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NIELSEN NUMBERS IN TORUS-BUNDLE SKEINS

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Abstract. For a torus bundle with hyperbolic monodromy γ, we compute the dimension of the empty-skein summand of its generic SL_N skein module. Each term in the Weyl centralizer average involves periodic Nielsen numbers and a count of classes visible modulo a divisor of N. For monodromies of the same trace and any $N \geq 2$, the rank-N dimensions agree exactly when the groups of N-torsion points fixed by the two monodromies are isomorphic. The dimensions in all ranks on all cyclic covers along the base circle determine, and are determined by, the signed trace and the torsion in the first homology of the double cover. We also construct infinitely many pairs of Sol torus bundles whose GL_N dimensions agree in every rank and whose finite-cover visibility profiles agree at every iterate, while their SL_3 empty-skein dimensions differ by six.

1. Introduction

Let f_γ be the automorphism of T^2 induced by a hyperbolic matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and let $M_\gamma = T^2 \times_\gamma S^1$ be its mapping torus. Bierent–Jordan–Vancraeynest–Vazirani showed that the generic GL_N skein dimensions of M_γ depend only on $\mathrm{tr}(\gamma)$ [2, Theorem 1.2]. Kinnear's formula for the full SL_2 skein dimension already involves the invariant factors of $I - \gamma$ and $I + \gamma$ [8, Theorem 1.2 and Remark 1.7].

Write $S_N(\gamma)$ for the dimension of the summand associated with the empty skein in Kinnear's decomposition,

$$S_N(\gamma) = \dim_{\mathbb{K}} \mathrm{HH}_0^\gamma(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)).$$

Kinnear asked how the Weyl centralizers act on the sectors of this summand [8, Question 1.8]. For hyperbolic monodromy, Theorem 3.3 gives the fixed-point count for every element g of each centralizer. The free part of the coinvariant lattice contributes a ratio of periodic Nielsen numbers $N_k = N(f_\gamma^k) = |\det(I - \gamma^k)|$ [3]. Its torsion contributes the number of Nielsen classes that remain distinguishable after reduction modulo e, the greatest common divisor of the cycle lengths of g. This is the visible Nielsen number of the cover associated with $e\mathbb{Z}^2$ [9]. Only $k \leq N$ and $e \mid N$ occur (Corollary 3.4).

What can one recover from a single dimension? At fixed trace, $S_N(\gamma)$ and the isomorphism type of the group of fixed points of f_γ on $T^2[N]$ determine one another (Theorem 4.1). The proof regroups the Weyl sum by orders of torsion elements. At a composite rank, different orders can contribute with opposite signs when two monodromies are compared. The GL_r dimensions grow fast enough that the contribution from the smallest order at which the counts differ outweighs all contributions of the opposite sign combined.

Passing to the degree-j cover along the base replaces γ by γ^j. Corollary 4.2 recovers the dimensions on every such cover from the rank-three dimension on the original bundle and, when needed, one further dimension on its double cover. In the rank-three examples of Theorem 5.9, the two groups $\mathrm{coker}(I - \gamma)$ have the same order. One has three elements killed by 3 and the other has nine. This accounts for the difference of six in the empty-skein dimensions.

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2. The quantum-torus summand

Fix $N \geq 2$ and a generic quantum parameter q. We work over a characteristic-zero field $\mathbb{K}$, adjoining the fractional powers of q in Kinnear's normalized monomials when necessary. Scalar extension preserves the dimensions computed below. Write

$$\Lambda_N = \mathbb{Z}^N/\mathbb{Z}(1,\dots,1),$$

the character lattice of a maximal torus of SL_N. The Weyl group $\mathfrak{S}_N$ permutes the coordinates. The SL_N skein algebra of T^2 is the $\mathfrak{S}_N$-invariant subalgebra of the quantum torus on $\Lambda_N \otimes H_1(T^2;\mathbb{Z})$ [4, 5] (see [8, Theorem 2.14]). For $w \in \mathfrak{S}_N$, let P_w be its permutation matrix on $\mathbb{Z}^N$, let M_w be the induced map on Λ_N, and put

$$C_w^{(N)}(\gamma) := \operatorname{coker}\bigl(I - M_w \otimes \gamma : \Lambda_N \otimes \mathbb{Z}^2 \longrightarrow \Lambda_N \otimes \mathbb{Z}^2\bigr).$$

These groups are finite for hyperbolic γ. Kinnear's decomposition [8, Theorem 1.4 and Corollary 3.12] gives

$$\mathrm{HH}_0^{\gamma}\bigl(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)\bigr) \cong \bigoplus_{[w]\in \mathrm{cl}(\mathfrak{S}_N)} \mathbb{K}[C_w^{(N)}(\gamma)]_{C(w)}, \tag{1}$$

where $C(w)$ is the centralizer of w in $\mathfrak{S}_N$. Before taking coinvariants, the classes of Kinnear's normalized monomials form a basis indexed by $C_w^{(N)}(\gamma)$ [8, Proposition 3.11 and Corollary 3.12]. The other summands of the full skein module come from endomorphism algebras of marked points [8, Corollary 3.6 and Question 1.8].

Lemma 2.1 (Phase-freeness of the centralizer action)**.** *Let γ be hyperbolic, let $w \in \mathfrak{S}_N$, and let $g \in C(w)$. Put $A_w = M_w \otimes \gamma$, let Ω denote the invariant skew form on $\Lambda_N \otimes \mathbb{Z}^2$, and let*

$$\widetilde{m}_x = q^{\frac{1}{2}\Omega((I-A_w)^{-1}x,\,x)}\, m_x, \qquad x \in \Lambda_N \otimes \mathbb{Z}^2,$$

denote the normalized monomials [8, Proposition 3.11]*. Then*

$$g(\widetilde{m}_x) = \widetilde{m}_{gx} \qquad \text{for all } x.$$

Consequently the $C(w)$-action on the w-summand is the permutation action induced by the natural action of $C(w)$ on $C_w^{(N)}(\gamma)$, and the dimension of the coinvariants is the number of $C(w)$-orbits.

Proof. The Weyl action $g(m_x) = m_{gx}$ preserves the multiplication $m_x m_y = q^{\frac{1}{2}\Omega(x,y)} m_{x+y}$ since it preserves Ω [8, Notation 2.15 and Lemma 2.19]. The eigenvalues of A_w are of the form $\zeta\lambda^{\pm 1}$ with ζ a root of unity and $|\lambda| > 1$. Thus $I - A_w$ is invertible over $\mathbb{Q}$ and the preimage in the normalization is unique [8, Remark 3.10]. The action of $g \in C(w)$ commutes with A_w, so

$$\Omega((I-A_w)^{-1}gx,\, gx) = \Omega(g(I-A_w)^{-1}x,\, gx) = \Omega((I-A_w)^{-1}x,\, x),$$

whence $g(\widetilde{m}_x) = \widetilde{m}_{gx}$.

The proof of [8, Theorem 3.5] identifies the residual relations in (1) with $a \sim g(a)$ for $g \in C(w)$. The coinvariants therefore have one basis vector per orbit. □

Lemma 2.2 (Cycle determinant)**.** *Let $w \in \mathfrak{S}_N$ have cycle lengths $c_1,\dots,c_s$. If $\det(I-\gamma^{c_i}) \neq 0$ for all i, then*

$$|C_w^{(N)}(\gamma)| = \frac{\prod_{i=1}^{s} |\det(I-\gamma^{c_i})|}{|\det(I-\gamma)|}.$$

Compare [2, Lemma 3.1]*, where the corresponding cycle decomposition is obtained on the permutation lattice used in the GL_N calculation.*

Proof. First use the permutation lattice $\mathbb{Z}^N$. On the block belonging to a cycle of length c, the eigenvalues of the permutation matrix are the cth roots of unity. If λ, λ^{-1} are the eigenvalues of γ, then

$$\prod_{\zeta^c=1} (1-\zeta\lambda)(1-\zeta\lambda^{-1}) = (1-\lambda^c)(1-\lambda^{-c}) = \det(I-\gamma^c).$$

Multiplying over the cycles gives

$$\det(I - P_w \otimes \gamma \mid \mathbb{Z}^N \otimes \mathbb{Z}^2) = \prod_i \det(I-\gamma^{c_i}).$$

Tensor the diagonal sequence

$$0 \longrightarrow \mathbb{Z}(1,\dots,1) \longrightarrow \mathbb{Z}^N \longrightarrow \Lambda_N \longrightarrow 0$$

with $\mathbb{Z}^2$. On the diagonal, $P_w \otimes \gamma$ restricts to γ. The determinant on the quotient is therefore the product above divided by $\det(I-\gamma)$, which is nonzero under the stated hypothesis. Taking absolute values gives the cokernel order. □

Lemma 2.3 (Fixed points on a difference cokernel)**.** *Let L be a lattice, let $F : L \to L$ be injective with finite cokernel, and let $g \in \mathrm{Aut}(L)$ commute with F. Then*

$$\left|\mathrm{Fix}(g \curvearrowright \operatorname{coker} F)\right| = [L : FL + (g-I)L].$$

Equivalently, the right-hand side is the order of the cokernel of the endomorphism induced by F on $L/(g-I)L$. For the corresponding cyclic-sector calculation, compare [2, Lemma 3.8].

Proof. Put $C = \operatorname{coker} F$. The endomorphism $g - I$ of the finite abelian group C has kernel and cokernel of the same order. Its cokernel is

$$C/(g-I)C \cong L/(FL + (g-I)L),$$

which proves the formula. Since F commutes with g, it descends to $L/(g-I)L$, giving the equivalent description. □

3. The dimension formula

For $k, m \geq 1$, write

$$N_k = |\det(I-\gamma^k)|, \qquad V_m = V_{m\mathbb{Z}^2}(f_\gamma).$$

The subgroup $m\mathbb{Z}^2$ defines a cover of degree m^2. Its visible Nielsen number V_m counts the classes in the support of the Reidemeister trace after reduction modulo m [9, Definitions 2.2–2.3]. If γ_m denotes reduction modulo m, then

$$V_m = \left|\operatorname{coker}(I-\gamma_m : (\mathbb{Z}/m\mathbb{Z})^2 \to (\mathbb{Z}/m\mathbb{Z})^2)\right| = \left|\ker(I-\gamma_m : (\mathbb{Z}/m\mathbb{Z})^2 \to (\mathbb{Z}/m\mathbb{Z})^2)\right|. \tag{2}$$

Indeed, the fixed-point classes are indexed by $\operatorname{coker}(I-\gamma)$, which maps onto $\operatorname{coker}(I-\gamma_m)$ under reduction. All classes have index $\operatorname{sgn}\det(I-\gamma)$, so every image class has a nonzero coefficient. The second equality holds for every endomorphism of a finite abelian group.

Let $g \in \mathfrak{S}_N$ have cycles $O_1, \dots, O_s$ of lengths $c_1, \dots, c_s$, and put

$$Q_g := \Lambda_N/(g-I)\Lambda_N, \qquad e(g) := \gcd(c_1,\dots,c_s).$$

Lemma 3.1 (The Weyl coinvariant lattice)**.** *There is an exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto (c_1,\dots,c_s)} \mathbb{Z}^s \longrightarrow Q_g \longrightarrow 0. \tag{3}$$

Consequently,

$$Q_g \cong \mathbb{Z}^{s-1} \oplus \mathbb{Z}/e(g)\mathbb{Z}.$$

If $w \in C(g)$, then w permutes the cycles of g and (3) is equivariant for the induced permutation $\bar{w}$ of $\{O_1,\dots,O_s\}$, with trivial action on the left-hand copy of $\mathbb{Z}$.

Proof. The coinvariants of the permutation lattice $\mathbb{Z}^N$ under g are freely generated by the cycles $O_1, \dots, O_s$. In these coordinates the diagonal vector $(1, \dots, 1)$ maps to $(c_1, \dots, c_s)$, which gives (3). Smith normal form gives the displayed decomposition of Q_g.

An element of $C(g)$ permutes cycles of equal length. Rotations within a cycle act trivially on its coinvariant generator, so $\bar{w}$ fixes $(c_1, \dots, c_s)$ and the source of the first map. □

Lemma 3.2 (Triviality on coinvariant torsion)**.** *For every $w \in C(g)$, the induced action of $\bar{w}$ on $\mathrm{Tor}(Q_g) \cong \mathbb{Z}/e(g)\mathbb{Z}$ is trivial.*

Proof. Inside $\mathbb{Z}^s$, the saturation of $\mathbb{Z}(c_1, \dots, c_s)$ is generated by $(c_1/e(g), \dots, c_s/e(g))$. Hence

$$\mathrm{Tor}(Q_g) \cong \frac{\mathbb{Z}(c_1/e(g), \dots, c_s/e(g))}{\mathbb{Z}(c_1, \dots, c_s)}.$$

The permutation $\bar{w}$ fixes the displayed generator because it permutes only cycles of equal length. It therefore acts trivially on the quotient. □

For commuting $w, g \in \mathfrak{S}_N$, let

$$d(w, g) = (d_1, \dots, d_\ell)$$

be the cycle lengths of the permutation induced by w on the set of cycles of g.

Theorem 3.3 (General SL_N empty-skein formula)**.** *Let $N \geq 2$ and let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For every $w \in \mathfrak{S}_N$ and $g \in C(w)$,*

$$\left|\mathrm{Fix}\left(g \curvearrowright C_w^{(N)}(\gamma)\right)\right| = \frac{\prod_{j=1}^{\ell} N_{d_j}}{N_1} V_{e(g)}. \tag{4}$$

Consequently,

$$\dim \mathrm{HH}_0^{\gamma}(\mathrm{SkAlg}_{\mathrm{SL}_N}(T^2)) = \sum_{[w] \in \mathrm{cl}(\mathfrak{S}_N)} \frac{1}{|C(w)|} \sum_{g \in C(w)} \frac{\prod_{j=1}^{\ell(w,g)} N_{d_j(w,g)}}{N_1} V_{e(g)}. \tag{5}$$

Proof. Fix $w \in \mathfrak{S}_N$ and $g \in C(w)$. Apply Lemma 2.3 with

$$L = \Lambda_N \otimes \mathbb{Z}^2, \qquad F = I - M_w \otimes \gamma.$$

The fixed-point count is the cokernel order of the induced map on $Q_g \otimes \mathbb{Z}^2$, namely

$$F_{w,g} = I - \bar{w} \otimes \gamma.$$

Tensoring (3) with the free lattice $\mathbb{Z}^2$ preserves exactness. The difference maps on its three terms are $I - \gamma$, $I - P_{\bar{w}} \otimes \gamma$ and $F_{w,g}$. The first two are injective because the eigenvalues of $P_{\bar{w}}$ have modulus one and those of γ do not. The snake lemma gives

$$0 \longrightarrow \ker F_{w,g} \longrightarrow \mathrm{coker}(I - \gamma) \longrightarrow \mathrm{coker}(I - P_{\bar{w}} \otimes \gamma) \longrightarrow \mathrm{coker}\, F_{w,g} \longrightarrow 0.$$

By the cycle determinant identity,

$$|\mathrm{coker}(I - P_{\bar{w}} \otimes \gamma)| = \prod_{j=1}^{\ell} N_{d_j}, \qquad |\mathrm{coker}(I - \gamma)| = N_1.$$

It follows that

$$|\,\mathrm{coker}\, F_{w,g}| = |\ker F_{w,g}|\, \frac{\prod_{j=1}^{\ell} N_{d_j}}{N_1}. \tag{6}$$

The same exact sequence shows that $\ker F_{w,g}$ is finite, hence it lies in the torsion subgroup of $Q_g \otimes \mathbb{Z}^2$. By Lemmas 3.1 and 3.2, this subgroup is $(\mathbb{Z}/e(g)\mathbb{Z})^2$ and $\bar{w}$ acts trivially on it. Therefore

$$|\ker F_{w,g}| = \left|\ker(I - \gamma_{e(g)} : (\mathbb{Z}/e(g)\mathbb{Z})^2 \to (\mathbb{Z}/e(g)\mathbb{Z})^2)\right| = V_{e(g)}$$

by (2). Substitution in (6) proves (4). Finally, Burnside's lemma applied to each summand of (1) gives (5). □

In rank three, Proposition 5.1 computes the three sector contributions separately.

Corollary 3.4 (Data dependence at rank N)**.** *For hyperbolic γ, the dimension of the rank-N empty-skein summand is determined by the periodic Nielsen numbers $N_1, \dots, N_N$ together with the visible Nielsen numbers $V_{e\mathbb{Z}^2}(f_\gamma)$ for $e \mid N$.*

Proof. Every $d_j(w,g)$ is at most N. Moreover, $e(g)$ divides every cycle length of g, hence it divides their sum N. These are precisely the data appearing in (5). □

Proposition 3.5 (No torsion corrections for GL_N)**.** *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $N \geq 1$. For every $w \in \mathfrak{S}_N$ and $g \in C(w)$,*

$$\left|\mathrm{Fix}\Big(g \curvearrowright \mathrm{coker}\big(I - P_w \otimes \gamma : \mathbb{Z}^N \otimes \mathbb{Z}^2 \to \mathbb{Z}^N \otimes \mathbb{Z}^2\big)\Big)\right| = \prod_{j=1}^{\ell(w,g)} N_{d_j(w,g)},$$

and consequently

$$\dim \mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma) = \sum_{[w]\in \mathrm{cl}(\mathfrak{S}_N)} \frac{1}{|C(w)|} \sum_{g \in C(w)} \prod_{j=1}^{\ell(w,g)} N_{d_j(w,g)}.$$

In particular the GL_N-skein dimension is determined by $N_1, \dots, N_N$ alone, hence by $\mathrm{tr}(\gamma)$ (Remark 5.7), and no visible-Nielsen correction occurs: the coinvariant lattice $\mathbb{Z}^N/(g-I)\mathbb{Z}^N$ is free, so the torsion mechanism of Lemmas 3.1 and 3.2 is absent.

Proof. For GL_N, Kinnear's decomposition on the permutation lattice $\mathbb{Z}^N$ gives the whole module [8, Corollary 3.6],

$$\mathrm{Sk}_{\mathrm{GL}_N}(M_\gamma) \cong \bigoplus_{[w]\in \mathrm{cl}(\mathfrak{S}_N)} \mathbb{K}\big[\mathrm{coker}(I - P_w \otimes \gamma)\big]_{C(w)}.$$

Lemma 2.1 applies to the permutation lattice, so Burnside's lemma gives the second formula from the first. For the first, apply Lemma 2.3 to $F = I - P_w \otimes \gamma$. The g-coinvariants of $\mathbb{Z}^N$ have one free generator per cycle of g (Lemma 3.1). On their tensor product with $\mathbb{Z}^2$, F induces $I - P_{\bar{w}} \otimes \gamma$. The free-lattice determinant calculation in Lemma 2.2 gives its cokernel order as $\prod_j |\det(I - \gamma^{d_j})| = \prod_j N_{d_j}$. Every d_j is at most N. □

Remark 3.6 (The $\mathrm{GL}_N/\mathrm{SL}_N$ contrast). Proposition 3.5 recovers the trace dependence in [2]. Passing from $\mathbb{Z}^N$ to Λ_N lowers the free rank of the g-coinvariants by one and introduces torsion $\mathbb{Z}/e(g)\mathbb{Z}$ (Lemma 3.1). In (4) these changes account for the divisor N_1 and the factor $V_{e(g)}$, respectively.

Remark 3.7 (Diagonal and off-diagonal terms). If $g = w$ has s cycles, then w induces the identity permutation on its own cycles. Thus $d(w,w) = (1, \dots, 1)$ and (4) reduces to

$$\left|\mathrm{Fix}(w \curvearrowright C_w^{(N)}(\gamma))\right| = N_1^{s-1} V_{e(w)}.$$

For $N = 3$, put $\sigma = (123)$. The diagonal pair $(w,g) = (\sigma,\sigma)$ contributes $V_3/3$ after Burnside averaging. The pair (σ,σ^2) contributes another $V_3/3$, and (e,σ) and (e,σ^2) together contribute the remaining $V_3/3$. Only one third of the correction is diagonal.

Example 3.8 (The rank-four formula)**.** At $N = 4$, Theorem 3.3 gives the following centralizer averages, with $V_e = V_{e\mathbb{Z}^2}(f_\gamma)$.

cycle type of w	contribution of the $[w]$-sector
1^4	$\frac{N_1^3}{24} + \frac{N_1^2}{4} + \frac{N_1V_2}{8} + \frac{N_1}{3} + \frac{V_4}{4}$
$2\,1^2$	$\frac{N_1^2}{4} + \frac{N_1N_2}{4} + \frac{N_1V_2}{4} + \frac{N_2}{4}$
2^2	$\frac{N_2^2}{8N_1} + \frac{N_2}{4} + \frac{N_1V_2}{8} + \frac{N_2V_2}{4N_1} + \frac{V_4}{4}$
$3\,1$	$\frac{2N_1}{3} + \frac{N_3}{3}$
4	$\frac{N_4}{4N_1} + \frac{N_2V_2}{4N_1} + \frac{V_4}{2}$

For example, if w is a 4-cycle, then $C(w) = \{e, w, w^2, w^3\}$. The four fixed-point terms in that sector are

$$\frac{N_4}{N_1}, \qquad V_4, \qquad \frac{N_2}{N_1}V_2, \qquad V_4,$$

whose average gives the last row. Summing the five sectors gives

$$\dim \mathrm{HH}_0^\gamma(\mathrm{SkAlg}_{\mathrm{SL}_4}(T^2)) = \frac{N_1^3}{24} + \frac{N_1^2}{2} + \frac{N_1N_2}{4} + \frac{N_1V_2}{2} + N_1 + \frac{N_2}{2} + \frac{N_3}{3} + V_4 + \frac{N_2^2}{8N_1} + \frac{N_2V_2}{2N_1} + \frac{N_4}{4N_1}.$$

The two nontrivial moduli are 2 and 4, as in Corollary 3.4. For the companion matrices $\gamma_t = \left(\begin{smallmatrix} 0 & -1 \\ 1 & t \end{smallmatrix}\right)$, the values at $t = 3, -3, 5, -5, 6$ are, respectively, $29, 45, 139, 163, 248$.

4. Torsion fixed points and cyclic covers

Let $T^2[N]$ be the subgroup of points killed by N in T^2. For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ write $A_\gamma = \mathrm{coker}(I - \gamma)$. When γ is hyperbolic with trace t, this group has order $a = |2 - t| = N_1$. Write s_γ for the positive gcd of the entries of $I - \gamma$. The Smith decomposition is

$$A_\gamma \simeq \mathbb{Z}/s_\gamma\mathbb{Z} \oplus \mathbb{Z}/(a/s_\gamma)\mathbb{Z}.$$

Thus s_γ divides $E(a)$, where $E(b)$ denotes the largest positive integer whose square divides b.

Theorem 4.1 (Detection at a single rank)**.** *Let $N \geq 2$. If $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ are hyperbolic and have the same trace t, then*

$$S_N(\gamma) = S_N(\gamma') \quad \Longleftrightarrow \quad \mathrm{Fix}(f_\gamma|_{T^2[N]}) \cong \mathrm{Fix}(f_{\gamma'}|_{T^2[N]}).$$

Here $\cong$ denotes isomorphism of abelian groups. Equivalently, the dimensions agree precisely when $\gcd(N, s_\gamma) = \gcd(N, s_{\gamma'})$. At fixed t, rank N determines A_γ for all such monodromies if and only if $E(|2 - t|) \mid N$.

Corollary 4.2 (Cyclic covers along the base)**.** *For hyperbolic $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the family $\left(S_N(\gamma^j)\right)_{N\geq 2,\, j\geq 1}$ and the pair*

$$(\mathrm{tr}(\gamma),\ \mathrm{Tor}\, H_1(M_{\gamma^2}; \mathbb{Z}))$$

determine one another. The rank-three dimension on M_γ gives the signed trace t. If $E(t^2-4) > 1$, one further dimension, at rank $E(t^2 - 4)$ on the double cover M_{γ^2}, suffices to recover the family. If $E(t^2 - 4) = 1$, the trace alone suffices. No finite set of ranks chosen independently of t determines the family for all hyperbolic monodromies, even if those ranks are known at every cover degree.

For the action on finite torus lattices, Baake–Roberts–Weiss give a criterion for two toral endomorphisms to be conjugate on $T^2[n]$ for every n [1, Theorem 2 and Corollary 3].

Proof of Theorem 4.1. Fix t, and let $P_r = \dim \mathrm{Sk}_{\mathrm{GL}_r}(M_\gamma)$ for $r \geq 1$, with $P_0 = 1$. These numbers depend only on t by Proposition 3.5. Write $J_A(d)$ for the number of elements of exact order d in $A = A_\gamma$. The map $[v] \mapsto (I-\gamma)^{-1}v \bmod \mathbb{Z}^2$ identifies A with $\mathrm{Fix}(f_\gamma)$. By (2), $V_e = |A[e]| = \sum_{d|e} J_A(d)$. We claim that (5) becomes

$$aS_N(\gamma) = \sum_{d|N} J_A(d) P_{N/d}. \tag{7}$$

To see this, write the centralizer average as $1/N!$ times the sum over commuting pairs (w, g) and group the pairs by the cycle type of g. A term with $J_A(d)$ occurs exactly when every cycle length of g is divisible by d. Suppose g has m_c cycles of length c. Each permutation of these cycles has c^{m_c} lifts in the corresponding centralizer factor, one for each choice of rotations. This cancels c^{m_c} in the order of that factor. The contribution from this length is

$$h_{m_c}, \qquad h_m = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \prod_{C \in \mathrm{cyc}(\sigma)} N_{|C|}, \qquad h_0 = 1,$$

where the product ranges over the cycles of σ. Putting $H(z) = \sum_{m \geq 0} h_m z^m$, the unrestricted average gives $\sum_{r \geq 0} P_r z^r = \prod_{c \geq 1} H(z^c)$ by Proposition 3.5, in agreement with the partition function of [2]. Restricting c to multiples of d replaces z by z^d, proving (7).

Let $T = |t|$, $\epsilon = \mathrm{sgn}(t)$ and $\rho = (T + \sqrt{T^2-4})/2$. Then $N_j = \rho^j + \rho^{-j} - 2\epsilon^j$. The cycle index of the symmetric groups gives

$$H(z) = \exp\left(\sum_{j \geq 1} N_j \frac{z^j}{j} \right) = \frac{(1-\epsilon z)^2}{1 - Tz + z^2} = 1 + \frac{az}{1 - Tz + z^2}.$$

For $r \geq 1$ this yields $h_r = a(\rho^r - \rho^{-r})/(\rho - \rho^{-1})$. In particular $h_{r+1}/h_r > \rho > a/2$, since $\sqrt{T^2-4} > 2$ and $a \leq T + 2$. At $r = 0$ we have $h_0 = 1$ and $h_1 = a$, so $h_1 \geq h_0$ and $2h_1 > ah_0$. The series $\prod_{c \geq 2} H(z^c)$ has nonnegative coefficients and constant term 1. Convolving its coefficients with those of H gives

$$P_{r+1} \geq P_r, \qquad 2P_{r+1} > aP_r \qquad (r \geq 0). \tag{8}$$

For a prime p, set $v = v_p(a)$ and $b = v_p(s_\gamma) \leq v/2$. The Smith decomposition gives

$$|A[p^j]| = p^{\min(j,b)+\min(j,v-b)}.$$

For $j \leq v_p(N)$ this depends only on $\min(b, v_p(N))$ and v. Thus equal values of $\gcd(N, s_\gamma)$ give equal $|A[p^j]|$ for all $p^j \mid N$. Successive differences give $J_A(p^j)$, and multiplicativity gives $J_A(d)$ for every $d \mid N$. Equation (7) proves equality of the dimensions.

For the converse, suppose the gcds differ and put $A' = A_{\gamma'}$. At a prime where the truncated exponents differ, denote the smaller and larger exponents of the small Smith factors by $b < b'$. The first differing order at this prime is $p^{b+1} \mid N$. Indeed $v \geq 2b + 2$, so the groups have p^{2b+1} and p^{2b+2} elements killed by p^{b+1}, respectively, and the same number killed by p^b. Thus the exact-order counts at p^{b+1} differ by $p^{2b+1}(p-1) \geq 2$. By multiplicativity, the least divisor d_0 of N with $J_A(d_0) \neq J_{A'}(d_0)$ is one of these prime powers.

Interchange A, A' if necessary so that the difference at d_0 is positive, and put $k = N/d_0$. Its contribution to the difference of the sums in (7) is at least $2P_k$. For $d > d_0$ one has $N/d \leq k - 1$

whenever $d \mid N$. The sum of the magnitudes of the negative contributions is at most

$$P_{k-1} \sum_{\substack{d \mid N \\ J_A(d) < J_{A'}(d)}} J_{A'}(d) \;\le\; aP_{k-1} \;<\; 2P_k$$

by (8). The dimensions therefore differ. A related first-difference argument for zero-sum multisets uses binomial weights [10, Theorem 3]. For equal group orders and a common degree at least the group order, see [6, Lemma 4.4 and Appendix A].

It remains to interpret the gcd. The group $\mathrm{Fix}(f_\gamma|_{T^2[N]})$ identifies with $A[N]$, whose invariant factors, in divisibility order, are $\gcd(N, s_\gamma)$ and $\gcd(N, a/s_\gamma)$. At fixed a the first determines the second. At any $p \mid N$ it either specifies b exactly or gives $b \ge v_p(N)$, in which case $v - b \ge b \ge v_p(N)$ as well. Consequently two such groups are isomorphic precisely when their first factors agree.

Finally, every s with $s^2 \mid a$ occurs as s_γ for

$$\gamma_{t,s} = \begin{pmatrix} 1 & (t-2)/s \\ s & t-1 \end{pmatrix}. \tag{9}$$

If $E(a) \mid N$, then $\gcd(N, s_\gamma) = s_\gamma$. Otherwise there is a prime p with $u = v_p(N) < v_p(E(a))$. The choices $s = p^u$ and $s = p^{u+1}$ give the same gcd and distinct Smith factors. □

Proof of Corollary 4.2. First we recover the signed trace. In the notation of the preceding proof, $P_1 = a$ and $P_3 = h_3 + h_1^2 + h_1 = a(T^2 + a)$. In rank three, (7) reads

$$S_3(\gamma) = T^2 + a + |A_\gamma[3]| - 1.$$

The last group has order $1, 3$ or 9. Hence the offset $S_3(\gamma) - (T^2 + T)$ lies in $\{-2, 0, 6\}$ for $t > 0$ and in $\{2, 4, 10\}$ for $t < 0$. For $T \ge 6$ the intervals $[T^2 + T - 2, T^2 + T + 10]$ for consecutive T are disjoint. For $T = 3, 4, 5$, substituting the possible groups of order $|2 - t|$ gives $\{10, 14\}$, $\{18, 24\}$ and $\{30, 32\}$, respectively. These sets are disjoint and lie below 40, the lower endpoint for $T = 6$. Thus the dimension specifies both T and the sign of t.

Write $\mathrm{cont}(B)$ for the positive gcd of the entries of a nonzero integral matrix B, and set $s_j = \mathrm{cont}(\gamma^j - I)$. The homology sequence of the mapping torus identifies $\mathrm{Tor}\, H_1(M_{\gamma^j}; \mathbb{Z})$ with A_{γ^j} [11, Corollary 3]. Its Smith factors are s_j and N_j/s_j. We shall recover them all from t and s_2. First,

$$s_1 = \frac{\gcd(s_2, t-2)}{\gcd(2, t)}. \tag{10}$$

To prove this, write $\gamma = I + s_1 B$ with $\mathrm{cont}(B) = 1$. Then $t - 2 = s_1 \operatorname{tr} B$ and $\operatorname{tr} B = -s_1 \det B$ by $\det \gamma = 1$. Right multiplication by γ^{-1} preserves content, so $s_2 = \mathrm{cont}(\gamma - \gamma^{-1}) = s_1 \,\mathrm{cont}(2B - (\operatorname{tr} B)I)$. Taking the gcd with $t - 2$ gives $s_1 \gcd(2, \operatorname{tr} B)$ because B is primitive. When s_1 is odd, t and $\operatorname{tr} B$ have the same parity. When s_1 is even, both are even. This proves (10).

Let $u_0 = 0$, $u_1 = 1$ and $u_{k+1} = tu_k - u_{k-1}$. Cayley–Hamilton gives $\gamma^k - \gamma^{-k} = u_k(\gamma - \gamma^{-1})$ and $\gamma^i + \gamma^{-i} = \operatorname{tr}(\gamma^i) I$ [1, Lemma 1]. Hence, for $k \ge 1$,

$$\gamma^{2k} - I = u_k\, \gamma^{k-1}(\gamma^2 - I),$$

$$\gamma^{2k+1} - I = \left(1 + \sum_{i=1}^{k} \operatorname{tr}(\gamma^i)\right) \gamma^k(\gamma - I).$$

The second identity follows by pairing the terms of $\sum_{r=-k}^{k} \gamma^r$. Taking contents gives every s_j, since the scalar factors depend only on t and the powers of γ are unimodular. Thus t and A_{γ^2} determine every A_{γ^j} and $\operatorname{tr}(\gamma^j)$, hence every dimension by Theorem 4.1.

Conversely, once $S_3(\gamma)$ gives t, the double cover has trace t^2-2 and torsion of order t^2-4. If $E(t^2-4)=1$ this group is cyclic. Otherwise Theorem 4.1 recovers it at rank $E(t^2-4)$. This proves the reconstruction in both directions. For some traces the second measurement is redundant even when $E(t^2-4)>1$. For instance, at $t=4$ one has $s_2 = 2\,\mathrm{cont}(\gamma-2I)=2$, since $\mathrm{cont}(\gamma-2I)^2$ divides $|\det(\gamma-2I)|=3$.

To see why no fixed finite set of ranks suffices, choose a prime p dividing none of them and take $t=2+p^2$. The matrices $\gamma_{t,1}$ and $\gamma_{t,p}$ of (9) are conjugate over $\mathbb{Z}[1/p]$ by $\mathrm{diag}(p,1)$. This diagonal matrix is invertible modulo every chosen rank, so at each such rank all powers of the two matrices have isomorphic fixed subgroups. Theorem 4.1 gives equal dimensions there. At rank p the two gcds on the base are 1 and p. □

5. Rank three and Sol torus bundles

5.1. **The three Weyl sectors.** Write $\Lambda=\Lambda_3$ and $C_w(\gamma)=C_w^{(3)}(\gamma)$. The three conjugacy classes of $\mathfrak{S}_3$ are represented by

$$e, \qquad \tau=(12), \qquad \sigma=(123).$$

In the basis $\bar{e}_1,\bar{e}_2$ of Λ, one may take

$$M_\tau=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad M_\sigma=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}.$$

The centralizer orders are 6, 2, and 3, respectively.

Write $T_3(\gamma)=|A_\gamma[3]|$ for the number of elements of A_γ killed by 3.

Proposition 5.1. *The orbit counts for the identity, transposition, and 3-cycle sectors are, respectively,*

$$d_e(\gamma)=\frac{N_1^2+3N_1+2T_3(\gamma)}{6}, \tag{11}$$

$$d_\tau(\gamma)=\frac{N_2+N_1}{2}, \tag{12}$$

$$d_\sigma(\gamma)=\frac{N_3/N_1+2T_3(\gamma)}{3}. \tag{13}$$

Proof. We use Burnside's lemma with Lemma 2.3.

Identity sector. Here $C(e)=\mathfrak{S}_3$. By Lemma 2.2, the identity element fixes all N_1^2 elements of $C_e(\gamma)$. For a transposition g, the Smith normal form of M_g-I on Λ is $\mathrm{diag}(1,0)$, so

$$\Lambda/(g-I)\Lambda\cong\mathbb{Z}.$$

The map $I-I_\Lambda\otimes\gamma$ induced on the quotient is $I-\gamma$, and Lemma 2.3 gives N_1 fixed points.

For a 3-cycle g, the matrix M_g-I has determinant 3, and

$$\Lambda/(g-I)\Lambda\cong\mathbb{Z}/3\mathbb{Z}.$$

The induced difference map is $I-\bar{\gamma}$ on $(\mathbb{Z}/3\mathbb{Z})^2$. Its cokernel and kernel have the same order. To identify the kernel, apply the snake lemma to $I-\gamma$ on the exact sequence $0\to\mathbb{Z}^2\xrightarrow{3}\mathbb{Z}^2\to\mathbb{F}_3^2\to 0$. Since $I-\gamma$ is injective on $\mathbb{Z}^2$, this gives

$$A_\gamma[3]\cong\ker(I-\bar{\gamma}:\mathbb{F}_3^2\to\mathbb{F}_3^2).$$

Averaging over the identity, three transpositions and two 3-cycles gives (11).

Transposition sector. The centralizer of τ is $\{e,\tau\}$. The identity fixes all $|C_\tau(\gamma)|=N_2$ elements. Since $\Lambda/(\tau-I)\Lambda\cong\mathbb{Z}$ and M_τ becomes the identity on this quotient, the map $I-M_\tau\otimes\gamma$ descends to $I-\gamma$. Thus τ has N_1 fixed points, proving (12).

3-*cycle sector.* In $C(\sigma) = \{e, \sigma, \sigma^2\}$, the identity fixes $|C_\sigma(\gamma)| = N_3/N_1$ elements. For $g = \sigma$ or σ^2, the quotient $\Lambda/(g-I)\Lambda$ is $\mathbb{Z}/3\mathbb{Z}$, on which M_σ acts trivially. In both cases the descended difference map is $I - \bar{\gamma}$ on $\mathbb{F}_3^2$, so the fixed-point count is $T_3(\gamma)$. This proves (13). □

Theorem 5.2 (The rank-three dimension)**.** *For hyperbolic* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$*,*

$$S_3(\gamma) = \frac{N_1^2}{6} + N_1 + \frac{N_2}{2} + \frac{N_3}{3N_1} + T_3(\gamma), \qquad T_3(\gamma) = V_3.$$

Proof. Adding the three orbit counts in Proposition 5.1 gives the formula for $S_3(\gamma)$. The identity-sector calculation gives $T_3(\gamma) = |\ker(I - \bar{\gamma})| = V_3$ by (2). □

Corollary 5.3 (Trace form)**.** *Let* $t = \mathrm{tr}(\gamma)$*, with* $|t| > 2$*. Then*

$$S_3(\gamma) = \begin{cases} t^2 + t - 3 + T_3(\gamma), & t > 2, \\ t^2 - t + 1 + T_3(\gamma), & t < -2. \end{cases}$$

Proof. The trace recurrence for powers of an SL_2 matrix gives

$$N_1 = |2 - t|, \qquad N_2 = t^2 - 4, \qquad \frac{N_3}{N_1} = (t+1)^2.$$

Substitute in Theorem 5.2 for each sign of t. □

5.2. **The mod-3 correction.** The subgroup $3\mathbb{Z}^2$ defines a degree-nine cover of the torus to which every toral automorphism lifts. The inclusion of the fiber identifies the group responsible for the correction with the 3-torsion in the first homology of the bundle.

Proposition 5.4 (Mod-3 observer identity)**.** *Let* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *with* $\det(I - \gamma) \neq 0$*, and write* $\bar{\gamma}$ *for its reduction modulo* 3*. Then*

$$V_{3\mathbb{Z}^2}(f_\gamma) = |\ker(I - \bar{\gamma})| = T_3(\gamma) = \big|H_1(M_\gamma; \mathbb{Z})[3]\big|.$$

Proof. Since $\det(I - \gamma) \neq 0$, all fixed-point classes have the same nonzero index. Reduction maps A_γ onto $\mathrm{coker}(I - \bar{\gamma})$, so every class in that cokernel is visible. Its order equals $|\ker(I - \bar{\gamma})|$. The snake lemma identifies this kernel with $A_\gamma[3]$, as in the identity-sector calculation above. The Wang sequence for the bundle gives

$$0 \longrightarrow A_\gamma \longrightarrow H_1(M_\gamma; \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the first map is induced by inclusion of the fiber. Since A_γ is finite, its image is the torsion subgroup of $H_1(M_\gamma; \mathbb{Z})$. □

Proposition 5.5. *Let* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *with* $\det(I - \gamma) \neq 0$*, and let* $\bar{\gamma}$ *be its reduction modulo* 3*. Then*

$$T_3(\gamma) = |\ker(I - \bar{\gamma} : \mathbb{F}_3^2 \to \mathbb{F}_3^2)| = \begin{cases} 1, & \mathrm{tr}(\gamma) \not\equiv 2 \pmod 3, \\ 3, & \mathrm{tr}(\gamma) \equiv 2 \pmod 3 \text{ and } \bar{\gamma} \neq I, \\ 9, & \bar{\gamma} = I. \end{cases}$$

Proof. The first equality follows from Proposition 5.4. Since $\det \bar{\gamma} = 1$,

$$\det(I - \bar{\gamma}) = 1 - \mathrm{tr}(\bar{\gamma}) + \det \bar{\gamma} = 2 - \mathrm{tr}(\bar{\gamma}).$$

If this determinant is nonzero, the kernel has order 1. Otherwise $I - \bar{\gamma}$ has rank zero or one according as $\bar{\gamma} = I$ or $\bar{\gamma} \neq I$. The kernel then has order 9 or 3, respectively. □

Theorem 5.6 (Classification at fixed Nielsen data)**.** *Let $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic with the same trace. Then*

$$N(f_\gamma^k) = N(f_{\gamma'}^k) \qquad (k \geq 1),$$

where $f_\gamma, f_{\gamma'} : T^2 \to T^2$ are the induced toral automorphisms. Their rank-three summands satisfy

$$S_3(\gamma) - S_3(\gamma') = T_3(\gamma) - T_3(\gamma').$$

The residue of the common trace modulo 3 gives the following cases.

(i) *If the common trace is not congruent to 2 modulo 3, the two dimensions agree.*
(ii) *If the common trace is congruent to 2 modulo 3, the dimensions differ by 6 in absolute value precisely when exactly one of γ, γ' reduces to the identity modulo 3. Otherwise they agree.*

Proof. For matrices in $\mathrm{SL}_2(\mathbb{Z})$, the sequence $a_k = \mathrm{tr}(\gamma^k)$ is determined by

$$a_0 = 2, \qquad a_1 = \mathrm{tr}(\gamma), \qquad a_{k+1} = \mathrm{tr}(\gamma)a_k - a_{k-1}.$$

Thus equal traces give equal traces of all powers. For a hyperbolic toral automorphism, every fixed-point class of every iterate is essential and

$$N(f_\gamma^k) = |\det(I - \gamma^k)| = |2 - \mathrm{tr}(\gamma^k)|.$$

This gives the same periodic Nielsen sequence for both automorphisms [7]. Subtract the formulas in Theorem 5.2 and apply Proposition 5.5. □

Remark 5.7 (Periodic Nielsen data and trace). Put $t = \mathrm{tr}(\gamma)$. For hyperbolic γ, the trace recurrence determines every $N(f_\gamma^k) = |2 - \mathrm{tr}(\gamma^k)|$. Conversely,

$$N(f_\gamma) = |2 - t|, \qquad N(f_\gamma^2) = t^2 - 4$$

determine $|t|$ and its sign.

5.3. **An infinite family of torus bundles.** We construct pairs of torus bundles with the same periodic Nielsen numbers and different rank-three summand dimensions. To compare the covers at each iterate, recall the visibility profile of a map $f : X \to X$ [9, Definition 5.1]

$$V_f(B) = \max\{V_H(f) : H \text{ is a finite observer with } [\pi_1(X) : H] \leq B\}$$

for a budget $B \geq 1$. On the torus the maximum runs over the f_*-invariant subgroups of $\mathbb{Z}^2$ of index at most B. For a positive integer D, put

$$\delta_D(B) = \max\{d : d \mid D,\ d \leq B\}.$$

For a hyperbolic toral automorphism η, the visibility profile is determined by the divisors of $|\det(I - \eta)|$ [9, Theorem 8.2].

Lemma 5.8 (Exact toral visibility profile)**.** *Let $\eta \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and put $D = |\det(I - \eta)|$. Then for every $B \geq 1$,*

$$V_{f_\eta}(B) = \delta_D(B).$$

Proof. Write $C = \mathrm{coker}(I - \eta)$, so $|C| = D$. Twisted conjugacy in the abelian group $\mathbb{Z}^2$ reduces to cosets of $\mathrm{im}(I - \eta)$, so the Reidemeister classes of f_η are canonically indexed by C. Every class is essential with the same local index $\mathrm{sgn}\det(I - \eta)$, so no cancellation occurs when the Reidemeister trace is pushed to any finite observer, and for an η-invariant subgroup $H \leq \mathbb{Z}^2$ of finite index, $V_H(f_\eta)$ is the order of the image of the natural homomorphism

$$C \longrightarrow \mathrm{coker}(I - \bar{\eta} : \mathbb{Z}^2/H \to \mathbb{Z}^2/H).$$

This image is a quotient of C, so its order divides D; it also has order at most $|\mathbb{Z}^2/H| = [\mathbb{Z}^2 : H] \le B$. Hence $V_{f_\eta}(B) \le \delta_D(B)$.

Conversely, let $d \mid D$. Choose a surjection $\pi_d : C \twoheadrightarrow Q_d$ onto an abelian group of order d, which exists because C is finite abelian, and let $H_d \le \mathbb{Z}^2$ be the preimage of $\ker \pi_d$ under the projection $\mathbb{Z}^2 \twoheadrightarrow C$. Since η acts as the identity on C (indeed $\eta x - x \in \mathrm{im}(I - \eta)$ for every x), the subgroup H_d is η-invariant of index d, the induced endomorphism on $\mathbb{Z}^2/H_d \cong Q_d$ is the identity, and the class map $C \to \mathrm{coker}(I - \mathrm{id}) = Q_d$ is the surjection π_d. Hence $V_{H_d}(f_\eta) = d$. Maximizing over divisors $d \le B$ gives the reverse inequality. □

Theorem 5.9. *For every $r \in \mathbb{Z} \setminus \{0\}$, set*

$$\gamma_{1,r} = \begin{pmatrix} 0 & -1 \\ 1 & 9r+2 \end{pmatrix}, \qquad \gamma_{2,r} = \begin{pmatrix} 1 & 3 \\ 3r & 9r+1 \end{pmatrix}.$$

These matrices have the following properties.

(i) *$\gamma_{1,r}$ and $\gamma_{2,r}$ are hyperbolic elements of $\mathrm{SL}_2(\mathbb{Z})$ with common trace $9r+2$;*

(ii) *for every $N \ge 1$,*

$$\dim \mathrm{Sk}_{\mathrm{GL}_N}(M_{\gamma_{1,r}}) = \dim \mathrm{Sk}_{\mathrm{GL}_N}(M_{\gamma_{2,r}});$$

(iii) *their SL_3 quantum-torus summands satisfy*

$$\dim \mathrm{HH}_0^{\gamma_{2,r}}(\mathrm{SkAlg}_{\mathrm{SL}_3}(T^2)) - \dim \mathrm{HH}_0^{\gamma_{1,r}}(\mathrm{SkAlg}_{\mathrm{SL}_3}(T^2)) = 6.$$

(iv) *their periodic Nielsen numbers and Nielsen zeta functions are identical; moreover, for every $k, B \ge 1$,*

$$V_{f^k_{\gamma_{1,r}}}(B) = V_{f^k_{\gamma_{2,r}}}(B),$$

so their finite-cover visibility profiles agree at every iterate.

The associated torus bundles are non-homeomorphic.

Proof. Both matrices have determinant one and trace $9r+2$, whose absolute value exceeds 2 for $r \neq 0$. Proposition 3.5 therefore gives (ii), also a consequence of the partition function in [2].

Modulo 3,

$$\bar{\gamma}_{1,r} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \neq I, \qquad \bar{\gamma}_{2,r} = I.$$

Proposition 5.5 therefore gives

$$V_{3\mathbb{Z}^2}(f_{\gamma_{1,r}}) = T_3(\gamma_{1,r}) = 3, \qquad V_{3\mathbb{Z}^2}(f_{\gamma_{2,r}}) = T_3(\gamma_{2,r}) = 9.$$

Their difference gives (iii) by Theorem 5.6.

The equality of all periodic Nielsen numbers follows from Theorem 5.6, and therefore their Nielsen zeta functions

$$\zeta_N(f; z) = \exp\left(\sum_{k \ge 1} \frac{N(f^k)}{k} z^k\right)$$

coincide. Applying Lemma 5.8 to each hyperbolic iterate η^k gives

$$V_{f^k_\eta}(B) = \delta_{|\det(I - \eta^k)|}(B).$$

The equal Nielsen numbers thus give equal profiles for all k and B.

Finally, the entries of the two difference matrices have greatest common divisors 1 and 3, respectively, and each determinant has absolute value $9|r|$. Their Smith normal forms give

$$\mathrm{coker}(I - \gamma_{1,r}) \cong \mathbb{Z}/(9|r|)\mathbb{Z}, \qquad \mathrm{coker}(I - \gamma_{2,r}) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/(3|r|)\mathbb{Z}.$$

Only the first group is cyclic. These groups are the torsion subgroups of H_1 of the two mapping tori, so the manifolds cannot be homeomorphic. □

Remark 5.10 (Mutant torus bundles). In the shear example of [2, Remark 1.9], the mapping tori of a shear and its negative have different GL_2-skein dimensions. The pairs in Theorem 5.9 have equal GL_N dimensions for all N. The distinction occurs in the SL_3 empty-skein summand.

Example 5.11. For $r = -1$, the matrices are

$$\gamma_{1,-1} = \begin{pmatrix} 0 & -1 \\ 1 & -7 \end{pmatrix}, \qquad \gamma_{2,-1} = \begin{pmatrix} 1 & 3 \\ -3 & -8 \end{pmatrix}.$$

Their common first three Nielsen numbers are

$$N_1 = 9, \qquad N_2 = 45, \qquad N_3 = 324.$$

Theorem 5.2 becomes

$$\dim \mathrm{HH}_0^{\gamma}(\mathrm{SkAlg}_{\mathrm{SL}_3}(T^2)) = 57 + V_{3\mathbb{Z}^2}(f_\gamma).$$

Both cokernels have order 9. The cyclic cokernel has three elements killed by 3, and all nine elements of the other cokernel are killed by 3. The empty-skein dimensions are therefore 60 and 66. Thus the degree-nine congruence cover detects a difference even though the optimized visibility profiles agree at every iterate.

The negatives of these same matrices illustrate Corollary 4.2. Both have trace 7, and each $I + \gamma_{i,-1}$ has cyclic cokernel of order 5. Their empty-skein dimensions therefore agree in every rank. On the double covers the common trace is 47. The contents of $\gamma_{1,-1}^2 - I$ and $\gamma_{2,-1}^2 - I$ are 1 and 3, respectively, so the torsion groups are $\mathbb{Z}/45\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$. Corollary 5.3 gives rank-three dimensions 2256 and 2262 on these covers.

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