omega_i)$ $(i=1,2,3)$. By a straightforward calculation, we have

$$\frac{d^2}{dx^2}=\wp'(x)^2\left(\frac{d^2}{dz^2}+\frac{1}{2}\left(\frac{1}{z-e_1}+\frac{1}{z-e_2}+\frac{1}{z-e_3}\right)\frac{d}{dz}\right).$$

Hence the equation (2.2) is equivalent to the equation

(2.3)
$$\left\{\frac{d^2}{dz^2}+\frac{1}{2}\left(\frac{1}{z-e_1}+\frac{1}{z-e_2}+\frac{1}{z-e_3}\right)\frac{d}{dz}-\frac{1}{4(z-e_1)(z-e_2)(z-e_3)}\cdot\right.$$
$$\left.\cdot\left(\tilde{C}+l_0(l_0+1)z+\sum_{i=1}^{3}l_i(l_i+1)\frac{(e_i-e_{i'})(e_i-e_{i''})}{z-e_i}\right)\right\}\tilde{f}(z)=0,$$

where $i',i''\in\{1,2,3\}$, $i'<i''$, i' and i'' are different from i, $\tilde{f}(\wp(x))=f(x)$ and $\tilde{C}=-E+\sum_{i=1}^{3}l_i(l_i+1)e_i$.

The equation have four singular points $e_1,\ e_2,\ e_3,\ \infty$ and all the singular points are regular. Therefore this equation is a kind of the Heun equation. The exponents at $z=e_i$ $(i=1,2,3)$ are $(l_i+1)/2$ and $-l_i/2$, and the exponents at ∞ are $(l_0+1)/2$ and $-l_0/2$. The parameter $\tilde{C}$ corresponds to the accessory parameter.

Conversely, if the differential equation of second order with four regular singular points on the Riemann sphere is given, we can transform it to the equation (2.3) with suitable l_i $(i=0,1,2,3)$ and $\tilde{C}$ by changing the variable $z\to\frac{az+b}{cz+d}$ and the transformation $f\mapsto(z-e_1)^{\alpha_1}(z-e_2)^{\alpha_2}(z-e_3)^{\alpha_3}f$.

In this article, we will find some property of the Heun equation by using the expression of elliptic function especially for the cases $l_i\in\mathbb{Z}_{\geq 0}$ $(i=0,1,2,3)$. We remark that the Heun equation with exponents (γ_i,δ_i) $(i=0,1,2,3)$ satisfying $\gamma_i-\delta_i\in\mathbb{Z}+\frac{1}{2}$ are transformed to the equation (2.3) with $l_i\in\mathbb{Z}_{\geq 0}$.

3. Bethe Ansatz equation for the Heun equation

3.1. **The invariant subspace of the doubly periodic functions.** In this subsection we will find the invariant space of the doubly periodic functions with respect to the Hamiltonian (2.1) for the cases $l_i \in \mathbb{Z}_{\geq 0}$ $(i = 0, 1, 2, 3)$ and count the dimension of it. In the terminology of the Heun's equation (2.3), the doubly periodic function in x corresponds to the algebraic function in the parameter $z(= \wp(x))$.

We mean the doubly periodic function up to signs as the function $f(x)$ s.t. $(f(x + 2\omega_1) = f(x)$ or $-f(x))$ and $(f(x + 2\omega_3) = f(x)$ or $-f(x))$, where $(2\omega_1, 2\omega_3)$ are basic periods of elliptic function.

Let k_i $(i = 0, 1, 2, 3)$ be the rearrangement of l_i such that $k_0 \geq k_1 \geq k_2 \geq k_3 (\geq 0)$. The dimension of the finite dimensional invariant space of the rational doubly periodic functions up to signs is given as follows.

Theorem 3.1. *(i) If the number $k_0 + k_1 + k_2 + k_3 (= l_0 + l_1 + l_2 + l_3)$ is even and $k_0 + k_3 \geq k_1 + k_2$, then the dimension of the finite dimensional invariant space of the rational doubly periodic functions up to signs w.r.t. the Hamiltonian (2.1) is $2k_0 + 1$.*

(ii) If the number $k_0 + k_1 + k_2 + k_3$ is even and $k_0 + k_3 < k_1 + k_2$, then the dimension is $k_0 + k_1 + k_2 - k_3 + 1$.

(iii) If the number $k_0 + k_1 + k_2 + k_3$ is odd and $k_0 \geq k_1 + k_2 + k_3 + 1$, then the dimension is $2k_0 + 1$.

(iv) If the number $k_0 + k_1 + k_2 + k_3$ is odd and $k_0 < k_1 + k_2 + k_3 + 1$, then the dimension is $k_0 + k_1 + k_2 + k_3 + 2$.

Proof. Let V be the finite dimensional invariant space of the rational doubly periodic functions up to signs and $T^{(i)}$ $(i = 0, 1, 2, 3)$ be operators on V defined by $T^{(i)} f(x) = f(-x + 2\omega_i)$. We set $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3} = \{f(x) \in V | T^{(i)} f(x) = \epsilon_i f(x) \text{ for } i = 0, 1, 2, 3\}$ $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 = \pm 1)$. From the definition of the operator H and the periodicity of the function which belongs to V, we have $(T^{(i)})^2 = 1$, $T^{(i)} T^{(j)} = T^{(j)} T^{(i)}$,

$T^{(i)}H = HT^{(i)}$ and $T^{(3)}T^{(2)}T^{(1)}T^{(0)} = 1$. Hence we have the decomposition,

$$V = \bigoplus_{\substack{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3=\pm 1 \\ \epsilon_0\epsilon_1\epsilon_2\epsilon_3=1}} V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}. \tag{3.1}$$

Let $f(x)$ be an element of $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}$. From the definition of $T^{(i)}$, we have the following local expansion

$$f(x-\omega_i) = \sum_{n\in\mathbb{Z}} c^{(i)}_{2n+(1-\epsilon_i)/2}(x-\omega_i)^{2n+(1-\epsilon_i)/2}, \tag{3.2}$$

where there exists M_i s.t. $c^{(i)}_{2n+(1-\epsilon_i)/2} = 0$ for $n < -M_i$. The exponents of the operator H at $x = \omega_i$ are $(-l_i, l_i+1)$. Let α_i be one of the exponent of the operator H at $x = \omega_i$ which has the same parity as $(1-\epsilon_i)/2$.

If $c^{(i)}_{2n+(1-\epsilon_i)/2} \neq 0$ for some n such that $2n+(1-\epsilon_i)/2 < \alpha_i$ then the functions $H^j f(x-\omega_i)$ $(j = 0, 1, 2, \dots)$ are linearly independent and contradict the finite dimensionality. Therefore we have $c^{(i)}_{2n+(1-\epsilon_i)/2} = 0$ for n such that $2n+(1-\epsilon_i)/2 < \alpha_i$.

Conversely if we have $c^{(i)}_{2n+(1-\epsilon_i)/2} = 0$ for all $i(\in \{0,1,2,3\})$ and n such that $2n+(1-\epsilon_i)/2 < \alpha_i$, the local expansions of the function $Hf(x)$ at $x = \omega_i$ satisfy the condition $\tilde{c}^{(i)}_{2n+(1-\epsilon_i)/2} = 0$ for all $i(\in \{0,1,2,3\})$ and n such that $2n+(1-\epsilon_i)/2 < \alpha_i$, where $\tilde{c}^{(i)}_{2n+(1-\epsilon_i)/2}$ are the coefficients of the function $Hf(x-\omega_i)$ defined similarly as (3.2).

Therefore the space $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}$ consist of the function whose coefficient of the local expansion (3.2) at $x = \omega_i$ $(i = 0, 1, 2, 3)$ satisfy $c^{(i)}_{2n+(1-\epsilon_i)/2} = 0$ for all $i(\in \{0,1,2,3\})$ and n such that $2n+(1-\epsilon_i)/2 < \alpha_i$.

From the decomposition (3.1) and the condition $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 = 1$, the number $\alpha_0+\alpha_1+\alpha_2+\alpha_3$ is even.

The dimension of the space $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}$ is $\max(1-(\alpha_0+\alpha_1+\alpha_2+\alpha_3)/2, 0)$. If $-(\alpha_0+\alpha_1+\alpha_2+\alpha_3) > 0$ then a basis of the space $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}$ is given by $\{\sigma_1(x)^{\alpha_1}\sigma_2(x)^{\alpha_2}\sigma_3(x)^{\alpha_3}\wp(x)^n\}_{0\leq n\leq -(\alpha_0+\alpha_1+\alpha_2+\alpha_3)/2}$, where $\sigma_i(x)$ are the co-sigma functions which are defined in the appendix.

The dimension of the space V is the summation of the dimension of the space $V_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}$ for $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 = 1$.

If $l_0+l_1+l_2+l_3$ is even, then the dimension of the finite dimensional invariant space of the doubly periodic functions up to signs w.r.t. the Hamiltonian (2.1) are

$$\left|\frac{l_0+l_1+l_2+l_3}{2}+1\right|+\left|\frac{l_0+l_1-l_2-l_3}{2}\right| + \left|\frac{l_0-l_1-l_2+l_3}{2}\right|+\left|\frac{l_0-l_1+l_2-l_3}{2}\right|. \tag{3.3}$$

If $l_0+l_1+l_2+l_3$ is odd, then the dimension of the finite dimensional invariant space of the doubly periodic functions up to signs w.r.t. the Hamiltonian (2.1) are

$$\left|\frac{l_0+l_1+l_2-l_3+1}{2}\right|+\left|\frac{l_0+l_1-l_2+l_3+1}{2}\right| + \left|\frac{l_0-l_1+l_2+l_3+1}{2}\right|+\left|\frac{-l_0+l_1+l_2+l_3+1}{2}\right|. \tag{3.4}$$

Hence we have the theorem. □

We will consider the multiplicity of the eigenvalues of the Hamiltonian H on the space of the finite dimensional invariant space of the doubly periodic functions up to signs. We have the following result, which will be used later.

Theorem 3.2. *If two of l_i $(i=0,1,2,3)$ are zero, then the roots of the characteristic polynomial of the operator H (2.1) on the invariant space of the finite dimensional doubly periodic function up to signs are distinct for generic ω_1 and ω_3.*

Proof. By shifting the variable $z \to z+\omega_i$ and rearranging the periods, we find that it is sufficient to show the $l_2=l_3=0$ and $l_0 \geq l_1$ cases. We set $\Phi(z)=(z-e_1)^{\tilde{\alpha}_1}(z-e_2)^{\tilde{\alpha}_2}(z-e_3)^{\tilde{\alpha}_3}$, where $\tilde{\alpha}_i=-l_i/2$ or $(l_i+1)/2$ for each $i(\in\{1,2,3\})$. We set $\tilde{f}(z)=\Phi(z)f(z)$. If the function $\tilde{f}(z)$

satisfy the differential equation (2.3), we have

(3.5)

$$\frac{d^2f(z)}{dz^2}+\sum_{i=1}^{3}\frac{2\tilde{\alpha}_i+\frac{1}{2}}{z-e_i}\frac{df(z)}{dz}+\left(\frac{(\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3-\frac{l_0}{2})(\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\frac{l_0+1}{2})z}{(z-e_1)(z-e_2)(z-e_3)}\right.$$
$$\left.+\frac{\frac{E}{4}-e_1(\tilde{\alpha}_2+\tilde{\alpha}_3)^2-e_2(\tilde{\alpha}_1+\tilde{\alpha}_3)^2-e_3(\tilde{\alpha}_1+\tilde{\alpha}_2)^2}{(z-e_1)(z-e_2)(z-e_3)}\right)f(z)=0,$$

The exponents of this equation at $z=e_2$ are $\begin{cases}(0,1/2) & (\tilde{\alpha}_2=0)\\ (-1/2,0) & (\tilde{\alpha}_2=1/2)\end{cases}$. We expand the function $f(z)$ around $z=e_2$, i.e., $f(z)=\sum_{r=0}^{\infty}c_r(z-e_2)^r$ and $c_0=1$. The recursive relations for the coefficients c_r are given as follows,

(3.6)

$$(e_2-e_1)(e_2-e_3)(r+2\tilde{\alpha}_2+\frac{1}{2})c_1$$
$$+(((2\tilde{\alpha}_2+2\tilde{\alpha}_3)(e_2-e_1)+(2\tilde{\alpha}_2+2\tilde{\alpha}_1)(e_2-e_3))r+q)c_0=0,$$

(3.7)

$$(r-1+\gamma_1)(r-1+\gamma_2)c_{r-1}+(e_2-e_1)(e_2-e_3)(r+1)(r+2\tilde{\alpha}_2+\frac{1}{2})c_{r+1}$$
$$+(((2\tilde{\alpha}_2+2\tilde{\alpha}_3+r)(e_2-e_1)+(2\tilde{\alpha}_2+2\tilde{\alpha}_1+r)(e_2-e_3))r+q)c_r=0,$$

where $r\geq 1$, $\gamma_1=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3-\frac{l_0}{2}$, $\gamma_2=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3+\frac{l_0+1}{2}$ and $q=\frac{E}{4}-e_1(\tilde{\alpha}_2+\tilde{\alpha}_3)^2-e_2(\tilde{\alpha}_1+\tilde{\alpha}_3)^2-e_3(\tilde{\alpha}_1+\tilde{\alpha}_2)^2+e_2\gamma_1\gamma_2$.

Either γ_1 or γ_2 is an integer. Let $\tilde{\gamma}$ be the integer which is either γ_1 or γ_2.

The coefficients c_r are polynomial in E of degree r. We denote c_r by $c_r(E)$. If E is a solution of the equation $c_r(E)=0$ for the positive integer $r=1-\tilde{\gamma}$, we have $c_{2-\tilde{\gamma}}(E)=0$ from the recursive relation (3.7). We also have $c_r(E)=0$ for $r>2-\tilde{\gamma}$ from the recursive relation (3.7).

If the power series $f(z)=\sum_r c_r(E)(z-e_2)^r$ is a finite summation, the function $f(\wp(x))$ is rational and doubly periodic.

If l_0+l_1 is even, it occurs the cases $(\tilde{\alpha}_1,\tilde{\alpha}_2,\tilde{\alpha}_3)=(-l_1/2,0,0)$, $(-l_1/2,1/2,1/2)$, $((l_1+1)/2,1/2,0)$, $((l_1+1)/2,0,1/2)$ and the degrees of the polynomial $c_{1-\tilde{\gamma}}(E)$ are $(l_0+l_1+1)/2$, $(l_0+l_1)/2$, $(l_0-l_1)/2$,

$(l_0-l_1)/2$ respectively. We denote these polynomials by $c^{(1)}(E)$, $c^{(2)}(E)$, $c^{(3)}(E)$, and $c^{(4)}(E)$.

Since $\tilde{\alpha}_i$ belong to $\frac{1}{2}\mathbb{Z}$, the function $\Phi(\wp(x))f(\wp(x))$ is doubly periodic up to signs. Let E be a solution of $c^{(i)}(E) = 0$ for some i and let $f(z)$ be the corresponding solution of the differential equation (3.5) determined by the recursive relations (3.6, 3.7). The signs of the periodicity of the function $\Phi(\wp(x))f(\wp(x))$, (i.e. the signs of $\Phi(\wp(x+\omega_i))f(\wp(x+\omega_i))/\Phi(\wp(x))f(\wp(x))$ $(i = 1, 3)$) are different for any two of four cases $c^{(j)}(E) = 0$ $(j = 1, 2, 3, 4)$.

We assume that E is a common solution of $c^{(i)}(E) = 0$ and $c^{(j)}(E) = 0$ for $i \neq j$. Let $\tilde{f}^{(i)}(z)$ (resp. $\tilde{f}^{(j)}(z)$) be the solution of the differential equation (3.5) related to $c^{(i)}(E) = 0$ (resp. $c^{(j)}(E) = 0$). Then the functions $\tilde{f}^{(i)}(\wp(x))$ and $\tilde{f}^{(j)}(\wp(x))$ are the basis of the solution of (2.2). However, it contradicts to the periodicity of the solutions by considering the exponents.

Hence, it has shown that the equation $c^{(i)}(E) = 0$ and $c^{(j)}(E) = 0$ $(i \neq j)$ does not have a common solution.

The degree of the polynomial $c^{(1)}(E)c^{(2)}(E)c^{(3)}(E)c^{(4)}(E)$ are equal to the dimension of the space of the finite dimensional invariant space of the doubly periodic functions up to signs.

If we show that the roots of the equation $c^{(i)}(E) = 0$ $(i = 1, 2, 3, 4)$ are distinct for generic ω_1 and ω_3, we obtain that all the eigenvalue of the invariant subspace is obtained as the solution of $c^{(i)}(E) = 0$ for some i, and we have the theorem.

It is enough to show that if $(e_2-e_1)(e_2-e_3) < 0$ and e_1, e_2, e_3 are real then the roots of the equation $c^{(i)}(E) = 0$ are real and distinct for each cases $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = (-l_1/2, 0, 0)$, $(-l_1/2, 1/2, 1/2)$, $((l_1+1)/2, 1/2, 0)$, $((l_1+1)/2, 0, 1/2)$.

We will show that the polynomial $c_r(E)$ have r real distinct roots $s_i^{(r)}$ $(i = 1, \ldots, r)$ such that $s_1^{(r)} < s_1^{(r-1)} < s_2^{(r)} < s_2^{(r-1)} < \cdots < s_{r-1}^{(r)} < s_{r-1}^{(r-1)} < s_r^{(r)}$ for $1 \leq r \leq 1-\tilde{\gamma}$ by induction.

The $r = 1$ case is trivial. We assume that the statement is true for the $r \leq k$ case. From the assumption of the induction, we have $s_1^{(k)} <$

$s_1^{(k-1)} < s_2^{(k)} < s_2^{(k-1)} < \cdots < s_{k-1}^{(k-1)} < s_k^{(k)}$. It is immediate from (3.7) that the leading term of $c_r(E)$ is positive. Since $s_i^{(k-1)}$ satisfy the equation $c_{k-1}(s_i^{(k-1)}) = 0$, the sign of $c_{k-1}(s_i^{(k)})$ is $(-1)^{k-i}$. By the recursion relation (3.7) and the equation $c_k(s_i^{(k)}) = 0$, the sign of $c_{k+1}(s_i^{(k)})$ is opposite to the one of $c_{k-1}(s_i^{(k)})$. Combining with the asymptotics of $c_{k+1}(E)$ for $E \to +\infty$ and $E \to -\infty$ and $\deg_E c_{k+1}(E) = k+1$, we obtain that the polynomial $c_{k+1}(E)$ have $k+1$ real distinct roots and satisfy $s_1^{(k+1)} < s_1^{(k)} < s_2^{(k+1)} < s_2^{(k)} < \cdots < s_k^{(k)} < s_{k+1}^{(k+1)}$.

The proof for the $l_0 + l_1$: odd cases is similar. □

The following proposition was proved in the proof of the previous theorem.

Proposition 3.3. *Suppose $l_2 = l_3 = 0$ and $l_0 \geq l_1$. If $(e_2 - e_1)(e_2 - e_3) < 0$ and e_1, e_2, e_3 are real, then the roots of the characteristic polynomial of the operator (2.1) on the space which appears in Theorem 3.2 are real and distinct.*

3.2. **Results from the method of monodromy.** Let us consider the monodromy of the equation (2.2) for the cases $l_i \in \mathbb{Z}_{\geq 0}$ $(i = 0, 1, 2, 3)$. The equation (2.2) was given as follows,

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) - E \right) f(x) = 0.$$

The solution of the equation (2.2) have singular points only at $n_1\omega_1 + n_3\omega_3$ $(n_1, n_3 \in \mathbb{Z})$. The exponents at the singularity $a = \omega_i + 2m_1\omega_1 + 2m_3\omega_3$ $(m_1, m_3 \in \mathbb{Z},\ i = 0, 1, 2, 3)$ are $-l_i$ and $l_i + 1$. The equation (2.2) does not have the first-order differential term, the functions $\wp(x+\omega_i)$ $(i = 0, 1, 2, 3)$ are even and the numbers $l_i + 1 - (-l_i)$ are odd. Therefore there exists solutions $f_{a,1}(x) = (x-a)^{-l_i}(1 + \sum_{k=1}^{\infty} a_k x^{2k})$ and $f_{a,2}(x) = (x-a)^{l_i+1}(1 + \sum_{k=1}^{\infty} a'_k x^{2k})$ which converge around $x = a(= \omega_i + 2m_1\omega_1 + 2m_3\omega_3)$ and every solution is expressed as a linear combination of $f_{a,1}(x)$ and $f_{a,2}(x)$. The monodromy matrix with respect to the functions $f_{a,1}(x)$ and $f_{a,2}(x)$ around the point $x = a(=$

$\omega_i + 2m_1\omega_1 + 2m_3\omega_3)$ is unit matrix. Hence the local monodromy matrices of the equation (2.2) around each singular point is unit matrices.

Let M_i $(i = 1, 3)$ be the transformations of the solution of the differential equation (2.2) obtained by the analytic continuation $x \to x + \omega_i$. Since all the local monodromy matrices are unit, the transformations M_i does not depend on the choice of the path. From the fact that the fundamental group of the torus is commutative, we have $M_1M_3 = M_3M_1$. The operators M_i act on the space of the solution of the equation (2.2) for each E, which is two dimensional. By the commutativity $M_1M_3 = M_3M_1$, there exists a joint eigenvector $\Lambda(x)$ for the operators M_1 and M_3. The function $\Lambda(x)$ satisfy the equations $(H-E)\Lambda(x) = 0$, $M_1\Lambda(x) = m_1\Lambda(x)$ and $M_3\Lambda(x) = m_3\Lambda(x)$ for some $m_1, m_3 \in \mathbb{C} \setminus \{0\}$. We immediately have the relations $(H - E)\Lambda(-x) = 0$, $M_1\Lambda(-x) = m_1^{-1}\Lambda(-x)$ and $M_3\Lambda(-x) = m_3^{-1}\Lambda(-x)$.

Let us consider the function $\Xi(x) = \Lambda(x)\Lambda(-x)$. The function $\Xi(x)$ is even, doubly periodic (i.e. $M_1\Xi(x) = M_3\Xi(x) = \Xi(x)$), and satisfy the equation

$$\left(\frac{d^3}{dx^3} - 4\left(\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) - E\right)\frac{d}{dx} - 2\left(\sum_{i=0}^{3} l_i(l_i+1)\wp'(x+\omega_i)\right)\right)\Xi(x) = 0, \tag{3.8}$$

because the product of any pair of solution of the equation (2.2) satisfy the equation (3.8).

Since the function $\Xi(x)$ satisfy the differential equation (3.8), the function $\Xi(x)$ does not have poles except for the points ω_i $(i = 0, 1, 2, 3)$ up to periods of the elliptic function. The exponents of the differential equation at the point $x = -\omega_i$ $(i = 0, 1, 2, 3)$ are $(-2l_i, 1, 2l_i+2)$. Since the function $\Xi(x)$ is even doubly periodic, the function $\Xi(x)$ have pole of degree $2l_i$ or zero of degree $2l_i + 2$ at $x = -\omega_i$ $(i = 0, 1, 2, 3)$

Therefore the function $\Xi(x)$ is even doubly periodic function which may have poles of degree $2l_i$ at $x = -\omega_i$ $(i = 0, 1, 2, 3)$. Hence the

function $\Xi(x)$ has the expression

$$\Xi(x) = c_0 + \sum_{i=0}^{3} \sum_{j=0}^{l_i - 1} a_j^{(i)} \wp(x + \omega_i)^{l_i - j}. \tag{3.9}$$

The coefficients $a_j^{(i)}$ and c_0 are calculated by solving the recursive relations which is obtained by substituting in the equation (3.8). We adopt the normalization of $\Xi(x)$ such that all the coefficients are polynomials of E, e_1, e_2, e_3 and relatively prime, where $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$. We have $\deg_E a_j^{(i)} = \deg_E a_{j-1}^{(i)} + 1$.

We will get the integral representation of $\Lambda(x)$ by the function $\Xi(x)$. The following argument is motivated by the book of Whittaker and Watson ([10, 23.7]).

We have $\Lambda(x)\frac{d}{dx}\Lambda(-x) - \Lambda(-x)\frac{d}{dx}\Lambda(x) = 2C$ for some constant C, which is confirmed by differentiating the equation. From the equations $\frac{d \log \Lambda(-x)}{dx} - \frac{d \log \Lambda(x)}{dx} = \frac{2C}{\Xi(x)}$ and $\frac{d \log \Lambda(-x)}{dx} + \frac{d \log \Lambda(x)}{dx} = \frac{1}{\Xi(x)} \frac{d\Xi(x)}{dx}$, we have

$$\frac{d \log \Lambda(\pm x)}{dx} = \frac{1}{2\Xi(x)} \frac{d\Xi(x)}{dx} \mp \frac{C}{\Xi(x)}. \tag{3.10}$$

By integrating, we have

$$\Lambda(\pm x) = \sqrt{\Xi(x)} \exp \int \frac{\mp C dx}{\Xi(x)}. \tag{3.11}$$

This formula is the integral representation of the solution of (2.2).

Now we are going to investigate the constant C. If we differentiate the equation (3.10), we have

$$\begin{aligned} &\frac{1}{\Lambda(x)} \frac{d^2\Lambda(x)}{dx^2} - \left(\frac{1}{\Lambda(x)} \frac{d\Lambda(x)}{dx} \right)^2 \\ &= \frac{1}{2\Xi(x)} \frac{d^2\Xi(x)}{dx^2} - \frac{1}{2} \left(\frac{1}{\Xi(x)} \frac{d\Xi(x)}{dx} \right)^2 + \frac{C}{\Xi(x)^2} \frac{d\Xi(x)}{dx}. \end{aligned} \tag{3.12}$$

By substituting (2.2) and (3.10) in (3.12), we obtain the formula

$$C^2 = \Xi(x)^2 \left(\sum_{i=0}^{3} l_i(l_i + 1) \wp(x + \omega_i) - E \right) - \frac{1}{2} \Xi(x) \frac{d^2\Xi(x)}{dx^2} + \frac{1}{4} \left(\frac{d\Xi(x)}{dx} \right)^2. \tag{3.13}$$

From the expression of $\Xi(x)$ (3.9), it follows that the constant C^2 is a polynomial of E. We denote C^2 by $P(E)$.

From the formula (3.11), we obtain that if E is the solution of the equation $P(E) = 0$, the corresponding function $\Lambda(x)$ is doubly periodic function up to signs.

Conversely, we suppose that the function $\Lambda(x)$ is doubly periodic function up to signs. We assume that the function $\Lambda(x)$ is even or odd. Then we have $\Xi(x) = \pm\Lambda(x)^2$, where the sign is determined whether $\Lambda(x)$ is even or odd. Substituting in (3.11), we have

$$\int_a^x \frac{-Cdy}{\Xi(y)} = \pm 1,$$

for all x. By differentiating it, we have $C = P(E) = 0$.

If the function $\Lambda(x)$ is not even nor odd, then $\Lambda(-x)$ is also a solution of (2.2), and the functions $\Lambda(x) + \Lambda(-x)$ and $\Lambda(x) - \Lambda(-x)$ are basis of the solution of (2.2). By repeating the previous argument, we have $P(E) = 0$.

Therefore we have

Theorem 3.4. *The equation (2.2) have a solution of doubly periodic function up to signs if and only if E satisfy the equation $P(E) = 0$.*

Let us count the degree of the polynomial $P(E)$ and discuss the relationship to the finite dimensional invariant subspace of doubly periodic function up to signs.

Conjecture 1. *(i) The degree of the polynomial $P(E)$ w.r.t E is equal to the dimension of the finite dimensional invariant space of doubly periodic function up to signs. (See Theorem 3.1)*

(ii) For generic ω_1 and ω_3, the roots of the equation $P(E) = 0$ are distinct.

Proposition 3.5. *Conjecture 1 is true if two of l_i $(i = 0, 1, 2, 3)$ are zero.*

Proof. It is sufficient to show the $l_2 = l_3 = 0$ and $l_0 \geq l_1$ cases.

The function $\Xi(x)$ admit the expression (3.9), but we now adopt the other expression,

$$\Xi(x) = \frac{1}{(z-e_1)^{l_1}} \sum_{j=0}^{l_0+l_1} a_j (z-e_1)^j, \tag{3.14}$$

where $z = \wp(x)$. We are going to have a recursive relation for the coefficients a_j.

Since $\Xi(x)$ satisfy the differential equation (3.8), we have

(3.15)

$$\begin{aligned}
&j(j-2l_0-1)(2j-2l_0-1)a_j \\
&+2(j-l_0-1)(-E+(3j^2-6(l_0+1)j+2l_0^2+5l_0-l_1^2-l_1+3)e_1)a_{j-1} \\
&+(2j-2l_0-3)(j-2-l_0-l_1)(j-1-l_0+l_1)(e_1-e_2)(e_1-e_3)a_{j-2}=0,
\end{aligned}$$

where $j = 1, 2, \ldots, l_0 + l_1 + 1$ and $a_{-1} = a_{l_0+l_1+1} = 0$.

By the recursive relation (3.15), we have
(a) All the coefficients a_j $(j = 1, \ldots l_0 + l_1)$ are divisible by a_0 as polynomials in E, e_1, e_2, e_3. Hence we may set $a_0 = 1$.
(b) The coefficient a_j is a polynomial in variables E, e_1, e_2, e_3 and the homogeneous degree of a_j w.r.t E, e_1, e_2, e_3 is j.
(c) $\deg_E a_j \leq j$.
(d) If $j \leq l_0$ then $\deg_E a_j = j$.

If we set $j = l_0+l_1+1$ in (3.15), we have $\deg_E a_{l_0+l_1-1} \leq \deg_E a_{l_0+l_1} + 1$. We also have $\deg_E a_{l_0+l_1-k-1} \leq \deg_E a_{l_0+l_1-k} + 1$ by setting $j = l_0 + l_1 - k + 1$ $(k = 1, \ldots, l_1 - 1)$ in (3.15).

Therefore, we obtain that $\deg_E a_j \leq l_0$ for all j, $\deg_E a_j = l_0 \Leftrightarrow j = l_0$, and the coefficient of E^{l_0} in a_{l_0} does not depend on e_1, e_2, e_3.

From the relation (3.13), we have $\deg_E P(E) = 2l_0 + 1$ and the coefficient of E^{2l_0+1} in $P(E)$ does not depend on e_1, e_2, e_3.

From Theorem 3.4, the equation (2.2) have a doubly periodic function up to signs iff E satisfy the equation $P(E) = 0$. From Theorem 3.1, the dimension of the doubly periodic function up to signs are $2l_0 + 1$. From Theorem 3.2, the roots of the characteristic polynomial of the

operator (2.1) on the invariant space of the finite dimensional doubly periodic function up to signs are distinct for generic ω_1 and ω_3.

Hence the characteristic polynomial of the operator (2.1) on the invariant space of doubly periodic functions up to signs coincide with the polynomial $P(E)$ up to constant multiplication.

Therefore we obtain the proposition. □

The following proposition is proved case by case.

Proposition 3.6. *Conjecture 1 is true for the $l_1+l_2+l_3+l_4 \leq 8$ cases.*

3.3. **Bethe Ansatz equation in terms of sigma function.** In the previous subsection, we introduced the elliptic function $\Xi(x)$ and the polynomial $P(E)$. In this subsection, we will propose the Bethe Ansatz method. The Bethe Ansats method replace the spectral problem with solving the transcendental equation for the finite variables. We will also discuss the completeness of the eigenfunction obtained by the Bethe Ansatz method for a given eigenvalue E. We remark that the Lamé case (i.e. $l_1 = l_2 = l_3 = 0$ case) was treated in ([10]).

Throughout this subsection, we assume $P(E) \neq 0$. As was discussed in section 3.2, the function $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent.

For simplicity, we assume $l_0 \neq 0$. The function $\Xi(x)(=\Lambda(x)\Lambda(-x))$ is even doubly periodic function which may have poles of degree $2l_i$ at $x = \omega_i$ $(i = 0, 1, 2, 3)$. Therefore we have the following expression

$$\Xi(x) = \frac{a_0^{(0)} \prod_{j=1}^{l} (\wp(x) - \wp(t_j))}{(\wp(x) - e_1)^{l_1} (\wp(x) - e_2)^{l_2} (\wp(x) - e_3)^{l_3}}, \tag{3.16}$$

for some values $t_1, \ldots, t_l$, where $l = l_0 + l_1 + l_2 + l_3$ and $a_0^{(0)}$ was defined in (3.9).

Proposition 3.7. *We assume $P(E) \neq 0$. The values t_j, $-t_j$, ω_i $(j = 1, \ldots, l,\ i = 0, 1, 2, 3)$ are mutually distinct up to the periods of the elliptic function, where the values t_j were defined in (3.16).*

Proof. (a) We claim that for each $j \in \{1, \ldots, l\}$ and $i \in \{0, 1, 2, 3\}$ the values t_j and ω_i are different up to the periods of the elliptic function.

Since the function $\Xi(x)$ is even doubly periodic function and satisfy the differential equation (3.8), the function $\Xi(x)$ have pole of degree $2l_i$ or zero of degree $2l_i+2$ at $x=\omega_i$ $(i=0,1,2,3)$.

Suppose that the function $\Xi(x)$ has zero of degree $2l_i+2$ at $x=\omega_i$ for some $i \in \{0,1,2,3\}$. Then the functions $\Lambda(x)$ and $\Lambda(-x)$ have zero of degree l_i+1 at $x=\omega_i$, because the functions $\Lambda(x)$ and $\Lambda(-x)$ satisfy the differential equation (2.2) and the exponents at $x=\omega_i$ are $(-l_i, l_i+1)$.

From the condition $P(E) \neq 0$, the functions $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent and form a basis of the solution of the differential equation (2.2). Hence every solution of the differential equation (2.2) has zero of degree l_i+1 at $x=\omega_i$ and it contradicts to the fact that one of the exponents at $x=\omega_i$ is $-l_i$.

Therefore the function $\Xi(x)$ have pole of degree $2l_i$ at $x=\omega_i$ $(i=0,1,2,3)$.

If $t_j=\omega_i$ or $-\omega_i$ for some j and i, it contradicts to the degree of the pole $x=\omega_i$.

(b) Now we show that $t_j \neq t_{j'}$ $(j \neq j',\ j,j' \in \{1,\dots,l\})$ and $t_j \neq -t_{j'}$ $(j,j' \in \{1,\dots,l\})$ up to the periods of the elliptic function.

We assume $t_j = t_{j'}$ $(j \neq j',\ j,j' \in \{1,\dots,l\})$ or $t_j=-t_{j'}$ $(j,j' \in \{1,\dots,l\})$. We do not need to consider the case $t_j=-t_j$, because we have $t_j=\omega_i$ for some i from the condition $2t_j=0 \mod 2\omega_1\mathbb{Z}+2\omega_3\mathbb{Z}$. From (a) and the fact $\wp(t)=\wp(-t)$ for all t, we obtain that the function $\Xi(x)$ have zeros of degree no less than 2 at two different points $x=t_j$ and $x=-t_j$.

The degree of zero of the functions $\Lambda(x)$ and $\Lambda(-x)$ at points $x=t_j$ and $x=-t_j$ are 0 or 1 respectively, because $\Lambda(x)$ and $\Lambda(-x)$ satisfy the differential equation (2.2) and the differential equation (2.2) is holomorphic at the points $x=\pm t_j$. Hence both the functions $\Lambda(x)$ and $\Lambda(-x)$ have zeros at $x=t_j$ and $x=-t_j$.

The functions $\Lambda(x)$ and $\Lambda(-x)$ form a basis of the solution of the differential equation (2.2). It follows that every solution of the differential

equation (2.2) has zero at $x = t_j$ and it contradicts to the existence of the solution of the ordinary differential equation with initial values.

Therefore we proved the proposition. □

We set $z = \wp(x)$ and $z_j = \wp(t_j)$. From the formula (3.13), we have

$$\left(\frac{d\Xi(x)}{dz}\right)^2_{z=z_j} = \frac{4C^2}{\wp'(t_j)^2}.$$

We fix the signs of t_j by taking

$$\left(\frac{d\Xi(x)}{dz}\right)_{z=z_j} = \frac{2C}{\wp'(t_j)}.$$

If we put $2C/\Xi(x)$ into partial fractions, we have

$$\frac{2C}{\Xi(x)} = \sum_{j=1}^{l} \frac{\frac{2C}{\left(\frac{d\Xi}{dz}\right)_{z=z_j}}}{\wp(x) - \wp(t_j)} = \sum_{j=1}^{l} \left(\zeta(x - t_j) - \zeta(x + t_j) + 2\zeta(t_j)\right),$$

where $\zeta(x)$ is the Weierstrass zeta function. It follows from the formula (3.11) that

$$\begin{aligned}(3.17)\quad \Lambda(x) &= \sqrt{\frac{a_0^{(0)} \prod_{j=1}^{l} (\wp(x) - \wp(t_j))}{(\wp(x) - e_1)^{l_1} (\wp(x) - e_2)^{l_2} (\wp(x) - e_3)^{l_3}}} \\ &\exp\left(\frac{1}{2} \sum_{j=1}^{l} \left(\log \sigma(t_j + x) - \log \sigma(t_j - x) - 2x\zeta(t_j)\right)\right) \\ &= \frac{\sqrt{a_0^{(0)}} \prod_{j=1}^{l} \sigma(x + t_j)}{\sigma(x)^{l_0} \sigma_1(x)^{l_1} \sigma_2(x)^{l_2} \sigma_3(x)^{l_3} \prod_{j=1}^{l} \sigma(t_j)} \exp\left(-x \sum_{i=1}^{l} \zeta(t_j)\right),\end{aligned}$$

where $\sigma(x)$ is the Weierstrass sigma function and $\sigma_i(x)$ $(i = 1, 2, 3)$ are Weierstrass co-sigma functions. If $\Lambda(x)$ is not doubly periodic up to signs, $\Lambda(x)$ and $\Lambda(-x)$ are linearly independent and the linear combination of $\Lambda(x)$ and $\Lambda(-x)$ exhaust the solution of the equation (2.2) for the eigenvalue E. Hence, if there does not exist the doubly periodic solution of (2.2) up to signs for the value E, the basis of the solution is written as $\Lambda(x)$ and $\Lambda(-x)$ where $\Lambda(x)$ was given as (3.17).

Conversely we assume that the differential equation (2.2) admit the solution written as

$$\tilde{\Lambda}(x) = \frac{\prod_{j=1}^{l} \sigma(x+t_j)}{\sigma(x)^{l_0}\sigma_1(x)^{l_1}\sigma_2(x)^{l_2}\sigma_3(x)^{l_3}} \exp(cx), \tag{3.18}$$

for some c and t_j $(j = 1, \dots, l)$ such that $t_j \neq t_{j'}$ $(j \neq j')$. By a straightforward calculation, we have

(3.19)

$$\begin{aligned}
\frac{d^2\tilde{\Lambda}(x)}{dx^2} &= \tilde{\Lambda}(x) \left(\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) + 2l_0\zeta(x) \left(-c - \sum_{j=1}^{l} \zeta(t_j) \right) \right. \\
&+ 2\sum_{i=1}^{3} l_i\zeta(x+\omega_i) \left(-c - l\zeta(\omega_i) - \sum_{j=1}^{l} \zeta(t_j-\omega_i) \right) \\
&+ 2\sum_{j=1}^{l} \zeta(x+t_j) \left(c + \sum_{k\neq j} \zeta(t_k-t_j) - l_0\zeta(-t_j) - \sum_{i=1}^{3} l_i(\zeta(-t_j+\omega_i) - \zeta(\omega_i)) \right) \\
&+ \sum_{j=1}^{l}\sum_{i=0}^{3} l_i \left(\wp(t_j-\omega_i) - \zeta(t_j-\omega_i)^2 \right) - \sum_{j<k} \left(\wp(t_j-t_k) - \zeta(t_j-t_k)^2 \right) \\
&\left. + c^2 - (l_0l_1+l_2l_3)e_1 - (l_0l_2+l_1l_3)e_2 - (l_0l_3+l_1l_2)e_3 + \sum_{i=1}^{3} l_i\eta_i(2c+l\eta_i). \right)
\end{aligned}$$

To derive the previous equation, we used the formula

(3.20)

$$(\zeta(x+a) - \zeta(x+b) - \zeta(a-b))^2 = \wp(x+a) + \wp(x+b) + \wp(a-b).$$

Hence we find that the function $\tilde{\Lambda}(x)$ (3.19) satisfy the equation (2.2) if and only if t_j $(j = 1, \dots, l)$ and c satisfy the following equations

(3.21)

$$\sum_{k\neq j} \zeta(-t_j+t_k) - l_0\zeta(-t_j) - \sum_{i=1}^{3} l_i(\zeta(-t_j+\omega_i) - \zeta(\omega_i)) = -c \ \ (j = 1, \dots, l),$$

$$(1-\delta_{l_0,0})(c + \sum_{j=1}^{l} \zeta(t_j)) = 0,$$

$$(1-\delta_{l_i,0})(c + l\zeta(\omega_i) + \sum_{j=1}^{l} \zeta(-\omega_i+t_j)) = 0, \ \ (i = 1, 2, 3).$$

The eigenvalue E is given by

(3.22)

$$E = -c^2 + (l_0l_1 + l_2l_3)e_1 + (l_0l_2 + l_1l_3)e_2 + (l_0l_3 + l_1l_2)e_3 - \sum_{i=1}^{3} l_i\eta_i(2c + l\eta_i)$$

$$- \sum_{j=1}^{l}\sum_{i=0}^{3} l_i(\wp(t_j - \omega_i) - \zeta(t_j - \omega_i)^2) + \sum_{j<k}(\wp(t_j - t_k) - \zeta(t_j - t_k)^2).$$

We remark that there are other expressions of E in terms of t_j $(j = 1, \dots, l)$ and c.

Hence we have

Theorem 3.8. *We set*

$$\tilde{\Lambda}(x) = \frac{\prod_{j=1}^{l}\sigma(x + t_j)}{\sigma(x)^{l_0}\sigma_1(x)^{l_1}\sigma_2(x)^{l_2}\sigma_3(x)^{l_3}}\exp(cx), \tag{3.23}$$

and assume $t_j \neq t_{j'}$ for $j \neq j'$.

The condition that the function $\tilde{\Lambda}(x)$ satisfy the equation (2.2) for some E is equivalent to that t_j $(j = 1, \dots, l)$ and c satisfy the relations (3.21). The eigenvalue E is given as (3.22).

If there does not exist the doubly periodic solution of (2.2) up to signs for the value E, the basis of the solution is written as $\tilde{\Lambda}(x)$ and $\tilde{\Lambda}(-x)$ with the condition $t_j \neq t_{j'}$ $(j \neq j')$.

Remark (i) For each l_0, l_1, l_2, l_3, the cardinality of the eigenvalue E which does not have a basis of solution of the form (3.23) is at most the dimension given in Theorem 3.1.

(ii) If $l_0 \neq 0$ and $l_1 \neq 0$ then the number of the equations (3.21) is strictly greater that the number of the variables t_j $(j = 1, \dots, l)$, although the equations (3.21) have solutions $(t_1, \dots, t_l)$ in general.

For the $l_1 = l_2 = l_3 = 0$ case (Lamé case), the equation $c + \sum_{j=1}^{l}\zeta(t_j) = 0$ is obtained by summing up the other equations (3.21). Hence the number of the equations is equal to the number of the variables t_j $(j = 1, \dots, l_0)$.

3.4. **Bethe Ansatz equation in terms of theta function.** We will translate Theorem 3.8 in terms of theta function.

We set $\tau = \omega_3/\omega_1$. The theta function is expressed with the sigma function as follows,

$$\sigma(2\omega_1 x) = 2\omega_1 \exp(2\eta_1\omega_1 x^2)\theta(x)/\theta'(0). \tag{3.24}$$

From now on, we set $\omega_1 := 1/2$.

In this case, the Hamiltonian (2.1) is rewritten as

$$\begin{aligned} H = &-\frac{d^2}{dx^2} + l_0(l_0+1)\wp(x) \\ &+ l_1(l_1+1)\wp\left(x+\frac{1}{2}\right) + l_2(l_2+1)\wp\left(x+\frac{1+\tau}{2}\right) + l_3(l_3+1)\wp\left(x+\frac{\tau}{2}\right). \end{aligned} \tag{3.25}$$

Replacing the sigma function with the theta function by the formula (3.24), we obtain that if there does not exist the doubly periodic solution of (2.2) up to signs for the value E, the basis of the solution is written as $\tilde{\Lambda}(x)$ and $\tilde{\Lambda}(-x)$, where

$$\tilde{\Lambda}(x) = \frac{\exp(\pi\sqrt{-1}cx)\prod_{j=1}^{l}\theta(x+t_j)}{\theta(x)^{l_0}\theta(x+1/2)^{l_1}\theta(x+(1+\tau)/2)^{l_2}\theta(x+\tau/2)^{l_3}}, \tag{3.26}$$

$t_j \neq t_{j'}$ $(j \neq j')$, and $l = l_0 + l_1 + l_2 + l_3$.

By the similar calculation as the case of the sigma function, we have the relations for t_j $(j = 1, \ldots, l)$ and c by which the function $\tilde{\Lambda}(x)$ become the eigenfunction of (3.25).

Theorem 3.9. *The function $\tilde{\Lambda}(x)$ (3.26) with the condition $t_j \neq t_{j'}$ $(j \neq j')$ is an eigenfunction of (3.25) if and only if t_j $(j = 1, \ldots, l($=*

$l_0+l_1+l_2+l_3))$ *and* c *satisfy the relations,*

(3.27)

$$c\pi\sqrt{-1}+\sum_{k\neq j}\frac{\theta'(t_k-t_j)}{\theta(t_k-t_j)}+l_0\frac{\theta'(t_j)}{\theta(t_j)}+l_1\frac{\theta'(t_j-1/2)}{\theta(t_j-1/2)}$$
$$+l_2\frac{\theta'(t_j-(1+\tau)/2)}{\theta(t_j-(1+\tau)/2)}+l_3\frac{\theta'(t_j-\tau/2)}{\theta(t_j-\tau/2)}=0,\quad (j=1,\dots,l)$$
$$(1-\delta_{l_0,0})\left(c\pi\sqrt{-1}+\pi\sqrt{-1}(l_2+l_3)+\sum_{j=1}^{l}\frac{\theta'(t_j)}{\theta(t_j)}\right)=0,$$
$$(1-\delta_{l_1,0})\left(c\pi\sqrt{-1}+\pi\sqrt{-1}(l_2+l_3)+\sum_{j=1}^{l}\frac{\theta'(t_j-1/2)}{\theta(t_j-1/2)}\right)=0,$$
$$(1-\delta_{l_2,0})\left(c\pi\sqrt{-1}-\pi\sqrt{-1}(l_0+l_1)+\sum_{j=1}^{l}\frac{\theta'(t_j-(1+\tau)/2)}{\theta(t_j-(1+\tau)/2)}\right)=0,$$
$$(1-\delta_{l_3,0})\left(c\pi\sqrt{-1}-\pi\sqrt{-1}(l_0+l_1)+\sum_{j=1}^{l}\frac{\theta'(t_j-\tau/2)}{\theta(t_j-\tau/2)}\right)=0.$$

The eigenvalue E *is given as*

(3.28)

$$E=\pi^2(c^2+(l_0+l_1)(l_2+l_3))+l(l+1)\eta_1-\sum_{j<k}\frac{\theta''(t_j-t_k)}{\theta(t_j-t_k)}$$
$$+(l_0l_1+l_2l_3)e_1+(l_0l_2+l_1l_3)e_2+(l_0l_3+l_1l_2)e_3$$
$$+\sum_{j=1}^{l}\left(l_0\frac{\theta''(t_j)}{\theta(t_j)}+l_1\frac{\theta''(t_j-1/2)}{\theta(t_j-1/2)}+l_2\frac{\theta''(t_j-(1+\tau)/2)}{\theta(t_j-(1+\tau)/2)}+l_3\frac{\theta''(t_j-\tau/2)}{\theta(t_j-\tau/2)}\right).$$

If there does not exist the doubly periodic eigenfunction of (3.25) up to signs for the value E*, the basis of the solution is written as* $\tilde{\Lambda}(x)$ *and* $\tilde{\Lambda}(-x)$*, where the function* $\tilde{\Lambda}(x)$ *was given in (3.26) and satisfy the condition* $t_j\neq t_{j'}$ $(j\neq j')$*.*

It is hopeless to solve the equations (3.27) explicitly for generic τ. Instead, it may be possible to solve them for the $\tau\to\sqrt{-1}\infty$ case and get some knowledges for the $\tau\neq\sqrt{-1}\infty$ case from this limit. We will investigate the solutions of the equations (3.27) for the trigonometric $(\tau\to\sqrt{-1}\infty)$ case in the next section.

4. Trigonometric limit

4.1. **Trigonometric limit of the polynomial $P(E)$.** In this section, we will consider the trigonometric limit $\tau \to \sqrt{-1}\infty$.

First, we consider the trigonometric limits of the elliptic functions. We have

$$\wp(x) \to -\frac{\pi^2}{3} + \frac{\pi^2}{(\sin \pi x)^2}, \quad \wp\left(x + \frac{1}{2}\right) \to -\frac{\pi^2}{3} + \frac{\pi^2}{(\cos \pi x)^2}, \tag{4.1}$$

$$\wp\left(x + \frac{\tau}{2}\right) \to -\frac{\pi^2}{3}, \quad \wp\left(x + \frac{1+\tau}{2}\right) \to -\frac{\pi^2}{3}, \quad \frac{\theta(x)}{\theta'(0)} \to \sin \pi x, \tag{4.2}$$

as $\tau \to \sqrt{-1}\infty$. They converge uniformly on compact sets.

By this limit, the terms $l_2(l_2+1)\wp(x+\frac{1+\tau}{2}) + l_3(l_3+1)\wp(x+\frac{\tau}{2})$ converge to a constant. From now on, we consider the $l_2 = l_3 = 0$ cases.

As $\tau \to \sqrt{-1}\infty$, we have $H \to H_T - C_T$, where

$$H_T = -\frac{d^2}{dx^2} + l_0(l_0+1)\frac{\pi^2}{(\sin \pi x)^2} + l_1(l_1+1)\frac{\pi^2}{(\cos \pi x)^2}, \tag{4.3}$$

$$C_T = (l_0(l_0+1) + l_1(l_1+1))\pi^2/3. \tag{4.4}$$

We will justify similar arguments which are performed in section 3.2 for the trigonometric case. We remark that we have $e_1 \to \frac{2\pi^2}{3}$, $e_2 \to \frac{-\pi^2}{3}$ and $e_3 \to \frac{-\pi^2}{3}$ as $\tau \to \sqrt{-1}\infty$.

The function $\Xi(x)$ (3.9) admit the trigonometric limit

$$\Xi_T(x) = \tilde{c}_0(E) + \sum_{j=0}^{l_0-1} \frac{\tilde{a}_j^{(0)}(E)}{(\sin \pi x)^{2(l_0-j)}} + \sum_{j=0}^{l_1-1} \frac{\tilde{a}_j^{(1)}(E)}{(\cos \pi x)^{2(l_1-j)}}, \tag{4.5}$$

and the function $\Xi_T(x)$ satisfy the trigonometric version of the equations (3.8) and (3.13). We will calculate the polynomial $P_T(E)$ which is the trigonometric version of $P(E)$. From the trigonometric version of the equation (3.13), we have $P_T(E) = -(E + C_T)\tilde{c}_0(E)^2$.

Proposition 4.1. *If $l_0 + l_1$ is even, we have*

(4.6)

$$P_T(E) = b_{l_0+l_1} \cdot (E + C_T) \prod_{i=1}^{\frac{l_0+l_1}{2}} (E - (2i)^2 + C_T)^2 \prod_{i=1}^{\frac{|l_0-l_1|}{2}} (E - (2i-1)^2 + C_T)^2,$$

where C_T is defined in (4.4) and $b_{l_0+l_1}$ is a constant which is independent on E.

If l_0+l_1 is odd, we have

(4.7)

$$P_T(E) = b_{l_0+l_1}\cdot(E+C_T)\prod_{i=1}^{\frac{|l_0-l_1|-1}{2}}(E-(2i)^2+C_T)^2\prod_{i=1}^{\frac{l_0+l_1+1}{2}}(E-(2i-1)^2+C_T)^2,$$

where C_T is defined in (4.4) and $b_{l_0+l_1}$ is a constant which is independent on E.

Proof. For simplicity, we prove for the $l_0 \geq l_1$ and l_0+l_1: even cases. The proofs for the other cases are similar.

From the proof of Proposition 3.5, it follows that the polynomial $P_T(E)$ is obtained by a suitable limit of the characteristic polynomial of the invariant subspace of the doubly periodic functions up to signs.

We set $\epsilon = (e_2-e_3)$. We have $\epsilon \to 0$ as $\tau \to \sqrt{-1}\infty$. We will solve the recursive relation (3.7) for the $\epsilon \to 0$ case. From now on, we will apply the notations which were used in the proof of Proposition 3.5.

The characteristic polynomial of the invariant subspace of the type $\left(\tilde{\alpha}_1 = -\frac{l_0}{2},\ \tilde{\alpha}_2 = 0,\ \tilde{\alpha}_3 = 0\right)$ is expressed as

$$\epsilon^{-\frac{l_0+l_1}{2}-1}c_1\prod_{i=0}^{\frac{l_0+l_1}{2}}(E-(2i)^2+C_T)+\epsilon^{-\frac{l_0+l_1}{2}}\star+\ldots, \tag{4.8}$$

where c_1 is a constant. The leading terms of characteristic polynomial in E of the invariant subspace of the other type are given as

$$\begin{aligned}
&\prod_{i=1}^{\frac{l_0+l_1}{2}}\left(E-(2i)^2+C_T\right) \quad \left(\tilde{\alpha}_1 = -\frac{l_0}{2},\ \tilde{\alpha}_2 = \frac{1}{2}, \tilde{\alpha}_3 = \frac{1}{2}\right)\\
&\prod_{i=1}^{\frac{l_0-l_1}{2}}\left(E-(2i-1)^2+C_T\right) \quad \left(\tilde{\alpha}_1 = \frac{l_0+1}{2},\ \tilde{\alpha}_2 = \frac{1}{2}, \tilde{\alpha}_3 = 0\right)\\
&\prod_{i=1}^{\frac{l_0-l_1}{2}}\left(E-(2i-1)^2+C_T\right) \quad \left(\tilde{\alpha}_1 = \frac{l_0+1}{2},\ \tilde{\alpha}_2 = 0, \tilde{\alpha}_3 = \frac{1}{2}\right)
\end{aligned} \tag{4.9}$$

up to constants.

By multiplying the leading terms for the four cases, we have the proposition. □

4.2. **Bethe Ansatz equation for the trigonometric model.** In this subsection, We will obtain the trigonometric version of the equations (3.27).

We set $X = \exp(2\pi\sqrt{-1}x)$ and $T_j = \exp(2\pi\sqrt{-1}t_j)$. We state the trigonometric version of Theorem 3.9.

We set

$$\tilde{\Lambda}_T(X) = \frac{\prod_{j=1}^{l_0+l_1}(XT_j - 1)}{(X-1)^{l_0}(X+1)^{l_1}} X^{c/2}, \tag{4.10}$$

and assume $T_j \neq T_{j'}$ $(j \neq j')$ and $T_j \neq 0, \pm 1$ $(j \in \{1, \ldots, l_0 + l_1\})$. Suppose that T_j and c satisfy the relations,

(4.11)

$$c + \sum_{k \neq j} \frac{T_k + T_j}{T_k - T_j} + l_0 \frac{T_j + 1}{T_j - 1} + l_1 \frac{T_j - 1}{T_j + 1} = 0 \quad (j = 1, \ldots, l_0 + l_1)$$

(4.12)

$$(1 - \delta_{l_0,0}) \left(c + \sum_{j=1}^{l_0+l_1} \frac{T_j + 1}{T_j - 1} \right) = 0, \quad (1 - \delta_{l_1,0}) \left(c + \sum_{j=1}^{l_0+l_1} \frac{T_j - 1}{T_j + 1} \right) = 0.$$

Then the function $\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x})$ is an eigenfunction of the operator H_T (4.3) with an eigenvalue $E = \pi^2 c^2$.

Conversely, if the function $\tilde{\Lambda}_T(X) = \frac{\prod_{j=1}^{l_0+l_1}(XT_j-1)}{(X-1)^{l_0}(X+1)^{l_1}} X^{c/2}$ ($T_j \neq T_{j'}$ for $j \neq j'$ and $T_j \neq 0, \pm 1$ for all j) satisfy the equation $H_T \tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x}) = E\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x})$, we have the equations (4.11), (4.12) and $E = \pi^2 c^2$.

Proposition 4.2. *We fix the value E. If $P_T(E) \neq 0$ then the basis of the solution of equation $(H_T - C_T - E)f(x) = 0$ is written as $\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x})$ and $\tilde{\Lambda}_T(e^{-2\pi\sqrt{-1}x})$ where $\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x})$ is given in (4.10) with the condition $c^2 = \frac{E+C_T}{\pi^2}$, $T_j \neq T_{j'}$ $(j \neq j')$ and $T_j \neq 0, \pm 1$ $(j \in \{1, \ldots, l_0 + l_1\})$.*

Proof. From the expression (4.5), we have

$$\Xi_T(x) = \frac{\tilde{c}_0(E)(\sin \pi x)^{2(l_0+l_1)} + (\text{lower terms in } (\sin \pi x)^2)}{(\sin \pi x)^{2l_0}(\cos \pi x)^{2l_1}}, \tag{4.13}$$

for some $t_1, \dots t_{l_0+l_1}$, where the polynomial $\tilde{c}_0(E)$ is common in (4.5) and (4.13). By the relation $P_T(E) = -(E+C_T)\tilde{c}_0(E)^2$, we have $\tilde{c}_0(E) \neq 0$. Hence we have the expression

$$\Xi_T(x) = \frac{\tilde{c}_0(E)\prod_{k=1}^{l_0+l_1}((\sin \pi x)^2 - (\sin \pi t_j)^2)}{(\sin \pi x)^{2l_0}(\cos \pi x)^{2l_1}} \tag{4.14}$$

for some $t_1, \dots t_{l_0+l_1}$. By a similar argument performed in section 3.3, we have

$$\Lambda_T(x) = \frac{\sqrt{\tilde{c}_0(E)}\prod_{k=1}^{l_0+l_1}\sin \pi(x+t_k)}{(\sin \pi x)^{l_0}(\cos \pi x)^{l_1}}\exp(\pi\sqrt{-1}c), \tag{4.15}$$

where c satisfy $c^2 = \frac{E+C_T}{\pi^2}$ and the signs of $t_1, \dots, t_{l_0+l_1}$ are chosen suitable.

If $\Lambda_T(x)$ satisfy the differential equation $(H_T - C_T - E)\Lambda_T(x) = 0$, $\Lambda_T(-x)$ also satisfy the same equation. From the trigonometric version of the formula (3.11), we have the linear independence of $\Lambda_T(x)$ and $\Lambda_T(-x)$. We also obtain that the values $\pm t_j$, $(j = 1, \dots, l_0 + l_1)$, 0, 1/2 are different mutually by the similar argument of the proof of Proposition 3.7.

By changing the variables $X = \exp(2\pi\sqrt{-1}x)$ and $T_j = \exp(2\pi\sqrt{-1}t_j)$, we have the proposition. □

4.3. **Solutions of the Bethe Ansatz equation.** We will solve the Bethe Ansatz equation for the trigonometric cases.

Theorem 4.3. *We assume $P_T(\pi^2c^2 - C_T) \neq 0$.*

Let $(T_1^0, \dots, T_l^0)$ $(l = l_0 + l_1)$ be a solution of the equations (4.11, 4.12), and let $\sigma_i = \sum_{j_1<\dots<j_i} T_{j_1}^0 \dots T_{j_i}^0$ be the i-th elementary symmetric function of the solution $(T_1^0, \dots, T_l^0)$.

There exists a solution of (4.11, 4.12) uniquely up to the permutation of $(T_1^0, \dots, T_l^0)$. The functions σ_i are determined by the following

recursive relation,

(4.16)

$$\sigma_0 = 1, \quad \sigma_1 = \frac{(l_0 - l_1)(c - l_0 - l_1)}{c+1},$$

(4.17)

$$\sigma_i = \frac{(l_0 - l_1)(c + 2i - 2 - l_0 - l_1)}{i(i+c)}\sigma_{i-1} + \frac{(i - 2 - l_0 - l_1)(c + i - 2 - l_0 - l_1)}{i(i+c)}\sigma_{i-2},$$

where $i = 2, 3, \ldots, l_0 + l_1$.

Proof. By the assumption $P_T(\pi^2 c^2 - C_T) \neq 0$ and Proposition 4.2, there exist the solution of $(H_T - \pi^2 c^2)\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x}) = 0$ of the form $\tilde{\Lambda}_T(X) = \frac{\prod_{j=1}^{l}(XT_j - 1)}{(X-1)^{l_0}(X+1)^{l_1}} X^{c/2}$. Then the values $(T_1, \ldots, T_l)$ satisfy the equations (4.11, 4.12).

We set

$$\tilde{\Lambda}_T(X) = \frac{1 + \sum_{j=1}^{l}(-1)^j \tilde{\sigma}_j X^j}{(X-1)^{l_0}(X+1)^{l_1}} X^{c/2}.$$

Then we have $\tilde{\sigma}_i = \sum_{j_1 < \cdots < j_i} T_{j_1} \ldots T_{j_i}$.

The function $\tilde{\Lambda}_T(X)$ satisfy the differential equation $(H_T - \pi^2 c^2)\tilde{\Lambda}_T(e^{2\pi\sqrt{-1}x}) = 0$. Hence we have the recursive relation (4.17) for $\tilde{\sigma}_i = \sigma_i$ by comparing the coefficients of X^j $(j = 1, \ldots, l)$ of the equation $(H_T - \pi^2 c^2)\tilde{\Lambda}_T(X) = 0$ where $X = e^{2\pi\sqrt{-1}x}$.

The existence of the solution of (4.11, 4.12) follow from the existence of the solution of the form $\tilde{\Lambda}_T(X) = \frac{\prod_{j=1}^{l}(XT_j - 1)}{(X-1)^{l_0}(X+1)^{l_1}} X^{c/2}$. □

We will exhibit the explicit expression of σ_m for some cases. By straightforward calculations, we have the following proposition.

Proposition 4.4. *(i) If* $l_1 = 0$ *then we have*

$$\sigma_m = \prod_{j=0}^{m-1} \frac{(l_0 - j)(c + j - l_0)}{(j+1)(c+j+1)}. \tag{4.18}$$

(ii) If $l_1 = 1$ *then we have*

$$\sigma_m = \frac{\prod_{j=0}^{m-2}(l_0 - j)\prod_{j=1}^{m-2}(c + j - l_0)}{m! \prod_{j=0}^{m-1}(c+j+1)} \cdot \tag{4.19}$$
$$\cdot (c - l_0 - 1)((l_0 - 2m + 1)c - l_0^2 - l_0 + 2ml_0 + 2m - 2m^2)$$

(iii) If $l_1 = 2$ then we have

(4.20)

$$\sigma_m = \frac{\prod_{j=0}^{m-3}(l_0-j)\prod_{j=2}^{m-3}(c+j-l_0)}{m!\prod_{j=0}^{m-1}(c+j+1)}(c-l_0)(c-l_0-2)\cdot$$
$$\cdot\,(c^2(l_0^2-(4m-3)l_0+4m^2-8m+2)$$
$$+2c(-l_0^3+(4m-3)l_0^2-(6m^2-10m+2)l_0+4m^3-12m^2+8m)$$
$$+l_0^4-(4m-3)l_0^3+(8m^2-12m+1)l_0^2$$
$$-(8m^3-20m^2+8m+3)l_0+4m^4-16m^3+16m^2-2).$$

(iv) If $l_1 = l_0$ then we have

$$\sigma_{2m-1}=0,\quad \sigma_{2m}=\prod_{j=0}^{m-1}\frac{(j-l_0)(c+2j-2l_0)}{(j+1)(c+2j+2)}. \tag{4.21}$$

(v) If $l_1 = l_0 - 1$ then we have

$$\sigma_{2m-1}=\frac{\prod_{j=1}^{m-1}(j-l_0)\prod_{j=0}^{m-1}(c+2j+1-2l_0)}{(m-1)!\prod_{j=0}^{m-1}(c+2j+1)}, \tag{4.22}$$
$$\sigma_{2m}=\frac{\prod_{j=1}^{m}(j-l_0)\prod_{j=0}^{m-1}(c+2j+1-2l_0)}{m!\prod_{j=0}^{m-1}(c+2j+1)}.$$

(vi) If $l_1 = l_0 - 2$ then we have

$$\sigma_{2m-1}=\frac{2\prod_{j=2}^{m}(j-l_0)\prod_{j=0}^{m-1}(c+2j+2-2l_0)}{(m-1)!(c+1)\prod_{j=0}^{m-2}(c+2j+2)}, \tag{4.23}$$
$$\sigma_{2m}=-\frac{\prod_{j=2}^{m}(j-l_0)\prod_{j=0}^{m-1}(c+2j+2-2l_0)}{m!(c+1)\prod_{j=0}^{m-1}(c+2j+2)}\cdot$$
$$\cdot\,(c(l_0-2m-1)+(4m+1)l_0-(2m+1)^2).$$

Remark Proposition 4.4 (i) reproduces Proposition 4.1. in ([8]).

5. Eigenstates for the BC_1 Inozemchev system

5.1. **Eigenstates for the trigonometric Hamiltonian.** Let us find the eigenstates $\phi(x)$ of the Hamiltonian H_T, which are square-integrable.

We set

$$\Phi(x)=(\sin\pi x)^{l_0+1}(\cos\pi x)^{l_1+1},\quad \mathcal{H}_T=\Phi(x)^{-1}H_T\Phi(x), \tag{5.1}$$

then we have the gauge transformed Hamiltonian.

(5.2)

$$\mathcal{H}_T = -\frac{d^2}{dx^2} - 2\pi\left(\frac{(l_0+1)\cos\pi x}{\sin\pi x} - \frac{(l_1+1)\sin\pi x}{\cos\pi x}\right)\frac{d}{dx} + (l_0+l_1+2)^2\pi^2.$$

By a straightforward calculation, we have

$$\begin{aligned}\mathcal{H}_T(\cos 2\pi x)^m &= \pi^2(2m+l_0+l_1+2)^2(\cos 2\pi x)^m \\ &\quad + 4\pi^2((-m(m-1)+m(l_0-l_1))(\cos 2\pi x)^{m-1}.\end{aligned} \tag{5.3}$$

Therefore the operator $\mathcal{H}_T$ acts triangularly on the space spanned by $\{1, \cos 2\pi x, \cos^2 2\pi x, \dots\}$. Since $(2m+l_0+l_1+2)^2 \neq (2m'+l_0+l_1+2)^2$ $(m, m' \in \mathbb{Z}_{>0},\ m \neq m')$, the operator $\mathcal{H}_T$ is diagonalizable and have

$$\mathcal{H}_T\psi_m(x) = \pi^2(2m+l_0+l_1+2)^2\psi_m(x), \tag{5.4}$$

where $\psi_m(x) = (\cos 2\pi x)^m$+(lower terms).

We will obtain more explicit expression of $\psi_m(x)$.

We set $w = (1-\cos 2\pi x)/2$. By a change of variable, we have

(5.5)

$$\mathcal{H}_T - \pi^2(2m+l_0+l_1+2)^2 = -4\pi^2 \cdot$$
$$\left\{w(1-w)\frac{d^2}{dw^2} + \left(\frac{2l_0+3}{2} - ((l_0+l_1+2)+1)w\right)\frac{d}{dw} + m(m+l_0+l_1+2)\right\}.$$

Hence the equation $\mathcal{H}_T - \pi^2(2m+l_0+l_1+2)^2$ is transformed to the hypergeometric equation. From the expression of (5.5), we have

$$\psi_m(x) = \tilde{c}_m G_m\left(l_0+l_1+2, \frac{2l_0+3}{2}; \frac{1-\cos 2\pi x}{2}\right), \tag{5.6}$$

where the function $G_m(\alpha, \beta; w)$ is the Jacobi polynomial of degree m and $\tilde{c}_m$ is a constant.

We define the inner product by

$$\langle f, g\rangle = \int_0^1 dx \overline{f(x)} g(x). \tag{5.7}$$

Then the values $\langle \Phi(x)\psi_m(x), \Phi(x)\psi_{m'}(x)\rangle$ are finite and we have

$$\begin{aligned}&\pi^2(2m+l_0+l_1+2)^2\langle \Phi(x)\psi_m(x), \Phi(x)\psi_{m'}(x)\rangle \\ &\langle H_T\Phi(x)\psi_m(x), \Phi(x)\psi_{m'}(x)\rangle = \langle \Phi(x)\psi_m(x), H_T\Phi(x)\psi_{m'}(x)\rangle \\ &= \pi^2(2m'+l_0+l_1+2)^2\langle \Phi(x)\psi_m(x), \Phi(x)\psi_{m'}(x)\rangle.\end{aligned} \tag{5.8}$$

Therefore the functions $\Phi(x)\psi_m(x)$ $(m \in \mathbb{Z}_{\geq 0})$ are orthogonal.

Remark (i) The values $\tilde{c}_m$ and $\langle \Phi(x)\psi_m(x), \Phi(x)\psi_m(x) \rangle$ are known explicitly.

(ii) The functions $\Phi(x)\psi_m(x)$ $(m \in \mathbb{Z}_{\geq 0})$ are complete orthogonal basis in $L^2(\mathbb{R}/\mathbb{Z})$. For the proof, see ([3]) or ([9]).

(iii) The functions $\psi_m(x)$ are nothing but the BC_1-Jacobi polynomial.

5.2. **Jacobi polynomial, Bethe Ansatz and perturbation.** In this subsection, we will show the correspondence between the Jacobi polynomial and the function obtained by the Bethe Ansatz method. After that, we will discuss the holomorphic perturbation with respect to the parameter $p = \exp(\pi\sqrt{-1}\tau) \sim 0$ for the Inozemtsev model (2.1) for the $l_2 = l_3 = 0$ cases. For the $l_1 = l_2 = l_3 = 0$ cases, the corresponding result is obtained in ([8]).

Let us remind the Bethe Ansatz method for the trigonometric case, which was discussed in section 4.

Let m be an non-negative integer. We set $c = l_0 + l_1 + 2 + 2m$. Then we have $P_T(\pi^2 c^2 - C_T) \neq 0$, where $P_T(E)$ and C_T was defined in section 4.1. Hence, the solution $T_1, \ldots, T_{l+1}$ of the Bethe Ansatz equation (4.11, 4.12) with the condition $c = l_0 + l_1 + 2 + 2m$ exists uniquely up to permutation. In this setting, the function

$$\Lambda_T(x) = \frac{\prod_{j=1}^{l_0+l_1} \sin \pi(x + t_j)}{(\sin \pi x)^{l_0} (\cos \pi x)^{l_1}} e^{\pi\sqrt{-1}cx}, \tag{5.9}$$

satisfy $H_T \Lambda_T(x) = \pi^2 c^2 \Lambda_T(x)$, where $\exp(2\pi\sqrt{-1}t_j) = T_j$ and the values $(T_1, \ldots T_{l_0+l_1})$ are a solution of the equations (4.11, 4.12). From the condition $P_T(\pi^2 c^2 - C_T) \neq 0$, it follows that the functions $\Lambda_T(x)$ and $\Lambda_T(-x)$ are linearly independent.

We set $\Lambda_T^{sym}(x) = \Lambda_T(x) - (-1)^{l_0}\Lambda_T(x)$. Then the function $\Lambda_T^{sym}(x)$ does not vanish and the degree of the pole of the function $\Lambda_T^{sym}(x)$ at $x = n$ (resp. $x = n + 1/2$) $(n \in \mathbb{Z})$ is less than l_0 (resp. l_1). Since the exponents of the differential equation $H_T - \pi^2 c^2$ at $x = n$ (resp. $x = n + 1/2$) are $-l_0$ and $l_0 + 1$ (resp. $-l_1$ and $l_1 + 1$), The

function $\Lambda_T^{sym}(x)$ has zeros of degree l_0+1 (resp. l_1+1) at $x=n$ (resp. $x=n+1/2$).

Since the eigenvalue of the function $\Phi(x)\psi_m(x)$ with respect to the Hamiltonian H_T is $\pi^2(l_0+l_1+2+2m)^2$ and the function $\Phi(x)\psi_m(x)$ is holomorphic at $x=0$, we have $\Lambda_T^{sym}(x) = B_m\Phi(x)\psi_m(x)$ for some non-zero constant B_m.

From now on, we will show that for each $c = l_0+l_1+2+2m$ ($m \in \mathbb{Z}_{\geq 0}$), there exist $\tilde{p} \in \mathbb{R}_{>0}$ such that the solution of the Bethe Ansatz equation (3.27) exists for each p ($|p| < \tilde{p}$), which converge analytically to the solution of the Bethe Ansatz equation for $p=0$ and $c = l_0+l_1+2+2m$. We set $c_m = l_0+l_1+2+2m$.

Since the coefficients of the polynomial $P(E)$ which was defined in section 3.2 are holomorphic in p, there exists $\epsilon_1, p_1 \in \mathbb{R}_{>0}$ such that $P(E) \neq 0$ for all (E,p) s.t. $|E+C_T-\pi^2c_m^2| < \epsilon_1,\ |p| < p_1$.

Let us remind that the function $\Xi(x)$ is expressed as

$$\Xi(x) = \frac{\prod_{j=1}^{l_0+l_1}(\wp(x)-\wp(t_j))}{\wp(x)^{l_0}\wp(x+1/2)^{l_1}} \tag{5.10}$$

We set $a_j = \wp(t_j)$ and $z = \wp(x)$, then the values a_j depend on the values E and p.

The numerator of the equation (5.10) are polynomial in z and the coefficients of z^j ($j=1,\dots,l_0+l_1$) are holomorphic in E and p. Since the zeros of the polynomial are simple, the values a_j are holomorphic in (E,p) s.t. $|E+C_T-\pi^2c_m^2| < \epsilon_1,\ |p| < p_1$.

We suppose $|E+C_T-\pi^2c_m^2| < \epsilon_1$ and $|p| < p_1$, then we have $P(E) \neq 0$ and there exists a solution of the Bethe Ansatz equation (3.27) for E and p. We denote the solution by $(t_1(E,p),\dots,t_{l_0+l_1}(E,p))$. If we choose the ordering suitably, we have $a_j = \wp(t_j(E,p))$.

From Proposition 3.7 and the proof of Proposition 4.2, we obtain that the values $t_j(E,p)$ are mutually distinct and do not contained in $\begin{cases} \mathbb{Z}/2 & (p=0) \\ \mathbb{Z}/2+\mathbb{Z}\tau/2 & (p\neq 0) \end{cases}$ for $(E, p(=e^{2\pi\sqrt{-1}\tau}))$ s.t. $|E+C_T-\pi^2c_m^2| < \epsilon_1,\ |p| < p_1$.

Since $t_j(E,p) \notin \frac{1}{2}\mathbb{Z} + \frac{1}{2}\tau\mathbb{Z}$, we have $\wp'(t_j(E,p)) \neq 0$ and the values $t_j(E,p)$ are holomorphic in E and p bacause the values a_j are holomorphic in E and p.

From (3.28), the value c satisfy the relation $\pi^2 c^2 = h(E,p)$, where

$$h(E,p) = E - (l_0+l_1)(l_0+l_1+1)\eta_1 - l_0 l_1 e_1 + \sum_{j<k} \frac{\theta''(t_j - t_k)}{\theta(t_j - t_k)} - \sum_{j=1}^{l_0+l_1} \left(l_0 \frac{\theta''(t_j)}{\theta(t_j)} + l_1 \frac{\theta''(t_j - 1/2)}{\theta(t_j - 1/2)} \right). \tag{5.11}$$

We are going to solve the Bethe Ansatz equation with the condition $c = c_m$ for each p ($|p|$: sufficiently small).

We have $h(E - C_T, 0) = E$ and $\frac{\partial h}{\partial E}(\pi^2 c_m^2 - C_T, 0) = 1$. By the implicit function theorem, there exists $\tilde{p} \in \mathbb{R}_{>0}$ such that there exists a holomorphic function $E(p)$ on $|p| < \tilde{p}$ such that $h(E(p),p) = \pi^2 c_m^2$ and $E(0) = \pi^2 c_m^2 - C_T$.

We set $\tilde{t}_j(p) = t_j(E(p),p)$ ($j = 1, \ldots, l_0+l_1$). Then the function $\tilde{t}_j(p)$ is the holomorphic in the variable p on $|p| < \tilde{p}$, $(\tilde{t}_1(p), \ldots, \tilde{t}_{l_0+l_1}(p))$ is a solution of the Bethe Ansatz equation (3.27) with the condition $c = c_m$, and the corresponding eigenvalue $E(p)$ is also holomorphic in p.

Hence we obtain that the solution of the Bethe Ansatz equation (3.27) at $p(= \exp(2\pi\sqrt{-1}\tau))$ such that $|p| < \tilde{p}$ are holomorphically connected to the solution at $p = 0$ with the condition $c = l_0+l_1+2+2m$ ($m \in \mathbb{Z}_{\geq 0}$).

We set

$$\Lambda_{(m)}(x) = \frac{\exp(\pi\sqrt{-1}c_m x) \prod_{j=1}^{l_0+l_1} \theta(x + \tilde{t}_j(e^{2\pi\sqrt{-1}\tau}))}{\theta(x)^{l_0}\theta(x+1/2)^{l_1}}, \tag{5.12}$$

$$\Lambda^{sym}_{(m)}(x) = \Lambda_{(m)}(x) - (-1)^{l_0}\Lambda_{(m)}(-x). \tag{5.13}$$

By a similar argument to the case of the function $\Lambda^{sym}_T(x)$, we obtain that the function $\Lambda^{sym}_{(m)}(x)$ is square-integrable on the interval $[0,1]$. The function $\Lambda^{sym}_{(m)}(x)$ converge to the function $B'_m \Phi(x)\psi_m(x)$ uniformly on compact sets for some non-zero constant B'_m as $p \to 0$.

Therefore we have the following theorem

Theorem 5.1. *Let m be an non-negative integer. For each m, there exists $\epsilon \in \mathbb{R}_{>0}$ such that if $|p| < \epsilon$ then for each p there exist eigenvalues $E_m(p)$ and eigenfunctions $g_m(x)$ of the Hamiltonian of the Inozemtsev model with $l_2 = l_3 = 0$ such that*

$$E_m(p) \to \pi^2(l_0 + l_1 + 2 + 2m)^2 - \frac{\pi^2}{3}\left(l_0(l_0+1) + l_1(l_1+1)\right), \tag{5.14}$$

$$g_m(x) \to \Phi(x)\psi_m(x) \qquad \text{on compact sets}, \tag{5.15}$$

as $p \to 0$ and $g_m(x)$ is square-integrable, i.e.

$$\int_0^1 |g_m(x)|^2 dx < \infty. \tag{5.16}$$

Here, the function $\psi_m(x)$ is defined in (5.6).

Remark For the $l_1 = l_2 = l_3 = 0$ cases, this theorem was established in ([8]).

6. Comments

6.1. After obtaining Theorem 5.1, we should discuss the completeness of the function $g_m(x)$ which appear in Theorem 5.1 and clarify the properties of the spectral and the eigenstates. It will be discussed in the forthcoming article ([9]). We remark that some results will be obtained by applying the result given in ([3]).

6.2. Roughly speaking, the problem to find the square-integrable eigenstates for the Inozemtsev Hamiltonian (2.1) is replaced with the problem to find the solutions of the Heun equation (2.3) whose monodromy matrix around two singular points satisfy certain conditions.

We hope that the Bethe Ansatz method would be of help to investigate the monodnomies of the Heun equation.

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Appendix A.

We note definitions and formulae of the elliptic functions.

Let ω_1 and ω_3 be complex numbers such that the value ω_3/ω_1 is an element of the upper half plane.

The Weierstrass $\wp$-function, the Weierstrass sigma-function and the Weierstrass zeta-function are defined as follows,

(A.1)

$$\begin{aligned}
&\wp(z) = \wp(z|2\omega_1, 2\omega_3) = \\
&\quad \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus(0,0)} \left(\frac{1}{(z-2m\omega_1-2n\omega_3)^2} - \frac{1}{(2m\omega_1+2n\omega_3)^2} \right), \\
&\sigma(z) = \wp(z|2\omega_1, 2\omega_3) = z \prod_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \left(1 - \frac{z}{2m\omega_1+2n\omega_3}\right) \cdot \\
&\quad \cdot \exp\left(\frac{z}{2m\omega_1+2n\omega_3} + \frac{z^2}{2(2m\omega_1+2n\omega_3)^2} \right), \\
&\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.
\end{aligned}$$

We set $\omega_2 = -\omega_1 - \omega_3$ and

$$e_i = \wp(\omega_i), \quad \eta_i = \zeta(\omega_i), \quad (i = 1, 2, 3). \tag{A.2}$$

We have the relations

$$\begin{aligned}
&e_1 + e_2 + e_3 = \eta_1 + \eta_2 + \eta_3 = 0, \\
&\wp(z) = -\zeta'(x), \quad (\wp'(z))^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3) \\
&\wp(z+2\omega_i) = \wp(z), \quad \zeta(z+2\omega_i) = \zeta(z) + 2\eta_i, \quad (i = 1, 2, 3) \\
&\frac{\wp''(z)}{(\wp'(z))^2} = \frac{1}{2}\left(\frac{1}{z-e_1} + \frac{1}{z-e_2} + \frac{1}{z-e_3} \right),
\end{aligned} \tag{A.3}$$

The co-sigma functions $\sigma_i(z)$ $(i = 1, 2, 3)$ are given by

$$\sigma_i(z) = \exp(-\eta_i z)\sigma(z+\omega_i)/\sigma(\omega_i). \tag{A.4}$$

We have

$$\left(\frac{\sigma_i(z)}{\sigma(z)} \right)^2 = \wp(z) - e_i \qquad (i = 1, 2, 3). \tag{A.5}$$

Let τ be an complex number on the upper complex plane. The elliptic theta function $\theta(x)$ is defined by

$$\theta(x) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\exp(\tau\pi\sqrt{-1}(n-(1/2))^2)\sin(2n-1)\pi x. \tag{A.6}$$

The theta function $\theta(x)$ have the quasi-periodicity, i.e.

$$\theta(x+1) = -\theta(x), \quad \theta(x+\tau) = -e^{-2\pi\sqrt{-1}x-\pi\sqrt{-1}\tau}\theta(x). \tag{A.7}$$

If we set $\tau = \omega_1/\omega_3$, we have the following relation

$$\sigma(2\omega_1 x) = 2\omega_1\exp(2\eta_1\omega_1 x^2)\theta(x)/\theta'(0). \tag{A.8}$$

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