arXiv:2610.04096v3 [math.RT] 10 Nov 2026

THE BENSON–ETINGOF–OSTRIK CONJECTURE FOR FINITE TENSOR CATEGORIES

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Abstract. We prove that every finite symmetric tensor category over an algebraically closed field of positive characteristic admits a symmetric tensor functor to a higher Verlinde category. This was conjectured for arbitrary symmetric tensor categories of moderate growth by Benson, Etingof and Ostrik [BEO23]. The proof uses higher Frobenius functors, introduced by the authors in [CF26]. The main effort is directed towards a proof of a conjecture from [CF26] about thick tensor ideals in the representation categories of elementary abelian p-groups, which implies that the higher Frobenius functors are symmetric monoidal. In fact, we prove a refinement of the latter conjecture that relates tensoring with restricted Steinberg modules to an iterated generalised cohomology of modules of elementary abelian p-groups.

1. Introduction

Let $\Bbbk$ be an algebraically closed field. We consider $\Bbbk$-linear abelian symmetric rigid monoidal categories satisfying some finiteness conditions, which are known as (symmetric) *tensor categories*, see [EGNO15].

If the characteristic of $\Bbbk$ is zero, then Deligne proved in [Del02] that every tensor category of *moderate growth* (which means that the length of tensor powers of a given object grows at most exponentially) is a representation category of an affine supergroup scheme. By the principle of tannakian reconstruction [Del90], this is equivalent to saying that every tensor category of moderate growth admits a tensor functor to (i.e., *fibres* over) the tensor category of super vector spaces. Yet equivalently, Deligne's theorem states that the category of super vector spaces and its subcategory of vector spaces are the only *incompressible* tensor categories of moderate growth, see [CEO24].

In this paper, we consider tensor categories (of moderate growth) over an algebraically closed field $\Bbbk$ of characteristic $p > 0$. A naive version of Deligne's theorem is known to be false. Indeed, by [BE19; BEO23; Cou21], there is an infinite sequence of incompressible tensor categories called *higher Verlinde categories*

$$\mathrm{Ver}_p \subsetneq \mathrm{Ver}_{p^2} \subsetneq \cdots \subsetneq \mathrm{Ver}_{p^\infty} = \bigcup_{r\geq 1} \mathrm{Ver}_{p^r},$$

where $(\mathrm{Ver}_{p^r})_{1\leq r<\infty}$ are defined as the abelian envelopes of the quotients of $\operatorname{Tilt}\mathrm{SL}_2$. The main conjecture of the field asserts that Ver_{p^∞} plays the role of super vector spaces in characteristic 0:

Conjecture 1.1 ([BEO23, Conjecture 1.4])**.** *Any symmetric tensor category of moderate growth over an algebraically closed field of positive characteristic p fibres over Ver_{p^∞}.*

The most important positive result yet towards Conjecture 1.1 is a characterisation in [CEO23] of those tensor categories that fibre over Ver_p, which built on the earlier special cases of fusion categories [Ost20] and finite tensor categories [EO21]. It is shown in [CEO23] that a tensor category

of moderate growth admits a tensor functor to Ver_p if and only if a certain Frobenius functor, originally defined by Ostrik in [Ost20], acts exactly. We say the Frobenius functor *detects* fibring over Ver_p. Crucial in the definition of Ostrik's functor was a realisation of Ver_p as the semisimplification of the category of representations of the cyclic p-group C_p, rather than of $\mathrm{Tilt}\, SL_2$.

In an attempt at generalising these ideas, the authors in [CF26] constructed realisations of every higher Verlinde category Ver_{p^r} in terms of categories of representations of the elementary abelian p-group C_p^r. We then constructed candidates for generalisations of Ostrik's functor to higher Frobenius functors for all r. However, some properties (collected using the notion **O**-*functor*) of our higher Frobenius functors remained conjectural. In fact, they were shown to rely on the following conjecture on representations of elementary abelian p-groups.

Conjecture 1.2 ([CF26, Conjecture 5.4.2(1)], [CF25])**.** *For any $r \geq 1$, fix an embedding of C_p^r into $\mathrm{SL}_2(\Bbbk)$. Let $S \in \mathrm{Rep}\, C_p^r$ be the restriction of the tilting module with highest weight $p^{r-1}-1$. Then the functor $S \otimes -$ on $\mathrm{Rep}\, C_p^r$ has values in the subcategory of restrictions of SL_2-tilting modules.*

In this paper, we resolve Conjecture 1.2 by proving a refined description of the functor $S \otimes -$. Using an iterated cohomology H_X (depending on an embedding of C_p^r into $\mathrm{SL}_2(\Bbbk)$) for modules X of C_p^r that we introduce in Section 3.3, and denoting restricted SL_2-simples by the symbols $L(i)$ according to their highest weight $i \in \mathbb{N}$, we show:

Theorem A (Theorem 3.21)**.** *Conjecture 1.2 is true; more precisely, for $X \in \mathrm{Rep}\, C_p^r$ and S as above:*

$$S \otimes X \cong \bigoplus_{0 \leq i < p^r} (S \otimes L(i))^{\oplus \dim H_X(i)}. \tag{1.1}$$

As a consequence, the functors Φ_C^V from [CF26] are **O***-functors.*

We then use these **O**-functors to show a finite version of Conjecture 1.1.

Theorem B (Theorem 4.7)**.** *Conjecture 1.1 is true for* ***finite*** *symmetric tensor categories; in fact, any such category fibres over Ver_{p^r} for some $1 \leq r < \infty$.*

Moreover, by Theorem 4.1, see Remark 4.10 for a more precise discussion, the functors Φ_C^V detect fibring over Ver_{p^r} for finite symmetric tensor categories.

Around the time we finished writing this article, Alexander Sherman independently found an alternative proof [She26] of Conjecture 1.2, also using AI. While his proof does not imply Equation (1.1), it is shorter and does seem conceptually illuminating to us.

Implications and new directions. We refer to Section 4.4 for some immediate applications of the main results in the paper. Here we briefly discuss the broader potential implications of our results on current and future research.

Tensor categories. Two emerging lines of research in (finite) tensor categories gain considerably in importance due to Theorem B. These are the study of polynomial functors in higher Verlinde categories, in the sense of [Cou26b], and the study of finite (or more specifically infinitesimal) group schemes in higher Verlinde categories. Indeed, the "classification" of the latter is now equivalent to the "classification" of all finite tensor categories. Moreover, via Theorem B, both topics together should play a major role in resolving [EO04, Conjecture 2.18] about finite generation of cohomology, in the symmetric case.

Theorem B constitutes significant progress towards Conjecture 1.1. Indeed, the latter now essentially reduces to the conjecture that any finitely generated tensor category of moderate growth

admits a tensor functor to a finite tensor category. Nonetheless, it seems that fundamentally new ideas and techniques will be required to prove the full Conjecture 1.1. Perhaps the most significant implication of Theorem B in this regard is the convincing evidence it provides for Conjecture 1.1. Indeed, when the authors of [BEO23] formulated this conjecture they described it as "*a (perhaps imprudently) bold conjecture*". We feel this description is now no longer applicable.

Modular representation theory. Beyond tensor categories, we expect that our results will open new directions in modular representation theory. Even the validity of Conjecture 1.2 seemed surprising to experts in the field. Its resolution via the explicit formula in Equation (1.1) should be a helpful new technique. It can also be seen as a first step towards a theory of (additive) tensor ideals in categories of modular representations, on which almost nothing seems to be known. The iterated cohomology H_X we introduce to formulate Equation (1.1) seems to be a new and interesting invariant of a module. As it can predict the tensor product with S, and in particular, carries information on the annihilation of extensions under the functor $S \otimes -$, it could be interpreted as a refinement of classical support theory. Note that S and hence all non-projective indecomposable summands in tensor products involving S have a classical support of one point in projective space, but classical support theory does not give any more insights into these tensor products. Such insights are provided by H_X, and in fact, we have a definition of H_X and a tensor product decomposition result for every point in a dense subset of projective space.

Another interpretation of the now proven Conjecture 1.2 is that any representation of an elementary abelian p-group is "just a tensor product with S away from" an SL_2-representation. This suggest new ways of using algebraic group representation theory to study representations of finite groups. It would be interesting to study if such a connection can be widened to include larger algebraic groups.

Note that while Theorem B is specific to elementary abelian p-groups, we expect it to be of relevance for modular representations of general finite groups, due to the classical insight in modular representation theory that the restrictions to elementary abelian subgroups control many aspects of the theory, such as projectivity, support, or nilpotence in cohomology.

Use of AI. The main ideas in the proofs in Section 3 and Section 4 were originally obtained using GPT-6 Astra, using as an input the formula in Equation (1.1), which was obtained previously using extensive numerical experiments with the help of Claude 4.6–5 Opus. This document was then written by hand and checked with GPT-6 Astra.

Remark. Four days after the first version of this paper was submitted to arXiv, a commercial AI company released a proof written by AI, of the same two conjectures that are resolved in this paper. More concretely, it announces a proof of Theorem B and a very different proof of Conjecture 1.2, which does not prove the concrete Equation (1.1).

Acknowledgments. We thank an anonymous referee of [CF26] whose questions essentially about a formula like Equation (1.1) triggered us to look into this again. J.F. was supported by the Hausdorff Center for Mathematics Bonn. J.F.'s Claude license was provided by the 2024 Max Planck–Humboldt Research Award (Max Planck Society / Alexander von Humboldt Foundation) bestowed on Geordie Williamson and hosted by Catharina Stroppel. K.C. was supported by ARC grants FT220100125 and DP250100762. We thank Alex Sherman for sharing his independent proof.

2. Preliminaries

2.0.1. *Notation.* Throughout the paper, $\Bbbk$ is an algebraically closed field of characteristic $p > 0$. All vector spaces are over $\Bbbk$, and tensor products are over $\Bbbk$ if not indicated otherwise. We write Vect for the category of finite-dimensional vector spaces. The symmetric group S_n is the group of permutations of $\{1, 2, \dots, n\}$. For a finite group G, we let $\operatorname{Rep} G$ denote its category of finite-dimensional representations over $\Bbbk$. We will use the same notation for an affine group scheme G and its category of finite-dimensional rational representations.

2.1. **Elementary abelian groups.** In this paper, the central role will be played by elementary abelian p-groups. Here we review how we will view them as subgroups of $\mathrm{SL}_2(\Bbbk)$ and of symmetric groups, and the basics of cohomological support theory. For an elementary abelian p-group C_p^r, for $r \in \mathbb{Z}_{>0}$, we denote the generator of the jth copy of C_p by g_j.

2.1.1. *Subgroups of* SL_2. We will use embeddings of C_p^r into the unipotent radical of the Borel subgroup of lower-triangular matrices of SL_2 (this radical is thus just a copy of $\mathbb{G}_a$). In other words, we consider homomorphisms

$$\iota_{\underline{\lambda}} : C_p^r \;\to\; (\Bbbk, +) \;\to\; SL_2(\Bbbk), \quad g_i \mapsto \begin{pmatrix} 1 & 0 \\ \lambda_i & 1 \end{pmatrix},$$

for non-zero $(\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r = \mathbb{A}^r(\Bbbk)$. Up to conjugacy, the above morphism only depends on the image $[\lambda_1 : \cdots : \lambda_r]$ in $\mathbb{P}^{r-1} = \mathbb{P}^{r-1}(\Bbbk)$, so we will usually reserve the notation $\underline{\lambda}$ for this element in $\mathbb{P}^{r-1}$. The following is [CF25, Lemma 3.2.2].

Lemma 2.1. *The following conditions are equivalent on $\underline{\lambda} = [\lambda_1 : \lambda_2 : \cdots : \lambda_r]$ in $\mathbb{P}^{r-1}(\Bbbk)$:*

1. *The homomorphism $\iota_{\underline{\lambda}}$ is injective.*
2. *The elements $\lambda_i \in \Bbbk$ are linearly independent over $\mathbb{F}_p$.*
3. *The determinant of the Moore matrix*

$$\mathbb{M}(\underline{\lambda}) := \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_r \\ \lambda_1^p & \lambda_2^p & \cdots & & \\ \cdots & & & & \\ \lambda_1^{p^{r-1}} & \cdots & & & \lambda_r^{p^{r-1}} \end{pmatrix}$$

is non-zero.

We denote the set of $\underline{\lambda}$ satisfying these equivalent properties by $\mathbb{P}_f^{r-1} \subset \mathbb{P}^{r-1}$.

Following [CF25, §4.2], for $\underline{\lambda} \in \mathbb{P}_f^{r-1}$, we can define $\boldsymbol{\lambda} \in \mathbb{P}^{r-1}$, uniquely determined by

$$\sum_{i=1}^{r} \lambda_i^{p^l} \boldsymbol{\lambda}_i = 0, \qquad \text{for all} \quad 0 \le l \le r-2.$$

Lemma 2.2. *The assignment $\underline{\lambda} \mapsto \boldsymbol{\lambda}$ gives a bijection $\mathbb{P}_f^{r-1} \to \mathbb{P}_f^{r-1}$.*

Proof. For $r = 1$ the assertion is immediate, so assume $r \ge 2$. First we need to establish that $\boldsymbol{\lambda} \in \mathbb{P}_f^{r-1}$. If we choose representatives in $\mathbb{A}^r$, then $\boldsymbol{\lambda}$ spans the orthogonal complement of $\underline{\lambda}, \underline{\lambda}^p, \dots, \underline{\lambda}^{p^{r-2}}$, where the powers are taken component-wise. If $\boldsymbol{\lambda} \notin \mathbb{P}_f^{r-1}$ then there are $a_i \in \mathbb{F}_p$,

not all zero, with $\sum_i a_i \boldsymbol{\lambda}_i = 0$, which thus in turn means that $(a_1, \dots, a_r) \in \mathbb{A}^r$ is in the linear span of $\underline{\lambda}, \underline{\lambda}^p, \dots, \underline{\lambda}^{p^{r-2}}$. In other words, there are $c_i \in \Bbbk$, not all zero, such that

$$\sum_{j=0}^{r-2} c_j \underline{\lambda}^{p^j} \ \in \ \mathbb{A}^r(\mathbb{F}_p).$$

By taking the p-th power, this means that

$$\sum_{j=0}^{r-2} c_j^p \underline{\lambda}^{p^{j+1}} \ = \ \sum_{j=0}^{r-2} c_j \underline{\lambda}^{p^j},$$

and by linear independence of the $\underline{\lambda}^{p^j}$ for $0 \leq j \leq r-1$ it follows quickly that this forces $c_j = 0$ for all j, a contradiction.

The defining relations for $\boldsymbol{\lambda}$ can be rewritten as $\sum_{i=1}^r \lambda_i^{p^{r-2}} \boldsymbol{\lambda}_i^{p^j} = 0$ for $0 \leq j \leq r-2$. In other words, applying the procedure $\underline{\lambda} \mapsto \boldsymbol{\lambda}$ to $\boldsymbol{\lambda}$ produces $\underline{\lambda}^{p^{r-2}}$. Hence $\underline{\lambda} \mapsto \boldsymbol{\lambda}$ has a two-sided inverse. □

2.1.2. *Subgroups of symmetric groups.* We let $E_r < S_{p^r}$ denote the copy of C_p^r embedded as in [CF26, §5.1.5]. It is uniquely determined up to conjugacy due to the following well-known lemma.

Lemma 2.3. *If $n > 1$ and S_n has a transitive elementary abelian p-subgroup E then $n = p^r$ for some $r \in \mathbb{Z}_{>0}$ and E is conjugate to E_r.*

Proof. It is clear that any transitive abelian subgroup of S_n must act regularly on $\{1, \dots, n\}$. In particular $E = C_p^r$ with $n = p^r$.

So now consider two transitive elementary abelian p-subgroups E, F of S_n and an isomorphism $\varphi : E \to F$. Define $\sigma \in S_n$ via

$$\sigma(e(1)) = \varphi(e)(1),$$

for all $e \in E$. That σ is a permutation follows from regularity of the actions. Now we can observe that $F = \sigma E \sigma^{-1}$. □

2.1.3. *Support theory.* We refer to [Ben17] for the basic notions of cohomological support theory and the rank variety for elementary abelian p-groups.

Set $X_i = g_i - 1 \ \in \ \Bbbk C_p^r$. For $M \in \operatorname{Rep} C_p^r$, we define its support $\operatorname{supp}(M) \subset \mathbb{P}^{r-1}$ as the projective variety that has as cone the rank variety $V^{\mathrm{rk}}(M) \subset \mathbb{A}^r$. Concretely, the point $[\mu_1 : \mu_2 : \cdots : \mu_r] \in \mathbb{P}^{r-1}$ is in $\operatorname{supp}(M)$ if and only if M is not projective as a module over the algebra $\Bbbk[t]/t^p$, where t acts on M as

$$\sum_{i=1}^r \mu_i X_i \ \in \ \Bbbk C_p^r.$$

Recall that

$$\operatorname{supp}(M \otimes N) \ = \ \operatorname{supp}(M) \cap \operatorname{supp}(N), \tag{2.1}$$

and that $M \in \operatorname{Rep} C_p^r$ is projective if and only if $\operatorname{supp}(M) = \emptyset$.

Lemma 2.4. *If $M \in \operatorname{Rep} C_p^r$ is not projective, but is projective over every proper subgroup of C_p^r, then $\operatorname{supp}(M) \cap \mathbb{P}_f^{r-1} \neq \varnothing$.*

Proof. Subgroups $C_p^{r-1} \cong G < C_p^r$ are in natural bijection with codimension one $\mathbb{F}_p$-subspaces of $\mathbb{F}_p^r$. So for every such G we have a corresponding $a^G \in \mathbb{P}^{r-1}(\mathbb{F}_p)$ describing the hyperplane. In particular, each such subgroup G determines an $\mathbb{F}_p$-rational hyperplane $P_G \subset \mathbb{P}^{r-1}$ comprising all $\underline{\lambda}$ with $\sum_i a_i^G \lambda_i = 0$. Moreover, by [Ben17, Lemma 1.9.8] we have

$$\operatorname{supp}(M|_G) \;=\; \operatorname{supp}(M) \cap P_G.$$

Since the complement of the unions of the P_G is precisely $\mathbb{P}_f^{r-1}$, the condition that M be projective over all proper subgroups implies $\operatorname{supp}(M) \subset \mathbb{P}_f^{r-1}$ (see [Ben17, Lemma 1.9.5]), while the condition that M not be projective shows $\operatorname{supp}(M) \neq \varnothing$, concluding the proof. □

2.2. **Tensor categories.**

2.2.1. *Definitions.* We define *tensor categories* to be essentially small finite-length abelian symmetric rigid monoidal $\Bbbk$-linear categories with one-dimensional endomorphism space of the tensor unit. In other words, they are symmetric tensor categories in the sense of [EGNO15]. *Tensor functors* are exact linear symmetric monoidal, and automatically faithful. A tensor category $\mathcal{C}$ is said to be *of moderate growth* if for any $X \in \mathcal{C}$ there is $K \in \mathbb{R}$ such that $\operatorname{len}(X^{\otimes n}) \leq K^n$, where len denotes the length of an object in an abelian category. A tensor category is *finite* if it has enough projectives and only finitely many simple objects up to isomorphism.

Remark 2.5. We will often encounter exact symmetric monoidal functors $F : \mathcal{C} \to \mathcal{D}$ between tensor categories that are not $\Bbbk$-linear, but 'twisted' $\Bbbk$-linear by a field automorphism. In practice this automorphism will be a Frobenius homomorphism $\lambda \mapsto \lambda^q$ for a power q of p, meaning $F(\lambda f) = \lambda^q F(f)$ for a morphism f in $\mathcal{C}$ and $\lambda \in \Bbbk$. Since $\Bbbk$ is perfect, we can twist the $\Bbbk$-linear structure on the target, to obtain a new tensor category (with same underlying abelian monoidal category) for which F is now an honest tensor functor. We will therefore largely ignore this subtlety and freely treat these functors as tensor functors.

In a tensor category with finitely many simple objects up to isomorphism, for any object X, the exact functor $X \otimes -$ corresponds to multiplication with an integer matrix in the Grothendieck group. By the Frobenius–Perron theorem, the spectral radius of this matrix is an eigenvalue. The *Frobenius-Perron dimension* $\operatorname{FPdim}(X) \in \mathbb{R}$ of X is defined as this maximal eigenvalue. FPdim is additive under direct sums, multiplicative under tensor products, and preserved by tensor functors, see [EGNO15]. As $\operatorname{len}(X^{\otimes n}) \leq \operatorname{FPdim}(X)^n$, tensor categories with finitely many isomorphism classes of simples, and in particular finite tensor categories, are automatically of moderate growth.

Since we are working over an algebraically closed field, for any two tensor categories $\mathcal{C}, \mathcal{D}$ in our sense, there is a *Deligne tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ (see [CF26, Section 3] and references therein for more information). It is a tensor category containing the naive tensor product category $\mathcal{C} \otimes_{\Bbbk} \mathcal{D}$, whose objects are pairs of objects from $\mathcal{C}$ and $\mathcal{D}$, and whose hom-spaces are tensor products of hom-spaces in $\mathcal{C}$ and $\mathcal{D}$, respectively, over $\Bbbk$. In fact, any object in $\mathcal{C} \boxtimes \mathcal{D}$ is a quotient of an object in $\mathcal{C} \otimes_{\Bbbk} \mathcal{D}$, and the simple objects of $\mathcal{C} \boxtimes \mathcal{D}$ are exactly the objects of the form $X \boxtimes Y = (X, Y)$, for simple objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$. If $\mathcal{C}$ is a finite tensor category, then we can realise $\mathcal{C} \boxtimes \mathcal{D}$ as the category of $\Bbbk$-linear functors $\mathcal{P}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{D}$, where $\mathcal{P}_{\mathcal{C}} \subset \mathcal{C}$ is the full subcategory of projective objects. In this realisation, an object of the form $X \boxtimes Y$ corresponds to the functor $\operatorname{Hom}(-, X) \otimes Y$.

2.2.2. *Higher Verlinde categories.* We recall the construction of the higher Verlinde categories Ver_{p^r} for $r \geq 1$. Let $\operatorname{Tilt} \mathrm{SL}_2$ be the category of tilting modules of SL_2 over $\Bbbk$. Its indecomposable objects T_i can be characterised by their highest weights $i \geq 0$. Then Ver_{p^r} is the monoidal abelian envelope

of the quotient category $\operatorname{Tilt}\mathrm{SL}_2/I_r$, where I_r is the tensor ideal of morphisms that factor through a direct sum of tilting modules T_i with $i \geq p^r - 1$. This ideal I_r is also the tensor ideal generated by the Steinberg representation T_{p^r-1}. As a special case, when $r = 1$, $\operatorname{Tilt}\mathrm{SL}_2/I_1$ is already semisimple, and hence its own abelian envelope, with simples $L_1, \dots, L_{p-1}$, where L_i is the image of T_{i-1}, following the notation from [Ost20], rather than [BEO23].

2.2.3. **V**-*functors and* **O**-*functors.* We will freely use that for a tensor category $\mathcal{C}$ and $X, Y \in \mathcal{C}$, the braiding produces algebra morphisms $\Bbbk S_n \to \operatorname{End}(X^{\otimes n})$, so that $\operatorname{Hom}(Y, X^{\otimes n})$ becomes a S_n-representation via composition.

We recall some definitions from [CF26]. For a permutation group G, a **V**-*functor* is a linear symmetric monoidal functor of the form

$$\Theta : \operatorname{Rep} G \to \mathcal{V},$$

for some tensor category $\mathcal{V}$, that annihilates objects of the form $\operatorname{Ind}_H^G \mathbf{1}$, for all non-transitive subgroups $H < G$.

Now assume that the permutation group G is defined as a subgroup of S_{p^r}, for some $r \geq 1$. To any **V**-functor Θ as above, and any tensor category $\mathcal{C}$, we can associate the additive symmetric monoidal functor

$$\Phi_{\mathcal{C}} \colon \mathcal{C} \xrightarrow{X \mapsto X^{\otimes p^r}} \mathcal{C} \boxtimes \operatorname{Rep} S_{p^r} \xrightarrow{\mathcal{C}\boxtimes\mathrm{Res}} \mathcal{C} \boxtimes \operatorname{Rep} G \xrightarrow{\mathcal{C}\boxtimes\Theta} \mathcal{C} \boxtimes \mathcal{V}.$$

Here, we view the p^r-th tensor power as an object in $\mathcal{C}$ with an action of S_{p^r} by permuting tensor factors. It was shown in [CF26, Section 5] that such functors intertwine tensor functors in a certain way, as a generalisation of this property for Ostrik's Frobenius functor in [Ost20; EO21; Cou20]. The term **O**-functor was coined in [CF26] for such functors. Such functors need not be exact, but when they are, they are tensor functors up to the comments in Remark 2.5.

Assume that $\mathcal{C}$ is a finite tensor category with subcategory $\mathcal{P}_{\mathcal{C}}$ of projective objects. Then using the description from Section 2.2.1 of the Deligne tensor product $\mathcal{C} \boxtimes \mathcal{V}$ as a functor category $\mathcal{P}_{\mathcal{C}}^{\mathrm{op}} \to \mathcal{V}$, we can think of the above **O**-functors alternatively as functors

$$\Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{V}, \quad X \mapsto \Theta \circ \operatorname{Hom}(-, X^{\otimes p^r}). \tag{2.2}$$

Here $\operatorname{Hom}(-, X^{\otimes p^r})$ is viewed as a functor $\mathcal{P}_{\mathcal{C}}^{\mathrm{op}} \to \operatorname{Rep} G$, using the permutation action of $G < S_{p^r}$ on tensor factors. This more explicit form unfortunately obscures the monoidal structure.

As a special case, take $G = C_p < S_p$, and take $\Theta\colon \operatorname{Rep} G \to \operatorname{Ver}_p$, the semisimplification functor. Then the above construction recovers Ostrik's Frobenius functor

$$\mathrm{Fr} : \mathcal{C} \to \mathcal{C} \boxtimes \operatorname{Ver}_p, \quad X \mapsto \mathrm{Fr}(X) = \bigoplus_{i=1}^{p-1} F_i(X) \boxtimes L_i, \tag{2.3}$$

where $F_i(X) = \frac{\ker(d) \cap \operatorname{im}(d^{i-1})}{\ker(d) \cap \operatorname{im}(d^i)}$, for d the action of a generator of C_p minus the identity on $X^{\otimes p}$. Using the same formula, we also set $F_p(X) = \operatorname{im}(d^{p-1})$.

Given an **O**-functor Φ, we say that a tensor category $\mathcal{C}$ is Φ-exact when $\Phi_{\mathcal{C}}$ is exact, and thus a tensor functor. In particular Fr-exactness is the same as 'Frobenius exactness'.

3. The refined Steinberg splitting conjecture

In this section, we prove Equation (1.1) and Conjecture 1.2. The proof strategy is as follows: First, it will turn out to be useful to consider all (not just $S = St_{r-1}$ in the statement) representations of elementary abelian p-groups as representations of the additive group. Specifically, by this

change of perspective, we will be able to use certain operators (called $(c_i)_i$ in Section 3.3) that are very natural for representations of the additive group. These operators define a generalised cohomology on representations, along which tensor products with (lifts of) Steinberg representations are shown to split in Section 3.2. This implies Conjecture 1.2.

In more detail, the statement of Conjecture 1.2 requires (up to conjugacy) an embedding $C_p^r < (\Bbbk, +) < \mathrm{SL}_2(\Bbbk)$, see 2.1.1. We thus have resulting monoidal functors

$$\mathrm{Tilt}\,\mathrm{SL}_2 \to \mathrm{Rep}\,\mathbb{G}_a \to \mathrm{Rep}\,C_p^r.$$

Denote by $\mathbb{G}_{a,j}$ the j-th Frobenius kernel of $\mathbb{G}_a$. The group algebra is given by

$$\Bbbk\mathbb{G}_{a,j} \cong \Bbbk[z_0, z_1, \dots, z_{j-1}]/(z_0^p, \dots, z_{j-1}^p) \cong \Bbbk\mathbb{G}_{a,j-1} \otimes \Bbbk[z_{j-1}]/z_{j-1}^p. \tag{3.1}$$

We can filter $\mathrm{Rep}\,\mathbb{G}_a$ by full (non-monoidal) subcategories

$$\mathrm{Vect} = (\mathrm{Rep}\,\mathbb{G}_a)_0 \subset (\mathrm{Rep}\,\mathbb{G}_a)_1 \subset (\mathrm{Rep}\,\mathbb{G}_a)_2 \subset \cdots,$$

where $(\mathrm{Rep}\,\mathbb{G}_a)_j$ contains the representations that have polynomial degree less than p^j. The monoidal restriction functors give rise to $\Bbbk$-linear equivalences

$$\mathrm{Rep}\,\mathbb{G}_{a,r} \xleftarrow{\sim} (\mathrm{Rep}\,\mathbb{G}_a)_r \xrightarrow{\sim} \mathrm{Rep}\,C_p^r.$$

Moreover, the restriction of St_{r-1} to $\mathbb{G}_a$ corresponds to the projective cover of $\mathbf{1}$ in $(\mathrm{Rep}\,\mathbb{G}_a)_{r-1}$.

To understand tensor products with St_{r-1} in $\mathrm{Rep}\,C_p^r$ it thus suffices to understand tensor products of objects in $(\mathrm{Rep}\,\mathbb{G}_a)_r$ with St_{r-1}, and these are objects in $(\mathrm{Rep}\,\mathbb{G}_a)_{r+1}$. Subsequently, it is sufficient to understand tensor products in $\mathrm{Rep}\,\mathbb{G}_{a,r+1}$ of objects in $\mathrm{Rep}\,\mathbb{G}_{a,r}$, where we have a non-monoidal inclusion $\mathrm{Rep}\,\mathbb{G}_{a,r} \subset \mathrm{Rep}\,\mathbb{G}_{a,r+1}$ by letting z_r act as zero, see (3.1), with $\Bbbk\mathbb{G}_{a,r-1}$.

Using (3.1), an obvious class of $\mathbb{G}_{a,r}$-representations are given by modules of the form

$$M = \bigoplus_{i=1}^{p} M_i \boxtimes J_i$$

where $M_i \in \mathrm{Rep}\,\mathbb{G}_{a,r-1}$ and J_i is the indecomposable $\Bbbk[z_{r-1}]/z_{r-1}^p$-module of dimension i. In general, for $V \in \mathrm{Rep}\,\mathbb{G}_{a,r}$, we can set

$$W_j(V) := \ker(z_{r-1}|_V) \cap \mathrm{im}((z_{r-1}|_V)^{j-1}), \quad B_j(V) := W_j(V)/W_{j+1}(V) \quad \in \mathrm{Rep}\,\mathbb{G}_{a,r-1}.$$

Clearly $B_j(M) \cong M_j$, for M as above. The main technical result (Proposition 3.15) is now that

$$\Bbbk\mathbb{G}_{a,r-1} \otimes V \;\cong\; \bigoplus_{j=1}^{p} \Bbbk\mathbb{G}_{a,r-1} \otimes (B_j(V) \boxtimes J_j)$$

inside $\mathrm{Rep}\,\mathbb{G}_{a,r+1}$. Using Nakayama's lemma it is actually sufficient to establish a morphism that is an isomorphism modulo the Jacobson radical. Constructing this morphism is the difficult part, and relies on a brute force construction.

3.1. **Normal form of Steinberg tensor products.** Our first goal is to derive, in Corollary 3.6, a description of a functor given by tensoring with representations that, under a suitable identification, will correspond to Steinberg representations.

Definition 3.1. Let $\mathscr{P}ol' := \mathrm{Rep}\,\mathbb{G}_a$ be the tensor category of finite-dimensional rational (or 'polynomial') representations over $\Bbbk$ of the additive group $\mathbb{G}_a = (\Bbbk, +)$. We think of its objects[1] as pairs

[1]Here, a pair (V, g) corresponds to the coaction of $\Bbbk[\mathbb{G}_a] = \Bbbk[x]$ given by $v \mapsto \sum_{i\geq 0} g_i(v) \otimes x^i$. On K-points, where K is any (possibly infinite-dimensional) commutative $\Bbbk$-algebra, it corresponds to the action of $(K, +)$ given by $\alpha \mapsto g(\alpha)$.

(V, g), where V is a vector space over $\Bbbk$ and $g = \sum_{i\geq 0} g_i T^i$ is an element in $\mathrm{End}_\Bbbk(V)[T]$, i.e., a matrix polynomial in one indeterminate T, satisfying

$$g(0) = 1 = \mathrm{id}_V, \quad g(T+S) = g(T)g(S),$$

as polynomials, where S is another indeterminate. The *degree* of a polynomial representation (V, g) is the degree $\deg(g)$ of g in T. For any $r \geq 0$, let $\mathcal{P}ol'_r = (\mathrm{Rep}\,\mathbb{G}_a)_r$ be the full subcategory of $\mathcal{P}ol'$ on objects (V, g) of degree less than p^r.

In the above description, the tensor product of (V, g) and (W, h) is given by $(V \otimes W, g \otimes h)$, where $g \otimes h \in \mathrm{End}(V \otimes W)[T]$ is based on the Kronecker product of matrices.

At the same time, polynomial representations as in the definition are nothing but tuples of commuting p-nilpotent operators[2]: For any vector space V and $x = (x_0, \dots, x_{r-1}) \in \mathrm{End}_\Bbbk(V)^r$ a tuple of pairwise commuting p-nilpotent operators, define

$$g^{\{x\}} := \prod_{0\leq i\leq r-1} \exp(x_i T^{p^i}) \quad \in \mathrm{End}_\Bbbk(V)[T], \quad \text{with } \exp(x) := \sum_{0\leq k\leq p-1} \frac{x^k}{k!}.$$

We will also use the notation

$$x^{[a]} := \prod_{0\leq i\leq r-1} \frac{x_i^{a_i}}{a_i!} \quad \text{for all } 0 \leq a < p^r,$$

using the p-adic expansion $a = a_0 + pa_1 + \cdots + p^{r-1}a_{r-1}$. We set $x^{[a]} = 0$ for $a \geq p^r$. In particular,

$$g^{\{x\}} = \sum_{0\leq a<p^r} x^{[a]} T^a$$

Set

$$A_r := \Bbbk[z_0, \dots, z_{r-1}]/(z_0^p, \dots, z_{r-1}^p),$$

let A_r-mod denote its category of finite-dimensional modules. Its objects can be thought of as pairs (V, x), where V is a vector space and x is an r-tuple of pairwise commuting p-nilpotent operators on V as before.

In the following, Ψ_r is the restriction $\mathrm{Rep}\,\mathbb{G}_a \to \mathrm{Rep}\,\mathbb{G}_{a,r}$ to the Frobenius kernel, expressed in terms of the algebra A_r (which can be identified with the group algebra of $\mathbb{G}_{a,r}$).

Lemma 3.2. *For any $r \geq 0$, there are exact functors*

$$\Psi_r \colon \mathcal{P}ol' \to A_r\text{-mod}, \quad (V, g) \mapsto (V, (g_{p^0}, \dots, g_{p^{r-1}})),$$
$$\text{and} \quad \Psi_r^- \colon A_r\text{-mod} \to \mathcal{P}ol', \quad (V, x) \mapsto (V, g^{\{x\}}),$$

that both commute with the faithful linear forgetful functors to vector spaces and that induce an equivalence of linear categories $\mathcal{P}ol'_r \xrightarrow{\simeq} A_r$-mod.

Proof. Starting with a polynomial representation (V, g), additivity of g means that for all $i, j \geq 0$,

$$g_i g_j = [g(T)g(S)]_{T^i S^j} = [g(T+S)]_{T^i S^j} = g_{i+j}\binom{i+j}{i}.$$

Since the right-hand side is symmetric with respect to swapping $i \leftrightarrow j$, this shows that all g_i commute pairwise. Also, for any $m \geq 0$,

$$g_{p^m}^p = g_{p^{m+1}}(\dots)\binom{p^{m+1}}{p^m} = 0,$$

[2]... which essentially means thinking of them as $\mathrm{Dist}\,\mathbb{G}_a$-modules.

omitting more binomial coefficients, and using Lucas' theorem for the last one. Hence, g_{p^m} is p-nilpotent, implying that Ψ_r is well-defined. The first displayed formula also implies that for any $m \geq 0$ and $i = \sum_{k\geq 0} i_k p^k$ with $i_m > 0$,

$$g_{p^m} g_{i-p^m} = g_i \binom{i}{p^m} = g_i i_m,$$

again using Lucas' theorem. This implies that, if $\deg(g) < p^s$, then $g = g^{\{x\}}$ for $x = (g_{p^0}, \dots, g_{p^{s-1}})$. In particular, for $(V, g) \in \mathcal{P}o\ell'_r$ we may take $s = r$.

For any tuple x of commuting p-nilpotent operators, $g^{\{x\}}(0) = 1$ and additivity of $g^{\{x\}}$ in the indeterminate follows from that of the polynomials $\exp(x_i T^{p^i})$, implying Ψ_r^- is well-defined. If $g = g^{\{x\}}$, then indeed $g_{p^i} = x_i$, completing the proof. □

For any $r \geq 1$, note that the subcategories $\mathcal{P}o\ell'_r \subset \mathcal{P}o\ell$ fail to be closed under tensor products. We want to control this failure at least for tensor products involving the regular representation in A_{r-1}-mod, i.e., under the above equivalence, the polynomial representation

$$(A_{r-1}, g^{\{z\}}) \in \mathcal{P}o\ell'_{r-1} \subset \mathcal{P}o\ell'_r, \quad \text{with } z := (z_0, \dots, z_{r-2}),$$

viewed as a tuple of commuting p-nilpotent endomorphisms of A_{r-1}. The offset in the indices is intended; working with A_{r-1} in $\mathcal{P}o\ell'_r$ will lead us to the phenomenon relevant for our purposes. A_{r-1} will correspond to a (higher) Steinberg representation under a suitable identification.

We start with some auxiliary notions.

Definition 3.3. For any vector space V, $r \geq 1$, and $(r-1)$-tuples x, y of commuting p-nilpotent operators on V such that operators from x and from y always commute pairwise, let $x * y$ and $C(x, y)$ be the first $(r-1)$ entries and the last entry, respectively, of the r-tuple corresponding by Lemma 3.2 to the polynomial representation $(V, g^{\{x\}} g^{\{y\}})$, i.e. such that

$$g^{\{x\}} g^{\{y\}} = g^{\{x*y\}} \exp(T^{p^{r-1}} C(x, y)). \tag{3.2}$$

The above symbolically means

$$\Psi_r((V, g^{\{x\}} g^{\{y\}})) = (V, (x * y, C(x, y))),$$

using the concatenation of tuples on the right-hand side. Note that the product $g^{\{x\}} g^{\{y\}}$ has degree less than p^r, as required.

Lemma 3.4. *For all x, y as above, $C(x, y)^2 = 0$, and we have*

$$(x * y) * (-y) = x, \quad C(x * y, -y) = -C(x, y).$$

Proof. If $p = 2$, then $C(x, y)^2 = 0$ immediately. Otherwise, $C(x, y)$ has to square to zero as $g^{\{x\}} g^{\{y\}}$ is of degree less than $2p^{r-1}$. Multiplying Equation (3.2) with $g^{\{-y\}} \exp(-T^{p^{r-1}} C(x, y))$ and using

$$g^{\{y\}} g^{\{-y\}} = 1 = \exp(T^{p^{r-1}} C(x, y)) \exp(-T^{p^{r-1}} C(x, y))$$

implies the remaining assertions. □

Proposition 3.5. *For any $r \geq 1$ and $(V, g) \in \mathcal{P}o\ell'_r$ the tensor product representation on $A_{r-1} \otimes V$ is isomorphic as a polynomial representation to the one given by*

$$g^{\{x\}} \exp(T^{p^{r-1}} D_1)(1 + T^{p^r} D_2),$$

using the definitions

$$x := (z_i \otimes \mathrm{id})_{0 \leq i \leq r-2}, \quad y := (\mathrm{id} \otimes g_{p^i})_{0 \leq i \leq r-2}, \quad R := -C(x, -y),$$

$$D_1 := (\mathrm{id} \otimes g_{p^{r-1}}) + R, \quad D_2 := -(\mathrm{id} \otimes g_{p^{r-1}})^{p-1} R \quad \in \mathrm{End}_{\Bbbk}(A_{r-1} \otimes V).$$

Proof. By definition, and using Lemma 3.2 and Lemma 3.4, the tensor product polynomial representation on $A_{r-1} \otimes V$ is given by

$$G(T) := g^{\{x\}} g^{\{y\}} \exp(T^{p^{r-1}}(\mathrm{id} \otimes g_{p^{r-1}})) = g^{\{x*y\}}(1 + T^{p^{r-1}} C(x,y)) \exp(T^{p^{r-1}}(\mathrm{id} \otimes g_{p^{r-1}})).$$

Consider the linear endomorphisms $\phi_\pm$ of $A_{r-1} \otimes V$ given by

$$\phi_\pm(z^{[a]} \otimes v) = (x * (\pm y))^{[a]}(1 \otimes v) \quad \text{for all } 0 \le a < p^{r-1}.$$

A computation using Lemma 3.4 shows that $\phi_- = \phi_+^{-1}$. Conjugation with ϕ_- turns $G(T)$ into

$$g^{\{(x*(-y))*y\}}(1+T^{p^{r-1}} C(x*(-y), y)) \exp(T^{p^{r-1}}(\mathrm{id} \otimes g_{p^{r-1}})) = g^{\{x\}}(1+T^{p^{r-1}} R) \exp(T^{p^{r-1}}(\mathrm{id} \otimes g_{p^{r-1}})),$$

again using Lemma 3.4. For any two commuting operators P_1, P_2 with $P_1^2 = 0 = P_2^p$ and an indeterminate S, we compute

$$(1 + SP_1) \exp(SP_2) = \exp(S(P_1 + P_2))(1 - S^p P_1 P_2^{p-1}),$$

using $(p-1)! = -1$. This shows that the conjugate of $G(T)$ above is the asserted polynomial. □

We can reformulate Proposition 3.5 as follows:

Corollary 3.6. *Set $\overline{A}_{r+1} := A_{r+1}/(z_{r-1} z_r, z_r^2)$. There is a diagram of exact functors, commutative up to natural isomorphism,*

$$\begin{array}{ccccc} A_r\text{-mod} & \xrightarrow{\simeq} & \mathcal{P}ol'_r & \xrightarrow{A_{r-1}\otimes -} & \mathcal{P}ol'_{r+1} \\ {\scriptstyle \mathcal{T}}\downarrow & & & & \downarrow{\scriptstyle \simeq} \\ \overline{A}_{r+1}\text{-mod} & & \longrightarrow & & A_{r+1}\text{-mod} \end{array},$$

with the equivalences from Lemma 3.2, the natural inclusion functor between the module categories of $\overline{A}_{r+1}$ and A_{r+1}, and a functor $\mathcal{T}$ given by

$$\mathcal{T}(X) = A_{r-1} \otimes X,$$

where $z_0, \dots, z_{r-2}$ act only on the first tensor factor and the action of z_{r-1}, z_r is given by D_1, D_2 as in Proposition 3.5.

3.2. **Steinberg splitting for polynomial representations.** Our goal is to derive an even more concrete description of the functor $A_{r-1} \otimes -$ on $\mathcal{P}ol'_r \simeq A_r$-mod, using the information from Corollary 3.6 as a starting point. More precisely, we will see that objects of the form $\mathcal{T}(X)$ as above can be decomposed as direct sums according to a suitable generalised cohomology on the objects X. This direct sum decomposition will be in the full subcategory on objects which are images under $\mathcal{T}$. Hence, Corollary 3.6 tells us that the decomposition can be detected and constructed in $\overline{A}_{r+1}$-mod.

We fix $r \ge 1$ and set $A := A_{r-1}$. To define the modules in the sought after decomposition and describe morphisms to them, we need some notation. Let

$$A\colon \mathrm{Vect} \to A\text{-mod}$$

be the free A-module functor, i.e., $A(V) = A \otimes V$ with A acting on the first factor. We abuse notation by using the same symbol for a $\Bbbk$-algebra and a functor. For any A-module X (in particular, for the restriction of an A_r-module), set $x := (z_i|_X)_{0 \le i \le r-2}$ and

$$R_X := -C((z_i \otimes \mathrm{id})_i, (-\mathrm{id} \otimes x_i)_i) = [-g^{\{z\}}(T) \otimes g^{\{x\}}(-T)]_{T^{p^{r-1}}} \quad \in \mathrm{End}_A(A(X)),$$

which is the operator R from Proposition 3.5. For any $N \in A$-mod and any $1 \leq j \leq p$, set $J_j := \Bbbk v_1 \oplus \cdots \oplus \Bbbk v_j$ and let

$$N \boxtimes J_j \quad \in A_r\text{-mod}$$

be the vector space $N \otimes J_j$ with $z_0, \dots, z_{r-2}$ acting on the first factor and z_{r-1} acting on the second factor J_j as a nilpotent Jordan block, by sending $v_j \mapsto 0$ and $v_k \mapsto v_{k+1}$ for $1 \leq k \leq j-1$.

Our goal will be to show that objects of the form $\mathcal{T}(Y)$ for $Y \in A_r$-mod are direct sums of objects of the form $\mathcal{T}(N \boxtimes J_j)$, for suitable $N \in A$-mod. Our first step is hence to parametrize morphisms between these kinds of objects. The idea is that such morphisms can be written with coefficient maps for $v_1, \dots, v_j$. Intertwining the action of $z_0, \dots, z_r$ is equivalent to a system of concrete conditions on these coefficient maps that can be solved explicitly.

Lemma 3.7. *For $Y \in A_r$-mod, $B \in A$-mod, and $1 \leq j \leq p$, we define*

$$H(Y,B,j) := \{f : Q^j(f) = 0 = fR_YA((z_{r-1}|_Y)^{p-1}) - R_BQ^{p-1}(f)\} \quad \subset \mathrm{Hom}_A(A(Y), A(B)),$$

with $Q(f) := fA(z_{r-1}|_Y) + fR_Y - R_Bf$. Then we have an isomorphism

$$\mathrm{Hom}_{\overline{A}_{r+1}}(\mathcal{T}(Y), \mathcal{T}(B \boxtimes J_j)) \cong H(Y,B,j), \quad h \mapsto \pi h,$$

where $\pi \colon A \otimes B \otimes J_j \to A \otimes B$ projects to the coefficient of v_j.

Proof. Set $M := \mathcal{T}(Y)$ and $N := \mathcal{T}(B \boxtimes J_j)$. By Proposition 3.5, M is an $\overline{A}_{r+1}$-module with underlying vector space $A \otimes Y$, with $z_0, \dots, z_{r-2}$ acting on the first factor A, and with

$$z_{r-1}|_M = R_Y + A(z_{r-1}|_Y), \quad z_r|_M = -R_YA(z_{r-1}|_Y)^{p-1}.$$

Similarly, N is an $\overline{A}_{r+1}$-module with underlying vector space $A \otimes B \otimes J_j$, with $z_0, \dots, z_{r-2}$ acting on the first factor A, and with

$$z_{r-1}|_N = R_B \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes d, \quad z_r|_N = -R_B \otimes d^{p-1},$$

where d is the action of the nilpotent Jordan block on J_j in its chosen basis. For any $f \in \mathrm{Hom}_{\Bbbk}(M,N)$, we can write

$$f = \sum_{1 \leq a \leq j} f_a \otimes v_a \quad \text{with } f_a \in \mathrm{Hom}_{\Bbbk}(M, A \otimes B).$$

f intertwining the action of $z_0, \dots, z_{r-2}$ is equivalent to $f_a \in \mathrm{Hom}_A(M, A(B))$ for all a. f intertwining the action of z_{r-1} is equivalent to

$$f_1 z_{r-1}|_M = R_Bf_1, \quad f_az_{r-1}|_M = R_Bf_a + f_{a-1} \quad \text{for all } 2 \leq a \leq j,$$

which is equivalent to $f_a = Q^{j-a}(f_j)$, for all a, and $Q^j(f_j) = 0$, which is one of the conditions in the definition of $H(Y,B,j)$. The maps f described by such coefficient maps $(f_a)_a$ are A-linear if and only if f_j is. For them, intertwining the z_r-action is equivalent to

$$f_az_r|_M = \begin{cases} 0 & a < p \\ -R_Bf_1 = -R_BQ^{p-1}(f_j) & a = p \end{cases} \quad \text{for all } 1 \leq a \leq j.$$

As $Q(\cdot)$ commutes with $(\cdot z_r|_M)$ and with left multiplication by R_B, and $Q^j(f_j) = 0$, this is equivalent to

$$f_jR_YA(z_{r-1}|_Y)^{p-1} = \begin{cases} 0 = R_BQ^{p-1}(f_j) & j < p \\ R_BQ^{p-1}(f_j) & j = p \end{cases}.$$

□

Recall that for any $Y \in A_r$-mod, we want to show that $\mathcal{T}(Y)$ is direct sum of objects of the form $\mathcal{T}(B \boxtimes J_j)$. More precisely, the object B corresponding to an index j should be given by a certain generalized cohomology $B_j(Y)$, as follows. The generalised cohomologies B_j were studied in [EO21, Section 3].

Definition 3.8. For $Y \in A_r$-mod, we define for all $j \geq 1$:

$$W_j(Y) := \ker(z_{r-1}|_Y) \cap \operatorname{im}((z_{r-1}|_Y)^{j-1}), \quad B_j(Y) := W_j(Y)/W_{j+1}(Y) \quad \in A_{r-1}\text{-mod} = A\text{-mod}.$$

To obtain the projections $\mathcal{T}(Y) \twoheadrightarrow \mathcal{T}(B_j(Y) \boxtimes J_j)$, we want to use Lemma 3.7. Hence, we need an element in the parametrising space $H(Y, B_j(Y), j)$. We can build such an element by starting with a morphism of the form $A(\beta)$, where $\beta\colon Y \to B_j(Y)$ is a choice of a linear morphism extending the defining quotient map $W_j(Y) \twoheadrightarrow B_j(Y)$. $A(\beta)$ itself might not lie in $H(Y, B_j(Y), j)$, but we will find a correction morphism ℓ that achieves this, in the following sense.

Lemma 3.9. *Fix $1 \leq j \leq p$. For $Y \in A_r$-mod, set $d := z_{r-1}|_Y$. Assume that there is a linear map $\beta\colon Y \to B := B_j(Y)$ such that*

$$\beta|_{W_j(Y)} = (W_j(Y) \twoheadrightarrow B), \quad \beta d^j = 0, \quad R_B A(\beta) R_Y = 0. \tag{3.3}$$

Assume also that there is an A-linear map $\ell\colon A(Y) \to A(B)$ such that

$$\ell R_Y = R_B \ell, \quad \ell A(d^j) = A(\beta j d^{j-1}) R_Y - R_B A(\beta j d^{j-1}). \tag{3.4}$$

Then $\eta := A(\beta) - \ell$ lies in $H(Y, B, j)$.

Proof. Set

$$\delta(h) := hR_Y - R_B h, \quad Q(h) := hA(d) + \delta(h), \quad \text{for all } h \in \operatorname{Hom}_A(A(Y), A(B)).$$

Note that Q as defined here agrees with Q as in Lemma 3.7.

As R_Y^2 and R_B^2 are zero by Lemma 3.4, using the assumptions we get

$$\delta^2(\eta) = \delta^2(A(\beta)) = 0.$$

As $\delta(\cdot)$ and $(\cdot A(d))$ commute, this implies

$$Q^j(\eta) = (\delta(\cdot) + (\cdot A(d)))^j(\eta) = \eta A(d^j) + \delta(A(\beta)) A(j d^{j-1}) = A(\beta d^j) = 0,$$

using the assumptions in the last two steps. Similarly, we compute

$$R_B Q^{p-1}(\eta) = R_B \eta A(d^{p-1}) - R_B \delta(A(\beta)) A(d^{p-2}) = \eta R_Y A(d^{p-1})$$

using $\delta(\eta) A(d^{p-1}) = 0$ and $R_B \delta(\eta) = 0$ in the last step; the first of these statements follows as $\operatorname{im}(d^{p-1}) = W_p \subset W_j$ and $\delta(\eta) = \delta(A(\beta))$, and the second follows from again $\delta(\eta) = \delta(A(\beta))$ and $R_B \delta(A(\beta)) = 0$, using the assumptions. This completes the proof. □

In order to find β and ℓ as in Lemma 3.9, we will need some machinery.

Definition 3.10. For $(X, x), (Y, y) \in A$-mod, we define a $\Bbbk$-linear mapping

$$\mathcal{K} = \mathcal{K}_{X,Y}\colon \operatorname{Hom}_{\Bbbk}(X, Y) \to \operatorname{Hom}_A(A(X), A(Y)), \quad f \mapsto A([-g^{\{y\}}(T) f g^{\{x\}}(-T)]_{T^{p^{r-1}}}).$$

In other words, $\mathcal{K}$ is the negative coefficient of $T^{p^{r-1}}$ in the natural action of the additive group on the internal hom-space $X^* \otimes Y$, transported along $A(\cdot)$, and R_X is one of the operators arising in Proposition 3.5. Note that $R_X^2 = 0$ by Lemma 3.4.

Definition 3.11. For any $(X, x) \in A\text{-mod}$, a *weight filtration* is a descending vector space filtration

$$X = \mathcal{F}_0 X \supset \mathcal{F}_1 X \supset \dots$$

such that $\mathcal{F}_{p^{r-1}} X = 0$ and $x^{[a]}(\mathcal{F}_i X) \subset \mathcal{F}_{i+a} X$ for all $a, i \geq 0$.

Lemma 3.12. (a) *For any composition bfa of $\Bbbk$-linear maps a, b, f between A-modules, where a, b are A-linear,*

$$\mathcal{K}(bfa) = A(b)\mathcal{K}(f)A(a).$$

(b) *For any $\Bbbk$-linear map $f\colon X \to Y$ between A-modules,*

$$R_Y \mathcal{K}(f) - \mathcal{K}(f) R_X = R_Y A(f) R_X.$$

(c) *For any $\Bbbk$-linear map $f\colon X \to Y$ between A-modules with weight filtrations,*

$$\mathcal{K}(f)(A(\mathcal{F}_i X)) \subset A(\mathcal{F}_{i+1} Y),$$

and if f is a filtered linear map, then $\mathcal{K}(f) = 0$.

(d) *For any A-linear map $f\colon X \to Y$ between A-modules,*

$$R_Y A(f) = A(f) R_X.$$

Proof. (a) follows directly from the definition of $\mathcal{K}$.

(b) We set $q := p^{r-1}$,

$$x' := (-\operatorname{id}_A \otimes x_i)_i, \quad y' := (-\operatorname{id}_A \otimes y_i)_i, \quad z'_X := (z_i \otimes \operatorname{id}_X)_i, \quad z'_Y := (z_i \otimes \operatorname{id}_Y)_i,$$

where $x = (x_i)_i$ and $y = (y_i)_i$ are the $(r-1)$-tuples specifying the A-action on X and Y. Then by Equation (3.2),

$$g^{\{z'_X\}} g^{\{x'\}} = g^{\{z'_X * x'\}}(1 - T^q R_X), \quad g^{\{y'\}} g^{\{z'_Y\}} = g^{\{z'_Y * y'\}}(1 - T^q R_Y),$$

hence,

$$g^{\{z'_Y * y'\}}(-T) A(f) g^{\{z'_X * x'\}}(T) = (1 - T^q R_Y) \underbrace{g^{\{y'\}}(-T) A(f) g^{\{x'\}}(T)}_{=:K(T)} (1 + T^q R_X).$$

Taking coefficients of T^{2q} and using $[K]_{T^q} = -\mathcal{K}(f)$ produces the desired formula.

(c) Up to scalar extension $A(\cdot)$, $\mathcal{K}(f)$ is a linear combination of terms of the form $y^{[a]} f x^{[q-a]}$, for $1 \leq a \leq q-1$. Such a term vanishes on $\mathcal{F}_i X$ for $a \leq i$, as $\mathcal{F}_q = 0$. The remaining terms send $\mathcal{F}_i X$ to $\mathcal{F}_{i+1} Y$. If f is filtered, then all terms send $\mathcal{F}_i X$ to $\mathcal{F}_q Y = 0$.

(d) With x, y as above, A-linearity means

$$f g^{\{x\}}(T) = g^{\{y\}}(T) f.$$

Evaluating at $-T$, tensoring with A, that is with $g^{\{z\}}$, and taking the coefficient of T^q yields the assertion. □

Set

$$\mathfrak{m} := (z_0, \dots, z_{r-2}) \quad \subset A,$$

the maximal ideal of codimension 1. Hence, for any $f\colon A(X) \to A(Y)$ in A-mod, we can consider the reduction modulo $\mathfrak{m}$, a linear map $\overline{f}\colon X \to Y$. Note that $\overline{R_X} = 0$, as $\deg(g^{\{x\}}) < p^{r-1}$.

We say that a map $f\colon X \to Y$ between filtered spaces X, Y with descending filtrations *strictly raises the filtration* if $f(\mathcal{F}_i X) \subset \mathcal{F}_{i+1} Y$ for all i.

Lemma 3.13. *Consider A-modules with weight filtrations X, Y, B, and a filtered A-linear map $\partial\colon X \to Y$. Let $f\colon X \to B$ be a filtered linear map whose restriction to $\ker(\partial)$ is A-linear. Then there is an A-linear map $\ell\colon A(Y) \to A(B)$ satisfying*

$$\ell R_Y = R_B \ell, \quad \ell A(\partial) = A(f)R_X - R_B A(f),$$

such that the reduction $\overline{\ell}\colon Y \to B$ factors through $Y/\operatorname{im}(\partial)$ and strictly raises the filtration.

Proof. We set

$$h := A(f)R_X - R_B A(f) :\ A(X) \to A(B).$$

By our assumptions, f restricted to $\ker(\partial)$ is A-linear, so $h(\ker A(\partial)) = 0$ by Lemma 3.12(d). As f is assumed filtered, using Lemma 3.12(c) and (b), we get $\mathcal{K}(f) = 0$ and hence $R_B A(f) R_X = 0$. This implies

$$R_B h = 0 = h R_X. \tag{3.5}$$

We claim that we can choose a linear (split) map $t\colon Y \to X$ such that $\partial t \partial = \partial$ and ∂t and $t\partial$ are both filtered linear maps. Indeed, pick filtered complements X' to $\ker\partial \subset X$ and Y' to $\operatorname{im}\partial \subset Y$, then t is defined by

$$t|_{Y'} = 0, \quad t|_{\operatorname{im}\partial} = (\partial|_{X'}^{\operatorname{im}\partial})^{-1},$$

where the upper and lower indicate (co-)restriction. This shows the claim.

Set $\ell_0 := hA(t)$. Then

$$\ell_0 A(\partial) = hA(t\partial) = h - hA(1 - t\partial) = h,$$

as $(1 - t\partial)(X) \subset \ker\partial$.

Using Lemma 3.12(a,c), we get that the following compositions are all zero:

$$A(\partial)\mathcal{K}(t), \quad \mathcal{K}(t)A(\partial), \quad \mathcal{K}(ft)A(\partial).$$

In particular, $\operatorname{im}\mathcal{K}(t) \subset \ker A(\partial)$, so by our assumption

$$A(f)R_X\mathcal{K}(t) = R_B A(f)\mathcal{K}(t). \tag{3.6}$$

Set $\ell := \ell_0 + A(f)\mathcal{K}(t) - \mathcal{K}(ft)$. It is an A-linear map by construction and $\ell A(\partial) = h$. We claim that $\delta(\ell) = 0$, for $\delta(H) := HR_Y - R_B H$. Indeed,

$$\delta(\ell_0) = A(f)R_X A(t)R_Y - R_B A(ft)R_Y = \delta(-A(f)\mathcal{K}(t) + \mathcal{K}(ft)),$$

using Equation (3.5) in the first step, and using Equation (3.6) and Lemma 3.12(b) in the second step.

Finally, $\overline{\ell_0} = 0$ as $\overline{R_X}$ and $\overline{R_B}$ are zero. The remaining terms of ℓ annihilate $\operatorname{im}(A(\partial))$, as noted before, and strictly raise the filtration by Lemma 3.12, as f is filtered. Hence, their reductions induce maps on $Y/\operatorname{im}(\partial)$ that strictly raise the filtration. □

Having set up the machinery, we can now show a lifting result on which the desired Steinberg splitting will rest.

For any A_r-module X, we can reduce the A_{r+1}-module $\mathcal{T}(X) = A \otimes X$ modulo $\mathfrak{m} = (z_0, \dots, z_{r-2})$. As the A-action on $\mathcal{T}(X)$ is on the first factor, this reduction can be identified with X as $\Bbbk[z_{r-1}]/(z_{r-1}^p)$-modules. Note that, as $\overline{R_X} = 0$, there is no other non-trivial action on the reduction.

Proposition 3.14. *For each $Y \in A_r$-mod and $1 \leq j \leq p$, there are weight filtrations of Y and of $B_j := B_j(Y)$ (as $A = A_{r-1}$-modules) together with an $\overline{A}_{r+1}$-morphism $F\colon \mathcal{T}(Y) \to \mathcal{T}(B_j \boxtimes J_j)$, such that the action of $B_j(-)$ on the reduction $Y \to B_j \boxtimes J_j$ modulo $\mathfrak{m}$ is a filtered map*

$$B_j = B_j(Y) \to B_j(B_j \boxtimes J_j) \cong B_j$$

whose associated graded is the identity.

Proof. Set

$$W_j := W_j(Y), \quad y_i := \begin{cases} z_i|_Y & i < r-1 \\ 0 & i \geq r-1 \end{cases}, \quad d := z_{r-1}|_Y \quad \text{for all } j \geq 1, i \geq 0.$$

Define $\mathcal{F}_i Y := \sum_{a \geq i} y^{[a]} Y$ for all $i \geq 0$. We can realise B_j as a subspace of $Y/\operatorname{im}(d^j)$, since $\operatorname{im}(d^j) \cap W_j = W_{j+1}$. Let $\mathcal{F}_i B_j$ denote the induced filtration on B_j. These are weight filtrations, and $d : Y \to Y$ is filtered.

Pick a filtered linear retraction of $B_j \hookrightarrow Y/\operatorname{im}(d^j)$. Pre-composing it with the quotient map yields a filtered linear map $\beta\colon Y \to B_j$ such that $\beta d^j = 0$ and $\beta|_{W_j}$ is the natural quotient map. Using Lemma 3.12(b,c) we hence see that β is as assumed in Lemma 3.9, i.e., satisfies Equation (3.3).

To apply Lemma 3.13 with

$$\partial := d^j, \quad f := \beta j d^{j-1}, \quad B := B_j,$$

we note that, indeed, ∂ and f are filtered, ∂ is A-linear, and the restriction $f|_{\ker(\partial)} = f|_{\ker(d^j)}$ is A-linear, as $d^{j-1}(\ker(d^j)) = W_j$, and $\beta|_{W_j}$ is the natural quotient map. Hence, from Lemma 3.13 and Lemma 3.9, we get an A-linear map $\ell\colon A(Y) \to A(B)$ such that

$$\eta := A(\beta) - \ell \quad \in \operatorname{Hom}_A(A(Y), A(B)) \cong \operatorname{Hom}_A(\mathcal{T}(Y), A(B))$$

lies in $H(Y, B, j)$ as defined in Lemma 3.7 and whose reduction $\overline{\ell}$ modulo $\mathfrak{m}$ factors via $Y/\operatorname{im}(d^j)$ and strictly raises the filtration. For $j = p$, we can take $\ell = 0$.

Hence, from Lemma 3.7, we get an $\overline{A}_{r+1}$-linear $F\colon \mathcal{T}(Y) \to \mathcal{T}(B \boxtimes J_j)$ such that $\pi F = \eta$, where $\pi\colon A \otimes B \otimes J_j \to A \otimes B$ projects to the coefficient of $v_j \in J_j$.

It remains to see that the reduction $\overline{\eta}$ of η modulo $\mathfrak{m}$, and hence also that of F by Lemma 3.7, induces the identity map on $\operatorname{gr}(B_j)$. Write $\overline{\ell} = Y \xrightarrow{\ell_1} Y/\operatorname{im}(d^j) \xrightarrow{\ell_2} B$, where ℓ_2 raises the filtration. As $\overline{\eta} = \beta - \overline{\ell}$, we see that $\overline{\eta}|_{W_j}$ induces the map $\mathrm{id}_B - \ell_2|_B$ on $B \subset Y/\operatorname{im}(d^j)$, which indeed has the identity as its associated graded. Modulo $\mathfrak{m}$, the maps $FA(d)$ and $A(z_{r-1})F$ agree, and $\overline{F}$ sends $W_j \subset \ker(d)$ into $B \otimes \Bbbk v_j$, with coefficient $\overline{\eta}|_{W_j}$. Under the natural identification $B_j(B \boxtimes J_j) \cong B$, it therefore induces the identity on $\operatorname{gr}(B_j)$. □

Combining Proposition 3.14 for the various values of j yields a first splitting result.

Proposition 3.15. *For all $Y \in A_r$-mod,*

$$\mathcal{T}(Y) \cong \bigoplus_{1 \leq j \leq p} \mathcal{T}(B_j(Y) \boxtimes J_j).$$

In particular, in $\mathcal{P}ol$, the tensor product $A \otimes Y$ decomposes as a direct sum of tensor products $A \otimes Y_j$, where Y_j corresponds to $B_j(Y) \boxtimes J_j$.

Proof. From Proposition 3.14 applied to all $1 \leq j \leq p$, we get an $\overline{A}_{r+1}$-linear morphism

$$F\colon \mathcal{T}(Y) \to \bigoplus_{1 \leq j \leq p} \mathcal{T}(B_j(Y) \boxtimes J_j),$$

whose reduction $\overline{F}$ modulo $\mathfrak{m}$ induces the identity on all $\mathrm{gr}(B_j(Y))$, with the appropriate corestriction on the target. Each induced map on B_j is an isomorphism, since its associated graded is the identity. The filtrations by W_j then show that $\overline{F}$ restricts to an isomorphism between the kernels of the respective z_{r-1}-actions. Every nonzero z_{r-1}-stable subspace intersects non-trivially with $\ker(z_{r-1})$, so $\overline{F}$ is injective. A dimension count shows that $\overline{F}$ is an isomorphism. Hence, by Nakayama's lemma, F is an isomorphism. The implication for $\mathscr{P}ol\ell$ follows from Lemma 3.2. □

We can now apply Proposition 3.15 recursively.

Definition 3.16. For any polynomial representation $(Y, g) \in \mathscr{P}ol\ell_{r-1}$ and $k \geq 0$, let $(Y^{(k)}, g^{(k)}) \in \mathscr{P}ol\ell_{r-1+k}$ be the *k-th Frobenius twist*, that is, the polynomial representation with $Y^{(k)} = Y$ and $g^{(k)}(T) = g(T^{p^k})$.

Let J_j be the polynomial representation corresponding to the A_1-module $(\Bbbk^j, d)$, where d is the nilpotent Jordan block of size j. Note that this is consistent with our use of the notation $- \boxtimes J_j$ so far.

Lemma 3.17. *For $Y \in A_{r-1}$-mod and $1 \leq j \leq p$, $Y \boxtimes J_j \cong Y \otimes J_j^{(r-1)}$ as polynomial representations.*

Proof. This follows by a direct computation from the definitions. □

For $0 \leq a < p^r$ with p-adic expansion $a = a_0 + pa_1 + \cdots + p^{r-1}a_{r-1}$, set

$$J(a) := J_{a_0+1} \boxtimes \cdots \boxtimes J_{a_{r-1}+1} = J^{(0)}_{a_0+1} \otimes \ldots \otimes J^{(r-1)}_{a_{r-1}+1} \quad \in A_r\text{-mod} \simeq \mathscr{P}ol\ell_r.$$

Here, by definition, $z_i \in A_r$ acts only on the factor J_{a_i+1}. We also set

$$H_X(a) := B_{a_0+1}(\ldots B_{a_{r-1}+1}(X)\ldots) \quad \in A_0\text{-mod} \simeq \mathrm{Vect} \quad \text{for all } X \in A_r\text{-mod} \simeq \mathscr{P}ol\ell_r.$$

Theorem 3.18. *For all $X \in \mathscr{P}ol\ell_r$,*

$$J(p^{r-1}-1) \otimes X \cong \bigoplus_{0 \leq a < p^r} (J(p^{r-1}-1) \otimes J(a))^{\oplus \dim H_X(a)}.$$

Proof. This follows immediately from Proposition 3.15 using Lemma 3.17 by an induction in r, noting that $A_{r-1} \cong J(p^{r-1}-1)$ and, if $r \geq 2$, $J(p^{r-1}-1) \cong J(p^{r-2}-1) \otimes J(p-1)^{(r-2)}$. □

3.3. **Steinberg splitting for elementary abelian groups.** The above results can be translated directly into results on representations of elementary abelian p-groups.

We continue with a fixed $r \geq 1$. Let C_p^r be the elementary abelian p-group of order p^r. Fix an embedding $\iota\colon C_p^r \hookrightarrow (\Bbbk, +)$. The results discussed will turn out to hold independently of this choice. For each $i \geq 0$, let $L(i)$ be the restriction to C_p^r of the simple $\mathrm{SL}_2 = \mathrm{SL}_2(\Bbbk)$-module with highest weight i along the inclusion

$$C_p^r \xrightarrow{\iota} (\Bbbk, +) \xrightarrow{\alpha \mapsto \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)} \mathrm{SL}_2.$$

We define a linear map

$$\phi\colon \Bbbk C_p^r \to A_r, \quad e \mapsto g^{\{z\}}(\iota(e)) = \prod_{0 \leq k \leq r-1} \exp(\iota(e)^{p^k} z_k) = \sum_{0 \leq a < p^r} \iota(e)^a z^{[a]} \quad \text{for all } e \in C_p^r.$$

Lemma 3.19. *(a) ϕ is an algebra isomorphism.*

(b) The monoidal restriction functor

$$\mathrm{Res}\colon \mathcal{P}ol' \simeq \mathrm{Rep}\,\mathbb{G}_a \to \mathrm{Rep}\,C_p^r, \quad (V,g) \mapsto (V, e \mapsto g(\iota(e))),$$

restricts on $\mathcal{P}ol'_r$ to the equivalence of abelian categories

$$\mathcal{P}ol'_r \xrightarrow{\Psi_r} A_r\text{-mod} \xrightarrow{\phi^*} \mathrm{Rep}\,C_p^r.$$

(c) For all $0 \leq a < p^r$, $\mathrm{Res}(J(a)) \cong L(a)$.

Proof. (a) Additivity of $g^{\{z\}}(T)$ in T implies that ϕ is an algebra map. Using bases $\{z^{[a]}\}_{0\leq a<p^r}$ of A_r and $\{e\}_{e\in C_p^r}$, ϕ is given by the matrix with entries $\iota(e)^a$. The matrix is invertible, as the p^r values $\iota(e)$ are all distinct.

(b) Restriction is clearly monoidal, a direct computation shows the two functors agree.

(c) follows from the Steinberg tensor product theorem for SL_2, since for $0 \leq b < p$, the restriction to $\mathbb{G}_a$ of the simple SL_2-representation with highest weight b is J_{b+1}. □

For any p-nilpotent operator d of a vector space and $1 \leq j \leq p$, we define generalised cohomologies

$$B_j(d) := \frac{\ker(d) \cap \mathrm{im}(d^{j-1})}{\ker(d) \cap \mathrm{im}(d^j)}.$$

Every operator commuting with d acts on these subquotients, while d itself acts as zero. Note that for any A_{r+1}-module Y and any j, $B_j(z_r|_Y)$ in this sense is $B_j(Y)$ in the sense of Definition 3.8.

We set $c_i := \phi^{-1}(z_i)$ in $\Bbbk C_p^r$ for $0 \leq i \leq r-1$.

Definition 3.20. For any $X \in \mathrm{Rep}\,C_p^r$ and any $0 \leq a < p^r$ with p-adic expansion $a = a_0 + pa_1 + \cdots + p^{r-1}a_{r-1}$, set

$$H_X(a) := B_{a_0+1}(c_0|\dots B_{a_{r-1}+1}(c_{r-1}|X)\dots)$$

(where $c_i|$ denotes the action of c_i on what follows).

Theorem 3.21. *For any $X \in \mathrm{Rep}\,C_p^r$, and $St_{r-1} := L(p^{r-1}-1)$,*

$$St_{r-1} \otimes X \cong \bigoplus_{0\leq a<p^r} (St_{r-1} \otimes L(a))^{\oplus \dim H_X(a)}.$$

In particular, the left-hand side is the restriction of a tilting module.

Proof. The isomorphism follows directly from Theorem 3.18 using Lemma 3.19. Note that in particular, $\mathrm{Res}(J(p^{r-1}-1)) \cong L(p^{r-1}-1) = St_{r-1}$.

The fact that the right-hand side, and hence also the left-hand side, is the restriction of a tilting module follows directly from [CF25, Lemma 3.1.6], which shows that $St_{r-1} \otimes L(a)$ is a restricted tilting module for $0 \leq a < p^r$. □

4. The Benson–Etingof–Ostrik conjecture

4.1. **V-functors lead to fibre functors.** For this subsection, we fix a $\mathbf{V}$-functor $\Theta\colon \mathrm{Rep}\,S_{p^r} \to \mathcal{V}$ to a finite tensor category $\mathcal{V}$, for some $r \in \mathbb{Z}_{>0}$. We also assume that there exists an additive symmetric monoidal equivalence $T : \mathcal{V} \to \mathcal{V}$ with the property that $T(\lambda f) = \lambda^{p^r} T(f)$ for $\lambda \in \Bbbk$ and all morphisms f in $\mathcal{V}$. For a tensor category $\mathcal{C}$ we denote the resulting $\mathbf{O}$-functor, see §2.2.3, as

$$\Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{V}.$$

Theorem 4.1. *Let $\mathcal{C}$ be a finite tensor category that is Φ-exact.*

(1) *If $p > 2$, then there exists a tensor functor $\mathcal{C} \to \mathrm{Ver}_p \boxtimes \mathcal{V}$.*
(2) *If $p = 2$, then there exists a tensor functor $\mathcal{C} \to \mathrm{Ver}_4^+ \boxtimes \mathcal{V}$.*

The rest of the subsection is devoted to the proof. We always let $\mathcal{C}$ be a finite tensor category. Recall from [EO21, §7] that there is a tensor subcategory $\mathcal{C}_{\mathrm{ex}} \subset \mathcal{C}$ which is the full subcategory of objects X with $\mathrm{FPdim}(\mathrm{Fr}(X)) = \mathrm{FPdim}(X)$, while all other objects satisfy $\mathrm{FPdim}(\mathrm{Fr}(X)) < \mathrm{FPdim}(X)$. We will use the following observation, for finite tensor categories $\mathcal{C}, \mathcal{D}$:

$$\text{Any tensor functor } \mathcal{C} \to \mathcal{D} \text{ restricts to } \mathcal{C}_{\mathrm{ex}} \to \mathcal{D}_{\mathrm{ex}}. \tag{4.1}$$

This follows immediately since tensor functors preserve Frobenius-Perron dimension, see [EGNO15, Proposition 3.3.13], and Fr commutes with tensor functors. The observation (4.1) can actually be generalised beyond finite tensor categories, by [CEO24, Theorem 8.2.1].

Lemma 4.2. *Consider $s \in \mathbb{Z}_{>0}$ and simple objects $V, W \in \mathcal{C}$ with $V, W \notin \mathcal{C}_{\mathrm{ex}}$. If*

$$[V^{\otimes p^s} : W] \not\equiv 0 \mod p,$$

then $\mathrm{FPdim}(W) < \mathrm{FPdim}(V)$.

Proof. Consider the p-th tensor power as a functor $\mathcal{C} \to \mathcal{C} \boxtimes \mathrm{Rep}\, C_p$. Consider arbitrary simple $X, Y \in \mathcal{C}$. It follows for instance from [Ben17, §2.2] that $X^{\otimes p}$ has a filtration whose subquotients are given by i times $F_i(X)$, for $1 \leq i \leq p$, where F_i for $i < p$ is as in (2.3). In particular

$$[X^{\otimes p} : Y] \equiv \sum_{i=1}^{p-1} i[F_i(X) : Y] \mod p.$$

By assumption $\mathrm{FPdim}(\mathrm{Fr}(V)) < \mathrm{FPdim}(V)$, so certainly $\mathrm{FPdim}(F_i(V)) < \mathrm{FPdim}(V)$. This proves a (stronger version of) the case $s = 1$. It also proves that if $X \in \mathcal{C}_{\mathrm{ex}}$ then $[X^{\otimes p} : Y] \neq 0$ modulo p forces $Y \in \mathcal{C}_{\mathrm{ex}}$ by (4.1) and the fact that $\mathcal{C}_{\mathrm{ex}}$ is Frobenius exact, see [EO21, Theorem 6.4].

Now assume that we have proved the lemma for some $s > 0$ and consider $s + 1$. For V, W as in the lemma, we have

$$[V^{\otimes p^{s+1}} : W] \equiv \sum_L [L^{\otimes p} : W][V^{\otimes p^s} : L] \mod p,$$

where L runs over all simples in $\mathcal{C}$, by vanishing of mixed terms modulo p. If $L \in \mathcal{C}_{\mathrm{ex}}$, then $[L^{\otimes p} : W]$ vanishes modulo p by last sentence of the previous paragraph. When $L \notin \mathcal{C}_{\mathrm{ex}}$ it follows that $[V^{\otimes p^s} : L]$ being non-zero modulo p forces $\mathrm{FPdim}(L) < \mathrm{FPdim}(V)$ by 'the case s' and $[L^{\otimes p} : W]$ being non-zero modulo p requires $\mathrm{FPdim}(W) < \mathrm{FPdim}(L)$ by the base case $s = 1$ of the lemma. This allows us to conclude the proof by induction on s. □

Lemma 4.3. *Assume that $\mathcal{C}$ is Φ-exact, then* ***at least one*** *of the following is true:*

(1) *$\mathcal{C}_{\mathrm{ex}}$ contains all simple objects of $\mathcal{C}$.*
(2) *$\Phi_{\mathcal{C}}$ takes values in $\mathcal{C}' \boxtimes \mathcal{V}$ for a proper tensor subcategory $\mathcal{C}' \subset \mathcal{C}$.*

Proof. Let us assume that $\mathcal{C}_{\mathrm{ex}}$ does not contain all simples. Then we can take a simple object $V_{\max} \notin \mathcal{C}_{\mathrm{ex}}$ that has maximal Frobenius-Perron dimension among all such simples. Consider an arbitrary simple $L \in \mathcal{C}$. We *claim* that no simple constituent of $\Phi_{\mathcal{C}}(L)$ is of the form $V_{\max} \boxtimes S$ for a simple $S \in \mathcal{V}$. We split the proof of this claim into two cases.

Assume first that $L \notin \mathcal{C}_{\mathrm{ex}}$. Since the Frobenius-Perron dimension of $\Phi_{\mathcal{C}}(L)$ equals that of L, the only hypothetical case to consider is $\mathrm{FPdim}(L) = \mathrm{FPdim}(V_{\max})$ and $\Phi_{\mathcal{C}}(L) = V_{\max} \boxtimes S$, for S an invertible object. The interpretation of $\Phi_{\mathcal{C}}$ as in (2.2) yields $\Theta(\mathrm{Hom}(P, L^{\otimes p^r})) \cong S$, with

P the projective cover of $V_{\max}$. By preservation of categorical dimension by the linear symmetric monoidal functor Θ, it follows that

$$\dim \mathrm{Hom}(P, L^{\otimes p^r}) = [L^{\otimes p^r} : V_{\max}] \equiv \pm 1 \mod p.$$

This leads to a contradiction with Lemma 4.2.

Secondly, assume that $L \in \mathcal{C}_{\mathrm{ex}}$. If $V_{\max} \boxtimes S$ is a simple constituent of $\Phi_{\mathcal{C}}(L)$, then we must have $V_{\max} \boxtimes S \in (\mathcal{C} \boxtimes \mathcal{V})_{\mathrm{ex}}$ by (4.1). This is contradicted by the following observation, where we abbreviate $V = V_{\max}$

$$\mathrm{FPdim}(\mathrm{Fr}(V \boxtimes S)) = \mathrm{FPdim}(\mathrm{Fr}(V))\,\mathrm{FPdim}(\mathrm{Fr}(S)) < \mathrm{FPdim}(V)\,\mathrm{FPdim}(S) = \mathrm{FPdim}(V \boxtimes S).$$

This concludes the proof of the claim.

Denote by $\mathcal{C}' \subset \mathcal{C}$ the Serre subcategory generated by all simple $V \in \mathcal{C}$ for which there exists simple $S \in \mathcal{V}$ such that $V \boxtimes S$ is a constituent of an object in the essential image of $\Phi_{\mathcal{C}}$. By construction, this is actually a tensor subcategory, since $V_{\max} \notin \mathcal{C}'$ it is a proper tensor subcategory. Finally, $\Phi_{\mathcal{C}}$ takes values in $\mathcal{C}' \boxtimes \mathcal{V}$, since $\mathcal{C}' \boxtimes \mathcal{V}$ is a Serre subcategory of $\mathcal{C} \boxtimes \mathcal{V}$ containing all simple objects that are constituents of objects in the essential image of $\Phi_{\mathcal{C}}$. □

Proof of Theorem 4.1. For a $\Bbbk$-linear category $\mathcal{A}$, let $\mathcal{A}^{(r)}$ denote the same underlying preadditive category, but where the new $\Bbbk$-linear structure satisfies $\lambda \cdot f = \lambda^{p^r} f$, where the right-hand side denotes the original action, for $\lambda \in \Bbbk$ and f a morphism. Since $\Bbbk$ is perfect, the dimension of the endomorphism algebra of $\mathbf{1}$ remains one when we apply this to a tensor category $\mathcal{C}$, so that $\mathcal{C}^{(r)}$ is again a tensor category. Moreover, by our assumed equivalence $T : \mathcal{V} \to \mathcal{V}$, we have an equivalence $\mathcal{V} \cong \mathcal{V}^{(r)}$ of tensor categories. Consequently, we can and will reinterpret $\Phi_{\mathcal{C}}$ as a ($\Bbbk$-linear) symmetric monoidal functor

$$\Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{(r)} \boxtimes \mathcal{V}.$$

Assume first that $p > 2$. Then $\mathcal{C}_{\mathrm{ex}} \subset \mathcal{C}$ is a Serre subcategory by [EO21, Corollary 9.5] and admits a tensor functor to Ver_p by [EO21, Theorem 8.1]. By Lemma 4.3, for some $N > 0$ (for example bounded by the number of simples in $\mathcal{C}$), the functor

$$(\Phi_{\mathcal{C}^{(r(N-1))}} \boxtimes \mathcal{V}^{\boxtimes(N-1)}) \circ \cdots \circ (\Phi_{\mathcal{C}^{(r)}} \boxtimes \mathcal{V}) \circ \Phi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{(rN)} \boxtimes \mathcal{V}^{\boxtimes N}$$

actually takes values in $\mathcal{C}_{\mathrm{ex}}^{(rN)} \boxtimes \mathcal{V}^{\boxtimes N}$. Using the product tensor functor $(\mathcal{V})^{\boxtimes N} \to \mathcal{V}$ and the above $\mathcal{C}_{\mathrm{ex}}^{(rN)} \to \mathrm{Ver}_p$, we obtain the desired tensor functor.

If $p = 2$, we need some minor adjustments. Let $\widetilde{\mathcal{C}}_{\mathrm{ex}}$ be the Serre closure of $\mathcal{C}_{\mathrm{ex}}$. By [EO21, Corollary 9.8], $\widetilde{\mathcal{C}}_{\mathrm{ex}}$ admits a tensor functor to Ver_4^+. We can now repeat the above argument for $p > 2$, but with replacements $\mathcal{C}_{\mathrm{ex}} \rightsquigarrow \widetilde{\mathcal{C}}_{\mathrm{ex}}$ and $\mathrm{Ver}_p \rightsquigarrow \mathrm{Ver}_4^+$. □

Remark 4.4. Theorem 4.1(1) actually remains true for pretannakian categories with finitely many simple objects up to isomorphism (but not necessarily enough projectives). The proof only requires very minor changes, for instance [EO21, Theorem 8.1] needs to be replaced with the more general [CEO23, Theorem 1.1]. Similarly extending Theorem 4.1(2) seems slightly more subtle.

4.2. **Higher Frobenius functors.** In this subsection, we fix $r \in \mathbb{Z}_{>0}$ and $\underline{\lambda} \in \mathbb{P}_f^{r-1}$. As in §2.1.1 this leads to an embedding $\iota_{\underline{\lambda}} : C_p^r \hookrightarrow \mathrm{SL}_2(\Bbbk)$ and $\boldsymbol{\lambda} \in \mathbb{P}_f^{r-1}$. We thus have a symmetric monoidal restriction functor from $\mathrm{Tilt}\,\mathrm{SL}_2$ to $\mathrm{Rep}\,C_p^r$ (by abuse of notation we use the same symbol for a

tilting module of SL_2 and its restriction under $\iota_{\underline{\lambda}}$). We denote its essential image by $\mathcal{D}_{\underline{\lambda}}$. It was proved in [CF26, Theorem 5.3.5] that the restriction functor induces an equivalence

$$\operatorname{Tilt}\mathrm{SL}_2/I_r \;\to\; \mathcal{D}_{\underline{\lambda}}/\mathcal{K}_{\underline{\lambda}},$$

where I_r is the kernel of the defining functor $\operatorname{Tilt}\mathrm{SL}_2 \to \mathrm{Ver}_{p^r}$ and $\mathcal{K}_{\underline{\lambda}}$ is some tensor ideal in $\mathcal{D}_{\underline{\lambda}}$. We invert this symmetric monoidal equivalence in the proposition below.

Proposition 4.5. *The $\Bbbk$-linear symmetric monoidal functor*

$$\mathcal{D}_{\underline{\lambda}} \to \mathcal{D}_{\underline{\lambda}}/\mathcal{K}_{\underline{\lambda}} \xrightarrow{\sim} \operatorname{Tilt}\mathrm{SL}_2/I_r \hookrightarrow \mathrm{Ver}_{p^r}$$

has a unique $\Bbbk$-linear symmetric monoidal extension $\Theta^{\underline{\lambda}} : \operatorname{Rep} C_p^r \to \mathrm{Ver}_{p^r}$. For C_p^r interpreted as the permutation group $E_r < S_{p^r}$, this is a $\mathbf{V}$-functor.

Proof. Uniqueness and existence of the extension along $\mathcal{D}_{\underline{\lambda}} \subset \operatorname{Rep} C_p^r$ follow from Theorem 3.21 and [CF26, Lemma 5.4.6] (for the morphism f in the latter lemma, we can take the evaluation $St_{r-1}^{\otimes 2} \to \mathbf{1}$). That the resulting $\Theta^{\underline{\lambda}}$ is actually a $\mathbf{V}$-functor was proved (assuming the validity of Conjecture 1.2, which is established in Theorem 3.21) in the proof of [CF26, Theorem 5.4.3]. □

Proposition 4.5 thus implies that we obtain an $\mathbf{O}$-functor $\Phi^{\underline{\lambda}}$, as predicted in [CF26, Conjecture 5.4.2(3)]. These are generalisations of the Frobenius functor Fr, which we recover for $r = 1$ (and the unique point in $\mathbb{P}^0$). To detect $\Phi^{\underline{\lambda}}$-exactness below, we add the following lemma.

Lemma 4.6. *The following conditions are equivalent on $M \in \operatorname{Rep} C_p^r$:*

(1) *$St_{r-1} \otimes M$ is not projective;*
(2) *$\boldsymbol{\lambda} \in \operatorname{supp}(M)$;*
(3) *$\Theta^{\underline{\lambda}}(M) \neq 0$.*

Proof. By [CF25, Corollary 4.2.7], we have $\operatorname{supp} St_{r-1} = \{\boldsymbol{\lambda}\}$, so that equivalence between (1) and (2) follows.

By construction above, the only indecomposable tilting modules $T(j)$ annihilated by $\Theta^{\underline{\lambda}}$ are those with $j \geq p^r - 1$. These are precisely the tilting modules that are projective over C_p^r, see [CF26, Theorem 5.3.5(1)]. Since $\Theta^{\underline{\lambda}}$ is monoidal, we have

$$\Theta^{\underline{\lambda}}(M) \otimes \Theta^{\underline{\lambda}}(St_{r-1}) \cong \Theta^{\underline{\lambda}}(M \otimes St_{r-1}).$$

Since $\Theta^{\underline{\lambda}}(St_{r-1})$ is not zero and $M \otimes St_{r-1}$ is the restriction of a tilting module by Theorem 3.21, the equivalence between (1) and (3) follows. □

4.3. **Proof of the conjecture.**

Theorem 4.7. *Every finite tensor category admits a tensor functor to Ver_{p^n} for some $n \in \mathbb{Z}_{>0}$. Every such tensor functor is unique up to isomorphism.*

We start the proof with some preliminary results. In this section, we let $\mathcal{C}$ be a finite tensor category.

Lemma 4.8. *Let P be a projective generator of $\mathcal{C}$ and let Q be the injective hull of $\mathbf{1}$ (Q is also projective). Then there exists $s \in \mathbb{Z}_{>0}$ such that the E_s-representation $\operatorname{Hom}(P, Q^{\otimes p^s})$ is non-projective, but its restriction to every proper subgroup of E_s is projective.*

Proof. For every $n \in \mathbb{Z}_{>0}$, we set

$$M_n := \mathrm{Hom}_{\mathcal{C}}(P, Q^{\otimes n}) \ \in \ \mathrm{Rep}\, S_n.$$

It has a non-zero S_n-invariant vector

$$P \twoheadrightarrow \mathbf{1} \hookrightarrow Q^{\otimes n},$$

where the right morphism is the n-th tensor power of the defining inclusion $\mathbf{1} \hookrightarrow Q$ and the left map is an arbitrary epimorphism. If M_n is projective, it thus needs to contain the projective cover $P_0(n)$ (which is also the injective hull) of the trivial representation in $\mathrm{Rep}\, S_n$ as a direct summand.

By [CEO23, Proposition A.1], for every $n \in \mathbb{Z}_{>0}$ we can choose $n' \geq n$ such that for the canonical inclusion $S_n < S_{n'}$

$$\Bbbk S_n \hookrightarrow \Bbbk S_{n'} \to \mathrm{End}_{\Bbbk}(P_0(n'))$$

is injective. Assume for a contradiction that all M_n are projective, then this last observation implies that for every n there exists $n' \geq n$ with

$$\Bbbk S_n \to \mathrm{End}_{\mathcal{C}}(Q^{\otimes n}) \to \mathrm{End}_{\mathcal{C}}(Q^{\otimes n'}) \to \mathrm{End}_{\Bbbk}(\mathrm{Hom}(P, Q^{\otimes n'}))$$

injective, so clearly the left-most map is injective too. This is contradicted for finite tensor categories (more generally, for tensor categories of moderate growth) by [CEO23, Corollary 4.11].

We can thus choose the minimal $m \in \mathbb{Z}_{\geq p}$ for which M_m is not projective. It follows from minimality that the restriction of M_m to any proper Young subgroup of S_m is projective. Indeed, by [Cou26b, Lemma 3.2.4], the restriction of M_m to a Young subgroup $S_\kappa < S_m$, for a partition κ of length l, is a direct sum of S_κ-representations

$$\mathrm{Hom}(P_{a_1}, Q^{\otimes \kappa_1}) \boxtimes \mathrm{Hom}(P_{a_2}, Q^{\otimes \kappa_2}) \boxtimes \cdots \boxtimes \mathrm{Hom}(P_{a_l}, Q^{\otimes \kappa_l}),$$

where P_{a_i} are indecomposable projectives.

By Chouinard's Theorem, see for instance [Ben17, Theorem 1.9.1], it follows that the restriction of M_m to some elementary abelian p-subgroup of S_m is not projective, while the restriction to every *non-transitive* (meaning contained in a conjugate of a proper Young subgroup) elementary abelian p-subgroup of S_m is projective. The conclusion thus follows from Lemma 2.3. □

Now we consider the $\mathbf{O}$-functors $\Phi^{\underline{\lambda}}$ resulting from §4.2.

Corollary 4.9. *$\mathcal{C}$ is $\Phi^{\underline{\lambda}}$-exact for some $r \in \mathbb{Z}_{>0}$ and $\underline{\lambda} \in \mathbb{P}_f^{r-1}$.*

Proof. Consider P, Q, r as in Lemma 4.8. The latter, combined with Lemma 2.4 and Lemma 2.2, implies that there exists $\underline{\lambda} \in \mathbb{P}_f^{r-1}$ such that $\boldsymbol{\lambda}$ is in the support of $\mathrm{Hom}(P, Q^{\otimes p^r})$, viewed as an E_r-representation. By Lemma 4.6 we find $\Theta^{\underline{\lambda}}(\mathrm{Hom}(P, Q^{\otimes p^r})) \neq 0$. The conclusion thus follows from [CF26, Proposition 5.2.4]. □

Proof of Theorem 4.7. In order to be able to use Theorem 4.1, we remark that Ver_{p^r} has the desired twisted linear autoequivalence T, see [BEO23, §4.9] or [CF26, Remark 2.3.8].

By the combination of Corollary 4.9 and Theorem 4.1, there is always some $r \in \mathbb{Z}_{>0}$ such that there exists a tensor functor

$$\mathcal{C} \to \mathrm{Ver}_{p^2} \boxtimes \mathrm{Ver}_{p^r}\,.$$

By taking $n = \max(2, r)$ and the tensor product tensor functor $\mathrm{Ver}_{p^2} \boxtimes \mathrm{Ver}_{p^r} \to \mathrm{Ver}_{p^n}$ the first assertion follows.

Uniqueness of the tensor functor is reduced to [EO21, Conjecture B.6][3] in [CEO24, Remark 6.1.9]. The conjecture was shown in [CSZ25, Theorem 7.1]. □

Remark 4.10. Theorem 4.1 shows that, for some $\iota_{\underline{\lambda}} : C_p^r \hookrightarrow (\Bbbk, +) \subset \mathrm{SL}_2(\Bbbk)$, if a finite tensor category $\mathcal{C}$ is $\Phi^{\underline{\lambda}}$-exact then it admits a tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^r}$. However, for most $\underline{\lambda}$, Ver_{p^r} itself will not be $\Phi^{\underline{\lambda}}$-exact. This is already explored for $r = 2 = p$ in [CF26]. As it happens Ver_{p^r} will be $\Phi^{\underline{\lambda}}$-exact for $\iota_{\underline{\lambda}}$ that identifies C_p^r with $\mathbb{F}_{p^r} \subset \Bbbk$, although we do not prove this in the current paper. Consequently, Theorem 4.1 can actually be upgraded to the statement that there exists a tensor functor $\mathcal{C} \to \mathrm{Ver}_{p^r}$ if and only if $\mathcal{C}$ is $\Phi^{\underline{\lambda}}$-exact for such a $\underline{\lambda}$. After the proof, it thus becomes clear that most of the **O**-functors we consider play no role.

4.4. **Some consequences.** We present some of the results that follow from the resolution of the finite Benson–Etingof–Ostrik conjectures.

Recall, see [EGNO15, Definition 6.1.13], that a finite (not necessarily symmetric) tensor category $\mathcal{C}$ is integral if $\mathrm{FPdim}(X) \in \mathbb{Z}$ for all $X \in \mathcal{C}$. Super tannakian tensor categories and tensor categories that fibre over Ver_4^+ are clearly integral, and the following result shows that these are the only (symmetric) ones.

Proposition 4.11. *Let $\mathcal{C}$ be an integral finite tensor category.*

(1) *If $p > 2$, then there is tensor functor $\mathcal{C} \to \mathrm{sVec}$.*
(2) *If $p = 2$, then there is tensor functor $\mathcal{C} \to \mathrm{Ver}_4^+$.*

Proof. By preservation of Frobenius-Perron dimension and Theorem 4.7 it suffices to observe that the only objects of integral Frobenius-Perron dimension in Ver_{p^∞} belong to the subcategory sVec, if $p > 2$, or Ver_4^+ if $p = 2$.

Assume $p > 2$. By [BEO23, Theorem 4.5(iv)], the only $X \in \mathrm{Ver}_{p^n}^+$ with $\mathrm{FPdim}(X) \in \mathbb{Z}$ are in Vec. Any $Y = A \oplus B \boxtimes \overline{\mathbf{1}}$ in $\mathrm{Ver}_{p^n}^+ \boxtimes \mathrm{sVec} = \mathrm{Ver}_{p^n}$ has the same Frobenius-Perron dimension as $A \oplus B \in \mathrm{Ver}_{p^n}^+$, so $\mathrm{FPdim}(Y) \in \mathbb{Z}$ forces $A \oplus B \in \mathrm{Vec}$.

The case $p = 2$ follows similarly from [BE19, Corollary 2.2] and the fact that the only objects in Ver_{2^∞} that only have trivial simple constituents belong to Ver_4^+, see [BE19, Proposition 2.5]. □

The following result gives an estimate on which higher Verlinde category a finite tensor category fibres over based on the Frobenius–Perron dimension of a projective generator.

Proposition 4.12. *Let $\mathcal{C}$ be a finite tensor category with projective generator P such that $\mathrm{FPdim}(P) < p^{r-1}$, for some $r \geq 2$. Then $\mathcal{C}$ admits a tensor functor for Ver_{p^r}.*

Proof. By Theorem 4.7, there is a tensor functor $F\colon \mathcal{C} \to \mathrm{Ver}_{p^s}$, for some $s \geq 1$. Pick the minimal such s. Let $\mathcal{D} \subset \mathrm{Ver}_{p^s}$ be the minimal tensor category containing the essential image of F. Then any object in $\mathcal{D}$ is a subquotient of an object in the essential image of F. In particular, any simple object is a subquotient of $F(P)$, so the number of simple objects in $\mathcal{D}$ up to isomorphism is at most $\mathrm{len}(F(P)) \leq \mathrm{FPdim}(P) < p^{r-1}$.

By minimality of s and by the classification of tensor subcategories of the higher Verlinde categories [BEO23, Corollary 4.61], $\mathcal{D}$ must be Ver_{p^s} or $\mathrm{Ver}_{p^s}^+$. In both cases, the number of simple objects up to isomorphism is at least p^{s-2} by [BEO23, Theorem 4.5]. This means $s \leq r$. □

The following proposition places significant restrictions on the types of non-abelian $\Bbbk$-linear symmetric rigid monoidal categories that can exist.

[3]Note that the numbering is different in the arxiv version.

Proposition 4.13. *Let $\mathcal{A}$ be an essentially small $\Bbbk$-linear additive idempotent complete rigid symmetric monoidal category such that* $\mathrm{End}(\mathbf{1})$ *is one-dimensional and all morphism spaces finite-dimensional. Assume that $\mathcal{A}$ has a unique minimal non-zero tensor ideal such that the set of isomorphism classes of indecomposable objects whose identities lie in the ideal is non-empty and finite. Then there exists a faithful $\Bbbk$-linear symmetric monoidal functor $\mathcal{A} \to \mathrm{Ver}_{p^n}$ for some $n \in \mathbb{Z}_{>0}$.*

Proof. By [CEO25, Theorem 2.4.1(1)], $\mathcal{A}$ admits an abelian envelope which, by construction *loc. cit.*, is a finite tensor category. The envelope thus admits a tensor functor to Ver_{p^n} by Theorem 4.7. □

We explain how our results resolve other open conjectures or questions.

Proposition 4.14. *Assume $p = 2$.*

1. *[CSZ25, Conjecture B.5] is true.*
2. *Every finite tensor category is geometrically reductive (in the sense of [Cou26a; Cou26c]).*
3. *[CEO24, Conjecture 9.3.5] is true.*
4. *[Cou26a, Conjecture 3.1.6(1)] is true for finite tensor categories.*

If $p > 2$, then all these results remain valid provided Ver_{p^∞} is maximally nilpotent (meaning [Cou26a, Conjecture 3.2.3] is true).

Proof. As explained in [CSZ25, Appendix B], the conjecture in (1) follows if, for an arbitrary finite tensor category $\mathcal{C}$, the unique incompressible finite tensor category that $\mathcal{C}$ admits a surjective tensor functor to (which by Theorem 4.7 is a tensor subcategory of some Ver_{p^n}) is maximally nilpotent (MN). Since Ver_{p^n} is conjectured to be MN in [Cou26a, Conjecture 3.2.3], and proved to be so for $p = 2$ in [CEO24, Theorem 9.2.1], the results follow.

Similarly, the other three results follow immediately from [Cou26c, Theorem 4.1.1] and Theorem 4.7. □

The following answers [CEO24, Question 7.1.4(2, 3)] for finite tensor categories. Recall from [CEO24] that the *symmetric growth dimension* $\mathrm{sgd}(X)$ of an object X in a tensor category is defined as

$$\mathrm{sgd}(X) := \limsup_{n\to\infty} \frac{\log(\mathrm{len}(\mathrm{Sym}^{\leq n} X))}{\log(n)},$$

where $\mathrm{Sym}^{\leq n}$ is the sum of the symmetric powers up to degree n.

Proposition 4.15. *Assume $p = 2$, let X be an object of a finite tensor category $\mathcal{C}$, let $F\colon \mathcal{C} \to \mathrm{Ver}_{p^r}$ be a tensor functor, see Theorem 4.7. Then*

$$\mathrm{sgd}(X) = \mathrm{sgd}(F(X)) = \dim \mathrm{Hom}(F(X), \mathbf{1}) \leq \mathrm{FPdim}(X),$$

and $\limsup$ *can be replaced by* $\lim$ *in the definition of* sgd*. The same is true for all p for which Ver_{p^∞} is maximally nilpotent.*

Proof. As explained in [CEO24, Remark 7.1.2], $\mathrm{len}(\cdot)$ can be replaced by $\mathrm{len}(F(\cdot))$ in the definition of sgd. This implies $\mathrm{sgd}(X) = \mathrm{sgd}(F(X))$. The identity $\mathrm{sgd}(F(X)) = \dim \mathrm{Hom}(F(X), \mathbf{1})$ holds by [CEO24, Theorem 7.2.2], as Ver_{2^∞} is maximally nilpotent by [CEO24] (or Ver_{p^∞} by the alternative assumption). Now $\dim \mathrm{Hom}(F(X), \mathbf{1})$ and $\mathrm{FPdim}(X) = \mathrm{FPdim}(F(X))$ can be compared with the multiplicity of the simple object $\mathbf{1}$ in $F(X)$, which shows the asserted inequality.

Pick an exact sequence

$$0 \to Z \to F(X) \to \mathbf{1}^h \to 0$$

in Ver_{p^r}, with $h = \dim \mathrm{Hom}(F(X), \mathbf{1})$. Let Sym denote the symmetric algebra of an object. Using maximal nilpotence of the target of F again, the image R of $\mathrm{Sym}\, Z \to \mathrm{Sym}\, F(X)$ is finite by [CEO24, Theorem 7.2.2], and we have an isomorphism $R \otimes \mathrm{Sym}(\mathbf{1}^h) \cong \mathrm{gr}(\mathrm{Sym}\, F(X))$ by [CEO24, Theorem 3.3.2], where the associated graded is taken with respect to the filtration $Z \subset F(X)$. Pick $d \geq 0$ such that $R_i = 0$ for all $i > d$, then for all $n \geq d$

$$\mathrm{len}(\mathrm{Sym}^{\leq n} F(X)) = \sum_{0 \leq i \leq d} \mathrm{len}(R_i) \binom{n-i+h}{h}$$

is a polynomial in n with positive leading coefficient.

Pick $K \geq 1$ greater than the length of the image of any simple object under F. It follows that

$$\liminf_n \frac{\log(\mathrm{len}(\mathrm{Sym}^{\leq n} X))}{\log(n)} \geq \lim_n \frac{\log(K^{-1}\,\mathrm{len}(\mathrm{Sym}^{\leq n} F(X)))}{\log(n)} = \limsup_n \frac{\log(K^{-1}\,\mathrm{len}(\mathrm{Sym}^{\leq n} F(X)))}{\log(n)},$$

which is again $\mathrm{sgd}(F(X)) = \mathrm{sgd}(X)$ by [CEO24, Remark 7.1.2]. □

We conclude the paper by pointing out how our results contribute towards the full conjecture of Benson, Etingof and Ostrik.

Proposition 4.16. *The following statements are equivalent:*

1. *Every finitely generated tensor category of moderate growth admits a tensor functor to* Ver_{p^n} *for some* $n \in \mathbb{Z}_{>0}$.
2. *Every finitely generated tensor category of moderate growth admits a tensor functor to a finite tensor category.*
3. *Every incompressible finitely generated tensor category of moderate growth is a finite tensor category.*

Proof. The implications (1)⇒(2)⇒(3) are obvious and do not require our results. The implication (3)⇒(1) follows immediately from Theorem 4.7 and [CEO24, Theorem 5.2.1]. □

We point out that for Conjecture 1.1 itself to follow as a consequence of the above equivalent properties, one would for instance need to prove that the categories Ver_{p^n} satisfy the property MN, so that uniqueness of tensor functors, see [CEO24, Theorem 7.3.1], would allow one to glue together tensor functors on finitely generated tensor subcategories.

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