arXiv:2603.26379v4 [math.CO] 18 Nov 2026

The Bollobás–Nikiforov inequality for complete multipartite graphs, with remarks on the K_4-free case

Piero Giacomelli[a,*]

[a] *IT Department, TENAX GROUP SPA, Verona, Italy*

Abstract

Bollobás and Nikiforov conjectured that every graph $G \neq K_n$ with m edges and clique number ω satisfies $\lambda_1^2(G)+\lambda_2^2(G) \leq 2(1-1/\omega)m$, where $\lambda_1(G) \geq \lambda_2(G)$ are the two largest adjacency eigenvalues. We give a short self-contained proof of the inequality for complete multipartite graphs, based on a quadratic-form identity which shows that $\lambda_2 \leq 0$, and we determine the equality cases in this class. We show that equality in the conjecture is attained by a wider family than the Turán graphs, namely by disjoint unions of two complete bipartite graphs, or of two balanced complete r-partite graphs of possibly different orders. For K_4-free graphs we record the cases that follow from known results, with corrected attributions, and we give examples showing that three natural lines of attack fail: a Hoffman-type eigenvalue bound for graphs with large independence number, a spectral stability statement for connected graphs, and Zykov symmetrization in an arbitrary direction.

Keywords: spectral graph theory, Bollobás–Nikiforov conjecture, adjacency eigenvalues, complete multipartite graphs, spectral Turán theorem
2020 MSC: 05C50, 05C35

1. Introduction

All graphs are finite and simple. For a graph G on n vertices with m edges we write $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ for the eigenvalues of its adjacency matrix $A(G)$ and $\omega(G)$ for its clique number.

*Corresponding author.
Email address: pgiacome@gmail.com (Piero Giacomelli)

Nosal [13] proved that $\lambda_1(G) \le \sqrt{m}$ for every triangle-free graph. Nikiforov [12] extended this to all graphs.

Theorem 1.1 (Nikiforov [12]). *Every graph G with m edges and $\omega(G) = \omega \ge 2$ satisfies*

$$\lambda_1^2(G) \;\le\; 2\Big(1 - \frac{1}{\omega}\Big)m. \tag{1}$$

Equality holds if and only if G, after removing isolated vertices, is a complete bipartite graph (when $\omega = 2$) or a complete regular ω-partite graph (when $\omega \ge 3$).

Bollobás and Nikiforov [2] proposed the following strengthening.

Conjecture 1.2 (Bollobás–Nikiforov [2]). *Every graph $G \ne K_n$ with m edges and clique number ω satisfies*

$$\lambda_1^2(G) + \lambda_2^2(G) \;\le\; 2\Big(1 - \frac{1}{\omega}\Big)m.$$

The complete graph must be excluded: K_n has $\lambda_1 = n-1$, $\lambda_2 = -1$ and $2(1-1/n)\binom{n}{2} = (n-1)^2$, so $\lambda_1^2 + \lambda_2^2 = (n-1)^2 + 1$ exceeds the bound.

Known cases.. Lin, Ning and Wu [10] proved the conjecture for triangle-free graphs and characterized the extremal graphs. For weakly perfect graphs, that is, graphs with $\chi(G) = \omega(G)$, the conjecture follows from the inertial bound $\chi(G) \ge 1 + s^+/s^-$ of Ando and Lin [1], where $s^{\pm}$ is the sum of the squares of the positive (resp. negative) eigenvalues; see Section 4.1. Zhang [15] proved the conjecture for regular graphs, with equality exactly for balanced Turán graphs and for disjoint unions of two balanced Turán graphs of the same order. Kumar and Pragada [9] proved a generalization due to Elphick, Linz and Wocjan [4] for graphs with $O(m^{3/2-\varepsilon})$ triangles, and Liu and Bu [11] showed that the conjecture holds asymptotically almost surely for the random graph $G(n,p)$. Kannan, Kumar and Pragada [8] proposed an edge-localized version and verified it for diamond-free graphs and random graphs. Most recently, Coutinho, Liu, Spier, Tang and Zhang [3] announced a proof of the conjecture for all graphs, in a stronger weighted form, accompanied by a formalization in the Lean proof assistant; at the time of writing this work has not been peer-reviewed.

Contents of this note.. Our aim is modest and complementary. In Section 3 we give a short self-contained proof of the conjecture for complete multipartite graphs (Theorem 3.4). The key step, $\lambda_2 \le 0$, is classical (it follows from a theorem of Smith [14]); we deduce it from an elementary quadratic-form identity. The inequality itself also follows from the weakly perfect case, since complete r-partite graphs satisfy $\chi = \omega = r$. We then exhibit a family of equality cases of Conjecture 1.2 that is strictly larger than the family of Turán graphs (Proposition 3.5). In Section 4 we collect what the known results give for K_4-free graphs, and we record three examples showing that some natural approaches to the K_4-free case cannot work.

The identity of Lemma 3.1, the spectral statement of Lemma 3.2, and the examples in Section 4.2 have been verified in the Lean 4 proof assistant with the Mathlib library; the files are available from the author.

2. Preliminaries

For a real symmetric $n \times n$ matrix M with eigenvalues $\mu_1 \ge \cdots \ge \mu_n$ and $\mathbf{x} \neq \mathbf{0}$, write $R_M(\mathbf{x}) = \mathbf{x}^\top M\mathbf{x}/\mathbf{x}^\top\mathbf{x}$. We use the following standard facts; see, e.g., [7].

Theorem 2.1 (Courant–Fischer)**.** *For $1 \le k \le n$,*

$$\mu_k = \min_{W} \max_{\mathbf{x}\in W\setminus\{\mathbf{0}\}} R_M(\mathbf{x}),$$

where W ranges over subspaces of $\mathbb{R}^n$ of dimension $n-k+1$.

Theorem 2.2 (Weyl)**.** *If M and E are real symmetric $n \times n$ matrices, then $\mu_k(M+E) \ge \mu_k(M) + \mu_n(E)$ for every k.*

Theorem 2.3 (Interlacing [5])**.** *If H is an induced subgraph of G on k vertices, then $\lambda_{n-k+i}(G) \le \lambda_i(H) \le \lambda_i(G)$ for $i = 1, \dots, k$.*

Since $A(G)$ has zero diagonal,

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{i=1}^{n} \lambda_i^2 = \operatorname{tr} A(G)^2 = 2m. \tag{2}$$

In particular $\lambda_1(G) > 0$ whenever $m > 0$. The spectrum of a disjoint union is the union (with multiplicities) of the spectra of its components.

Lemma 2.4. *If G has $n \ge 2$ vertices and is not complete, then $\lambda_2(G) \ge 0$.*

Proof. Two non-adjacent vertices induce $2K_1$, whose eigenvalues are $0, 0$. By Theorem 2.3, $\lambda_2(G) \ge \lambda_2(2K_1) = 0$. □

3. Complete multipartite graphs

Let $G = K_{n_1,\dots,n_r}$ with parts $V_1, \dots, V_r$, $r \geq 1$, $n_i \geq 1$, and $n = n_1 + \cdots + n_r$. Two vertices are adjacent if and only if they lie in different parts. For $\mathbf{x} \in \mathbb{R}^n$ put $S_i(\mathbf{x}) = \sum_{v \in V_i} x_v$.

Lemma 3.1. *For every $\mathbf{x} \in \mathbb{R}^n$,*

$$\mathbf{x}^{\top} A(G)\, \mathbf{x} \;=\; \sum_{i \neq j} S_i(\mathbf{x}) S_j(\mathbf{x}) \;=\; \Big(\sum_{v} x_v\Big)^2 - \sum_{i=1}^{r} S_i(\mathbf{x})^2 .$$

Proof. $\mathbf{x}^{\top} A(G) \mathbf{x} = \sum_{u,v} A_{uv} x_u x_v$, and $A_{uv} = 1$ exactly when $u \in V_i$, $v \in V_j$ with $i \neq j$. Grouping the terms by parts gives $\sum_{i \neq j} S_i S_j$, and $\big(\sum_i S_i\big)^2 = \sum_i S_i^2 + \sum_{i \neq j} S_i S_j$. □

Lemma 3.2. *Let $G = K_{n_1,\dots,n_r}$. Then $\lambda_2(G) \leq 0$, so G has exactly one positive eigenvalue if $r \geq 2$. If moreover $n > r$, then $\lambda_2(G) = 0$.*

Proof. Let $W = \{\mathbf{x} : \mathbf{1}^{\top}\mathbf{x} = 0\}$, a subspace of dimension $n - 1$. For $\mathbf{x} \in W$, Lemma 3.1 gives $\mathbf{x}^{\top} A(G) \mathbf{x} = -\sum_i S_i(\mathbf{x})^2 \leq 0$. By Theorem 2.1 with $k = 2$, $\lambda_2(G) \leq \max_{\mathbf{x} \in W \setminus \{\mathbf{0}\}} R_{A(G)}(\mathbf{x}) \leq 0$. If $r \geq 2$ then $m > 0$, so $\lambda_1(G) > 0$ by (2). If $n > r$, some part contains two vertices $u \neq w$; the rows of u and w in $A(G)$ coincide, so $\mathbf{e}_u - \mathbf{e}_w$ lies in the kernel of $A(G)$. Thus 0 is an eigenvalue, and since $\lambda_2(G) \leq 0$ and only $\lambda_1(G)$ is positive, we get $\lambda_2(G) = 0$. □

Remark 3.3. Lemma 3.2 is a special case of a theorem of Smith [14]: a graph has exactly one positive eigenvalue if and only if its non-isolated vertices induce a complete multipartite graph. The proof above avoids computing the spectrum; for completeness we note that the nonzero eigenvalues of $K_{n_1,\dots,n_r}$ other than $-p$ (with multiplicity one less than the number of parts of size p) are the roots of $\sum_i n_i/(\lambda + n_i) = 1$.

Theorem 3.4. *Let $G = K_{n_1,\dots,n_r}$ with $r \geq 2$, $n \geq r + 1$ and m edges. Then*

$$\lambda_1^2(G) + \lambda_2^2(G) \;\leq\; 2\Big(1 - \frac{1}{r}\Big) m .$$

If $r = 2$, equality holds for every complete bipartite graph K_{n_1,n_2} with $n_1 + n_2 \geq 3$. If $r \geq 3$, equality holds if and only if $n_1 = \cdots = n_r$.

Proof. We have $\omega(G) = r$ and, by Lemma 3.2, $\lambda_2(G) = 0$. Hence $\lambda_1^2 + \lambda_2^2 = \lambda_1^2$, and the inequality is Theorem 1.1. Since G has no isolated vertices, Theorem 1.1 gives equality exactly when $r = 2$, or when $r \geq 3$ and G is regular, that is, all parts have the same size. Directly: $\lambda_1(K_{a,b}) = \sqrt{ab}$ and $m = ab$; and for $n_1 = \cdots = n_r = p$ the all-ones vector is an eigenvector with eigenvalue $(r-1)p$, while $2(1-1/r)\binom{r}{2}p^2 = (r-1)^2p^2$. □

The equality cases of Conjecture 1.2 are not limited to Turán graphs and complete bipartite graphs.

Proposition 3.5. *Fix $r \geq 2$ and call a graph r-*extremal *if it is a complete bipartite graph $K_{a,b}$ $(a, b \geq 1)$ when $r = 2$, or a balanced complete r-partite graph $K_{p,\ldots,p}$ $(p \geq 1)$ when $r \geq 3$. Let $G = H_1 \cup H_2 \cup sK_1$ $(s \geq 0)$ be a disjoint union, where H_1 is r-extremal and H_2 is either empty or r-extremal. If G is not complete, then $\omega(G) = r$ and*

$$\lambda_1^2(G) + \lambda_2^2(G) = 2\Big(1 - \frac{1}{r}\Big)m.$$

Proof. Clearly $\omega(G) = r$. For an r-extremal graph H we have $\lambda_1(H)^2 = 2(1-1/r)\,e(H)$ (proof of Theorem 3.4, which also covers K_r, the case $p = 1$), and by Lemma 3.2 all other eigenvalues of H are at most 0. If H_2 is non-empty, the two positive eigenvalues of G are $\lambda_1(H_1)$ and $\lambda_1(H_2)$, so $\lambda_1^2(G) + \lambda_2^2(G) = 2(1-1/r)\big(e(H_1) + e(H_2)\big) = 2(1-1/r)m$. If H_2 is empty, then $\lambda_1(G) = \lambda_1(H_1)$, and $\lambda_2(G) = 0$: either $s \geq 1$, or H_1 has a part with at least two vertices (otherwise $G = K_r$ is complete), and in both cases 0 is an eigenvalue while no other eigenvalue is positive. □

Example 3.6. The graphs $2K_2$, $K_{2,3} \cup K_{1,4}$, $K_4 \cup K_4$, $K_4 \cup K_1$, $K_{2,2,2} \cup K_{3,3,3}$ and $K_{3,3,3} \cup 3K_1$ all attain equality in Conjecture 1.2. The first is triangle-free but not complete bipartite, and the $\omega = 3$ examples are not Turán graphs. On the other hand, equality fails for $K_{2,2,3} \cup K_{2,2,2}$ (unbalanced parts), for $K_{2,2,2} \cup K_{3,3}$ (components with different clique numbers) and for $3K_{2,2,2}$ (three components).

4. The K_4-free case

For a K_4-free graph, $\omega(G) \leq 3$, and Conjecture 1.2 reads $\lambda_1^2 + \lambda_2^2 \leq 4m/3$ whenever G is not complete.

4.1. Cases that follow from known results

Theorem 4.1 (Ando–Lin [1])**.** *For every graph G with at least one edge, $\chi(G) \geq 1+s^+/s^-$, where s^+ and s^- are the sums of the squares of the positive and of the negative eigenvalues of $A(G)$.*

Corollary 4.2. *Conjecture 1.2 holds for every weakly perfect graph.*

Proof. Let $G \neq K_n$ with $\chi(G) = \omega(G) = \omega$. If $m = 0$ there is nothing to prove, so assume $m > 0$; then $n \geq 2$ and $\lambda_2(G) \geq 0$ by Lemma 2.4, whence $\lambda_1^2+\lambda_2^2 \leq s^+$. By (2), $s^++s^- = 2m$, and Theorem 4.1 gives $s^+ \leq (\chi-1)s^- = (\chi-1)(2m-s^+)$, that is, $s^+ \leq 2(1-1/\chi)m = 2(1-1/\omega)m$. □

Theorem 4.3. *Let G be a K_4-free graph with $\chi(G) \leq 3$ that is not complete. Then $\lambda_1^2(G)+\lambda_2^2(G) \leq 4m/3$.*

Proof. If $\omega(G) = 3$ then $\chi(G) = \omega(G) = 3$ and Corollary 4.2 applies. If $\omega(G) \leq 2$, the result of Lin, Ning and Wu [10] (or Corollary 4.2 when $\chi(G) = 2$) gives $\lambda_1^2+\lambda_2^2 \leq m \leq 4m/3$; graphs with $m = 0$ are trivial. □

Remark 4.4. The hypothesis that G is not complete cannot be dropped: K_2 is K_4-free with $\chi = 2$, but $\lambda_1^2+\lambda_2^2 = 2 > 4m/3$.

Corollary 4.5 (Zhang [15])**.** *If G is a regular K_4-free graph that is not complete, then $\lambda_1^2(G)+\lambda_2^2(G) \leq 4m/3$.*

4.2. Three approaches that fail

We record three examples showing that some natural strategies for the K_4-free case cannot succeed as stated.

Hoffman-type bounds.. For a regular graph, Hoffman's ratio bound [6] states $\alpha(G) \leq -n\lambda_n/(\lambda_1-\lambda_n)$. It is tempting to combine it with $\alpha(G) \geq n/3$ to obtain $|\lambda_n| \geq \lambda_1/2$ and then use $\lambda_1^2+\lambda_2^2 \leq 2m-\lambda_n^2$. Both steps fail for irregular graphs.

Example 4.6. (a) The star $K_{1,3}$ has $\lambda_1 = \sqrt{3}$ and $\lambda_4 = -\sqrt{3}$, so the ratio bound would give $\alpha \leq 2$, while $\alpha(K_{1,3}) = 3$.

(b) Let G be the graph on $\{0,\dots,5\}$ with edges 01, 03, 05, 12, 13, 14, 15, 24, 25, 34. Then G is K_4-free with $\alpha(G) = 2 = n/3$, and its characteristic polynomial is

$$(x^2-2x-5)(x^2+x-1)^2.$$

Hence $\lambda_1 = 1+\sqrt{6}$ and $\lambda_6 = -(1+\sqrt{5})/2$, so $|\lambda_6| < \lambda_1/2$.

Even when $|\lambda_n| \geq \lambda_1/2$ does hold, the resulting bound $\lambda_1^2+\lambda_2^2 \leq 2m-\lambda_1^2/4$ is at most $4m/3$ only if $\lambda_1^2 \geq 8m/3$, which contradicts (1) for $\omega \leq 3$. So the approach is too weak in any case.

Spectral stability.. One might hope that if $\lambda_1^2+\lambda_2^2$ is close to $4m/3$, then G is close in edit distance to a complete tripartite graph. Write $d_{\text{edit}}(G, \mathcal{T}_3)$ for the minimum number of edge insertions and deletions turning G into a complete tripartite graph on the same vertex set (parts may be empty). The disjoint union $K_{k,k,k} \cup kK_1$ satisfies $\lambda_1^2+\lambda_2^2 = 4m/3$ but is far from $\mathcal{T}_3$. The next proposition shows that requiring connectivity does not help.

Proposition 4.7. *For $k \geq 1$ let G_k consist of two disjoint copies of $K_{k,k,k}$ joined by a single edge. Then G_k is connected and K_4-free, with $n = 6k$ and $m = 6k^2+1$, and*

$$\lambda_1^2(G_k)+\lambda_2^2(G_k) \;\geq\; 2(2k-1)^2, \qquad d_{\text{edit}}(G_k, \mathcal{T}_3) \;\geq\; 2k^2-1.$$

In particular $(\lambda_1^2+\lambda_2^2)/m \to 4/3$, while $d_{\text{edit}}(G_k, \mathcal{T}_3) \geq n^2/18-1$.

Proof. $A(G_k) = A(2K_{k,k,k}) + E$, where E is the adjacency matrix of a single edge, with smallest eigenvalue -1. Since $\lambda_1(2K_{k,k,k}) = \lambda_2(2K_{k,k,k}) = 2k$, Theorem 2.2 gives $\lambda_1(G_k), \lambda_2(G_k) \geq 2k-1$.

Let A and B be the vertex sets of the two copies, and let H be a complete tripartite graph with parts P_1, P_2, P_3. Put $a_j = |A \cap P_j|$, $b_j = |B \cap P_j|$ and $t = \min(\max_j a_j, \max_j b_j)$; by symmetry assume $\max_j a_j = t$, attained at $j = 1$. The number of edges of H between A and B is $\sum_{i \neq j} a_i b_j = 9k^2 - \sum_j a_j b_j \geq 9k^2 - 3kt$, and all but at most one of them must be inserted. If $t < k$, this already gives at least $9k^2-3k^2-1 \geq 2k^2-1$ edits. If $t \geq k$, the set $A \cap P_1$ consists of t vertices of a copy of $K_{k,k,k}$, say $s_1, s_2, s_3 \leq k$ of them in its three parts, and it spans $\bigl(t^2 - \sum_i s_i^2\bigr)/2$ edges of G_k, all of which must be deleted. By convexity, $\sum_i s_i^2 \leq k^2 + (t-k)^2$ if $t \leq 2k$ and $\sum_i s_i^2 \leq 2k^2 + (t-2k)^2$ if $t > 2k$; in both cases at least $k(t-k)$ edges must be deleted. In total we need at least $9k^2-3kt-1+k(t-k) = 8k^2-2kt-1 \geq 2k^2-1$ edits, because $t \leq 3k$. □

Zykov symmetrization.. For non-adjacent vertices u, v of G, let $Z(G; u, v)$ be the graph obtained by replacing the neighbourhood of u with that of v. This operation does not increase the clique number. It is sometimes used as if it never decreased λ_1, but the direction matters: if G is $K_{1,3}$ together with an isolated vertex v, and u is the centre of the star, then $\lambda_1(G) = \sqrt{3}$ while $Z(G; u, v)$ has no edges. The correct statement is the following.

Lemma 4.8. *Let $\mathbf{x} \geq 0$ be a unit eigenvector of $A(G)$ for $\lambda_1(G)$, and let u, v be non-adjacent vertices with $\sum_{w \in N(v)} x_w \geq \sum_{w \in N(u)} x_w$. Then $\lambda_1(Z(G; u, v)) \geq \lambda_1(G)$.*

Proof. Since $u \notin N(v)$ and $v \notin N(u)$, only the terms involving u change, and $\mathbf{x}^\top A(Z(G; u, v))\mathbf{x} - \mathbf{x}^\top A(G)\mathbf{x} = 2x_u\big(\sum_{w \in N(v)} x_w - \sum_{w \in N(u)} x_w\big) \geq 0$. By the Rayleigh principle, $\lambda_1(Z(G; u, v)) \geq \mathbf{x}^\top A(Z(G; u, v))\mathbf{x} \geq \lambda_1(G)$. □

Repeated symmetrization in the right direction turns every K_4-free graph into a complete multipartite graph with at most three parts (together with isolated vertices). However, both $\lambda_1^2 + \lambda_2^2$ and m change along the way, and no monotonicity of their ratio is known. So this does not reduce Conjecture 1.2 to Theorem 3.4.

5. Concluding remarks

If the announced proof [3] is confirmed, Conjecture 1.2 is settled. Natural questions remain:

1. Characterize all graphs attaining equality in Conjecture 1.2. Proposition 3.5 gives a family that properly contains the Turán graphs; the triangle-free case is settled in [10], and the regular case in [15].

2. The generalization of Elphick, Linz and Wocjan [4], which replaces $\lambda_1^2 + \lambda_2^2$ by the sum of the squares of the ω largest eigenvalues (when positive).

3. The edge-localized version of Kannan, Kumar and Pragada [8].

4. A spectral stability theorem for Conjecture 1.2. Proposition 4.7 shows that closeness of $\lambda_1^2 + \lambda_2^2$ to $4m/3$ does not force a connected K_4-free graph to be close to a complete tripartite graph, so any such statement must allow for the extremal unions of Proposition 3.5.

Acknowledgements

The author thanks Dr. Shengtong Zhang for helpful discussions.

References

[1] T. Ando, M. Lin, Proof of a conjectured lower bound on the chromatic number of a graph, *Linear Algebra Appl.* **485** (2015) 480–484.

[2] B. Bollobás, V. Nikiforov, Cliques and the spectral radius, *J. Combin. Theory Ser. B* **97** (2007) 859–865.

[3] G. Coutinho, Y. Liu, T. Jung Spier, Q. Tang, S. Zhang, A proof of the Bollobás–Nikiforov conjecture in Lean, preliminary research note and formalization, 2026, https://github.com/shengtongz/BN.

[4] C. Elphick, W. Linz, P. Wocjan, Two conjectured strengthenings of Turán's theorem, *arXiv:2101.05229*.

[5] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226–228** (1995) 593–616.

[6] A.J. Hoffman, On eigenvalues and colorings of graphs, in: *Graph Theory and its Applications* (B. Harris, Ed.), Academic Press, 1970, pp. 79–91.

[7] R.A. Horn, C.R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

[8] M.R. Kannan, H. Kumar, S. Pragada, Localization of spectral Turán-type theorems, *arXiv:2512.01409*, 2025.

[9] H. Kumar, S. Pragada, Bollobás–Nikiforov conjecture for graphs with not so many triangles, *arXiv:2407.19341*, 2024.

[10] H. Lin, B. Ning, B. Wu, Eigenvalues and triangles in graphs, *Combin. Probab. Comput.* **30** (2021) 258–270.

[11] C. Liu, C. Bu, Bollobás–Nikiforov conjecture holds asymptotically almost surely, *arXiv:2501.07137*, 2025.

[12] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combin. Probab. Comput.* **11** (2002) 179–189.

[13] E. Nosal, *Eigenvalues of Graphs*, Master's Thesis, University of Calgary, 1970.

[14] J.H. Smith, Some properties of the spectrum of a graph, in: *Combinatorial Structures and their Applications* (Proc. Calgary Internat. Conf., 1969), Gordon and Breach, New York, 1970, pp. 403–406.

[15] S. Zhang, On the first two eigenvalues of regular graphs, *Linear Algebra Appl.* **686** (2024) 102–110.