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FEIGIN-SEMIKHATOV DUALITY AT THE CRITICAL LEVEL

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Abstract. The Feigin-Semikhatov duality asserts that the Heisenberg cosets of the subregular W-algebra of $\mathfrak{sl}_n$ at level k and the one of the principal W-superalgebra of $\mathfrak{sl}_{n|1}$ at level ℓ coincide when the levels satisfy the Feigin-Frenkel relation $(k+n)(\ell+n-1)=1$. A similar duality holds between the subregular W-algebra of $\mathfrak{so}_{2n+1}$ and the principal W-superalgebra of $\mathfrak{osp}_{2|2n}$.

We study these dualities in the critical/large level limit. We describe the centerless subregular W-algebra at the critical level as an orbifold of the large level limit of the principal W-superalgebra times a lattice VOA. Our construction yields a functor between certain categories of the two involved vertex algebras. We show that in this set-up one in fact gets block-wise equivalences of categories. Studying the principal block of the large level limit of the principal W-superalgebra then gives us the structure of the principal blocks of the subregular W-algebras in the category of weight modules (which is much larger than the more common category of lower bounded modules).

1. Introduction

Let G be a connected complex simple algebraic group whose Lie algebra is $\mathfrak{g}$, we normalize the Killing form in the usual way so that long roots have norm two. To any complex number k one can then construct the affine vertex algebra $V^k(\mathfrak{g})$ of $\mathfrak{g}$ at level k. Let f be a nilpotent element in $\mathfrak{g}$, then one can further get another vertex algebra via quantum Hamiltonian reduction, $W^k(\mathfrak{g},f)$, the W-algebra of $\mathfrak{g}$ associated to f at level k. It only depends on the nilpotent orbit of f. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. It then turns out that the center of $V^k(\mathfrak{g})$ is trivial unless k is at the critical level $-h^\vee$. In this case the center coincides with $W^{-h^\vee}(\mathfrak{g},f_{\rm prin})$, the principal W-algebra at the critical level. Principal W-algebras satisfy Feigin-Frenkel duality [45], that is

$$W^k(\mathfrak{g},f_{\rm prin})\cong W^\ell({}^L\mathfrak{g},f_{\rm prin})$$

for

$$r(k+h^\vee)(\ell+{}^Lh^\vee)=1$$

and ${}^L\mathfrak{g}$ the Langlands dual of $\mathfrak{g}$, that is the Lie algebra whose root system is the coroot system of $\mathfrak{g}$. Here r is the lacing number of $\mathfrak{g}$. At the critical level the dual level goes to infinity and this correspondence becomes the statement that the center $W^{-h^\vee}(\mathfrak{g},f_{\rm prin})$ is isomorphic to $\mathrm{Fun}(\mathrm{Op}_{{}^LG})$, the algebra of functions on the moduli space of ${}^LG$-opers on the formal disc, see Theorem 4.3.1 of [43]. This Theorem is called Langlands correspondence for loop groups and the identification of $V^{-h^\vee}(\mathfrak{g})$-mod with the category of quasi-coherent sheaves on $\mathrm{Op}_{{}^LG}$ is a major aspect of this Langlands correspondence. It should be viewed as a loop group analogue of the Beilinson-Bernstein Theorem saying that taking global sections gives an equivalence between D-modules on the flag variety of G and the block of $\mathfrak{g}$ with trivial central character [19].

The duality of Feigin and Frenkel has an analogue for many nilpotent elements in type A [32, 30] and types B, C, D [33]. However the role of the dual Lie algebra is then played by a Lie superalgebra. The type A case of [32] is as follows. Let $n,m\in\mathbb{Z}_{>0}$ and $\mathfrak{g}=\mathfrak{sl}_{n+m}$. Nilpotent orbits in $\mathfrak{sl}_{n+m}$ are parameterized by partitions of the integer $n+m$ and we set $f_{n,m}$ the nilpotent element whose orbit corresponds to the partition $(n,1,\dots,1)$. The W-algebra $W^k(\mathfrak{sl}_{n+m},f_{n,m})$ has $V^{k+n-1}(\mathfrak{gl}_m)$

as subalgebra. On the dual side we consider $\mathfrak{g} = \mathfrak{sl}_{n+m|m}$ and the nilpotent element $f_{n+m|m}$ that corresponds to the pair of partitions $(n+m|1,\dots,1)$ of the pair of integers $(n+m|m)$. The W-algebra $W^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|m})$ has $V^{-(\ell+n+m-1)}(\mathfrak{gl}_m)$ as subalgebra and the analogue of Feigin-Frenkel duality says that for generic levels

$$\mathrm{Com}\left(V^{k+n-1}(\mathfrak{gl}_m), W^k(\mathfrak{sl}_{n+m}, f_{n,m})\right) \cong \mathrm{Com}\left(V^{-(\ell+n+m-1)}(\mathfrak{gl}_m), W^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|m})\right)$$

where the levels satisfy the Feigin-Frenkel type duality relation

$$(k+n+m)(\ell+n) = 1.$$

Note that the dual Coxeter numbers of $\mathfrak{sl}_{n+m}$ and $\mathfrak{sl}_{n+m|m}$ are $n+m$ and n respectively. This isomorphism is true as one-parameter families of vertex algebras, i.e. it holds for generic levels. This isomorphism had appeared in the physics literature in the case $n = m = 1$ and there it firmates under the name Kazama-Suzuki duality [55]. This duality alone is already significant, e.g. it was key to prove that the category of weight modules of the affine vertex algebra of $\mathfrak{sl}_2$ at any admissible level has a vertex tensor category structure [20]. For general n, but still $m = 1$ this has been a conjecture by Feigin and Semikhatov [51] and it had been first proven in [28].

In type B there is an analogous Feigin-Frenkel type duality. Consider $\mathfrak{g} = \mathfrak{so}_{2m+2n+1}$ and set $f_{n,m}$ the nilpotent element whose orbit corresponds to the partition $(2n+1, 1, \dots, 1)$ of $2m+2n+1$. Then $W^k(\mathfrak{so}_{2m+2n+1}, f_{n,m})$ has $V^{k+2n}(\mathfrak{so}_{2m})$ as subalgebra. On the dual side consider $\mathfrak{g} = \mathfrak{osp}_{2m|2(n+m)}$ and the nilpotent element $f_{n+m|m}$ corresponding to the partition $(1,\dots,1|2m+2n)$ of $(2m|2m+2n)$. Then $W^\ell(\mathfrak{osp}_{2m|2(n+m)}, f_{n+m|m})$ has $V^{-2(\ell+m+n)+1}(\mathfrak{so}_{2m})$ as subalgebra and the Feigin-Frenkel type duality [33] is

$$\mathrm{Com}\left(V^{k+2n}(\mathfrak{so}_{2m}), W^k(\mathfrak{so}_{2(m+n)+1}, f_{n,m})\right) \cong \mathrm{Com}\left(V^{-2(\ell+m+n)+1}(\mathfrak{so}_{2m}), W^\ell(\mathfrak{osp}_{2m|2(n+m)}, f_{n+m|m})\right)$$

where the levels satisfy the Feigin-Frenkel type duality relation

$$2(k+2n+2m-1)(\ell+n+1) = 1.$$

These are thus not isomorphisms of W-algebras, but of their coset subalgebras. Nonetheless there are convolution operations between these W-algebras that provide quite useful functors between representation categories [36]. The convolution kernels are certain shifted versions of the vertex algebra of chiral differential operators. Their existences has been conjectured in [24] and these algebras have been named quantum geometric Langlands kernel VOAs. This conjecture has been proven in few low rank cases in [24, 25] and then in generality for types A, D by Yuto Moriwaki [62]. For $\mathfrak{g} = \mathfrak{gl}_m$ it has the following form

$$\mathfrak{A}^n[\mathfrak{gl}_m, k] = \bigoplus_{\lambda \in P^+} V^k_\lambda \otimes V^\ell_{\lambda^*} \otimes V_{\sqrt{nm}\mathbb{Z}+\frac{n\omega(\lambda)}{\sqrt{nm}}} \otimes \pi$$

$$\mathfrak{A}^n[\mathfrak{so}_{2m}, k] = \bigoplus_{\lambda \in R^+} V^k_\lambda \otimes V^\ell_{\lambda^*}..$$

Here P^+ is the set of dominant weights of $\mathfrak{sl}_m$ and R^+ is the set of dominant weights of $\mathfrak{so}_{2m}$ that are non-spin (they lie in the lattice $Q \cup (Q+\omega_1)$ with Q the root lattice of $\mathfrak{so}_{2m}$ and ω_1 the first fundamental weight). V^k_λ is the Weyl module of $\widehat{\mathfrak{sl}}_m$ of highest-weight λ at level k. λ^* is the weight dual to λ, π is a simple rank one Heisenberg vertex algebra and the map $\omega : P \to \mathbb{Z}/m\mathbb{Z}$ satisfies $\omega(\lambda) = i$ if $\lambda = \omega_i \mod Q$ with ω_i the i-the fundamental weight, the convention that $\omega_0 = 0$ and Q the root lattice of $\mathfrak{sl}_m$. Finally the upper index n tells us that the level ℓ satisfies

$$\frac{1}{k+h^\vee} + \frac{1}{\ell+h^\vee} = n$$

with $h^\vee = m$, resp. $2m-2$ the dual Coxeter number of $\mathfrak{sl}_m$, resp $\mathfrak{so}_{2m}$. In the case of $m=1$, this is simply $\mathfrak{A}^n[\mathfrak{gl}_1, k] \cong V_{\sqrt{n}\mathbb{Z}} \otimes \pi \cong \mathfrak{A}^n[\mathfrak{so}_2, k]$. Set $\tilde{k} = -k-n-2m+1$ and $\tilde{\ell} = \ell + n - m - 1$ in the case of type A and $\tilde{k} = -k - 2n - 4m + 2$ and $\tilde{\ell} = 2(\ell + n - m + 1) + 1$ in the case of type B. Then the convolution of W-algebra statement for generic levels and for our cases is [36]

$$\begin{aligned}
H^{\mathrm{rel}}(\mathfrak{gl}_m, \mathfrak{A}^1[\mathfrak{gl}_m, \tilde{k}] \otimes W^k(\mathfrak{sl}_{n+m}, f_{n,m})) &\cong W^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|m}) \\
H^{\mathrm{rel}}(\mathfrak{gl}_m, \mathfrak{A}^{-1}[\mathfrak{gl}_m, \tilde{\ell}] \otimes W^\ell(\mathfrak{sl}_{n+m|m}, f_{n+m|m})) &\cong W^k(\mathfrak{sl}_{n+m}, f_{n,m}) \\
H^{\mathrm{rel}}(\mathfrak{so}_{2m}, \mathfrak{A}^1[\mathfrak{so}_{2m}, \tilde{k}] \otimes W^k(\mathfrak{so}_{2(n+m)+1}, f_{n,m})) &\cong W^\ell(\mathfrak{osp}_{2m|2(n+m)}, f_{n+m|m}) \\
H^{\mathrm{rel}}(\mathfrak{so}_{2m}, \mathfrak{A}^{-1}[\mathfrak{so}_{2m}, \tilde{\ell}] \otimes W^\ell(\mathfrak{osp}_{2m|2(n+m)}, f_{n+m|m})) &\cong W^k(\mathfrak{so}_{2(n+m)+1}, f_{n,m}).
\end{aligned}$$

Here H^{rel} denotes the relative semi-infinite Lie algebra cohomology of [46] that we will discuss in section 3 in the instance of $\mathfrak{g} = \mathfrak{gl}_1$. For more details on this see [36]. Replacing the W-algebras by their modules in the cohomology leads to a functor between representation categories. In particular in the case of $\mathfrak{g} = \mathfrak{gl}_1$ it yields block-wise equivalences of categories and the functor also respects the associativity of intertwining operators, see [29].

The aim of this work is to generalize the Feigin-Semikhatov duality to the critical level and hence the dual level goes to ∞. We now present the general philosophy. The first idea is that one can take the large level limit of $\mathfrak{A}^n[\mathfrak{g}, k]$ in such a way that the limit admits a large center and that the quotient is a simple vertex algebra, that decomposes as

$$\begin{aligned}
\mathrm{FT}[\mathfrak{gl}_m, n] &= \bigoplus_{\lambda \in P^+} \rho_\lambda \otimes V_{\lambda^*}^{-m+\frac{1}{n}} \otimes V_{\sqrt{nm}\mathbb{Z} + \frac{n\omega(\lambda)}{\sqrt{mn}}} \\
\mathrm{FT}[\mathfrak{so}_{2m}, n] &= \bigoplus_{\lambda \in R^+} \rho_\lambda \otimes V_{\lambda^*}^{-2m+2+\frac{1}{n}}.
\end{aligned}$$

Here ρ_λ is the integrable representation of highest-weight λ of $\mathfrak{sl}_m$ resp $\mathfrak{so}_{2m}$ and FT stands for (affine) Feigin-Tipunin (algebra) as they first suggested principal W-algebra analogues of these algebras [52]. These types of algebras have obtained recent attention [66, 21, 23, 47, 48, 49]. For $m=1$ the affine Feigin Tipunin algebra is just a lattice VOA, $\mathrm{FT}[\mathfrak{gl}_1, n] \cong V_{\sqrt{n}\mathbb{Z}} \cong \mathrm{FT}[\mathfrak{so}_2, n]$. For $n=-1$ it should be closely related to the chiral Hecke algebra [42], however the latter is to our knowledge not known to be simple. Finally for $\mathfrak{sl}_2$ and n positive these algebras have been constructed [38] and then later also in [5]. In particular $\mathrm{FT}[\mathfrak{gl}_2, 1] \cong L_1(\mathfrak{psl}_{2|2})$. For the construction and proof of simplicity of $\mathrm{FT}[\mathfrak{gl}_2, -1]$ the work [11] is probably quite useful.

Let $(\mathfrak{a}, \mathfrak{g}_1, \mathfrak{g}_2, G) \in \{(\mathfrak{gl}_m, \mathfrak{sl}_{n+m}, \mathfrak{sl}_{n+m|m}, GL(m)), (\mathfrak{so}_{2m}, \mathfrak{so}_{2(n+m)+1}, \mathfrak{osp}_{2m|2(n+m)}, SO(2m))\}$ and let $h_1^\vee, h_2^\vee$ be the dual Coxeter numbers of $\mathfrak{g}_1, \mathfrak{g}_2$. We then conjecture that at the critical level/large level limit the dualities take the form

$$(W_\infty(\mathfrak{g}_1, f_{n,m}) \otimes \mathrm{FT}[\mathfrak{a}, 1])^G \cong W_{-h_2^\vee}(\mathfrak{g}_2, f_{n+m|m}), \tag{1.1}$$

$$H^{\mathrm{rel}}\Big(\mathfrak{a},\, \mathrm{FT}[\mathfrak{a}, -1] \otimes W_{-h_2^\vee}(\mathfrak{g}_2, f_{n+m|m})\Big) \cong W_\infty(\mathfrak{g}_1, f_{n,m}), \tag{1.2}$$

$$(W_\infty(\mathfrak{g}_2, f_{n+m|m}) \otimes \mathrm{FT}[\mathfrak{a}, -1])^G \cong W_{-h_1^\vee}(\mathfrak{g}_1, f_{n,m}), \tag{1.3}$$

$$H^{\mathrm{rel}}\Big(\mathfrak{a},\, \mathrm{FT}[\mathfrak{a}, 1] \otimes W_{-h_1^\vee}(\mathfrak{g}_1, f_{n,m})\Big) \cong W_\infty(\mathfrak{g}_2, f_{n+m|m}). \tag{1.4}$$

where $W_{-h_1^\vee}(\mathfrak{g}_1, f_{n,m}), W_{-h_2^\vee}(\mathfrak{g}_2, f_{n+m|m})$ denote the centerless critical level W-algebras. Moreover $W_\infty(\mathfrak{g}_1, f_{n,m}), W_\infty(\mathfrak{g}_2, f_{n+m|m})$ are certain large level limits, see Section 5 for those relevant for us. These algebras are (generalized) free field algebras, see [32, 33] and our Definition 2.1. The representation theory of the free field algebras is non semi-simple and in particular the principal

block can be well studied. It is natural to expect that this duality gives in particular interesting information about the abelian category of the critical level W-algebras.

Remark 1.1. *There are different choices on taking the large level limit, Section 5. If $\mathfrak{g}$ is not a Lie algebra but only a Lie superalgebra then it seems as there are also at least two choices of defining the critical level W-superalgebra. Firstly as the quantum Hamiltonian reduction of the affine vertex superalgebra and secondly by viewing the W-superalgebra as a one-parameter family of vertex superalgebras (with the parameter being the level) obtained from the one-parameter family of affine vertex superalgebras via quantum Hamiltonian reduction and then choosing a specialization to the critical level. These two procedures need not to coincide and for example for the W-superalgebra of $\mathfrak{sl}_{2|1}$ the first procedure leads to a vertex superalgebra without Virasoro field and with a center, while the maybe most natural specialization in the second case leads to a simple vertex operator superalgebra.*

The conjecture should work for either case as long as one takes the appropriate large level limit on the dual side. Drazen Adamovic, Shigenori Nakatsuka and Boris Feigin already related the centers of critical level principal W-superalgebras of $\mathfrak{gl}_{n|1}$ to commutative algebras associated to large level limits of the dual side [9, 13].

1.1. **Feigin-Semikhatov duality at the critical level.** We focus on the subregular case, $m = 1$. In this case the following statements are known in type A: the OPE algebra is known [53]; the critical level subregular W-algebra is realized as a coset inside n^2 $\beta\gamma$-VOAs [26, 27], in particular using Zhu's algebra (which is a Smith algebra) simple lower-bounded modules have been classified in [27].

One of our main results is to show the duality in the critical level /large level limit (1.3) and (1.4) for $m = 1$.

Theorem 1.2. *Let $(\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{sl}_{n+1}, \mathfrak{sl}_{n+1|1})$ or $(\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n})$, and let $h_1^\vee$ be the dual Coxeter number of $\mathfrak{g}_1$ and f_{sub} be subregular nilpotent element of $\mathfrak{g}_1$. The Kazama-Suzuki type isomorphisms hold for the large-critical level duality:*

$$(W_\infty(\mathfrak{g}_2) \otimes V_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \cong W_{-h_1^\vee}(\mathfrak{g}_1, f_{\mathrm{sub}}),$$
$$H^{\mathrm{rel},0}\Big(\mathfrak{gl}_1, W_{-h_1^\vee}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\Big) \cong W_\infty(\mathfrak{g}_2).$$

As a corollary of Theorem 1.2, we prove that the two large level limit W-algebras $W_\infty(\mathfrak{sl}_{2n|1})$ and $W_\infty(\mathfrak{osp}_{2|2n})$ are isomorphic and hence

Corollary 1.3. *There is an isomorphism of vertex algebras*

$$W_{-2n}(\mathfrak{sl}_{2n}, f_{\mathrm{sub}}) \cong W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}).$$

Andrew Linshaw made us aware of the possibility of such isomorphisms and in fact he explained to us that it is even believed that there is a surjection of universal critical level subregular W-algebras, $W^{-2n}(\mathfrak{sl}_{2n}, f_{\mathrm{sub}}) \twoheadrightarrow W^{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$.

1.1.1. *Consequences for representation theory.* The subregular W-algebras contain a Heisenberg vertex subalgebra and admit a Hamiltonian H even at the critical level, where the standard conformal vector is unavailable. In analogy with [6, Definition 3.1], we introduce the following categories of modules with semisimple Heisenberg zero mode admitting a compatible Hamiltonian, subject to the conditions in Definition 6.1.

Let V be a vertex algebra with a Hamiltonian and a Heisenberg element. Then we define

(1) V-wtmod: the category of finitely generated V-modules with finite-dimensional weight spaces, that is an object M is finitely generated and graded by conformal weight and Heisenberg weight, that is

$$M = \bigoplus_{\lambda,\Delta} M_{\lambda,\Delta}$$

and $\dim M_{\lambda,\Delta} < \infty$ for any (λ, Δ) and for each λ there exists h_λ, such that $M_{\lambda,\Delta} = 0$ for $\mathrm{Re}(\Delta) < \mathrm{Re}(h_\lambda)$.

(2) V-wtmod$_{\geq}$: the full subcategory of V-wtmod consisting of objects that are finitely generated lower bounded weight modules, that is M in V-wtmod$_{\geq}$, if it is in V-wtmod and if Hamiltonian weight is lower bounded.

(3) V-wtmod$_{\mathcal{O}}$: the full subcategory of V-wtmod$_{\geq}$: consisting of objects such that for every weight Δ there exists an $N \in \mathbb{Z}$, such that $M_{\lambda,\Delta} = 0$ for $\mathrm{Re}(\lambda) \geq N$.

(4) V-wtmod$_{KL}$: the full subcategory of V-wtmod$_{\mathcal{O}}$ consisting of objects with finite-dimensional Hamiltonian weight spaces.

If V is a vertex operator algebra, that is, it has a conformal vector, then one takes the Hamiltonian as the Virasoro zero-mode and recovers [6, Definition 3.1] for V an affine vertex algebra at non-critical level.

The only case where modules of the category of weight modules are classified is the case of the simple affine vertex algebra of $\mathfrak{sl}_2$, $V_k(\mathfrak{sl}_2)$, at any admissible level [6], building on many earlier works [12, 57, 1]. Subsequently it was then possible to show that this category is a ribbon category [64, 22, 20, 37]. The proofs of the existence of vertex tensor category [20] as well as the computation of fusion rules of [64] relied on the Kazama-Suzuki/Feigin-Semikhatov duality with the $N = 2$ super conformal algebra. Note that the latter is the principal W-superalgebra of $\mathfrak{sl}_{2|1}$.

Our next aim is to provide a framework that upgrades the Feigin-Semikhatov duality at the critical level to an equivalence of certain blocks of representation categories. We formalize this situation as follows.

Let L be a non-degenerate integral lattice of rank r, and L^- be the lattice $\sqrt{-1}L$. For any L-graded vertex algebra V^1, and lattice vertex algebra V_L, the torus $T := \mathrm{Hom}(L, \mathbb{C}^*)$ acts diagonally on $V^1 \otimes V_L$. The pair of vertex algebras $(W_\infty(\mathfrak{g}_2), W_{-h_1^\vee}(\mathfrak{g}_1, f_{\mathrm{sub}}))$ in Theorem 1.2 appears to be a special example for a more general framework between V^1 and the orbifold vertex algebra

$$V^2 \cong (V^1 \otimes V_L)^T \tag{1.5}$$

and the reverse relation

$$H^{\mathrm{rel},0}(\mathfrak{gl}_1^{\oplus r}, V^2 \otimes V_{L^-}) \cong V^1,$$

is just a consequence of Theorem 4.1.

Let $\mathfrak{h}$ be the Heisenberg algebra associated to L and L^* the dual lattice of L. We also studied the abelian categories between the two vertex algebras V^1 and V^2. Let $\mathcal{C}_1$ be the category of L-graded V^1-modules with grading-preserving morphisms. For fixed $\mu \in L^*$, let $\mathcal{C}_2^\mu$ be the category of V^2-modules admitting the Heisenberg decomposition

$$N = \bigoplus_{\lambda \in L} N_\lambda \otimes \pi_{\mu-\lambda},$$

with V^2-module homomorphisms. The isomorphisms mentioned above can be lifted to mutually inverse functors between abelian categories, namely for $\mu \in L^*$, one can define the orbifold functor F_μ and relative cohomology functor G_μ (see Corollary 4.6 for the precise definition) between the

two categories $\mathcal{C}_1$ and $\mathcal{C}_2^\mu$,

$$F_\mu : \mathcal{C}_1 \to \mathcal{C}_2^\mu, \qquad G_\mu : \mathcal{C}_2^\mu \to \mathcal{C}_1.$$

We then show that the functors F_μ and G_μ are exact functors, which induces the equivalence for the categories. The shifting of the Heisenberg weight μ by the functor F_μ coincides with the spectral flow induced from the Li's Δ-operator [60], on the corresponding module by F_0. Thus we obtain the following main theorem.

Theorem 1.4. *For $\mu \in L^*$, let σ_μ be the spectral flow induced by the Δ-operator associated with μ.*

(1) The categories $\mathcal{C}_1$ and $\mathcal{C}_2^\mu$ are equivalent as abelian categories.
(2) For any $M \in \mathcal{C}_1$, there is a V^2-module isomorphism

$$\sigma_\mu^*(F_0(M)) \cong F_\mu(M).$$

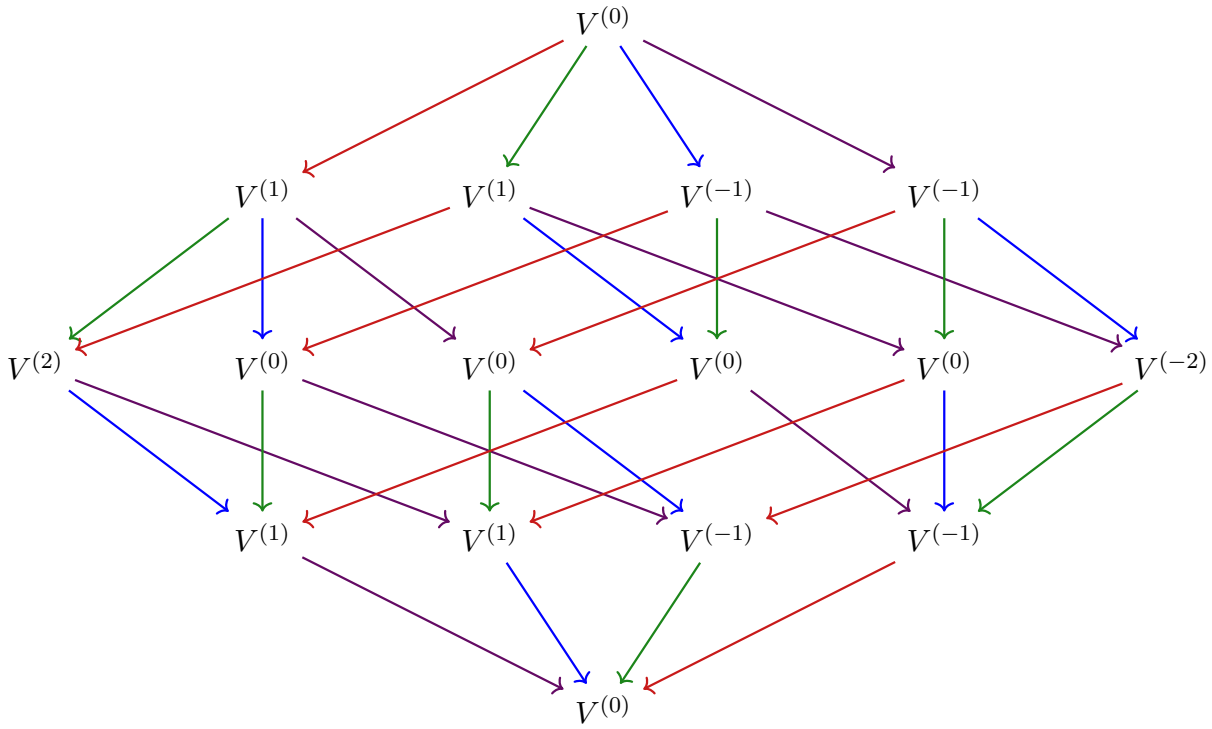
Theorems 1.2 and 1.4 identify the principal block of the centerless critical-level W-algebra $W_{-n-1}(\mathfrak{sl}_{n+1}, f_{\mathrm{sub}})$ (resp. $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$) with the principal block of the centerless large-level limit $F_{n+1} := W_\infty(\mathfrak{sl}_{n+1|1})$ (resp. $F_{2n} := W_\infty(\mathfrak{osp}_{2|2n})$), which is a free field algebra generated by two odd fields $X^\pm(z)$ of conformal weight $\frac{n+2}{2}$ (resp. $\frac{2n+1}{2}$).

Let $\mathfrak{g}_n$ be the mode algebra of F_n, i.e. the $\mathbb{Z}$-graded Lie superalgebra spanned by the odd modes $X_m^\pm$ together with an even center. Let $Z(\mathfrak{g}_n)$ denote the supercenter of the universal enveloping algebra $U(\mathfrak{g}_n)$, which is an exterior algebra generated by $2n$ elements. As explained in Section 6.2, one can construct mutually inverse functors between the abelian category of finitely generated $\mathbb{Z}$-graded F_n-modules and the category of finite-dimensional $\mathbb{Z}$-graded $Z(\mathfrak{g}_n)$-modules, either by taking vacuum spaces or by induction. Corollary 6.17 then shows that these two abelian categories are equivalent, and Remark 6.18 implies that the Loewy structure of any finitely generated F_n-module is completely determined by the Loewy structure of its vacuum space as a $Z(\mathfrak{g}_n)$-module. Combining this with Theorems 1.2 and 1.4, we obtain a complete description of the principal block of the centerless critical-level W-algebra $W_{-h_1^\vee}(\mathfrak{g}_1, f_{\mathrm{sub}})$. For example, the algebra of self-extensions of $W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}})$ (resp. $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$) is isomorphic to the coordinate ring of the variety of $n \times n$ (respectively $2n \times 2n$) matrices whose 2×2 minors vanish, see Remark 6.20. Intersecting this determinantal variety with $\mathfrak{sl}_n \subset \mathrm{Mat}_n(\mathbb{C})$ (respectively, $\mathfrak{sp}_{2n} \subset \mathrm{Mat}_{2n}(\mathbb{C})$) gives the closure of the minimal nilpotent orbit of $\mathfrak{sl}_n$ (respectively $\mathfrak{sp}_{2n}$). [1] In type A, the subregular and minimal nilpotent orbits are exchanged by Lusztig–Spaltenstein duality.

Since in particular the centerless subregular W-algebra $W_{-2}(\mathfrak{sl}_2, f_{\mathrm{sub}})$ is the centerless affine vertex algebra $V_{-2}(\mathfrak{sl}_2)$, we have the Loewy structure of its projective cover $P(V_{-2}(\mathfrak{sl}_2))$.

Theorem 1.5. *Let $V = V_{-2}(\mathfrak{sl}_2)$ be the centerless critical-level affine vertex algebra, regarded as its vacuum module, and let $V^{(m)} = \sigma^{m,*}(V)$, where σ is the spectral-flow transformation fixed in Section 4.2 and $\sigma^{m,*}$ denotes inverse pullback by σ^m. Then the projective cover $P\big(V_{-2}(\mathfrak{sl}_2)\big)$ in $\mathscr{B}_2$, defined in Definition 6.15, has the following Loewy diagram.*

[1] We thank S. Nakatsuka for pointing this out.



Loewy diagram of $P(V_{-2}(\mathfrak{sl}_2))$ in $\mathscr{B}_2$

Moreover, $V^{(\pm 1)} \cong D^{\pm}_{\mp\alpha}$, where $D^{\pm}_{\lambda}$ denote the simple module whose top level is the highest (+) resp. lowest (-) weight module of $\mathfrak{sl}_2$ of highest/lowest weight λ, and α is the simple root. Let H be the standard Hamiltonian on V, assigning weight one to the affine currents, and put $J = h/2$. Equip $V^{(m)}$ with the compatible Hamiltonian $H^{(m)} = H + mJ_0$, where the operators on the right act on the original module V. Then

- $(V^{(0)}, H^{(0)})$ *belongs to* V-wtmod$_{KL}$*;*
- $(V^{(1)}, H^{(1)})$ *belongs to* V-wtmod$_{\mathcal{O}}$ *but not to* V-wtmod$_{KL}$*;*
- $(V^{(-1)}, H^{(-1)})$ *belongs to* V-wtmod$_{\geq}$ *but not to* V-wtmod$_{\mathcal{O}}$*;*
- $(V^{(\pm 2)}, H^{(\pm 2)})$ *belongs to* V-wtmod *but is not lower bounded.*

Previously the structure of representations of centerless affine vertex algebras had been studied in the much smaller category $V_{-h^\vee}(\mathfrak{g})$-wtmod$_{\mathcal{O}}$, see the works of Arakawa and Fiebig [15, 16, 41]. Beyond this category there are studies by Adamovic et al with a focus on realizing simple modules [2, 4, 10].

We conclude with a list of problems and questions that are connected to our studies.

1.1.2. *Open questions and future directions.*

(1) At present we can describe the principal block of the centerless subregular W-algebras and our intention is to obtain similar results for other blocks and to include representations with non-trivial action of the center. The idea is to replace the free field superalgebras by larger vertex superalgebras that contain central extensions of the free field superalgebras as quotients.

(2) The categorical framework to study the critical level Feigin-Semikhatov dualities deserves further attention. This framework should apply to more interesting examples and it is important to study the involved functors on intertwining operators.

(3) The overall goal is to identify $W_{-h^\vee}(\mathfrak{g}, f_{\rm sub})$-wtmod or a larger subcategory of $W^{-h^\vee}(\mathfrak{g}, f_{\rm sub})$-wtmod with a category of modules of a certain quantum supergroup. $V_k(\mathfrak{sl}_2)$-wtmod for k admissible is braided tensor equivalent to a so-called partial semisimplification of a twisted Deligne product of categories of quantum $\mathfrak{sl}_2$ and $\mathfrak{sl}_{2|1}$ at roots of unities related to k [34]. The techniques of [34, 35] hopefully also apply to these cases and one naturally expects quantum supergroups that are somehow related to degenerate versions of $\mathfrak{sl}_{n|1}$ or $\mathfrak{osp}_{2|2n}$. A hint comes from the mirabolic/orthosymplectic Satake equivalence: equivalences between certain categories of equivariant perverse sheaves on affine Grassmannians and categories of degenerate versions of supergroups [7, 8].

(4) Only recently the first results on the center of affine vertex superalgebras at the critical level have been obtained [3, 13, 9] and using our methods it should be possible to say more about principal W-superalgebras of $\mathfrak{sl}_{n|1}$ at the critical level.

(5) For $n = 1$, the affine Feigin-Tipunin algebra $FT[\mathfrak{gl}_m, 1]$ is also expected to be related to global sections of the sheaf $\mathcal{D}^{\rm ch}_{X,-m+1}$ of chiral differential operators on the base affine space $X := SL_m/N$ at the level $-m+1$, where N is the maximal unipotent subgroup of SL_m. More precisely, we expect the following isomorphism

$$FT[\mathfrak{gl}_m, 1] \cong H^{\rm rel,0}(\mathfrak{gl}_1^{\oplus m-1}, \Gamma(X, \mathcal{D}^{\rm ch}_{X,-m+1}) \otimes \mathcal{E}(m)), \tag{1.6}$$

where $\mathcal{E}(m)$ is the rank m bc-system. We note that $V_m := \Gamma(X, \mathcal{D}^{\rm ch}_{X,-m+1})$ is isomorphic to the rank two $\beta\gamma$-system when $m = 2$ [17, 18], and hence the right hand side of (1.6) coincides with $FT[\mathfrak{gl}_2, 1] \cong L_1(\mathfrak{psl}_{2|2})$. Furthermore the associated variety X_{V_m} of V_m is shown to be isomorphic to the affine closure of the cotangent bundle $T^*(X)$ [18], which is a symplectic singularity. In particular V_m is quasi-lisse. It is natural to conjecture that the associated variety of $FT[\mathfrak{gl}_m, 1]$ is isomorphic to the Hamiltonian reduction (equivalently, $(\mathbb{C}^*)^{m-1}$-reduction) of X_{V_m}, and $FT[\mathfrak{gl}_m, 1]$ is also quasi-lisse.

2. Background

2.1. **Vertex-superalgebra conventions.** We work over $\mathbb{C}$ and use the conventions of [54]. A vertex superalgebra is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ equipped with an even vacuum vector $\mathbf{1}$, an even translation operator T, and a parity-preserving state-field map

$$Y(\,\cdot\,, z) : V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]], \qquad Y(a, z) = \sum_{m\in\mathbb{Z}} a_{(m)} z^{-m-1}.$$

These data satisfy the truncation condition $a_{(m)}b = 0$ for $m >> 0$, the vacuum and translation axioms

$$Y(\mathbf{1}, z) = \mathrm{id}_V, \qquad Y(a, z)\mathbf{1} \in a + zV[[z]], \qquad T\mathbf{1} = 0, \qquad [T, Y(a, z)] = \partial_z Y(a, z),$$

and locality: for homogeneous $a, b \in V$, there exists $N \geq 0$ such that

$$(z - w)^N [Y(a, z), Y(b, w)] = 0.$$

Here and below, brackets denote supercommutators, and $p(a) \in \mathbb{Z}_2$ denotes the parity of a homogeneous element a. A vertex algebra is the purely even case.

All modules are untwisted weak modules unless further conditions are stated; in particular, their vertex operators satisfy the field truncation condition. Morphisms of supermodules preserve parity.

We will use the iterate identity

$$\begin{aligned}(a_{(n)}b)_{(m)} = &\sum_{i\geq 0}(-1)^i\binom{n}{i}a_{(n-i)}b_{(m+i)}\\ &-(-1)^{p(a)p(b)+n}\sum_{i\geq 0}(-1)^i\binom{n}{i}b_{(m+n-i)}a_{(i)},\end{aligned} \tag{2.1}$$

valid on V and its modules, with the sums interpreted after application to a vector.

A Hamiltonian is an even diagonalizable operator H on V satisfying

$$[H, Y(a,z)] = z\partial_z Y(a,z) + Y(Ha,z). \tag{2.2}$$

If $Ha = \Delta a$, then

$$[H, a_{(m)}] = (\Delta - m - 1)a_{(m)}.$$

A *compatible module Hamiltonian* is an even operator H_M satisfying

$$[H_M, Y_M(a,z)] = z\partial_z Y_M(a,z) + Y_M(Ha,z).$$

When used below, H_M is required to be locally finite, so that M is the direct sum of its generalized eigenspaces. In general, a compatible module Hamiltonian is additional structure. When $H = L_0$ comes from a specified conformal vector, the module action supplies the corresponding operator L_0^M, although its local finiteness is an additional condition. Unless stated otherwise, our category equivalences and extension groups are taken without a chosen module Hamiltonian.

A strong generating set spans V by normally ordered monomials in its derivatives, where

$$:ab: = a_{(-1)}b.$$

A free strong generating set gives a PBW basis after ordering the pairs consisting of a generator and a derivative order. Identical odd derivative generators cannot be repeated, whereas different derivatives of the same odd generator may occur together. Polynomial algebras on supervariables are understood to be supercommutative.

2.2. **Free fields and lattices.**

Definition 2.1. *A free-field vertex superalgebra is generated by fields with scalar singular OPE coefficients. For generators X^i homogeneous under a Hamiltonian, we write*

$$X^i(z)X^j(w) \sim a_{ij}(z-w)^{-\operatorname{wt}X^i - \operatorname{wt}X^j}, \qquad a_{ij}\in\mathbb{C},$$

where $a_{ij} = 0$ unless $\operatorname{wt}X^i + \operatorname{wt}X^j$ is a positive integer, and the coefficients satisfy the symmetry imposed by parity and skew-symmetry. We use the usual vacuum free-field algebras with their PBW bases.

The generalized orthogonal and symplectic free-field algebras used below are those of [32]. In particular, F_N has two odd generators $X^\pm$ of charges ± 1 and weight $(N+1)/2$, with

$$X^+(z)X^-(w) \sim (z-w)^{-N-1}.$$

For odd N the scalar pairing is symplectic, and for even N it is orthogonal. The nondegenerate oscillator pairing gives simplicity of F_N.

A rank-one Heisenberg field a of level $\kappa \neq 0$ satisfies

$$a(z)a(w) \sim \kappa(z-w)^{-2}.$$

We denote its Fock module of zero-mode eigenvalue λ by π_λ, with chosen highest-weight vector e^λ satisfying

$$a_{(0)}e^\lambda = \lambda e^\lambda, \qquad a_{(n)}e^\lambda = 0 \quad (n > 0).$$

An even current a satisfying $a(z)a(w) \sim 0$ is called an *isotropic current*.

The rank-one charged free-fermion system is generated by odd fields b, c with

$$b(z)c(w) \sim (z-w)^{-1}.$$

For BRST ghosts we use the mode expansions

$$b(z) = \sum_{m\in\mathbb{Z}} b_m z^{-m-1}, \qquad c(z) = \sum_{m\in\mathbb{Z}} c_m z^{-m},$$

and Hamiltonian weights $1, 0$, respectively. The same underlying vertex superalgebra also admits weights $1/2, 1/2$; we use this grading for the copy identified with $V_\mathbb{Z}$.

Let L be a nondegenerate integral lattice and put $\mathfrak{h} = L \otimes_\mathbb{Z} \mathbb{C}$. We write

$$V_L = \bigoplus_{\lambda\in L} \pi_\lambda$$

for its lattice vertex superalgebra. Here the vector label λ specifies the Fock momentum:

$$h_{(0)}e^\lambda = (h,\lambda)e^\lambda, \qquad h_{(n)}e^\lambda = 0 \quad (h \in \mathfrak{h},\ n > 0).$$

The parity of e^λ is $(\lambda,\lambda) \bmod 2$, and positive definiteness of L is not assumed. Cocycles for L and the opposite lattice $L^- = iL$ are chosen so that, after a coboundary adjustment, the restricted cocycle on the even isotropic diagonal subgroup of $L \oplus L^-$ is trivial. This normalization is used in the BRST reconstruction maps.

2.3. **Affine vertex algebras.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra equipped with an invariant bilinear form $(\cdot,\cdot)$. The associated affine Lie algebra is

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K,$$

with commutation relations for any $\xi, \eta \in \mathfrak{g}$

$$[\xi \otimes t^m, \eta \otimes t^n] = [\xi,\eta] \otimes t^{m+n} + m\,\delta_{m,-n}(\xi,\eta)K.$$

Fix a level $k \in \mathbb{C}$. The universal affine vertex algebra $V^k(\mathfrak{g})$ is defined as the vacuum module of $\widehat{\mathfrak{g}}$ at level k,

$$V^k(\mathfrak{g}) = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g}[t]\oplus\mathbb{C}K}\mathbb{C},$$

where $\mathfrak{g}[t]$ acts trivially on $\mathbb{C}$ and K acts by scalar k. It is strongly generated by fields $J^\xi(z)$ for $\xi \in \mathfrak{g}$ with OPEs

$$J^\xi(z)J^\eta(w) \sim \frac{k(\xi,\eta)}{(z-w)^2} + \frac{J^{[\xi,\eta]}(w)}{z-w}.$$

2.4. **Affine W-algebras and their Hamiltonians.** Let $f \in \mathfrak{g}_{\bar{0}}$ be nilpotent and choose a good grading $\mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ with $f \in \mathfrak{g}_{-1}$. Write x for its grading element, so $[x,u] = ju$ for $u \in \mathfrak{g}_j$, and put $\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j$. For a homogeneous basis $u_\alpha \in \mathfrak{g}_{j_\alpha}$, the charged ghosts b^α, c_α have parity $p(u_\alpha)+1$. Thus in the superalgebra case the charged ghost system need not consist entirely of odd fields. The neutral system on $\mathfrak{g}_{1/2}$ uses the form $(f,[u,v])$ and fields Φ_u of parity $p(u)$.

The quantum Drinfeld–Sokolov complex is

$$C_{\mathrm{DS}} = V^k(\mathfrak{g}) \otimes \mathcal{E}_{\mathrm{ch}}(\mathfrak{g}_+) \otimes \mathcal{E}_{\mathrm{ne}}(\mathfrak{g}_{1/2}).$$

We use the charged-ghost pairings and signs of [58, 59]. With these conventions its differential is the zero mode of the odd BRST field

$$d = \sum_{\alpha}(-1)^{p(u_\alpha)} : J^{u_\alpha} c_\alpha : -\frac{1}{2}\sum_{\alpha,\beta}(-1)^{p(u_\alpha)(1+p(u_\beta))} : b^{[u_\alpha,u_\beta]} c_\alpha c_\beta : \\ + \sum_{\alpha}(f,u_\alpha)c_\alpha + \sum_{j_\alpha = 1/2} : c_\alpha \Phi_{u_\alpha} : .$$

Here b^u depends linearly on u. The differential satisfies $d_0^2 = 0$, and

$$W^k(\mathfrak{g}, f) = H^0(C_{\mathrm{DS}}, d_0).$$

The PBW theorem of [59] gives free strong generators indexed by a homogeneous basis of $\mathfrak{g}^f$, with the corresponding supercommutative associated graded algebra.

We define the grading uniformly in k. Let D_{aff} be the standard grading operator on $V^k(\mathfrak{g})$, characterized by

$$D_{\mathrm{aff}}\mathbf{1} = 0, \qquad [D_{\mathrm{aff}}, J^u_{(n)}] = -nJ^u_{(n)}.$$

On the affine factor, use $D_{\mathrm{aff}} - x_{(0)}$; thus J^u has weight $1-j$ for $u \in \mathfrak{g}_j$. For each charged ghost pair associated with $u \in \mathfrak{g}_j$, $j > 0$, assign b^u and c_{u^*} weights $1-j$ and j, respectively. Each neutral ghost Φ_u, with $u \in \mathfrak{g}_{1/2}$, has weight $1/2$. These assignments respect the OPEs and define a Hamiltonian H_{DS} on the complex. Each term of d has weight one: the first and cubic terms have total weight one, $(f, u_\alpha) \neq 0$ implies $j_\alpha = 1$, and the neutral term has weight $1/2 + 1/2$. Therefore

$$[H_{\mathrm{DS}}, d_0] = 0.$$

The induced operator H on $W^k(\mathfrak{g}, f)$ is consequently defined for all k, including $k = -h^\vee$. A generator corresponding to $u \in \mathfrak{g}^f \cap \mathfrak{g}_j$ has weight $1-j$. At noncritical levels this is the grading of the usual DS conformal vector. At the critical level the grading remains meaningful even though that conformal vector is unavailable. The augmentation ideal of the critical center is homogeneous for H, so H descends to the centerless quotient. All weights of critical-level fields below refer to this Hamiltonian.

3. Relative semi-infinite cohomology

Let $K(z) = \sum_{m\in\mathbb{Z}} K_m z^{-m-1}$ be an even isotropic current in a vertex superalgebra V, and let $\mathcal{E}$ be the BRST ghost system of Section 2. For a V-module M put

$$C(M) = M \otimes \mathcal{E}, \qquad Q = \mathrm{Res}_z : c(z)K(z) := \sum_{m\in\mathbb{Z}} c_{-m}K_m.$$

The OPE $K(z)K(w) \sim 0$ gives $Q^2 = 0$. Give c cohomological degree 1 and b degree -1. Since $[Q, b_0] = K_0$, the relative subcomplex

$$C^{\mathrm{rel}}(M) = \ker K_0 \cap \ker b_0 \subset C(M)$$

is stable under Q. Its cohomology is denoted by $H^{\mathrm{rel},p}(M)$. The zero-mode differential is a vertex-superalgebra derivation on $C(V)$; consequently $H^{\mathrm{rel},0}(V)$ is a vertex superalgebra and the cohomology of M is naturally a module for it. For a rank-r isotropic system we use r ghost pairs and the sum of the commuting rank-one differentials.

Lemma 3.1 (Fock-module cohomology)**.** *Let J^+, J^- be commuting Heisenberg fields of levels $\kappa, -\kappa$, where $\kappa \neq 0$, and set $K = J^+ + J^-$. Label their Fock modules by their zero-mode eigenvalues. Write e^λ and e^μ for chosen highest-weight vectors in π^+_λ and π^-_μ, respectively. Then*

$$H^{\mathrm{rel},p}(\pi^+_\lambda \otimes \pi^-_\mu) = \delta_{p,0}\delta_{\lambda+\mu,0}\,\mathbb{C}[e^\lambda \otimes e^\mu].$$

The corresponding statement holds for opposite nondegenerate Heisenberg forms of any finite rank.

Proof. If $\lambda + \mu \neq 0$, the relative subcomplex is zero. Otherwise put $A = (J^+ - J^-)/(2\kappa)$, so $K(z)A(w) \sim (z-w)^{-2}$. The relative complex is the tensor product of the highest-weight line with

$$\mathbb{C}[K_{-n}, A_{-n} \mid n \geq 1] \otimes \Lambda(b_{-n}, c_{-n} \mid n \geq 1).$$

The relative ghost condition excludes c_0. On the displayed generators,

$$Qb_{-n} = K_{-n}, \qquad QA_{-n} = nc_{-n}, \qquad QK_{-n} = Qc_{-n} = 0.$$

Define an odd operator s by

$$sK_{-n} = b_{-n}, \qquad sc_{-n} = A_{-n}/n, \qquad sA_{-n} = sb_{-n} = 0,$$

extending as a superderivation. Then $Qs + sQ$ is the total polynomial/exterior-degree operator. Every vector is a finite sum of monomials, so division by that degree contracts its positive-degree part. Only the highest-weight line survives, in cohomological degree zero. The higher-rank case is the tensor product of these contractions after choosing dual bases. This is the standard Fock calculation underlying [46, 29]. □

Proposition 3.2. *Suppose V contains the commuting Heisenberg tensor product $\pi^+ \otimes \pi^-$ and has a decomposition*

$$V = \bigoplus_{\lambda,\mu} \pi^+_\lambda \otimes \pi^-_\mu \otimes M_{\lambda,\mu}$$

into ordinary Fock modules with multiplicity spaces $M_{\lambda,\mu}$. Write

$$V^{K[t]} = \{v \in V : K_n v = 0 \text{ for all } n \geq 0\}.$$

The map

$$V^{K[t]} \longrightarrow H^{\mathrm{rel},0}(V), \qquad v \longmapsto [v \otimes \mathbf{1}], \tag{3.1}$$

is a surjective vertex-superalgebra homomorphism whose kernel is the ideal generated by K. Moreover $H^{\mathrm{rel},p}(V) = 0$ for $p \neq 0$.

Proof. The differential acts trivially on every multiplicity space. Lemma 3.1 therefore gives

$$H^{\mathrm{rel},p}(V) = \delta_{p,0} \bigoplus_{\lambda} \mathbb{C}[e^\lambda \otimes e^{-\lambda}] \otimes M_{\lambda,-\lambda}.$$

These highest-weight representatives are annihilated by all K_n, $n \geq 0$, proving surjectivity. To identify the kernel, use the variables K_{-n}, A_{-n} in the proof of the lemma. On a matching Fock summand, K_n for $n > 0$ acts as $n\partial/\partial A_{-n}$. Hence

$$V^{K[t]} = \bigoplus_{\lambda} \mathbb{C}[K_{-1}, K_{-2}, \ldots](e^\lambda \otimes e^{-\lambda}) \otimes M_{\lambda,-\lambda}.$$

The map (3.1) is evaluation at $K_{-n} = 0$ on each summand. Its kernel is the span of the positive-degree K-monomials. Since

$$K_{-n}v = \frac{(T^{n-1}K)_{(-1)}v}{(n-1)!},$$

this span is precisely the vertex ideal generated by K in the commutant. The multiplication and translation on the quotient agree with cohomology because Q is a vertex-algebra derivation. This proves all assertions. □

4. Lattice reconstruction and equivalence of categories

4.1. **Equivalence of L-graded categories.** Let L be a nondegenerate integral lattice, $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, and $T = \mathrm{Hom}(L, \mathbb{C}^*)$. Let $V^1 = \bigoplus_{\lambda \in L} V^1_\lambda$ be an L-graded vertex superalgebra. The torus acts on V^1_λ and $\pi_\lambda \subset V_L$ by the character λ, and

$$V^2 = (V^1 \otimes V_L)^T = \bigoplus_{\lambda \in L} V^1_\lambda \otimes \pi_{-\lambda}.$$

In tensoring with V_{L^-}, the BRST current is the sum of the two oppositely paired Heisenberg currents. Opposite momenta are identified so that $e^{-\lambda} \otimes e^{i\lambda}$ has zero total charge. Use the compatible cocycles fixed in Section 2. Ghost factors are implicit in $H^{\mathrm{rel},p}$.

Theorem 4.1. *There are isomorphisms*

$$H^{\mathrm{rel},p}(V^2 \otimes V_{L^-}) \cong \delta_{p,0} V^1,$$

where the isomorphism in degree zero is an isomorphism of vertex superalgebras.

Proof. The tensor product is the direct sum of $V^1_\lambda \otimes \pi_{-\lambda} \otimes \pi_{i\nu}$, $\lambda, \nu \in L$. The relative charge condition selects $\nu = \lambda$, and Lemma 3.1 contracts the nonzero modes and the ghosts. Thus the linear map

$$\Phi(v_\lambda) = [v_\lambda \otimes e^{-\lambda} \otimes e^{i\lambda}]$$

is bijective, and higher cohomology vanishes.

We check the product, rather than only its underlying vector space. For $v_\lambda \in V^1_\lambda$ and $w_\nu \in V^1_\nu$, the two lattice vertex operators contribute powers $z^{(\lambda,\nu)}$ and $z^{-(\lambda,\nu)}$, which cancel. The tensor-product super sign and the lattice cocycles cancel with our isotropic-diagonal normalization. The remaining creation factor is an exponential of negative modes of the diagonal isotropic currents. Each such mode is Q-exact by $Qb_{-n} = K_{-n}$. Therefore, coefficientwise in the formal variable,

$$Y(\Phi(v_\lambda), z)\Phi(w_\nu) = [Y(v_\lambda, z) w_\nu \otimes e^{-\lambda-\nu} \otimes e^{i(\lambda+\nu)}] = \Phi(Y(v_\lambda, z) w_\nu).$$

The lattice contributions to translation are likewise diagonal isotropic creation modes and hence exact. The vacuum is preserved. This proves the vertex-superalgebra assertion. □

Theorem 4.2. *Suppose V contains the Heisenberg algebra of L and decomposes as $V = \bigoplus_{\lambda \in L} C_\lambda \otimes \pi_\lambda$. Then*

$$\left(H^{\mathrm{rel},p}(V \otimes V_{L^-}) \otimes V_L\right)^T \cong \delta_{p,0} V.$$

The torus acts on the cohomology class from $C_\lambda \otimes \pi_\lambda \otimes \pi_{-i\lambda}$ with character $-\lambda$.

Proof. For $v \in C_\lambda$ and $s \in \pi_\lambda$, the vector-space isomorphism is

$$v \otimes s \longmapsto [v \otimes e^{\lambda} \otimes e^{-i\lambda}] \otimes s.$$

For a Heisenberg-highest vector a of momentum α, the identity

$$[h_{(n)}, Y(a, z)] = (h, \alpha) z^n Y(a, z)$$

determines its Heisenberg creation and annihilation factors. Indeed, removing these factors gives

$$\exp\left(-\sum_{n>0}\frac{\alpha_{(-n)}}{n}z^n\right)Y(a,z)\exp\left(\sum_{n>0}\frac{\alpha_{(n)}}{n}z^{-n}\right),$$

which commutes with all nonzero Heisenberg modes and hence acts on the Fock multiplicity spaces. Tensoring with the opposite lattice cancels the Heisenberg factors in relative cohomology, as in Theorem 4.1; tensoring back with V_L restores them, including the zero-mode powers and the fixed cocycles. Thus the displayed map intertwines the fields of all Heisenberg-highest vectors. Their Heisenberg descendants span V, so it intertwines every field. The vacuum and translation are preserved as in Theorem 4.1, and higher cohomology vanishes summandwise. □

We now lift the reconstruction isomorphisms to equivalences of module categories.

Definition 4.3. *Let $\mathcal{C}_1$ be the category of L-graded V^1-modules*

$$M=\bigoplus_{\lambda\in L}M_\lambda$$

such that

$$a_{(n)}M_\lambda\subseteq M_{\lambda+\alpha},\qquad a\in V^1_\alpha,\quad n\in\mathbb{Z}.$$

Morphisms are grading-preserving V^1-module homomorphisms.

Definition 4.4. *For a fixed $\mu\in L^*$, let $\mathcal{C}_2^\mu$ be the category of V^2-modules whose restriction to the Heisenberg vertex algebra admits a decomposition*

$$N=\bigoplus_{\lambda\in L}N_\lambda\otimes\pi_{\mu-\lambda},$$

where N_λ denotes the corresponding Fock multiplicity space. Morphisms are V^2-module homomorphisms.

Since the Heisenberg Fock modules with distinct momenta are nonisomorphic, every morphism in $\mathcal{C}_2^\mu$ preserves this decomposition and acts on each summand as a map of multiplicity spaces tensored with the identity. Moreover,

$$\left(V^1_\alpha\otimes\pi_{-\alpha}\right)_{(n)}\left(N_\lambda\otimes\pi_{\mu-\lambda}\right)\subseteq N_{\lambda+\alpha}\otimes\pi_{\mu-\lambda-\alpha}.$$

Thus the fixed representative μ determines a compatible L-grading on N.

Both categories are abelian, i.e. kernels and cokernels are computed on the graded components in $\mathcal{C}_1$ and on the Fock multiplicity spaces in $\mathcal{C}_2^\mu$. All exactness and projectivity statements below refer to these categories. Their definitions require no choice of a compatible module Hamiltonian.

For $\mu\in L^*$, the lattice module

$$V_{\mu+L}=\bigoplus_{\beta\in L}\pi_{\mu+\beta}$$

is an untwisted V_L-module. Give the summand $\pi_{\mu+\beta}$ torus character β,

$$t\cdot u=t(\beta)u,\qquad u\in\pi_{\mu+\beta}.$$

We use compatible lattice cocycles and highest-weight vector normalizations as in Theorem 4.1.

Proposition 4.5. *Let M be an L-graded V^1-module and let $\mu\in L^*$. Then $M\otimes V_{\mu+L}$ is a T-equivariant $V^1\otimes V_L$-module. Consequently,*

$$(M\otimes V_{\mu+L})^T$$

carries a natural V^2-module structure.

Proof. The diagonal torus action is

$$t \cdot (m_\lambda \otimes u_{\mu+\beta}) = t(\lambda+\beta)\, m_\lambda \otimes u_{\mu+\beta},$$

where $m_\lambda \in M_\lambda$ and $u_{\mu+\beta} \in \pi_{\mu+\beta}$. The module actions respect the charge gradings. Therefore

$$t \cdot Y(a,z)w = Y(t \cdot a, z)(t \cdot w)$$

for $a \in V^1 \otimes V_L$ and $w \in M \otimes V_{\mu+L}$. If both a and w are T-invariant, so is every coefficient of $Y(a,z)w$. Hence the invariant subspace is stable under $V^2 = (V^1 \otimes V_L)^T$. □

Corollary 4.6. *For fixed $\mu \in L^*$, there is a functor*

$$F_\mu : \mathcal{C}_1 \to \mathcal{C}_2^\mu, \quad M \mapsto (M \otimes V_{\mu+L})^T,$$

and one has a vector space isomorphism

$$F_\mu(M) \cong \bigoplus_{\lambda \in L} M_\lambda \otimes \pi_{-\lambda+\mu}.$$

Similarly there is a functor

$$G_\mu : \mathcal{C}_2^\mu \longrightarrow \mathcal{C}_1, \qquad N \longmapsto H^{\mathrm{rel},0}(N \otimes V_{-i\mu+L^-}).$$

Proof. Proposition 4.5 gives the V^2-module structure on $F_\mu(M)$. A summand $M_\lambda \otimes \pi_{\mu+\beta}$ is invariant precisely when $\lambda+\beta = 0$. This proves the displayed Fock decomposition and shows that $F_\mu(M) \in \mathcal{C}_2^\mu$.

For $N \in \mathcal{C}_2^\mu$, the tensor product $N \otimes V_{-i\mu+L^-}$ is a module for $V^2 \otimes V_{L^-}$. Its relative cohomology is therefore a module for

$$H^{\mathrm{rel},0}(V^2 \otimes V_{L^-}) \cong V^1$$

by Theorem 4.1. Lemma 3.1 gives

$$H^{\mathrm{rel},p}\big(N \otimes V_{-i\mu+L^-}\big) \cong \begin{cases} \bigoplus_{\lambda \in L} N_\lambda, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

More explicitly, the summand N_λ is represented by

$$\left[u_\lambda \otimes e^{\mu-\lambda} \otimes e^{i\lambda - i\mu}\right], \qquad u_\lambda \in N_\lambda.$$

Assign this class degree λ. The action of V^1_α shifts this degree by α, so $G_\mu(N) \in \mathcal{C}_1$.

The constructions commute with module homomorphisms. Finally, F_μ is exact componentwise, and the displayed description of G_μ shows that it is exact on Fock multiplicity spaces. □

By Theorem 4.1, when $\mu = 0$, we have isomorphisms of graded vertex algebras $G_\mu \circ F_\mu(V^1) \cong V^1$ and $F_\mu \circ G_\mu(V^2) \cong V^2$. Similarly, we have the following isomorphisms for L-graded vertex algebra modules.

Theorem 4.7. *For any $\mu \in L^*$, $M \in \mathcal{C}_1$, there is an isomorphism*

$$M \;\cong\; G_\mu \circ F_\mu(M) = H^{\mathrm{rel},0}\big((M \otimes V_{\mu+L})^T \otimes V_{-i\mu+L^-}\big) \tag{4.1}$$

of L-graded V^1-modules. Similarly, for any $N \in \mathcal{C}_2^\mu$, there is an isomorphism

$$N \;\cong\; F_\mu \circ G_\mu(N) = \big(H^{\mathrm{rel},0}(N \otimes V_{-i\mu+L^-}) \otimes V_{\mu+L}\big)^T \tag{4.2}$$

of L-graded V^2-modules.

Proof. We define a linear map

$$\Phi_M :\ M \longrightarrow H^{\mathrm{rel},0}\big((M \otimes V_{\mu+L})^T \otimes V_{-i\mu+L^-}\big),$$
$$m_\lambda \longmapsto \big[\, m_\lambda \otimes e^{-\lambda+\mu} \otimes e^{i\lambda-i\mu} \,\big].$$

which is an isomorphism of L-graded V^1-modules similar to the proof of Theorem 4.1.

By definition, $N = \oplus_{\lambda\in L} N_\lambda \otimes \pi_{-\lambda+\mu}$, where N_λ is defined to be the multiplicity of $\pi_{-\lambda+\mu}$ in N. Then the isomorphism in (4.2) can be proved similarly by constructing the map

$$\Psi_N :\ N \longrightarrow \big(H^{\mathrm{rel},0}(N \otimes V_{-i\mu+L^-}) \otimes V_{\mu+L}\big)^T,$$
$$v_\lambda \otimes s_{-\lambda+\mu} \longmapsto \big[\, v_\lambda \otimes e^{-\lambda+\mu} \otimes e^{i\lambda-i\mu} \,\big] \otimes s_{-\lambda+\mu}.$$

This map is bijective on each Fock summand. The field-factorization argument in the proof of Theorem 4.2, applied to module fields, shows that it intertwines the V^2-action. Since $\mathcal{C}_1$ and $\mathcal{C}_2^\mu$ are abelian, the equivalences F_μ and G_μ are exact. □

Theorem 4.8. *For $\mu \in L^*$, the functors F_μ and G_μ form mutually inverse functors, namely there exists natural exact equivalences of functors*

$$G_\mu \circ F_\mu \ \cong\ \mathrm{Id}_{\mathcal{C}_1}, \qquad F_\mu \circ G_\mu \ \cong\ \mathrm{Id}_{\mathcal{C}_2^\mu}. \tag{4.3}$$

In particular, we have an equivalence of $\mathcal{C}_1$ and $\mathcal{C}_2^\mu$ as abelian categories.

Proof. By Theorem 4.7, it suffices to show that the functors constructed therein are natural. This follows by tracing the elements in the commutative diagrams below,

$$\begin{array}{ccc} M & \xrightarrow{\Phi_M} & G_\mu \circ F_\mu(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G_\mu \circ F_\mu(f)} \\ M' & \xrightarrow[\Phi_{M'}]{} & G_\mu \circ F_\mu(M') \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\Psi_N} & F_\mu \circ G_\mu(N) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F_\mu \circ G_\mu(g)} \\ N' & \xrightarrow[\Psi_{N'}]{} & F_\mu \circ G_\mu(N'). \end{array}$$

The commutativity of these diagrams shows that the isomorphisms Φ_M and Ψ_N are natural in M and N, respectively, hence establishing the claimed equivalences of functors. □

4.2. **Charge shifts and spectral flow.** We fix the convention for spectral flow, since its sign depends on whether a twist is defined by pullback or inverse pullback. For $\mu \in L^* \subset L \otimes \mathbb{C}$, let σ_μ be the mode transformation induced by Li's operator [60]

$$\Delta(-\mu, z) = z^{-\mu_{(0)}} \exp\left(\sum_{j\geq 1} \frac{\mu_{(j)}}{j}(-z)^{-j}\right).$$

We write $\sigma_\mu^* N$ for inverse pullback, so its fields are $Y_N(\Delta(\mu, z)v, z)$. For a Heisenberg vector h and a Heisenberg highest vector v_α of momentum α, this gives

$$Y_{\sigma_\mu^* N}(h, z) = Y_N(h, z) + (h, \mu)z^{-1}, \qquad Y_{\sigma_\mu^* N}(v_\alpha, z) = z^{(\mu,\alpha)} Y_N(v_\alpha, z).$$

The integral pairing with L ensures that these are untwisted module fields.

Theorem 4.9. *For $M \in \mathcal{C}_1$ and $\mu \in L^*$, there is a natural isomorphism*

$$\sigma_\mu^* F_0(M) \cong F_\mu(M).$$

Proof. As vector spaces, identify $\pi_{-\lambda}$ with $\pi_{\mu-\lambda}$ by preserving the nonzero-mode creation operators and shifting the highest momentum. Fix the lattice-module cocycle on $\mu + L$ by this transport; a different coherent choice changes the map by a nonzero scalar on each momentum summand. The zero mode of h changes by (h, μ). For $a_\alpha \otimes e^{-\alpha} \in V^2$, the zero-mode power in the lattice vertex operator changes by $z^{(-\alpha,\mu)}$; the creation and annihilation exponentials remain the same. These are precisely the two changes prescribed by $\Delta(\mu, z)$. Heisenberg descendants of these vectors span V^2, so the resulting map intertwines all fields. It is natural because it acts identically on M and by the fixed transport on the Fock factor. □

In the applications to critical-level subregular W-algebras in Section 6.1, we take $L = \mathbb{Z}\lambda$ with $(\lambda, \lambda) = -1$ and identify the Heisenberg field with $J = -\lambda$. For the charged generators $G^{\pm}$, normalized by

$$J(z)G^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{z - w},$$

write $\sigma = \sigma_{-\lambda}$. This mode transformation is induced by $\Delta(\lambda, z) = \Delta(-J, z)$ and satisfies

$$\sigma(G^+_{(j)}) = G^+_{(j-1)}, \qquad \sigma(G^-_{(j)}) = G^-_{(j+1)}, \qquad \sigma(J_{(j)}) = J_{(j)} + \delta_{j,0}.$$

We use this integer spectral-flow convention throughout the paper, in particular in Theorem 1.5.

4.3. **Equivalence of categories with compatible Hamiltonian.** Retain the notation of Section 4.1. Fix an even Hamiltonian H_1 on V^1 preserving its L-grading, and equip

$$V^2 = (V^1 \otimes V_L)^T$$

with the Hamiltonian

$$H_2 = (H_1 \otimes 1 + 1 \otimes L_0^L)\big|_{V^2}, \tag{4.4}$$

where L_0^L is the standard lattice Hamiltonian. In particular, the Heisenberg generators of V^2 have H_2-weight 1. We refine the equivalence of Theorem 4.8 by including a chosen compatible Hamiltonian in the module data.

Definition 4.10. *Let $\mathcal{C}_{1,H}$ be the category whose objects are pairs (M, H_M), where $M \in \mathcal{C}_1$ and H_M is an even locally finite operator preserving the L-grading and satisfying*

$$[H_M, a_{(n)}] = (H_1 a)_{(n)} - (n+1)a_{(n)}, \qquad a \in V^1, \quad n \in \mathbb{Z}. \tag{4.5}$$

Morphisms $f : (M, H_M) \to (M', H_{M'})$ are morphisms in $\mathcal{C}_1$ such that

$$f H_M = H_{M'} f.$$

For $\mu \in L^*$, define $\mathcal{C}^{\mu}_{2,H}$ similarly: its objects are pairs (N, H_N) with $N \in \mathcal{C}^{\mu}_2$ and an even locally finite operator H_N satisfying

$$[H_N, a_{(n)}] = (H_2 a)_{(n)} - (n+1)a_{(n)}, \qquad a \in V^2, \quad n \in \mathbb{Z}.$$

Morphisms are morphisms in $\mathcal{C}^{\mu}_2$ intertwining the chosen Hamiltonians. Write $\mathcal{C}^{\mathrm{ss}}_{1,H}$ and $\mathcal{C}^{\mu,\mathrm{ss}}_{2,H}$ for the full subcategories in which the chosen Hamiltonians are semisimple.

For a Heisenberg generator h, compatibility gives

$$[H_N, h(n)] = -nh(n). \tag{4.6}$$

Thus H_N commutes with the Heisenberg zero modes and preserves the corresponding highest-weight spaces. Both categories in Definition 4.10 are abelian: kernels and cokernels in the underlying categories inherit locally finite compatible Hamiltonians. The same assertion holds for their semisimple Hamiltonian subcategories.

Set

$$q(\xi) = \frac{1}{2}(\xi, \xi).$$

The lattice Hamiltonian acts on the oscillator-degree k subspace of π_ξ by $q(\xi) + k$.

Lemma 4.11. *Let $(N, H_N) \in \mathcal{C}^{\mu}_{2,H}$ and write*

$$N = \bigoplus_{\lambda \in L} N_\lambda \otimes \pi_{\mu-\lambda}.$$

There is a unique family of even locally finite operators D_λ on N_λ such that

$$H_N\big|_{N_\lambda \otimes \pi_{\mu-\lambda}} = D_\lambda \otimes 1 + 1 \otimes L_0^L. \tag{4.7}$$

Under the identification of N_λ with the Heisenberg-highest space $N_\lambda \otimes e^{\mu-\lambda}$, one has

$$D_\lambda = H_N\big|_{N_\lambda \otimes e^{\mu-\lambda}} - q(\mu - \lambda). \tag{4.8}$$

Moreover, H_N is semisimple if and only if every D_λ is semisimple.

Proof. Equation (4.6) shows that H_N preserves each Heisenberg-highest space. Define D_λ by (4.8). For $n_j > 0$, compatibility gives

$$\begin{aligned} &H_N\big(h_1(-n_1)\cdots h_r(-n_r)(u \otimes e^{\mu-\lambda})\big) \\ &\quad = h_1(-n_1)\cdots h_r(-n_r)\left(D_\lambda u \otimes e^{\mu-\lambda} + \left(q(\mu-\lambda) + \sum_{j=1}^{r} n_j\right) u \otimes e^{\mu-\lambda}\right). \end{aligned}$$

This proves (4.7) and uniqueness. Local finiteness follows by restriction to the highest-weight space and subtraction of a scalar. Since L_0^L is semisimple, the last assertion follows from the same decomposition. □

We use the standard Hamiltonian $L_0^{L^-}$ on the opposite lattice and the ghost Hamiltonian H_{gh} for which the b- and c-fields have weights 1 and 0, respectively. The ghost vacuum has weight zero. If K_j are the diagonal isotropic currents, the BRST current

$$\sum_j : K_j c^j :$$

has weight 1. Consequently, its zero mode Q commutes with the total Hamiltonian. This Hamiltonian also preserves the relative subcomplex, since it commutes with the relative zero-mode constraints.

Proposition 4.12. *The functors F_μ and G_μ lift to exact functors*

$$F_\mu^H : \mathcal{C}_{1,H} \longrightarrow \mathcal{C}^{\mu}_{2,H}, \qquad G_\mu^H : \mathcal{C}^{\mu}_{2,H} \longrightarrow \mathcal{C}_{1,H},$$

defined by

$$F_\mu^H(M, H_M) = \Big((M \otimes V_{\mu+L})^T,\ (H_M \otimes 1 + 1 \otimes L_0^L)\big|_{(M\otimes V_{\mu+L})^T}\Big), \tag{4.9}$$

$$G_\mu^H(N, H_N) = \Big(H^{\mathrm{rel},0}(N \otimes V_{-i\mu+L^-}),\ \overline{H}_{\mathrm{tot}}\Big), \tag{4.10}$$

where $\overline{H}_{\mathrm{tot}}$ is induced by

$$H_{\mathrm{tot}} = H_N \otimes 1 \otimes 1 + 1 \otimes L_0^{L^-} \otimes 1 + 1 \otimes 1 \otimes H_{\mathrm{gh}}.$$

Under the identification $G_\mu(N) \cong \bigoplus_{\lambda\in L} N_\lambda$, the induced Hamiltonian is

$$H_{G_\mu^H(N)}\big|_{N_\lambda} = D_\lambda. \tag{4.11}$$

Both functors preserve semisimplicity of the chosen Hamiltonians.

Proof. The operator in (4.9) preserves the invariant subspace because H_M preserves charge and L_0^L preserves lattice momentum. Its compatibility with H_2 follows from (4.4). It is locally finite, being the sum of commuting locally finite operators on the tensor product.

For the inverse construction, the preceding observation gives

$$[H_{\mathrm{tot}}, Q] = 0,$$

so H_{tot} induces an operator on relative cohomology. The reconstruction map

$$a_\alpha \longmapsto [a_\alpha \otimes e^{-\alpha} \otimes e^{i\alpha}]$$

from Theorem 4.1 intertwines H_1 with the induced algebra Hamiltonian. Indeed,

$$q(-\alpha) + q(i\alpha) = 0,$$

where the second quadratic form is computed in the opposite lattice. The induced module operator is therefore compatible with H_1.

For $u \in N_\lambda$, the corresponding cohomology class is

$$[u \otimes e^{\mu-\lambda} \otimes e^{i\lambda - i\mu}].$$

By Lemma 4.11, the contribution of the first two factors is $D_\lambda + q(\mu - \lambda)$, whereas the opposite lattice contributes $-q(\mu - \lambda)$. Thus

$$\overline{H}_{\mathrm{tot}}[u \otimes e^{\mu-\lambda} \otimes e^{i\lambda-i\mu}] = [D_\lambda u \otimes e^{\mu-\lambda} \otimes e^{i\lambda-i\mu}],$$

which proves (4.11) and local finiteness.

On morphisms, set

$$F_\mu^H(f) = (f \otimes \mathrm{id})^T, \qquad G_\mu^H(g) = H^{\mathrm{rel},0}(g \otimes \mathrm{id}).$$

These maps intertwine the induced Hamiltonians. Exactness follows from exactness of F_μ and G_μ and the inherited Hamiltonians on kernels and cokernels. Preservation of semisimplicity follows from (4.9) and Lemma 4.11. □

Theorem 4.13. *For every $\mu \in L^*$, the functors F_μ^H and G_μ^H are mutually quasi-inverse exact equivalences:*

$$\mathcal{C}_{1,H} \simeq \mathcal{C}_{2,H}^{\mu}.$$

They restrict to exact equivalences

$$\mathcal{C}_{1,H}^{\mathrm{ss}} \simeq \mathcal{C}_{2,H}^{\mu,\mathrm{ss}}.$$

The natural isomorphisms are the reconstruction maps of Theorem 4.7.

Proof. It remains to check that the natural isomorphisms Φ_M and Ψ_N in Theorem 4.7 intertwine the chosen Hamiltonians.

For $m_\lambda \in M_\lambda$,

$$\Phi_M(m_\lambda) = [m_\lambda \otimes e^{\mu-\lambda} \otimes e^{i\lambda-i\mu}].$$

The two lattice ground-state energies cancel, so

$$H_{G_\mu^H F_\mu^H(M)} \Phi_M(m_\lambda) = [H_M m_\lambda \otimes e^{\mu-\lambda} \otimes e^{i\lambda-i\mu}] = \Phi_M(H_M m_\lambda).$$

Next, for $u \in N_\lambda$ and $s \in \pi_{\mu-\lambda}$,

$$\Psi_N(u \otimes s) = [u \otimes e^{\mu-\lambda} \otimes e^{i\lambda - i\mu}] \otimes s.$$

Equations (4.7) and (4.11) give

$$H_{F^H_\mu G^H_\mu(N)} \Psi_N(u \otimes s) = \Psi_N(D_\lambda u \otimes s + u \otimes L_0^L s) = \Psi_N(H_N(u \otimes s)).$$

Thus both reconstruction maps are morphisms in the refined categories. Their naturality follows from the naturality of the underlying reconstruction maps. Proposition 4.12 gives the restriction to semisimple Hamiltonians. □

Corollary 4.14. *The equivalence in Theorem 4.13 restricts to the full subcategories satisfying the following conditions:*

(1) every joint charge and generalized Hamiltonian weight space is finite-dimensional;
(2) at each fixed charge, the real parts of the Hamiltonian weights are bounded below.

The same assertion holds with semisimple Hamiltonians.

Proof. Let $M_{\lambda,\Delta}$ be the generalized H_M-eigenspace of eigenvalue Δ in M_λ, and let $\pi_{\mu-\lambda}[k]$ denote the oscillator-degree k subspace. The generalized weight space of energy E in the summand of momentum $\mu - \lambda$ is

$$\big(F^H_\mu(M)\big)_{\lambda,E} = \bigoplus_{k \geq 0} M_{\lambda, E - q(\mu-\lambda) - k} \otimes \pi_{\mu-\lambda}[k].$$

If $\operatorname{Re}\Delta \geq b_\lambda$ on M_λ, only finitely many k contribute for fixed E, and each oscillator space is finite-dimensional. Moreover, the real parts of the resulting energies are bounded below by $b_\lambda + \operatorname{Re} q(\mu-\lambda)$.

Conversely, the Heisenberg-highest space is a Hamiltonian-stable subspace of the fixed-momentum summand. Subtracting $q(\mu-\lambda)$ from its Hamiltonian preserves finite-dimensional generalized weight spaces and lower boundedness. The assertion follows from (4.11). □

5. Kazama-Suzuki duality and quasi-classical limit

5.1. **Kazama-Suzuki duality at the generic level.** Let $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{g} = \mathfrak{so}_{2n+1}$, and let $f = f_{\text{sub}}$ be a subregular nilpotent element. The subregular W-algebra $W^k(\mathfrak{g}, f_{\text{sub}})$ is strongly and freely generated by a Heisenberg field J, two even fields $G^\pm$, and additional even fields W_i as follows:

- for $\mathfrak{g} = \mathfrak{sl}_n$, the fields $G^\pm$ have conformal weight $\frac{n}{2}$, and there is one field
$$W_2, W_3, \dots, W_{n-1}$$
of each conformal weight $2, 3, \dots, n-1$;
- for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the fields $G^\pm$ have conformal weight n, and there is one field
$$W_2, W_4, \dots, W_{2n-2}$$
of each conformal weight $2, 4, \dots, 2n-2$.

All generators are neutral with respect to the Heisenberg field J, except for $G^\pm$, which have J-charge ± 1. This determines the normalization of J. Its level is

$$-\frac{n-(n-1)(k+n)}{n}$$

for $\mathfrak{g} = \mathfrak{sl}_n$ [32, Lemma 3.4], and

$$k + 2n - 2$$

for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ [33, Section 3.3 (3)]. In particular, at the critical level $k+n=0$ for $\mathfrak{sl}_n$, respectively $k+2n-1=0$ for $\mathfrak{so}_{2n+1}$, the field J has level -1.

For $\mathfrak{g} = \mathfrak{sl}_{n|1}$ or $\mathfrak{osp}_{2|2n}$, the principal W-superalgebra $W^k(\mathfrak{g})$ is strongly generated by a Heisenberg field J, two odd fields $G^{\pm}$, and additional even fields W_i as follows:

- for $\mathfrak{g} = \mathfrak{sl}_{n|1}$, the fields $G^{\pm}$ have conformal weight $\frac{n+1}{2}$, and there is one field
$$W_2, W_3, \dots, W_n$$
of each conformal weights $2, 3, \dots, n$;
- for $\mathfrak{g} = \mathfrak{osp}_{2|2n}$, the fields $G^{\pm}$ have conformal weight $n+\frac{1}{2}$, and there is one field
$$W_2, W_4, \dots, W_{2n}$$
of each conformal weights $2, 4, \dots, 2n$.

Again, all generators are neutral with respect to J, except for $G^{\pm}$, which have J-charge ± 1. This fixes the normalization of J, whose level is
$$1 + \frac{n(k+n-1)}{1-n}$$
for $\mathfrak{g} = \mathfrak{sl}_{n|1}$ [32]. In particular, when $n=2$, the algebra $W^k(\mathfrak{sl}_{2|1})$ coincides with the $N=2$ superconformal vertex algebra.

We recall a Kazama–Suzuki type coset duality for
$$(\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{sl}_n, \mathfrak{sl}_{n|1}) \quad \text{or} \quad (\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n}).$$
Let $h_i^\vee$ be the dual Coxeter number of $\mathfrak{g}_i$, and let r denote the lacity of $\mathfrak{g}_1$. The relation between the corresponding levels k_1 and k_2 is given by
$$r\,(k_1 + h_1^\vee)(k_2 + h_2^\vee) = 1. \tag{5.1}$$

Theorem 5.1. [28] *Let $(\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{sl}_n, \mathfrak{sl}_{n|1})$ or $(\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n})$. Let $k_1 \in \mathbb{C} \setminus \{-h_1^\vee\}$ and $k_2 \in \mathbb{C} \setminus \{-h_2^\vee\}$ be generic levels satisfying (5.1). Then there are mutually inverse Kazama-Suzuki type coset isomorphisms*
$$W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \cong \mathrm{Com}\Big(\pi_{\widetilde{H}_2},\, W^{k_2}(\mathfrak{g}_2) \otimes V_{\mathbb{Z}\sqrt{-1}}\Big)\,,$$
$$W^{k_2}(\mathfrak{g}_2) \cong \mathrm{Com}\Big(\pi_{\widetilde{H}_1},\, W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\Big)\,,$$
and also for the corresponding simple quotients
$$W_{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \cong \mathrm{Com}\Big(\pi_{\widetilde{H}_2},\, W_{k_2}(\mathfrak{g}_2) \otimes V_{\mathbb{Z}\sqrt{-1}}\Big)\,,$$
$$W_{k_2}(\mathfrak{g}_2) \cong \mathrm{Com}\Big(\pi_{\widetilde{H}_1},\, W_{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\Big)\,.$$

Let π and π^- denote the Heisenberg vertex algebras of levels 1 and -1, respectively. For any Heisenberg element H, we write π^H for the Heisenberg vertex algebra generated by H. The W-superalgebras $W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}})$ and $W^{k_2}(\mathfrak{g}_2)$ contain Heisenberg vertex subalgebras π^{H^1} and π^{H^2}, respectively, with corresponding pairs of levels
$$\begin{cases} \left(-\dfrac{n-(n-1)(k_1+n)}{n},\ 1+\dfrac{n(k_2+n-1)}{1-n}\right), & (\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{sl}_n, \mathfrak{sl}_{n|1}), \\ (k_1+2n-2,\ 1-2(k_2+n))\,, & (\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n}). \end{cases}$$

By [29], one has the decomposition

$$V_{\mathbb{Z}} \otimes \pi^- \cong \bigoplus_{\lambda\in\mathbb{Z}} \pi_{\lambda}^{\sqrt{-1}H^1} \otimes \pi_{-\lambda}^{H^2}, \tag{5.2}$$

as a $\pi^{\sqrt{-1}H^1}\otimes\pi^{H^2}$-module. Since $W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}})\otimes\pi^{\sqrt{-1}H^1}$ contains a degenerate Heisenberg element, hence one can consider the relative semi-infinite cohomology of $W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}})\otimes V_{\mathbb{Z}}\otimes\pi^-$ with respect to $\mathfrak{gl}_1$. Using the decomposition (5.2) and Lemma 3.1, we deduce the isomorphisms

$$H^{\mathrm{rel},\,p}\Big(W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \otimes \pi^-\Big) \cong \delta_{p,0}\, W^{k_2}(\mathfrak{g}_2)$$

of vertex algebra modules over $\pi^{\sqrt{-1}H^1} \otimes \pi^{H^2}$. Similarly by decomposing $V_{\sqrt{-1}\mathbb{Z}} \otimes \pi^+$, one can obtain an isomorphism for reverse direction and hence reformulate the Kazama–Suzuki type coset isomorphism in terms of relative semi-infinite cohomology.

Theorem 5.2. [29, 63] *There are isomorphisms*

$$H^{\mathrm{rel},\,p}\Big(W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \otimes \pi^-\Big) \cong \delta_{p,0}\, W^{k_2}(\mathfrak{g}_2), \tag{5.3}$$

$$H^{\mathrm{rel},\,p}\Big(W^{k_2}(\mathfrak{g}_2) \otimes V_{\sqrt{-1}\mathbb{Z}} \otimes \pi^+\Big) \cong \delta_{p,0}\, W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}). \tag{5.4}$$

We use the standard notation for the infinite q-Pochhammer symbol $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$. Consider the vertex superalgebra $\mathcal{C} := W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \otimes \pi^- \otimes \mathcal{E}$. We identify $V_{\mathbb{Z}}$ with the bc-system with the field generated by ϕ and ψ. Let

$$L = L^1 + L_{V_{\mathbb{Z}}} + L_{\pi^-} + L_{\mathcal{E}} \in \mathcal{C}$$

be the conformal vector of $\mathcal{C}$, where L^1 is the standard conformal vector in $W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}})$, $L_{V_{\mathbb{Z}}} = \frac{1}{2}(\partial\psi\phi - \psi\partial\phi)$, $L_{\pi^-} = -\frac{1}{2}h_{(-1)}h$, $L_{\mathcal{E}} = -b\partial c$ are the conformal vectors in $V_{\mathbb{Z}}$, π^-, and $\mathcal{E}$, respectively. We derive the following identities from Theorem 5.2 by comparing ordinary graded Euler characters of the relative BRST complexes. The alternating sign refers to cohomological degree, independently of vertex-superalgebra parity.

Corollary 5.3. *The following combinatorial identities hold for* $|q| < 1$

$$\frac{(-y^{-1}zq^{\frac{1}{2}};q)_\infty(-yz^{-1}q^{\frac{1}{2}};q)_\infty}{(zq^{\frac{n}{2}};q)_\infty\,(z^{-1}q^{\frac{n}{2}};q)_\infty}\bigg|_{z^0} = \frac{(-yq^{\frac{n+1}{2}};q)_\infty(-y^{-1}q^{\frac{n+1}{2}};q)_\infty}{(q;q)_\infty\,(q^n;q)_\infty}, \tag{5.5}$$

$$\frac{(-y^{-1}zq^{\frac{1}{2}};q)_\infty(-yz^{-1}q^{\frac{1}{2}};q)_\infty}{(zq^{n};q)_\infty\,(z^{-1}q^{n};q)_\infty}\bigg|_{z^0} = \frac{(-yq^{n+\frac{1}{2}};q)_\infty(-y^{-1}q^{n+\frac{1}{2}};q)_\infty}{(q;q)_\infty\,(q^{2n};q)_\infty}, \tag{5.6}$$

where $|_{z^0}$ *refers to the* z^0*-coefficient.*

Proof. Choose generic levels satisfying the duality relation. We treat the two cases simultaneously by setting

$$(d, I) = \begin{cases} \big(n, \{1,2,\dots,n-1\}\big), & (\mathfrak{g}_1,\mathfrak{g}_2) = (\mathfrak{sl}_n, \mathfrak{sl}_{n|1}), \\ \big(2n, \{1\}\cup\{2j : 1\le j\le n-1\}\big), & (\mathfrak{g}_1,\mathfrak{g}_2) = (\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n}). \end{cases}$$

Thus the neutral strong generators of $W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}})$ have weights in I, and its two charged even generators have weight $d/2$.

Consider the BRST complex

$$C = W^{k_1}(\mathfrak{g}_1, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}} \otimes \pi^- \otimes \mathcal{E},$$

equipped with the sum of the standard conformal vectors of its four factors. Let H^3 denote the auxiliary Heisenberg current generating the factor denoted by $\pi^{\sqrt{-1}H^1}$ in (5.2). The currents H^1 and H^3 have opposite levels, so $K = H^1 + H^3$ is isotropic. The residual current H^2 commutes with K. Hence the BRST differential preserves both L_0 and $H^2_{(0)}$. We use the ordinary graded Euler character

$$\chi_{\mathrm{Eul}}(C^{\mathrm{rel}}; q, y) = \sum_{p\in\mathbb{Z}} (-1)^p \operatorname{tr}_{C^{\mathrm{rel},p}} \big(q^{L_0} y^{H^2_{(0)}}\big).$$

Every fixed conformal-weight subspace is finite-dimensional, and only finitely many cohomological degrees occur at that weight. Hence the Euler–Poincaré identity applies coefficientwise.

The condition $K_0 = 0$ is imposed by taking the z^0-coefficient, where z records K_0-charge. With the charge conventions of (5.2), the charged even generators contribute

$$\frac{1}{(zq^{d/2}; q)_\infty (z^{-1}q^{d/2}; q)_\infty},$$

and the two lattice fermions contribute

$$(-y^{-1}zq^{1/2}; q)_\infty (-yz^{-1}q^{1/2}; q)_\infty.$$

The auxiliary Heisenberg vacuum algebra contributes $(q; q)_\infty^{-1}$. The relative ghost condition excludes c_0; the remaining ghost creators have weights $m \geq 1$ and cohomological degrees 1 and -1. They are neutral under both charge operators, so their Euler character is

$$\prod_{m\geq 1} (1 - q^m)^2 = (q; q)_\infty^2.$$

Consequently,

$$\chi_{\mathrm{Eul}}(C^{\mathrm{rel}}; q, y) = \frac{(q; q)_\infty}{\prod_{i\in I} (q^i; q)_\infty} \left. \frac{(-y^{-1}zq^{1/2}; q)_\infty (-yz^{-1}q^{1/2}; q)_\infty}{(zq^{d/2}; q)_\infty (z^{-1}q^{d/2}; q)_\infty} \right|_{z^0}.$$

By Theorem 5.2, the relative cohomology vanishes outside degree zero and its degree-zero part is $W^{k_2}(\mathfrak{g}_2)$. The conformal Kazama–Suzuki realization identifies the induced conformal grading and residual Heisenberg charge with the standard ones on this algebra. Its neutral PBW generators have weights in $I \cup \{d\}$, and its two odd generators have weight $(d+1)/2$ and charges ± 1. Therefore

$$\chi_{\mathrm{Eul}}(C^{\mathrm{rel}}; q, y) = \operatorname{tr}_{W^{k_2}(\mathfrak{g}_2)} \big(q^{L_0} y^{H^2_{(0)}}\big) = \frac{(-yq^{(d+1)/2}; q)_\infty (-y^{-1}q^{(d+1)/2}; q)_\infty}{(q^d; q)_\infty \prod_{i\in I} (q^i; q)_\infty}.$$

Comparing the two expressions and cancelling the common neutral factors gives

$$\left. \frac{(-y^{-1}zq^{1/2}; q)_\infty (-yz^{-1}q^{1/2}; q)_\infty}{(zq^{d/2}; q)_\infty (z^{-1}q^{d/2}; q)_\infty} \right|_{z^0} = \frac{(-yq^{(d+1)/2}; q)_\infty (-y^{-1}q^{(d+1)/2}; q)_\infty}{(q; q)_\infty (q^d; q)_\infty}.$$

Taking $d = n$ and $d = 2n$ proves (5.5) and (5.6), respectively. □

5.2. **Rescaled large-level limits.** Let $\mathcal{V}_\epsilon$ be a vertex superalgebra over $\mathbb{C}[\epsilon]$, or over a localization consisting of functions regular at $\epsilon = 0$. Assume it has a PBW basis in finitely many homogeneous strong generators X^α, and that all singular OPE coefficients are divisible by ϵ. The specialization $\mathcal{V}_\epsilon/\epsilon\mathcal{V}_\epsilon$ is then commutative. A different rescaling can retain noncommutative OPEs.

Suppose the generators are charge graded, with neutral generators indexed by S^0 and charged generators by S^{ch}. Make the formal base change $\epsilon = \sigma^2$, and inside the localization at σ set

$$Y^\alpha = \sigma^{-1} X^\alpha \quad (\alpha \in S^{\mathrm{ch}}), \qquad Y^\alpha = X^\alpha \quad (\alpha \in S^0).$$

Let $\mathcal{V}_\sigma$ be their $\mathbb{C}[\sigma]$-span under normally ordered products and derivatives. The divisibility assumption makes this span closed under all products. Its ordered rescaled PBW monomials are a free basis, so specialization introduces no additional linear relations. We call $V^\infty = \mathcal{V}_\sigma / \sigma \mathcal{V}_\sigma$ the (rescaled) large-level limit. This is not, in general, the commutative specialization of the unrescaled family.

Proposition 5.4. *Suppose $[X^\alpha{}_\lambda X^\beta] = \epsilon R_{\alpha\beta}(X, \epsilon, \lambda)$. All neutral generators become central in V^∞. For two charged generators, their limiting bracket is obtained from $R_{\alpha\beta}$ by putting $\epsilon = 0$, setting every charged X to zero, and replacing the remaining neutral X by Y. In particular the bracket vanishes unless the two charges sum to zero.*

Proof. For two charged fields the factor σ^{-2} cancels $\epsilon = \sigma^2$. Substituting $X^{\mathrm{ch}} = \sigma Y^{\mathrm{ch}}$ then removes every monomial containing a charged generator. With one or two neutral inputs an additional factor σ or σ^2 remains, so the bracket vanishes at $\sigma = 0$. Charge conservation gives the final assertion. This is the rescaling argument of [32, Proposition 3.2], with the charged and neutral generators scaled differently. □

The applications below use the regular PBW families of [32, 33]. Their neutral limiting fields generate a commutative differential subalgebra; quotienting by its augmentation is the additional operation producing the centerless free fields. We write W^∞ for the limit retaining these fields and W_∞ for its indicated centerless quotient. The choice of regular family and rescaling is part of this notation.

Example 5.5. *Let $\mathfrak{g}$ be a Lie superalgebra with the normalized invariant supersymmetric bilinear form κ. For any complex number l, we denote by $V^l(\mathfrak{g})$ the corresponding universal affine vertex algebra at level l with strong generating set $\{J^\xi\}_{\xi \in \mathfrak{g}}$ such that*

$$J^\xi(z)\, J^\eta(w) \;\sim\; \frac{l\,\kappa(\xi,\eta)}{(z-w)^2} + \frac{J^{[\xi,\eta]}(w)}{z-w}, \qquad \text{for any } \xi, \eta \in \mathfrak{g}.$$

Fix a nonzero $k \in \mathbb{C}$ and let $l = \epsilon^{-1} k$. We recall the regular family $V^k_\epsilon(\mathfrak{g})$ by scaling the strong generators by ϵ as in [32], and it is the vertex superalgebra strongly generated by $\{J^\xi_\epsilon\}_{\xi \in \mathfrak{g}}$ such that

$$J^\xi_\epsilon(z)\, J^\eta_\epsilon(w) \;\sim\; \frac{\epsilon k\,\kappa(\xi,\eta)}{(z-w)^2} + \frac{\epsilon J^{[\xi,\eta]}_\epsilon(w)}{z-w}, \qquad \text{for any } \xi, \eta \in \mathfrak{g}.$$

Let $\{X^\alpha\}_{\alpha \in S}$ be a collection of strong generators of $V^k_\epsilon(\mathfrak{g})$ with a finite index set $S = S^+ \cup S^0 \cup S^-$, which coincides with the generalized root system of $\mathfrak{g}$. Set $Y^\alpha = \sigma^{-1} X^\alpha$ for $\alpha \in S^\pm$ and $Y^\alpha = X^\alpha$ for $\alpha \in S^0$ and denote by $V^k_\sigma(\mathfrak{g})$ the vertex superalgebra generated by $\{Y^\alpha\}_{\alpha \in S}$. We obtain $V^\infty(\mathfrak{g}) := V^k_\sigma(\mathfrak{g}) / \sigma V^k_\sigma(\mathfrak{g})$ the large level limit of the affine vertex algebra.

Below, we describe explicitly the large level limit of the affine vertex algebra associated with $\mathfrak{sl}_2$. Let e, f, h be the canonical basis of $\mathfrak{sl}_2$, satisfying that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{5.7}$$

Let $V^k(\mathfrak{sl}_2)$ be the universal affine vertex algebra of $\mathfrak{sl}_2$ at the level k. We define

$$e^{(k)} := \frac{1}{\sqrt{k}} e, \quad f^{(k)} := \frac{1}{\sqrt{k}} f, \quad h^{(k)} := \frac{1}{k} h. \tag{5.8}$$

By taking the limit $k \to \infty$, we obtain the large limit $\mathfrak{sl}_2^\infty$, which is a three-dimensional Lie algebra with a one-dimension center, spanned by $e^\infty, f^\infty, h^\infty$ and satisfying the nontrivial relation

$$[e^\infty, f^\infty] = h^\infty.$$

The corresponding affine Lie algebra is defined to be

$$\widehat{\mathfrak{sl}_2}^\infty := \mathfrak{sl}_2^\infty[t, t^{-1}] \oplus \mathbb{C}K,$$

with the nontrivial Lie bracket

$$[e^\infty \otimes t^m, f^\infty \otimes t^n] = h^\infty \otimes t^{m+n} + m\delta_{m+n,0}K.$$

The large-level limit of $V^k(\mathfrak{sl}_2)$, denoted by $V^\infty(\mathfrak{sl}_2)$, is the vacuum representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}^\infty$ at level 1. Namely,

$$V^\infty(\mathfrak{sl}_2) \cong U(\widehat{\mathfrak{sl}_2}^\infty) \otimes_{U(\mathfrak{sl}_2^\infty[t] \oplus \mathbb{C}K)} \mathbb{C},$$

where $\mathfrak{sl}_2^\infty[t]$ acts trivially on $\mathbb{C}$, and K acts as the identity. We note that the vertex algebra $V^\infty(\mathfrak{sl}_2)$ contains a big center, and its maximal ideals are not unique. We denote by $V_\infty(\mathfrak{sl}_2)$ its quotient vertex algebra by the augmented ideal generated by the center, which is a free field algebra generated by two elements e^∞ and f^∞ with the nontrivial OPE

$$e^\infty(z)f^\infty(w) \sim \frac{1}{(z-w)^2}.$$

In particular, the quotient vertex algebra $V_\infty(\mathfrak{sl}_2)$ is simple.

Example 5.6. *Let $\mathcal{W}_\epsilon^k(\mathfrak{g}, f)$ be the regular family of $\mathcal{W}$-algebras defined in [32], together with a collection of strong generators $\{X^\alpha\}_{\alpha \in S}$, indexed by a set S. Assume that there exists a Heisenberg element X^{α_0} for some $\alpha_0 \in S$. Moreover, suppose that the index set S admits a decomposition $S = S^+ \sqcup S^0 \sqcup S^-$, which is compatible with the Heisenberg grading induced by X^{α_0}. Set $Y^\alpha = \sigma^{-1}X^\alpha$ for $\alpha \in S^\pm$, and $Y^\alpha = X^\alpha$ for $\alpha \in S^0$. Similarly, we denote by $W_\sigma^k(\mathfrak{g}, f)$ the vertex superalgebra generated by $\{Y^\alpha\}_{\alpha \in S}$. We define $W^\infty(\mathfrak{g}, f) := W_\sigma^k(\mathfrak{g}, f)/\sigma W_\sigma^k(\mathfrak{g}, f)$ to be the large level limit of the regular family, and denote by $W_\infty(\mathfrak{g}, f)$ its quotient by the augmented ideal generated by the center. When f is a principal nilpotent element, we simplify the notation $W^k(\mathfrak{g}, f)$ (resp. $W_k(\mathfrak{g}, f)$) to $W^k(\mathfrak{g})$ (resp. $W_k(\mathfrak{g})$).*

For $W^k(\mathfrak{sl}_{2|1})$, consider the rescaled fields

$$J^{(k)} = k^{-1}J, \qquad L^{(k)} = k^{-1}L, \qquad G^{\pm,(k)} = k^{-1/2}G^\pm.$$

Their large-level limit $W^\infty(\mathfrak{sl}_{2|1})$ is freely strongly generated by the central even fields J^∞, L^∞ and the odd fields $G^{\pm,\infty}$, with the only nontrivial generating OPE, up to skew-symmetry,

$$G^{+,\infty}(z)G^{-,\infty}(w) \sim -\frac{2}{(z-w)^3} + \frac{J^\infty(w)}{(z-w)^2} + \frac{L^\infty(w) + \frac{1}{2}\partial J^\infty(w)}{z-w}. \tag{5.9}$$

Quotienting by the ideal generated by J^∞, L^∞ gives the simple free-field algebra $W_\infty(\mathfrak{sl}_{2|1})$, with

$$G^{+,\infty}(z)G^{-,\infty}(w) \sim -\frac{2}{(z-w)^3}. \tag{5.10}$$

5.3. **Application to the $N = 2$ superconformal algebra.** We now consider the reverse-limit construction. Write $e = e^\infty$, $f = f^\infty$ for the generators of $D = V_\infty(\mathfrak{sl}_2)$, and realize $V_\mathbb{Z}$ as the charged free-fermion algebra generated by b, c. Their nontrivial generating OPEs are

$$e(z)f(w) \sim \frac{1}{(z-w)^2}, \qquad b(z)c(w) \sim \frac{1}{z-w}.$$

Let

$$B = (D \otimes V_\mathbb{Z})^{\mathbb{C}^*}, \qquad t \cdot (e, f, b, c) = (te, t^{-1}f, tb, t^{-1}c).$$

The following fields belong to B:

$$\begin{aligned} G^+ &=: ec :, \qquad G^- =: fb :, \qquad J = - : bc :, \\ L &=: ef : + \frac{1}{2}\big(: \partial b\, c : - : b\, \partial c :\big). \end{aligned} \tag{5.11}$$

Let $\mathcal{U}_3$ denote the universal $N = 2$ superconformal vertex algebra of central charge 3.

Theorem 5.7. *The fields* (5.11) *define an isomorphism of conformal vertex superalgebras*

$$\mathcal{U}_3 \xrightarrow{\sim} \big(V_\infty(\mathfrak{sl}_2) \otimes V_\mathbb{Z}\big)^{\mathbb{C}^*}.$$

In particular, $\mathcal{U}_3$ is simple.

Proof. Wick contraction gives the $N = 2$ OPEs at central charge 3, in the normalization

$$G^+(z)G^-(w) \sim \frac{1}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w) + \frac{1}{2}\partial J(w)}{z-w}.$$

Thus we obtain a conformal homomorphism $\varphi : \mathcal{U}_3 \to B$.

Let $A = \operatorname{im}\varphi$. The fields

$$L_F = \frac{1}{2} : JJ := \frac{1}{2}\big(: \partial b\, c : - : b\, \partial c :\big), \qquad L_D = L - L_F =: ef :$$

belong to A, and their zero modes act as translation on the respective tensor factors. Hence, for $i, j \geq 0$,

$$\begin{aligned} P_{i,j} &= (L_D)^i_{(0)}(L_F)^j_{(0)}G^+ =: \partial^i e\, \partial^j c :\in A, \\ Q_{i,j} &= (L_D)^i_{(0)}(L_F)^j_{(0)}G^- =: \partial^i f\, \partial^j b :\in A. \end{aligned}$$

All fields $: \partial^i b\, \partial^j c :$ lie in the lattice-charge-zero subalgebra of $V_\mathbb{Z}$, which is the Heisenberg algebra generated by J. Moreover,

$$(P_{i,0})_{(0)}Q_{j,0} =: \partial^i e\, \partial^j f : + (-1)^i : \partial^{i+j+1} c\, b :,$$

so $: \partial^i e\, \partial^j f :$ also belongs to A.

These four families of quadratic fields generate B. Indeed, taking torus invariants commutes with passing to the associated graded algebra for the free-field length filtration. Every invariant monomial has equally many factors of charges 1 and -1, and thus factors into quadratic invariants. Since their lifts belong to A, induction on the filtration degree gives $A = B$.

Finally, (5.5) with $n = 2$, $y = 1$, and the PBW theorem give

$$\begin{aligned} \operatorname{ch} B &= \frac{(-zq^{1/2};q)_\infty(-z^{-1}q^{1/2};q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}\bigg|_{z^0} \\ &= \frac{(-q^{3/2};q)_\infty^2}{(q;q)_\infty(q^2;q)_\infty} = \operatorname{ch}\mathcal{U}_3. \end{aligned}$$

Here characters are ordinary graded characters. Since φ is surjective and the homogeneous components are finite-dimensional, it is an isomorphism.

The ambient algebra $D \otimes V_{\mathbb{Z}}$ is simple, and its $\mathbb{C}^*$-invariants equal its $U(1)$-invariants. The fixed-point simplicity argument of [40], valid also for vertex superalgebras, therefore shows that B, and hence $\mathcal{U}_3$, is simple. □

6. Large and critical level duality

6.1. **Duality theorem.** We first record the Hamiltonian conditions used to compare particular representations with familiar weight-module settings.

Definition 6.1. *Let V be a vertex algebra with Hamiltonian H and Heisenberg field J of H-weight 1. A Hamiltonian weight module is a pair (M, H_M), where M is a finitely generated V-module on which J_0 acts semisimply, and H_M is an even locally finite operator satisfying*

$$[H_M, a_{(n)}] = (Ha)_{(n)} - (n+1)a_{(n)}, \qquad a \in V, \quad n \in \mathbb{Z}.$$

In particular, $[H_M, J_0] = 0$. We require

$$M = \bigoplus_{\lambda,\Delta \in \mathbb{C}} M_{\lambda,\Delta}, \qquad \dim M_{\lambda,\Delta} < \infty,$$

where

$$M_{\lambda,\Delta} = \{m \in M \mid J_0 m = \lambda m, \quad (H_M - \Delta)^r m = 0 \text{ for some } r \geq 1\}.$$

For each fixed λ, we further require

$$\inf\{\operatorname{Re}\Delta \mid M_{\lambda,\Delta} \neq 0\} > -\infty.$$

Denote the category of these pairs by V-wtmod. A morphism $f : (M, H_M) \to (M', H_{M'})$ is a V-module homomorphism satisfying

$$f H_M = H_{M'} f.$$

Such morphisms preserve the joint generalized weight spaces.

Define the following full subcategories:

(1) V-wtmod$_{\geq}$ consists of objects for which the real parts of all Hamiltonian weights have a common lower bound:

$$\inf\{\operatorname{Re}\Delta \mid M_{\lambda,\Delta} \neq 0 \text{ for some } \lambda\} > -\infty.$$

(2) V-wtmod$_{\mathcal{O}}$ consists of objects of V-wtmod$_{\geq}$ such that, for every $\Delta \in \mathbb{C}$,

$$\sup\{\operatorname{Re}\lambda \mid M_{\lambda,\Delta} \neq 0\} < \infty.$$

(3) V-wtmod$_{KL}$ consists of objects of V-wtmod$_{\geq}$ such that every total generalized Hamiltonian weight space

$$M_{[\Delta]} = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda,\Delta}$$

is finite-dimensional.

Let $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{so}_{2n+1}$, and let $f_{\mathrm{sub}} \in \mathfrak{g}$ be a subregular nilpotent element. The universal affine W-algebra $W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}})$ at the critical level admits strong generators $J, G^{\pm}, W_i$, where J has level -1 and $G^{\pm}$ have J_0-charges ± 1. The generators W_i have charge zero and Hamiltonian weight i, with

$$i = 2, 3, \ldots, n-1 \quad \text{in type } A, \qquad i = 2, 4, \ldots, 2n-2 \quad \text{in type } B.$$

Here the weights refer to the DS Hamiltonian constructed in Section 2. The W_i may be chosen central. Together with one additional central field of weight n in type A and $2n$ in type B, they freely generate the center as a differential algebra. This additional field can be normalized as a nonzero multiple of $:G^+G^-:$ plus a normally ordered polynomial in J, the W_i, and their derivatives.

By [14, Theorem 1.1], the full center Z is the Feigin–Frenkel center and the universal algebra is free over Z. Write Z_+ for its homogeneous augmentation ideal and set

$$W_{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}) := W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}})/Z_+W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}).$$

This is the simple quotient in [14, Theorem 1.2]. We use only the augmentation specialization, for which translation and the Hamiltonian descend.

Remark 6.2. *Both the universal algebra and the centerless quotient have a nonnegative Hamiltonian grading with finite-dimensional homogeneous spaces and semisimple integral J_0-action. The positive Heisenberg modes therefore act locally nilpotently, and their oscillator decomposition is a direct sum of ordinary Fock modules; see also* [50, Theorem 1.7.3]. *Tensoring with $V_{\mathbb{Z}}$ preserves the Fock decompositions required in Proposition 3.2.*

The quotient map $W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}) \to W_{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}})$ induces a canonical homomorphism

$$(W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})^{\tilde{J}[t]} \longrightarrow (W_{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})^{\tilde{J}[t]}. \tag{6.1}$$

From now on, we will not distinguish between elements and their images under the projection map (6.1). Let $\tilde{J} := J - :bc:$. The following result is an immediate corollary of Proposition 3.2.

Corollary 6.3. *Let W be $W^{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}})$ or $W_{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}})$. We have a surjective vertex algebra morphism*

$$(W \otimes V_{\mathbb{Z}})^{\tilde{J}[t]} \twoheadrightarrow H^{\mathrm{rel},0}(W \otimes V_{\mathbb{Z}}), \tag{6.2}$$

where the kernel is generated by $\tilde{J}$.

It is known that $W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}})$ is freely generated by the fields J, $G^\pm$ and central fields $W_2, \cdots, W_{n-1}$ satisfying the following nontrivial OPEs [53, 61]

$$J(z)J(w) \sim -(z-w)^{-2}, \tag{6.3}$$

$$G^+(z)G^-(w) \sim \frac{n!}{(z-w)^n} - \frac{n!J(w)}{(z-w)^{n-1}} + \cdots, \tag{6.4}$$

$$J(z)G^\pm(w) \sim \frac{\pm G^\pm}{z-w}, \tag{6.5}$$

where the remaining terms in second OPE are normally ordered monomials in $J, W_2, \cdots, W_{n-1}$ and their derivatives. Set in $W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}$

$$X :=: G^+b :, \qquad Y :=: G^-c : .$$

It is straightforward to verify that both X and Y are annihilated by $\tilde{J}[t]$, and hence belong to the invariant subalgebra $\left(W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}\right)^{\tilde{J}[t]}$ with OPE

$$X(z)Y(w) \sim \frac{n!}{(z-w)^{n+1}} - \frac{n!\,\tilde{J}(w)}{(z-w)^n} + \cdots.$$

Lemma 6.4. *The odd classes*

$$X = [G^+ \otimes b], \qquad Y = [G^- \otimes c]$$

in $H^{\mathrm{rel},0}(W_{-h^\vee}(\mathfrak{g}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})$ *satisfy*

$$X(z)Y(w) \sim \frac{n!}{(z-w)^{n+1}}, \qquad X(z)X(w) \sim 0, \qquad Y(z)Y(w) \sim 0.$$

Proof. Put $a =: bc :$, and denote the Heisenberg vertex subalgebras generated by J and a by π_0^J and π_0^a, respectively. In the central augmentation quotient, all singular coefficients of $G^+(z)G^-(w)$ belong to π_0^J. Moreover, the relation obtained by setting the highest-weight central generator equal to zero gives $: G^+G^- :\in \pi_0^J$. This also accounts for the possible simple pole after multiplication by $b(z)c(w)$. Thus every singular coefficient of $X(z)Y(w)$ belongs to $\pi_0^J \otimes \pi_0^a$.

The representatives of X and Y are annihilated by $\widetilde{J}[t]$, so their OPE coefficients belong to $(\pi_0^J \otimes \pi_0^a)^{\widetilde{J}[t]}$. On the polynomial realization of this Heisenberg algebra,

$$\widetilde{J}_m = -m\left(\frac{\partial}{\partial J_{-m}} + \frac{\partial}{\partial a_{-m}}\right) \qquad (m > 0).$$

Consequently,

$$(\pi_0^J \otimes \pi_0^a)^{\widetilde{J}[t]} = \mathbb{C}[\widetilde{J}_{-m} \mid m \geq 1]\mathbf{1}.$$

Except for the leading scalar $n!$, all singular coefficients have positive weight and hence lie in the ideal generated by $\widetilde{J}$. They therefore vanish in relative cohomology by Proposition 3.2.

Finally, the same-sign OPEs of $G^\pm$ are regular by charge and weight. Together with the regularity of $b(z)b(w)$ and $c(z)c(w)$, this proves the remaining assertions. □

Lemma 6.4 shows that the fields X and Y generate an odd free field algebra of orthogonal or symplectic type, which is simple. Consequently, $H^{\mathrm{rel},0}(W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})$ contains a free field algebra. The following theorem shows that this inclusion is in fact an isomorphism.

Theorem 6.5. *The Kazama–Suzuki type isomorphisms hold for the large-critical level duality:*

$$H^{\mathrm{rel},p}(W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}) \cong \delta_{p,0} W_\infty(\mathfrak{sl}_{n|1}). \tag{6.6}$$

Proof. Lemma 6.4 gives an embedding of F_n into the degree-zero reduction. We show that it is onto by an ordinary-character calculation. Put

$$\operatorname{ch} V(q,z) = \operatorname{tr}_V(q^H z^{J_0}).$$

The PBW character of the universal critical subregular algebra is

$$\operatorname{ch} W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) = \frac{1}{\prod_{i=1}^{n-1}(q^i;q)_\infty (zq^{n/2};q)_\infty (z^{-1}q^{n/2};q)_\infty}.$$

By [14, Theorem 1.1] this algebra is free over its center, whose generators have weights $2, \dots, n$. Its augmentation quotient therefore has character

$$\operatorname{ch} B_n(q,z) = \frac{(q^n;q)_\infty}{(q;q)_\infty (zq^{n/2};q)_\infty (z^{-1}q^{n/2};q)_\infty}.$$

Here the character division uses freeness over the center, not merely the identification of the center. The standard lattice fermions have weights $1/2, 1/2$ and opposite charges. The relative ghosts have no zero-mode factor and have Euler character $(q;q)_\infty^2$. Thus the relative complex has Euler character

$$\frac{(q^n;q)_\infty (q;q)_\infty^2}{(q;q)_\infty} \frac{(-zq^{1/2};q)_\infty (-z^{-1}q^{1/2};q)_\infty}{(zq^{n/2};q)_\infty (z^{-1}q^{n/2};q)_\infty}\Bigg|_{z^0}.$$

The total Hamiltonian is $H + L_0^{\mathrm{lat}} + L_0^{\mathrm{gh}}$; the BRST current has weight one, so the differential preserves it. Every homogeneous space is finite-dimensional and the Euler sum is well defined degree by degree. Lemma 3.1 shows that cohomology is concentrated in degree zero. By (5.5) at $y = 1$, the displayed expression equals

$$(-q^{(n+1)/2}; q)_\infty^2 = \operatorname{ch} F_n(q).$$

It compares ordinary graded dimensions. The injective map from F_n is therefore surjective in every degree, proving (6.6) and the higher-cohomology vanishing. □

Together with Remark 6.2 and Theorem 4.2 with $V = W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}})$, we obtain the following vertex algebra isomorphism.

Corollary 6.6. *We have the vertex algebra isomorphism*

$$(W_\infty(\mathfrak{sl}_{n|1}) \otimes V_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \cong W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}). \tag{6.7}$$

Corollary 6.7. *The invariant vertex algebra $(W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})^{\tilde{J}[t]}$ is strongly generated by $\tilde{J}$, X and Y.*

Proof. Set

$$V = W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}, \qquad C = V^{\widetilde{J}[t]}.$$

By Proposition 3.2, the natural graded homomorphism

$$\rho : C \longrightarrow H^{\mathrm{rel},0}(V)$$

is surjective, with

$$\ker \rho = (\widetilde{J}) = \sum_{r \geq 1} \widetilde{J}_{(-r)} C.$$

Here $\widetilde{J}$ is central in C, so its generated ideal consists precisely of these finite sums.

Let A be the linear span of normally ordered monomials in $\widetilde{J}, X, Y$ and their derivatives. Since these fields belong to C, we have $A \subset C$. We prove the reverse inclusion by induction on the nonnegative half-integral Hamiltonian weight.

Let $v \in C$ be homogeneous of weight d. By Theorem 6.5, $H^{\mathrm{rel},0}(V)$ is strongly generated by $\rho(X)$ and $\rho(Y)$. Hence there is a homogeneous normally ordered polynomial $P(X, Y)$, allowing derivatives, such that

$$\rho(v) = \rho(P(X, Y)).$$

The kernel description gives

$$v - P(X, Y) = \sum_{r \geq 1} \widetilde{J}_{(-r)} v_r, \qquad v_r \in C_{d-r},$$

with only finitely many nonzero terms. Since $\mathrm{wt}(\widetilde{J}) = 1$, every v_r has strictly smaller weight. By induction, $v_r \in A$, and therefore

$$\widetilde{J}_{(-r)} v_r = \frac{1}{(r-1)!} : \partial^{r-1} \widetilde{J}\, v_r :\in A.$$

For $d = 0$, all C_{d-r} vanish, providing the base case. Thus $v \in A$, and $C = A$, proving the asserted strong generation. □

Since $Z = Z(W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}))$ is freely generated by the central elements W_i for $2 \leq i \leq n$, we have the following result.

Corollary 6.8. *The commutant* $(W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}})^{\widetilde{J}[t]}$ *is strongly generated by* $\widetilde{J}, X, Y$ *and the central generators* $W_2, \dots, W_n$.

Proof. Set

$$V = W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}, \qquad C = V^{\widetilde{J}[t]}.$$

Let Z_+ be the augmentation ideal of the center $Z = \mathbb{C}[\partial^m W_i \mid 2 \le i \le n,\ m \ge 0]$. Equip V and C with the decreasing filtrations

$$\mathcal{F}^r V = (Z_+)^r V, \qquad \mathcal{F}^r C = C \cap \mathcal{F}^r V.$$

Since Z is central, this filtration is compatible with translation and all vertex products. Moreover, freeness of W over Z gives an isomorphism of graded vertex algebras

$$\mathrm{gr}_{\mathcal{F}} V \cong \overline{V} \otimes \mathrm{gr}_{Z_+} Z, \qquad \overline{V} = W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}.$$

Here $\mathrm{gr}_{Z_+} Z$ is the polynomial differential algebra generated by the symbols of $W_2, \dots, W_n$, each of filtration degree one.

We next justify passing to $\widetilde{J}[t]$-invariants. The ordinary-Fock decomposition of V makes it semisimple as a module for the commuting nondegenerate Heisenberg fields J and $:bc:$. The spaces $\mathcal{F}^r V$ are stable under both fields, so the sequences

$$0 \longrightarrow \mathcal{F}^{r+1} V \longrightarrow \mathcal{F}^r V \longrightarrow \mathcal{F}^r V / \mathcal{F}^{r+1} V \longrightarrow 0$$

split as modules for their Heisenberg tensor product. Taking $\widetilde{J}[t]$-invariants is therefore exact on these sequences. Since the Heisenberg fields act trivially on Z, we obtain

$$\mathrm{gr}_{\mathcal{F}} C \cong \overline{V}^{\widetilde{J}[t]} \otimes \mathrm{gr}_{Z_+} Z.$$

By Corollary 6.7, this associated graded vertex algebra is strongly generated by the symbols of

$$\widetilde{J}, \quad X, \quad Y, \quad W_2, \dots, W_n.$$

Finally, lifting from the associated graded terminates in each Hamiltonian weight. Indeed, every generator of Z_+ has weight at least two, while V has nonnegative weights. Hence

$$(\mathcal{F}^r V)_d = 0 \qquad \text{whenever } 2r > d.$$

For a homogeneous $v \in C_d$, express its class modulo $\mathcal{F}^1 C$ as a normally ordered polynomial in the indicated symbols and subtract the corresponding polynomial in their lifts. The remainder belongs to $(\mathcal{F}^1 C)_d$. Repeating this procedure in successive filtration degrees gives zero after finitely many steps. Thus v is a normally ordered polynomial in $\widetilde{J}, X, Y, W_2, \dots, W_n$ and their derivatives, proving the assertion. □

For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, the OPEs of the strong generators remain unknown, in particular those involving G^+ and G^-.

Theorem 6.9. *The subregular* W*-algebra* $W^{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ *satisfies the following OPE relations:*

$$J(z)J(w) \sim -(z-w)^{-2}, \tag{6.8}$$

$$G^+(z)G^-(w) \sim \frac{c}{(z-w)^{2n}} - \frac{c\,J(w)}{(z-w)^{2n-1}} + \cdots, \tag{6.9}$$

$$J(z)G^{\pm}(w) \sim \frac{\pm G^{\pm}(w)}{z-w}, \tag{6.10}$$

where c is a nonzero constant and the omitted terms in the second OPE are normally ordered polynomials in J, $W_2, \dots, W_{2n-2}$, and their derivatives.

Proof. By choosing a suitable normalization of J, it suffices to show the relation (6.9). According to the conformal weight computation, the OPE for $G^+(z)$ and $G^-(w)$ has the form

$$G^+(z)G^-(w) \sim \frac{c}{(z-w)^{2n}} + \frac{c'\,J(w)}{(z-w)^{2n-1}} + \cdots$$

for certain constants $c, c' \in \mathbb{C}$. One deduces that $c = -c'$, thanks to the relation that

$$-c' = J_{(1)}G^+_{(2n-2)}G^- = [J_{(1)}, G^+_{(2n-2)}]G^- = G^+_{(2n-1)}G^- = c.$$

Assume that $c = 0$. We consider the corresponding centerless quotient $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$, which is simple by [14]. The OPE for (6.8) and (6.10) remains the same and the OPE for (6.9) turns into

$$G^+(z)G^-(w) \sim \frac{c}{(z-w)^{2n}} - \frac{c\,J(w)}{(z-w)^{2n-1}} + \sum_{i=1}^{2n-2} \frac{f_i(w)}{(z-w)^{2n-1-i}},$$

where f_i is a polynomial on the variables J and its derivatives with each monomials of conformal weight $i+1$. Set $f_1(w) = a_1 J^2(w) + a_2 \partial J(w)$. Then by a simple computation

$$-2a_2 = J_{(2)}G^+_{(2n-3)}G^- = [J_{(2)}, G^+_{(2n-3)}]G^- = G^+_{(2n-1)}G^- = 0,$$

and

$$2a_1 = J^2_{(1)}G^+_{(2n-3)}G^- = J_{(1)}[J_{(1)}, G^+_{(2n-3)}]G^- = J_{(1)}G^+_{(2n-2)}G^- = 0,$$

which implies $G^+_{(2n-3)}G^- = 0$. By applying the the positive modes of $J(z)$ recursively, one can show that all of $f_i = 0$ and hence the OPE for $G^+(z)$ and $G^-(w)$ in $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ is regular. Hence G^+ and G^- generate a nontrivial proper ideal in the centerless quotient, which contradicts to the simplicity of $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$. Hence $c \neq 0$. □

Using Theorem 6.9, one obtains an analogous Kazama-Suzuki type isomorphism in the large and critical level duality setting, by mimicking the proof of Theorem 6.5.

Theorem 6.10. *The Kazama-Suzuki type isomorphisms for the pair $(\mathfrak{g}_1, \mathfrak{g}_2) = (\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n})$ hold in the large-critical level duality, namely*

$$H^{\mathrm{rel},p}(W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \otimes V_{\mathbb{Z}}) \cong \delta_{p,0} W_\infty(\mathfrak{osp}_{2|2n}),$$
$$(W_\infty(\mathfrak{osp}_{2|2n}) \otimes V_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \cong W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}).$$

Proof. Normalize one charged generator so that the nonzero scalar in Theorem 6.9 is 1. The same Heisenberg cancellation as in type A shows that the BRST classes of $:G^+b:$ and $:G^-c:$ have only the scalar pole of order $2n+1$. They therefore generate an embedded copy of F_{2n}.

The universal algebra has even PBW generators of weights $1, 2, 4, \dots, 2n-2$ and two charged generators of weight n. Dividing its character by the character of the freely acting center, with weights $2, 4, \dots, 2n$, gives

$$\mathrm{ch}\, W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})(q,z) = \frac{(q^{2n};q)_\infty}{(q;q)_\infty (zq^n;q)_\infty (z^{-1}q^n;q)_\infty}.$$

The relative Euler character is obtained by multiplying by the lattice-fermion factors and $(q;q)^2_\infty$ and taking the charge-zero coefficient. By (5.6) at $y = 1$, it is $(-q^{n+1/2};q)^2_\infty = \mathrm{ch}\, F_{2n}$. Cohomology is concentrated in degree zero by Lemma 3.1, so the embedding is surjective. Proposition 5.4

identifies this free field with $W_\infty(\mathfrak{osp}_{2|2n})$. The inverse orbifold statement follows from Theorem 4.2 and the Fock decomposition in Remark 6.2. □

By the proofs of Theorem 6.5 and 6.10, both the $W_\infty(\mathfrak{sl}_{2n|1})$ and $W_\infty(\mathfrak{osp}_{2|2n})$ are isomorphic to the free field algebra generated by two odd fields of weight $n+\frac{1}{2}$. Hence the corresponding centerless critical level subregular W-algebras are also isomorphic.

Corollary 6.11. *There is an isomorphism of vertex algebras*

$$W_{-2n}(\mathfrak{sl}_{2n}, f_{\mathrm{sub}}) \cong W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}).$$

In the large-critical level setting, the Kazama-Suzuki type isomorphism for universal affine W-algebras does not hold in general; nevertheless, one still obtains a canonical embedding map between the corresponding vertex algebras. We need the following lemma.

Lemma 6.12. *The charged generators of $W^\infty(\mathfrak{sl}_{n|1})$ can be normalized so that*

$$G^{+,\infty}(z)G^{-,\infty}(w) \sim -\frac{1}{(z-w)^{n+1}} + \sum_{i=1}^{n} \frac{A_i(w)}{(z-w)^{n+1-i}}.$$

The fields $A_1, \dots, A_n$ are central, and

$$A_1, \dots, A_n, \quad G^{+,\infty}, G^{-,\infty}$$

freely strongly generate $W^\infty(\mathfrak{sl}_{n|1})$.

Proof. Choose neutral PBW generators $w_1, \dots, w_n$ corresponding to

$$h_0 = \sum_{a=1}^{n} E_{a,a} + nE_{n+1,n+1}, \qquad f, f^2, \dots, f^{n-1}, \qquad f = \sum_{a=1}^{n-1} E_{a+1,a}.$$

The large-level construction makes these fields central. By weight considerations,

$$A_i = c_i w_i + P_i(w_1, \dots, w_{i-1}),$$

where P_i is a differential polynomial. It remains to verify that $c_i \neq 0$.

The regular family identifies the limiting charged bracket with the classical Drinfeld–Sokolov bracket after setting the charged variables equal to zero by Proposition 5.4. We apply [65, Theorem 2.4] to the charged generators corresponding to $E_{n,n+1}$ and $E_{n+1,1}$. The two families of odd matrices

$$E_{r+1,n+1}, \qquad E_{n+1,r+1}, \qquad 0 \le r \le n-1,$$

are dual with respect to the supertrace form. The scalar contractions give a nonzero coefficient of ξ^n. In the coefficient of ξ^{n-i}, the terms linear in the neutral generators and containing no derivatives arise by replacing one scalar contraction with a matrix bracket. These terms have a common nonzero scalar factor, and their bracket contributions sum to

$$\sum_{r=0}^{n-i} [E_{n+1,r+1}, E_{r+i,n+1}] = \begin{cases} h_0, & i = 1, \\ f^{i-1}, & 2 \le i \le n. \end{cases}$$

Hence $c_i \neq 0$ for every i. The change from $w_1, \dots, w_n$ to $A_1, \dots, A_n$ is therefore an invertible triangular change of differential generators, and the assertion follows from the PBW theorem. □

Theorem 6.13. *There exists an embedding of the universal affine W-algebra associated to $\mathfrak{sl}_n$ at the critical level into the orbifold vertex algebra of $W^\infty(\mathfrak{sl}_{n|1})$, namely*

$$W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \hookrightarrow (W^\infty(\mathfrak{sl}_{n|1}) \otimes V_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \tag{6.11}$$

Proof. Let $A_0 = -\mathbf{1}$. Define central fields Q_i by

$$\sum_{i=0}^{n} Q_i t^i \equiv E_{A_1}(t) \sum_{i=0}^{n} A_i t^i \pmod{t^{n+1}},$$

where, for a even field a, $E_a(t) := \exp\left(\sum_{r\geq 1} \frac{a_{(-r)}}{r} t^r\right)$. The first three coefficients are

$$Q_0 = -\mathbf{1}, \qquad Q_1 = 0, \qquad Q_2 = A_2 + \frac{1}{2}A_1^2 - \frac{1}{2}\partial A_1.$$

More generally, $Q_i - A_i$ is a differential polynomial in $A_1, \dots, A_{i-1}$ for $2 \leq i \leq n$. It follows from Lemma 6.12 that $A_1, Q_2, \dots, Q_n$ freely generate the central differential algebra.

Let $(\lambda, \lambda) = -1$, and consider $B = \left(W^\infty(\mathfrak{sl}_{n|1}) \otimes V_{\mathbb{Z}\lambda}\right)^{\mathbb{C}^*}$. The fields

$$H = A_1 - \lambda, \qquad X = G^{+,\infty} \otimes e^{\lambda}, \qquad Y = -\epsilon(\lambda, -\lambda)^{-1} G^{-,\infty} \otimes e^{-\lambda}$$

belong to B. The lattice vertex-operator formula, with the tensor-product super sign, gives

$$Y_B(X,t)Y = t^{-n} E_{-H}(t) \left(-\mathbf{1} + \sum_{i=2}^{n} Q_i t^i\right) + O(t). \tag{6.12}$$

Here we have used

$$E_\lambda(t) \sum_{i=0}^{n} A_i t^i \equiv E_{\lambda - A_1}(t) \sum_{i=0}^{n} Q_i t^i \pmod{t^{n+1}}.$$

Since A_1 is central, the remaining OPEs are

$$H(z)H(w) \sim -\frac{1}{(z-w)^2}, \qquad H(z)X(w) \sim \frac{X(w)}{z-w}, \qquad H(z)Y(w) \sim -\frac{Y(w)}{z-w}.$$

The same-sign OPEs of X and Y are regular.

The OPE comparison in [61, Lemma 4.1 and Theorem 4.2] now gives a homomorphism

$$\varphi : W^{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \longrightarrow B, \qquad J \longmapsto H, \quad G^+ \longmapsto X, \quad G^- \longmapsto Y,$$

where the source charged generators are normalized to have leading mixed-OPE coefficient -1. The central generators $W_2, \dots, W_n$ of the source are normalized by

$$t^n E_J(t) Y(G^+, t) G^- \equiv -\mathbf{1} + \sum_{i=2}^{n} W_i t^i \pmod{t^{n+1}}.$$

Here W_n denotes the additional central generator of weight n described above. Since $\varphi(J) = H$ and $\varphi(G^\pm) = X, Y$, respectively, applying φ to this identity gives

$$t^n E_H(t) Y_B(X,t) Y \equiv -\mathbf{1} + \sum_{i=2}^{n} \varphi(W_i) t^i \pmod{t^{n+1}}.$$

Comparison with (6.12) therefore yields $\varphi(W_i) = Q_i$ for $2 \leq i \leq n$.

It remains to prove injectivity. Set

$$R = \mathbb{C}[\partial^m Q_i \mid 2 \leq i \leq n,\ m \geq 0].$$

The PBW basis and the neutrality of the Q_i show that B is free over R. Thus B/R_+B is nonzero, and the induced map

$$\overline{\varphi} : W_{-n}(\mathfrak{sl}_n, f_{\mathrm{sub}}) \longrightarrow B/R_+B$$

is injective because its source is simple. The map φ identifies the source center with R. Since both sides are free over this algebra, the associated graded map for the central augmentation filtrations

is $\overline{\varphi} \otimes \mathrm{id}_{\mathrm{gr}\, R}$, hence injective. The source filtration is separated because its central generators have positive weights. Therefore φ is injective. $\square$

Conjecture 6.14. *Similar embedding (6.13) exists for the pair* $(\mathfrak{so}_{2n+1}, \mathfrak{osp}_{2|2n})$, *namely*

$$W^{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}}) \hookrightarrow (W^{\infty}(\mathfrak{osp}_{2|2n}) \otimes V_{\sqrt{-1}\mathbb{Z}})^{\mathbb{C}^*} \tag{6.13}$$

6.2. **Representation-theoretic correspondence.** We describe the modules of the free-field algebra F_N and apply the equivalence of Section 4.1 to the centerless critical-level W-algebras. For $N \geq 2$, write

$$B_N := W_{-N}(\mathfrak{sl}_N, f_{\mathrm{sub}}).$$

By Theorem 1.2, the same description applies to $W_{-2n+1}(\mathfrak{so}_{2n+1}, f_{\mathrm{sub}})$ with $N = 2n$.

Definition 6.15. *Let* π^J *be the Heisenberg vertex subalgebra of* B_N *generated by* J, *and let* π^J_m *denote its Fock module of* J_0*-charge* m. *Define* $\mathscr{B}_N$ *to be the full subcategory of weak* B_N*-modules consisting of finite-length modules* M *whose restriction to* π^J *has the form*

$$M\big|_{\pi^J} \cong \bigoplus_{m \in \mathbb{Z}} \pi^J_m \otimes E_m$$

for some multiplicity spaces E_m. *Morphisms are* B_N*-module homomorphisms.*

A chosen compatible Hamiltonian is not part of the data of an object of $\mathscr{B}_N$. Our block, projective-cover, and extension statements will be made in this category. At the end of the section, we relate these results to B_N-wtmod, whose objects carry chosen compatible Hamiltonians and whose morphisms intertwine them, as in Definition 6.1.

Fix $N \geq 1$ and put

$$d = \frac{N+1}{2}, \qquad a = \frac{N-1}{2}.$$

The odd generators of F_N have expansions and OPE

$$X^{\pm}(z) = \sum_{s \in d+\mathbb{Z}} X^{\pm}_s z^{-s-d}, \qquad X^+(z)X^-(w) \sim (z-w)^{-N-1}.$$

Let $\mathfrak{g}_N$ be the associated mode Lie superalgebra. After fixing its even central element to act as 1, its relations are

$$\{X^+_s, X^-_t\} = \binom{s+a}{N} \delta_{s+t,0}, \qquad \{X^+_s, X^+_t\} = \{X^-_s, X^-_t\} = 0. \tag{6.14}$$

The $2N$ modes

$$X^{\pm}_s, \qquad s = -a, -a+1, \dots, a,$$

are odd central elements. We retain the notation

$$Z(\mathfrak{g}_N) := \Lambda(X^{\pm}_s \mid s = -a, -a+1, \dots, a), \qquad \deg X^{\pm}_s = \pm 1.$$

Thus $Z(\mathfrak{g}_N)$ denotes the supercenter of the associative mode algebra obtained from $U(\mathfrak{g}_N)$ by specializing its scalar central element to 1. In particular,

$$Z(\mathfrak{g}_1) = \Lambda(X^+_0, X^-_0), \qquad Z(\mathfrak{g}_2) = \Lambda\big(X^+_{1/2}, X^+_{-1/2}, X^-_{-1/2}, X^-_{1/2}\big).$$

Set

$$\mathfrak{g}_{N,+} = \mathrm{span}\{X^{\pm}_s : s > a\}, \qquad \mathfrak{g}_{N,-} = \mathrm{span}\{X^{\pm}_s : s < -a\}.$$

The pairing between these spaces in (6.14) is nondegenerate. Choose paired odd bases, consisting of scalar multiples of the modes, $p_i \in \mathfrak{g}_{N,+}$ and $q_i \in \mathfrak{g}_{N,-}$ with

$$\{p_i, q_j\} = \delta_{ij}.$$

For a weak F_N-module M, define

$$\Omega(M) = \{m \in M : \mathfrak{g}_{N,+} m = 0\}.$$

This is a $Z(\mathfrak{g}_N)$-supermodule. Conversely, for a $Z(\mathfrak{g}_N)$-supermodule U, let

$$\mathcal{I}_N(U) = \Lambda(\mathfrak{g}_{N,-}) \otimes U,$$

where the q_i act by exterior multiplication, the p_i by contraction, and the central modes act on U with the usual tensor-product signs. The resulting restricted mode representation is a weak F_N-module.

Theorem 6.16. *For every weak F_N-module M, the action map*

$$\mathcal{I}_N(\Omega(M)) \longrightarrow M, \qquad q_{i_1} \cdots q_{i_k} \otimes u \longmapsto q_{i_1} \cdots q_{i_k} u,$$

is an isomorphism. The functors Ω and $\mathcal{I}_N$ are mutually inverse and preserve charge gradings. Moreover, M is finitely generated if and only if $\Omega(M)$ is finite-dimensional.

Proof. Put $P_i = 1 - q_i p_i$. These are commuting even projections and $p_i P_i = 0$. For $m \in M$, restrictedness gives a finite set I such that every p_i with $i \notin I$ annihilates m. Expanding

$$1 = \prod_{i \in I} (P_i + q_i p_i)$$

expresses m as a finite sum of vectors $q_J u_J$ with $u_J \in \Omega(M)$. Modes outside I still annihilate these coefficients, since they anticommute with the modes indexed by I. This proves surjectivity.

For a finite relation $\sum_J q_J u_J = 0$, choose I containing all its oscillator indices. Suitable products of the p_i, followed by $\prod_{i \in I} P_i$, extract each coefficient u_J, up to sign. Hence the action map is injective. The construction is natural and gives

$$\Omega(\mathcal{I}_N(U)) = U.$$

If M is finitely generated, expand a finite set of generators in this decomposition. Their finitely many vacuum coefficients generate $\Omega(M)$ over the finite-dimensional algebra $Z(\mathfrak{g}_N)$, so $\Omega(M)$ is finite-dimensional. Conversely, a basis of a finite-dimensional U generates $\mathcal{I}_N(U)$. All maps preserve charge because the modes are charge homogeneous. □

Corollary 6.17. *The category of finitely generated charge-graded F_N-supermodules is equivalent to the category of finite-dimensional charge-graded $Z(\mathfrak{g}_N)$-supermodules, via Ω and $\mathcal{I}_N$. Its simple objects are the vacuum module with charge and parity shifts.*

Proof. The equivalence follows from Theorem 6.16. The augmentation ideal of $Z(\mathfrak{g}_N)$ is nilpotent and its quotient is $\mathbb{C}$. Thus every simple graded supermodule is a one-dimensional trivial module concentrated in a single charge and parity. Its induced module is the vacuum with the corresponding shifts. □

For comparison with B_N-modules, we restrict to the parity convention $p(u) = \deg u \pmod 2$. Write $U\langle r \rangle$ for the shift increasing charge by r and parity by $r \pmod 2$.

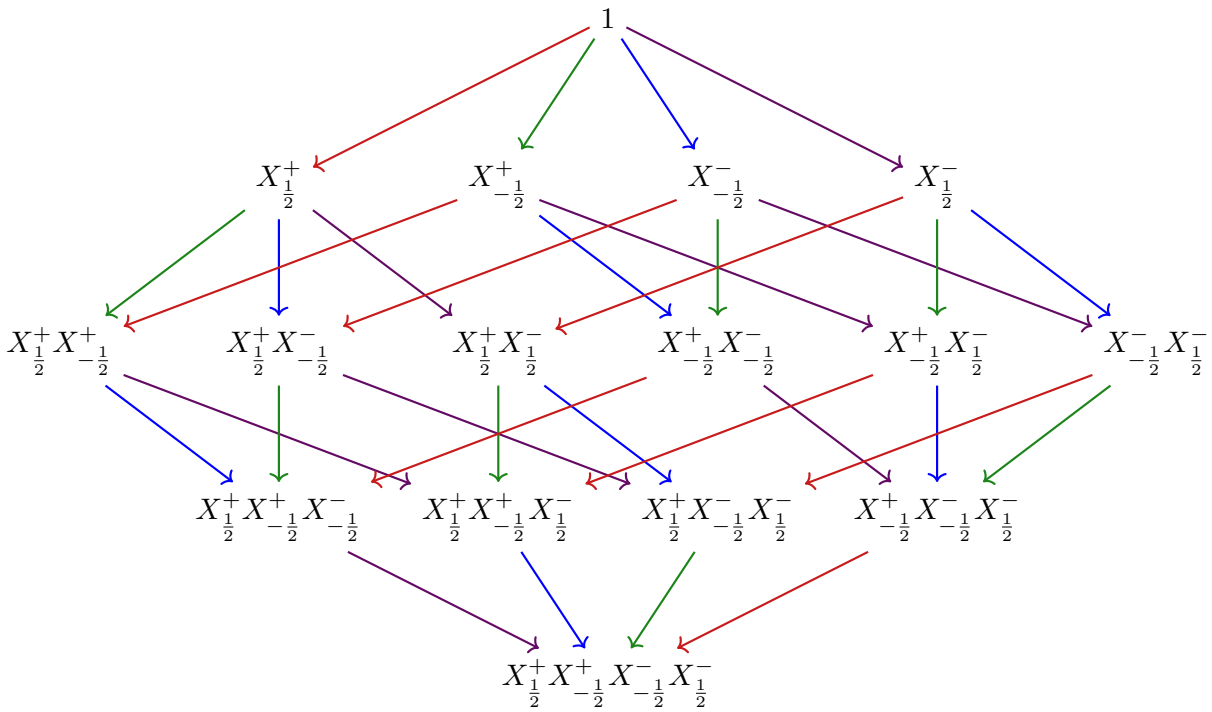
Remark 6.18. *The Loewy structure of a finitely generated F_N-module is completely determined by the Loewy structure of its vacuum space as a $Z(\mathfrak{g}_N)$-module. More precisely, the exact equivalence above identifies their radical and socle filtrations.*

The regular module $Z(\mathfrak{g}_N)$ is projective. Since $Z(\mathfrak{g}_N)$ is local, its augmentation map onto the trivial module is a projective cover. If $\mathfrak{m}$ is its augmentation ideal, then

$$\operatorname{rad}^j Z(\mathfrak{g}_N) = \mathfrak{m}^j, \qquad \mathfrak{m}^j/\mathfrak{m}^{j+1} \cong \Lambda^j\big(\mathbb{C}^N_{+1} \oplus \mathbb{C}^N_{-1}\big).$$

In particular, its Loewy length is $2N+1$.

The monomial diagram for $Z(\mathfrak{g}_2)$ *is displayed below. Each node denotes a monomial class in a radical layer; the coloured arrows record multiplication by the four generators, with exterior signs suppressed. Its rows have dimensions* $1, 4, 6, 4, 1$.



Loewy diagram for $Z(\mathfrak{g}_2)$

Theorem 6.19. *For* $N \geq 2$, *the functor* $F_0\mathcal{I}_N$ *gives an exact equivalence*

$$Z(\mathfrak{g}_N)\text{-gmod}_{\mathrm{fd}} \xrightarrow{\sim} \mathscr{B}_N,$$

where the category on the left uses the parity convention above. The category $\mathscr{B}_N$ *is also the finitely generated part of the integral-charge Fock category, and it is an indecomposable block.*

Its simple objects and their projective covers are

$$S_r = F_0\big(F_N\langle r\rangle\big), \qquad P_r = F_0\mathcal{I}_N\big(Z(\mathfrak{g}_N)\langle r\rangle\big), \qquad r \in \mathbb{Z}.$$

Here $S_0 = B_N$, *and the* S_r *are the spectral-flow images of the vacuum. The radical layers of* P_r *correspond to*

$$(\mathfrak{m}^j/\mathfrak{m}^{j+1})\langle r\rangle \cong \Lambda^j\big(\mathbb{C}^N_{+1} \oplus \mathbb{C}^N_{-1}\big)\langle r\rangle.$$

In particular, every P_r *has Loewy length* $2N+1$.

Proof. The equivalence of Section 4.1, applied to $B_N \cong (F_N \otimes V_{i\mathbb{Z}})^{\mathbb{C}^*}$, identifies the integral-charge Fock category with charge-graded F_N-modules in the specified parity sector. Corollary 6.17 identifies its finite-length subcategory with the displayed finite-dimensional exterior-algebra category.

Every finite-length module is finitely generated. Conversely, any charge-graded F_N-module is the directed union of its finitely generated graded submodules, which have finite length by Theorem 6.16. The functor F_0 preserves these directed unions. If its image is finitely generated, it

is already the image of one such finite-length submodule. This proves the assertion about finite generation.

The projective covers and their radical filtrations follow from Remark 6.18 and exactness. The charge shifts are identified with spectral flow by Theorem 4.9. A nonzero action of a generator of charge 1 gives a nonsplit extension linking the trivial modules in two adjacent charges. Hence all the simple objects lie in one block. □

Theorem 6.20 (Vacuum extension algebra)**.** *For $N \geq 2$, there is an isomorphism of graded associative algebras*

$$\operatorname{Ext}^{\bullet}_{\mathscr{B}_N}(B_N, B_N) \cong \mathbb{C}[x_1, \dots, x_N, y_1, \dots, y_N]^{\mathbb{C}^*} \cong \mathbb{C}[z_{ij} \mid 1 \leq i, j \leq N]/I_2.$$

Here the variables x_i, y_i have cohomological degree one and charges $1, -1$, respectively; $z_{ij} = x_i y_j$ has cohomological degree two, and I_2 is the ideal generated by the 2×2 minors of (z_{ij}). In particular, the odd-degree self-extension groups vanish.

Proof. Let

$$Z(\mathfrak{g}_N) = \Lambda(E), \qquad E = \mathbb{C}^N_{+1} \oplus \mathbb{C}^N_{-1}.$$

The charge-graded Koszul resolution of the trivial module has terms

$$\Lambda(E) \otimes \operatorname{Sym}^j(E)$$

in homological degree j. Here Sym denotes the ordinary symmetric algebra, with $\operatorname{Sym}^j(E)$ placed in parity $j \pmod 2$. The differential is the sum of right multiplication by an exterior generator on the first factor followed by differentiation in the corresponding symmetric variable on the second.

Applying charge- and parity-preserving $\operatorname{Hom}_{A_N}(-, \mathbb{C})$ kills the differential and gives the charge-zero part of $\operatorname{Sym}^j(E^*)$. Charge zero already forces even parity in this convention. The Koszul product identifies the Yoneda algebra with $\operatorname{Sym}(E^*)^{\mathbb{C}^*}$. Choose x_i dual to the charge -1 generators and y_i dual to the charge 1 generators, so dualization gives the stated charges.

Every invariant monomial is a product of the $x_i y_j$. The relations among these generators are

$$z_{ij} z_{kl} = z_{il} z_{kj}.$$

These exchanges identify precisely the products with the same row and column multiplicities. They generate I_2, proving the second isomorphism. The equivalence in Theorem 6.19 gives the assertion for B_N. □

Proof of Theorem 1.5. By Theorem 6.19, the projective cover of $V = B_2$ in $\mathscr{B}_2$ is

$$P_0 = F_0 \mathcal{I}_2(Z(\mathfrak{g}_2)).$$

The exact equivalence preserves radical filtrations, and Theorem 4.9 identifies charge shifts with spectral flow. Thus Remark 6.18 gives the layers

$$\begin{gathered} V^{(0)}, \\ (V^{(1)})^{\oplus 2} \oplus (V^{(-1)})^{\oplus 2}, \\ V^{(2)} \oplus (V^{(0)})^{\oplus 4} \oplus V^{(-2)}, \\ (V^{(1)})^{\oplus 2} \oplus (V^{(-1)})^{\oplus 2}, \\ V^{(0)}. \end{gathered}$$

The arrows are transported from multiplication by the four exterior generators, proving the asserted diagram in $\mathscr{B}_2$.

For the Hamiltonian assertions, the minimum F_2 energy in charge ℓ is

$$e_2(\ell) = \sum_{j=0}^{|\ell|-1} \left(\frac{3}{2} + j\right) = \frac{\ell^2}{2} + |\ell|,$$

attained by the lowest distinct creators of one sign. The negative-lattice contribution is $-\ell^2/2$. Hence the corresponding vector in V has minimum energy $|\ell|$ and J_0-charge $-\ell$.

On $V^{(r)}$, inverse pullback gives

$$J_0^{(r)} = J_0 - r, \qquad H^{(r)} = H + rJ_0.$$

The latter operator is compatible with the spectral-flow module action. Consequently, the resulting charge is $q = -(\ell + r)$, and the attained minimum energy at charge q is

$$|\ell| - r\ell = |q + r| + r(q + r).$$

The free-field realization gives finite-dimensional joint charge–energy spaces and a lower bound at each fixed charge. Since the spectral-flow modules are simple, they are finitely generated. Thus $(V^{(r)}, H^{(r)})$ belongs to V-wtmod.

For $r = 0$, only finitely many charges occur at each energy, so $(V^{(0)}, H^{(0)})$ belongs to V-wtmod$_{KL}$. For $r = 1$, all energies are nonnegative, and at fixed energy Δ the charges satisfy $q \leq \Delta/2 - 1$. However, every $q \leq -1$ occurs at energy zero. Thus $(V^{(1)}, H^{(1)})$ belongs to V-wtmod$_{\mathcal{O}}$ but not to V-wtmod$_{KL}$. For $r = -1$, all energies are nonnegative, but every $q \geq 1$ occurs at energy zero. Hence $(V^{(-1)}, H^{(-1)})$ belongs to V-wtmod$_{\geq}$ but not to V-wtmod$_{\mathcal{O}}$. For $r = 2$, the minimum is $q + 2$ when $q \leq -2$; for $r = -2$, it is $-q + 2$ when $q \geq 2$. Both are unbounded below.

Finally, the horizontal zero modes on $V^{(1)}$ and $V^{(-1)}$ are respectively

$$(e_1, f_{-1}, h_0 - 2), \qquad (e_{-1}, f_1, h_0 + 2).$$

Their energy-zero spaces are generated by the original vacuum as a highest-weight module of weight $-\alpha$ and a lowest-weight module of weight α, respectively. Simplicity therefore gives

$$V^{(1)} \cong D^+_{-\alpha}, \qquad V^{(-1)} \cong D^-_{\alpha}.$$

□

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