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STRETCHING AND ROTATION OF PLANAR QUASICONFORMAL MAPPINGS ON A LINE

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Abstract. In this article we examine stretching and rotation of planar quasiconformal mappings on a line. We show that for almost every point on the line the set of complex stretching exponents is contained in the disk $\overline{B}(1/(1-k^4), k^2/(1-k^4))$, yielding a quadratic improvement in comparison to the known optimal estimate on a general set with Hausdorff dimension 1. Our proof is based on holomorphic motions and known dimension estimates for quasicircles. In addition we establish a lower bound for the dimension of the quasiconformal image of a 1-dimensional subset of a line.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Then the classical local Hölder continuity result

$$C^{-1}|z - z_0|^K \leq |f(z) - f(z_0)| \leq C|z - z_0|^{\frac{1}{K}}$$

where $C = C(f, z_0, K)$, known as Mori's theorem (see e.g. [4, Theorem 3.10.2]) gives the local stretching properties of f at every point $z_0 \in \mathbb{C}$. Bounds for local rotation are obtained from [5, Theorem 3.1]:

$$\limsup_{z \to z_0} \frac{|\arg(f(z) - f(z_0))|}{|\log|f(z) - f(z_0)||} \leq \frac{1}{2}\left(K - \frac{1}{K}\right).$$

Moreover, the multifractal spectra estimates from [5] give the optimal upper bound for the Hausdorff dimension of the set where a prescribed stretching and rotation can happen. Namely, if $E \subset \mathbb{C}$ has the property that if for every $z \in E$ there is a sequence $(r_j) \to 0$ such that

$$\lim_{j \to \infty} \frac{\log(f(z + r_j) - f(z))}{\log r_j} = \alpha(1 + i\gamma), \quad \alpha > 0, \gamma \in \mathbb{R},$$

then by [5, Theorem 1.4]

$$\dim E \leq 1 + \alpha - \frac{1}{k}\sqrt{(1-\alpha)^2 + (1-k^2)\alpha^2\gamma^2},$$

and this upper bound is sharp. Here $k = (K-1)/(K+1)$. We call the quantity $\alpha(1 + i\gamma)$ complex stretching exponent, see the definition in Section 2. In order to give for the stated bound a more accessible geometric interpretation, we note that equivalently for any $0 \leq s \leq 2$ outside a set of zero s-dimensional Hausdorff measure all possible limits

$$\lim_{j \to \infty} \frac{\log(f(z + r_j) - f(z))}{\log r_j} = \alpha(1 + i\gamma)$$

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lie in the closed disk with the (geometric) diameter

$$\left[\frac{1-k}{1+k}+\frac{k}{1+k}s, \frac{1+k}{1-k}-\frac{k}{1-k}s\right]. \tag{1}$$

In this paper we study behaviour of quasiconformal maps on lines, in which situation it is natural to expect a more constrained behaviour than on general 1-dimensional sets. Indeed, this is quantified by our main result as follows:

Theorem 1. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. For almost every $x \in \mathbb{R}$ with respect to 1-dimensional Lebesgue measure we have that if (r_j) is a real sequence tending to zero such that*

$$\alpha = \lim_{j\to\infty} \frac{\log|\phi(x+r_j)-\phi(x)|}{\log r_j}$$

and

$$\gamma = \lim_{j\to\infty} \frac{\arg(\phi(x+r_j)-\phi(x))}{\log|\phi(x+r_j)-\phi(x)|},$$

then $\alpha(1+i\gamma) \in \overline{B}(1/(1-k^4), k^2/(1-k^4))$. Here $k = (K-1)/(K+1)$.

One should note that the bound (1) in case $s = 1$ only yields that $\alpha(1+i\gamma)$ has to lie in circle with the diameter $[1/(1+k), 1/(1-k)]$, while Theorem 1 gives as geometric diameter the segment $[1/(1+k^2), 1/(1-k^2)]$. In terms of pure rotation, the following corollary follows immediately from Theorem 1.

Corollary 2. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. For almost every $x \in \mathbb{R}$ with respect to 1-dimensional Lebesgue measure we have that if (r_j) is a real sequence tending to zero such that*

$$\gamma = \lim_{j\to\infty} \frac{\arg(\phi(x+r_j)-\phi(x))}{\log|\phi(x+r_j)-\phi(x)|},$$

then $|\gamma| \le k^2/\sqrt{1-k^4}$. Here $k = (K-1)/(K+1)$.

We also obtain the following estimate for the Hausdorff dimension of the images of subsets of line. The sharp estimate for the dimension of the image of a general 1-dimensional set A was given in [2] as $\dim f(A) \ge 1-k$.

Theorem 3. *For any K-quasiconformal mapping $f\colon \mathbb{C} \to \mathbb{C}$ and $A \subset \mathbb{R}$ with the Hausdorff dimension 1, $\dim f(A) \ge 1-k^2$ for $k = (K-1)/(K+1)$.*

This estimate generalizes results from [11] and [13] where it was assumed that $f(\mathbb{R}) \subset \mathbb{R}$, and gives a natural counterpart of the estimate $\dim f(\mathbb{R}) \le 1+k^2$ from [16]. In a similar manner, Theorem 1 can be viewed as a 'rotational' counterpart of [16].

The proof of Theorems 1 and 3 follows the general line of argument of [2, 13] based on holomorphic motions and pressure estimates adapted to our situation. It is given in Section 3 below. Before that, we recall the basic notions and some auxiliary results in Section 2. Finally, Section 4 contains further discussion, and e.g. considers the sharpness of our results and mentions possible avenues for further research.

2. Prerequisites

We say that an orientation preserving homeomorphism $f : \Omega \to \Omega'$ is K-quasiconformal for $K \geq 1$ if $f \in W^{1,2}_{loc}(\Omega)$ and for almost every $z \in \Omega$ the directional derivatives satisfy

$$\max_{\alpha} |\partial_\alpha f(z)| \leq K \min_{\alpha} |\partial_\alpha f(z)|.$$

Equivalently, we could define quasiconformal mapping using the Beltrami equation: a homeomorphic mapping $f : \Omega \to \Omega'$ between two planar domains is a K-quasiconformal mapping if it lies in the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the Beltrami equation

$$f_{\overline{z}} = \mu f_z$$

for some measurable function $\mu : \Omega \to \mathbb{C}$ with $|\mu(z)| \leq k < 1$ where $k = (K-1)/(K+1)$. As $k < 1$, we can say that f is k-quasiconformal without ambiguity.

We also consider the notion of quasisymmetry. If $\eta : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism, then $f : \Omega \to \Omega'$ is η-quasisymmetric if for any distinct $z_0, z_1, z_2 \in \Omega$ we have

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right).$$

Of particular interest is the case where $\Omega = \mathbb{C}$. In this case, any K-quasiconformal mapping is η-quasisymmetric, where η depends only on K. The measurable Riemann mapping theorem implies that for any measurable function μ with $|\mu(z)| \leq k < 1$ there is a unique normalized solution f solving the corresponding Beltrami equation and having $f(0) = 0$ and $f(1) = 1$.

For more information about quasiconformal mappings, see for example [4].

We will consider local stretching and rotation of quasiconformal mappings. To motivate this, consider mappings of the form $f(z) = f(z_0) + \omega \cdot \frac{z-z_0}{|z-z_0|}|z - z_0|^{\alpha(1+i\gamma)}$ defined in the disk $B(z_0, r)$, where $\omega \in \mathbb{C} \setminus \{0\}$, $\alpha > 0$ and $\gamma \in \mathbb{R}$. For these mappings, $|f(z) - f(z_0)| = |\omega||z - z_0|^\alpha$ and $\arg(f(z) - f(z_0)) = \arg\omega + \arg(z - z_0) + \alpha\gamma \log|z - z_0|$. This means that α measures the rotation of the map near z_0, while γ determines the local geometric rotation. Moreover, for any path τ in $B(z_0, r)$ with $\tau(0) = z_0$ and $\tau(t) \neq z_0$ for $t \in (0, 1]$ we have

$$\lim_{t \to 0} \frac{\log(f(\tau(t)) - f(z_0))}{\log(\tau(t) - z_0)} = \alpha(1 + i\gamma)$$

and this limit does not depend on the branch of the logarithms defined on $\tau((0, 1])$ and on the image $f(\tau((0, 1]))$.

Let $f : \Omega \to \Omega'$ be quasiconformal and $w \in \Omega$ be a point. Define the set of *complex stretching exponents* $\mathcal{X}_f(w) \subset \widehat{\mathbb{C}}$ by setting

$$\mathcal{X}_f(w) = \bigcap_{0 < t_0 \leq 1} \overline{\left\{ \frac{\log(f(w + t) - f(w))}{\log(t)} : 0 < t < t_0 \right\}},$$

or in other words, the limit points of the quotient $\log(f(w + t) - f(w))/\log(t)$ as $t \to 0$. This definition again does not depend on the chosen branch of the complex logarithm on $f((w, w+t_0[)$. With this definition, we are only approaching w along one ray, but this does not affect the limits. We refer to [5] for a more thorough discussion of defining the concept of rotation for quasiconformal maps.

3. Proofs of Theorems 1 and 3

We first prove an auxiliary estimate for holomorphic functions in Lemma 4. That allows us to prove the main technical result, Proposition 5. During a part of the proof of Proposition 5, we provide details of the quite standard argument in Lemma 6 for reader's convenience. Finally, we prove both Theorem 3 and Theorem 1.

We begin by proving the following auxiliary lemma for holomorphic mappings.

Lemma 4. *Let $h : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $h(0) = \varepsilon$ and for all $r \in (0,1)$ we have the inclusion $h(r\mathbb{D}) \subset B((1-r^2)/2, (1+r^2)/2)$. Then for any fixed $0 \leq k < 1$ we have*

$$|h(k)| \leq k^2 + o(1)$$

as $\varepsilon \to 0$.

Proof. Consider the following family $\mathcal{F}_\varepsilon$ of holomorphic functions

$$\mathcal{F}_\varepsilon = \{f : \mathbb{D} \to \mathbb{D} : |f(z) - (1-|z|^2)/2| \leq (1+|z|^2)/2, |f(0)| \leq \varepsilon\}.$$

Let $\psi(\varepsilon) := \sup\{|f(k)| : f \in \mathcal{F}_\varepsilon\}$. As the family $\mathcal{F}_\varepsilon$ is normal by Montel's theorem, for each $\varepsilon > 0$ we may find a sequence (f_n) in $\mathcal{F}_\varepsilon$, converging uniformly on compact subsets, such that $|f_n(k)| \to \psi(\varepsilon)$. It is easy to see that the limit function $h_\varepsilon := \lim_{n\to\infty} f_n$ belongs to $\mathcal{F}_\varepsilon$.

Clearly ψ is decreasing in ε and we need to show that $\lim_{\varepsilon\to 0} \psi(\varepsilon) \leq k^2$, or equivalently that $\lim_{m\to\infty} |h_{1/m}(k)| \leq k^2$. Using the fact that $\mathcal{F}_1$ is normal, we can further assume that $(h_{1/m})$ converges uniformly on compact subsets to a limit function g. Especially, then $\lim_{m\to\infty} h_{1/m}(k) = g(k)$.

The limit function $g : \mathbb{D} \to \mathbb{D}$ is analytic and satisfies $g(0) = 0$ together with $|g(z) - (1 - |z|^2)/2| \leq (1+|z|^2)/2$ for any $z \in \mathbb{D}$. This implies that $g'(0) = 0$ as otherwise we would obtain a contradiction by considering small values of z to the direction $-\overline{g'(0)}$. Finally, the Schwarz lemma yields that $|g(z)/z| \leq |z|$. In particular, $|g(k)| \leq k^2$, which proves the lemma. □

The following Proposition is crucial, as it allows us to prove both Theorem 1 and Theorem 3.

Proposition 5. *Let $0 \leq k < 1$ and let $\phi : \mathbb{C} \to \mathbb{C}$ be a k-quasiconformal mapping with $\phi(0) = 0$ and $\phi(1) = 1$. Assume that $k < \rho < 1$ and $0 < \delta \leq 1$. There exists a constant $a = a(\rho) > 0$ with the following property: for any (finite or countable) collection of disjoint disks $B(x_j, r_j)$ contained in $\mathbb{D}$ with $x_j \in \mathbb{R}$ and $\sum_j (ar_j)^\delta \geq 1$, there exists a probability distribution $p = (p_j)$ such that*

$$\operatorname{Re}\left(\frac{-\sum_j p_j \log p_j}{-\sum_j p_j \log\Big(a\big(\phi(x_j + r_j) - \phi(x_j)\big)\Big)}\right) \geq 1 - (k/\rho)^2 - R(1-\delta) \tag{2}$$

and

$$\frac{\sum_j p_j \log\Big(a\big(\phi(x_j + r_j) - \phi(x_j)\big)\Big)}{\sum_j p_j \log(ar_j)} \in \bigcup_{b\in[\delta,1]} \overline{B}(b/(1-s^2), bs/(1-s^2)), \tag{3}$$

where $s = (k/\rho)^2 + R(1-\delta)$ for some function R with $R(t) \to 0$ as $t \to 0$.

Proof. Let us first assume that the given collection of disks is finite. Let μ be the Beltrami coefficient of ϕ, and set

$$\mu_\lambda = \lambda \frac{\mu}{k}.$$

This defines Beltrami coefficients that depend analytically on $\lambda \in \mathbb{D}$. If we denote by ϕ_λ the unique solutions of the corresponding Beltrami equations with fixed points 0 and 1, then ϕ_0 is the identity mapping of $\mathbb{C}$. Then ϕ_λ also depends analytically on λ by Ahlfors-Bers theorem [1]. Moreover, the original ϕ can obtained from $\phi_k = \phi$.

Recall that $\rho \in (k, 1)$. The mappings ϕ_λ have Beltrami coefficients bounded uniformly above by ρ for all $|\lambda| < \rho$. Thus by uniform quasiconformality, the mappings ϕ_λ are uniformly η-quasisymmetric for some $\eta = \eta_\rho$, see [4, Theorem 3.5.3]. This implies weak quasisymmetry: whenever $|x - z| \le |y - z|$ and $|\lambda| < \rho$, we have

$$\frac{|\phi_\lambda(x) - \phi_\lambda(z)|}{|\phi_\lambda(y) - \phi_\lambda(z)|} \le \eta\left(\frac{|x - z|}{|y - z|}\right) \le \eta(1) = C = C(\rho).$$

Then the disks $B(\phi_\lambda(x_j), \frac{1}{C}|\phi_\lambda(x_j + r_j) - \phi_\lambda(x_j)|)$ are disjoint because they are contained in disjoint sets $\phi_\lambda(B(x_j, r_j))$. Moreover, these disks are contained in $B(0, C)$ as $|\phi_\lambda(z) - \phi_\lambda(0)|/|\phi_\lambda(1) - \phi_\lambda(0)| \le \eta(|z|) \le C$ for $z \in \mathbb{D}$, and hence we have the following holomorphic family of disks in unit disk:

$$B\left(\frac{1}{C}\phi_\lambda(x_j), \frac{1}{C^2}|\phi_\lambda(x_j + r_j) - \phi_\lambda(x_j)|\right). \tag{4}$$

We fix the constant in the theorem by setting $a = 1/C^2$. Let C_λ be the Cantor set generated by the above collection (4) of disks, having self-similarity inside each of these disks with rotation in directions given by $\phi_\lambda(x_j + r_j) - \phi_\lambda(x_j)$. Then C_λ lies on the image of the real line under a $|\lambda|/\rho$-quasiconformal mapping. We refer to Lemma 6 below for this fact and a more precise definition of the Cantor set.

Let us denote the '*complex radii*' by $r_j(\lambda) = a(\phi_\lambda(x_j + r_j) - \phi_\lambda(x_j)) \in \mathbb{C} \setminus \{0\}$. We use Jensen's inequality on the pressure function to obtain the following auxiliary result for any strictly positive probability distribution p (meaning that $p_j > 0$, $\sum_j p_j = 1$) and $d \in (0, 2]$:

$$P_\lambda(d) := \log \sum_j |r_j(\lambda)|^d = \log \sum_j p_j \frac{|r_j(\lambda)|^d}{p_j} \ge \sum_j p_j \log \frac{|r_j(\lambda)|^d}{p_j} = I_p - d \operatorname{Re} \Lambda_p(\lambda), \tag{5}$$

with the equality reached when all $|r_j(\lambda)|^d/p_j$ have the same value. Above, I_p is the *entropy*

$$I_p = -\sum_j p_j \log p_j$$

and Λ_p is the '*complex Lyapunov exponent*'

$$\Lambda_p(\lambda) = -\sum_j p_j \log r_j(\lambda).$$

By choosing a holomorphic branch of the logarithm, the function $\Lambda_p(\lambda)$ becomes holomorphic in λ. By Remark 2.3 of [5] this choice of the branch is consistent with our geometric definition in Section 2.

Let d_0 be the Hausdorff dimension of C_λ. It easily follows that $P_\lambda(d_0) = 0$. In fact, when this observation is combined with the basic dimension formula for self-similar fractals (see e.g. [10, Theorem 4.14]). we see that the following are equivalent:

(i) $d \leq d_0$.

(ii) $P_\lambda(d) \geq 0$.

(iii) There is a probability distribution p such that $I_p - d \operatorname{Re} \Lambda_p(\lambda) \geq 0$.

By assumption of the Proposition we have $\sum (ar_j)^\delta \geq 1$, or in other words, $P_0(\delta) \geq 0$. Let p be the maximizer of $I_p - \delta \operatorname{Re} \Lambda_p(0)$ in the variational principle and define the holomorphic function

$$\Phi(\lambda) = 1 - \frac{I_p}{\Lambda_p(\lambda)}, \quad \lambda \in \rho\mathbb{D}$$

We choose the branches of the logarithms in the sum for Λ_p so that $\operatorname{Im} \log r_j(0) = 0$. As obviously $\dim C_\lambda \leq 2$, it follows that $I_p \leq 2 \operatorname{Re} \Lambda_p(\lambda)$ for any $|\lambda| < \rho$, and thus $\Phi(\rho\mathbb{D}) \subset \mathbb{D}$.

For any λ with $|\lambda| < \rho$, Lemma 6 below verifies that C_λ lies on a $|\lambda|/\rho$-quasicircle. As any such quasicircle has a Hausdorff dimension at most $1 + |\lambda|^2/\rho^2$ by [16], it follows that $I_p \leq (1 + |\lambda|^2/\rho^2) \operatorname{Re} \Lambda_p(\lambda)$. This means that

$$\Phi(\lambda) \in \overline{B}((1 - |\lambda|^2/\rho^2)/2, (1 + |\lambda|^2/\rho^2)/2),$$

or in other words, Φ maps a disk of radius r centered at origin into a disk with (geometric) diameter $[-r^2/\rho^2, 1]$.

The logarithms in the sum for $\Lambda_p(0)$ are real. Then $I_p - \delta \Lambda_p(0) \geq 0$, or $I_p/\Lambda_p(0) \geq \delta$. It follows that $0 \leq \Phi(0) \leq 1 - \delta$. Lemma 4 applied to $\lambda \mapsto \Phi(\rho\lambda)$ then implies that

$$|\Phi(k)| \leq (k/\rho)^2 + R(1 - \delta), \tag{6}$$

where R is a mapping from $[0, 1]$ that has $R(t) \to 0$ as $t \to 0$. This proves the first statement (2) of the theorem. We have

$$\frac{\sum_j p_j \log r_j(k)}{\sum_j p_j \log r_j(0)} = \frac{1 - \Phi(0)}{1 - \Phi(k)},$$

where $\delta \leq 1 - \Phi(0) \leq 1$, and $1 - \Phi(k) \in B(1, (k/\rho)^2 + R(1 - \delta))$. Setting $s = (k/\rho)^2 + R(1 - \delta)$, this implies that

$$\frac{\sum_j p_j \log r_j(k)}{\sum_j p_j \log r_j(0)} \in \overline{B}(b/(1 - s^2), bs/(1 - s^2))$$

for some $b \in [\delta, 1]$. This proves the second statement (3) of the theorem. Finally, the proof is finished by observing that the case of countably many disks is obtained by a simple limiting argument since our estimates are uniform in the number of the disks. □

During the proof of the preceding proposition, the Cantor set determined by the disks in holomorphic motion lies always on a quasicircle if the disks are centered on the real line for $\lambda = 0$, see [11]. We provide the details of this for the convenience of the reader.

Lemma 6. *In the proof of Proposition 5, C_λ lies on a $|\lambda|/\rho$-quasiconformal image of the real line.*

Proof. The disks

$$B\left(\frac{1}{C}\phi_\lambda(x_j), \frac{1}{C^2}|\phi_\lambda(x_j+r_j)-\phi_\lambda(x_j)|\right)$$

have disjoint closures for every $|\lambda|<\rho$. Letting $w_j(\lambda)$ and $r_j(\lambda) := \frac{1}{C^2}(\phi_\lambda(z_j+r_j)-\phi_\lambda(z_j))$ be their centers and complex radii, we observe that both are holomorphic functions. All similarities of the form $\gamma_{j,\lambda}(z) = r_j(\lambda)z + w_i(\lambda)$ are strict contractions, and C_λ is the unique non-empty compact set such that

$$C_\lambda = \bigcup_j \gamma_{j,\lambda}(C_\lambda).$$

Any iterated map $\gamma_{j_1,\lambda}\circ\gamma_{j_2,\lambda}\circ\ldots\circ\gamma_{j_k,\lambda}$ has a unique fixed point that belongs to the Cantor set C_λ by the basic construction of C_λ as a self-similar fractal. The set of such fixed points is easily seen to be dense in C_λ since their closure F is a non-empty compact set such that

$$F = \bigcup_j \gamma_{j,\lambda}(F). \tag{7}$$

This equality follows from the fact that if z_0 is a fixed point of some iterated map γ and γ_0 is any of the contractions, the sequence of fixed points of mappings $\gamma_0\circ\gamma^k$ converges to $\gamma_0(z_0)$ as k goes to infinity. By uniqueness of the Cantor set with property (7), $F = C_\lambda$.

For any such fixed point $z = z(\lambda)$, we observe that $z(0)\mapsto z(\lambda)$ defines a holomorphic mapping. Using continuity, we can define a mapping $\Psi_\lambda : C_0 \to C_\lambda$. We have found a holomorphic motion $\Psi : \rho\mathbb{D}\times C_0 \to \mathbb{C}$. Extended λ-lemma [15, 4] allows us to extend this motion as $\Psi : \rho\mathbb{D}\times\mathbb{C}\to\mathbb{C}$, and as $C_0\subset\mathbb{R}$, the claim follows. □

Let A be a subset of real line with Hausdorff dimension 1. As a first application of Proposition 5, we obtain the lower bound for the dimension of the image of A under k-quasiconformal mapping, i.e. our Theorem 3.

Proof of Theorem 3. We may assume that $f(0)=0$ and $f(1)=1$. Let $0<\delta<1$ and $k<\rho<1$ be arbitrary. Then as A has Hausdorff dimension larger than δ, some intersection $A\cap[n,n+1)$ must have infinite δ-dimensional Hausdorff measure. Without loss of generality, we may therefore assume that $A\subset\mathbb{D}$.

Fix $\varepsilon>0$. Let $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ be a covering of $f(A)$ with disks of small enough diameters that $f^{-1}(B_\alpha)$ have a diameter at most ε. Then these preimages are contained in disks of radius equal to the diameter of the original preimage set, let us call these sets D_α. An application of Vitali's covering theorem allows us to take a disjoint countable subcollection $\{D_j = B(x_j,r_j)\}_{j\in\mathbb{N}}$ of these disks such that any of the disks $\{D_\alpha\}_{\alpha\in\mathcal{A}}$ is contained in some disk $5D_j = B(x_j,5r_j)$.

Let a be the constant from Proposition 5 for k, δ and ρ. By choosing ε small enough, we have $\sum_{j\in\mathbb{N}}(10r_j)^\delta \geq 10^\delta a^{-\delta}$ since the set A has infinite δ-dimensional Hausdorff measure. Proposition 5 can therefore be applied to this collection of disks, and estimate (2) yields that for a suitable probability distribution (p_j) we have

$$\operatorname{Re}\left(\frac{-\sum_j p_j\log p_j}{-\sum_j p_j\log\Big(a\big(f(x_j+r_j)-f(x_j)\big)\Big)}\right) \geq 1-(k/\rho)^2 - R(1-\delta).$$

It follows that

$$-\sum_j p_j \log p_j - \big(1-(k/\rho)^2 - R(1-\delta)\big)\Big(-\sum_j p_j \log\big|a\big(f(x_j+r_j)-f(x_j)\big)\big|\Big) \geq 0,$$

or equivalently

$$\sum_j p_j \log\Big(p_j^{-1}\big|a\big(f(x_j+r_j)-f(x_j)\big)\big|^{1-(k/\rho)^2-R(1-\delta)}\Big) \geq 0,$$

and thus by Jensen's inequality

$$\sum \big|a\big(f(x_j+r_j)-f(x_j)\big)\big|^{1-(k/\rho)^2-R(1-\delta)} \geq 1.$$

We conclude that $\dim f(A) \geq 1-(k/\rho)^2-R(1-\delta)$ as the original covering of $f(A)$ was arbitrary, only chosen to have small enough diameters. This holds for any $k<\rho<1$, so letting $\rho \to 1$ shows that $\dim f(A) \geq 1-k^2-R(1-\delta)$. Finally letting $\delta \to 1$ yields $\dim f(A) \geq 1-k^2$, finishing the proof. □

Next, we prove our main result.

Proof of Theorem 1. It is enough to show that given a k-quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$, then for almost every $x \in \mathbb{R}$ with respect to 1-dimensional Lebesgue measure we have

$$\mathcal{X}_\phi(x) \subset \overline{B}(1/(1-k^4), k^2/(1-k^4)). \tag{8}$$

We can assume without loss of generality that $\phi(0)=0$ and $\phi(1)=1$. Let $E \subset \mathbb{C}$ be any open convex set with positive distance from $B := \overline{B}(1/(1-k^4), k^2/(1-k^4))$, and $A \subset \mathbb{R}$ be the set of those points x with $\mathcal{X}_\phi(x) \cap E \neq \emptyset$. It suffices to show that A has measure 0.

As E and B have positive distance, we can find $0<\delta<1$ and $k<\rho<1$ such that

$$F := \bigcup_{\delta \leq b \leq 1} b\overline{B}(1/(1-s^2), s/(1-s^2))$$

has positive distance from E. Here $s=(k/\rho)^2+R(1-\delta)$ and R is from the conclusion of Proposition 5. Let a be the constant in this theorem corresponding to this choice of ρ.

Fix $0<\varepsilon<1$. For any $x \in A$, we can find $0<r_x<\varepsilon$ such that

$$\frac{\log\big(\phi(x+r_x)-\phi(x)\big)+\log a}{\log(r_x)+\log a} \in E.$$

Using Vitali covering theorem, we can find a countable collection of disjoint disks $B(x_j, r_{x_j})$ such that for any $y \in A$ there is j such that $B(y,r_y) \subset B(x_j, 5r_{x_j})$. We observe that $m(A) \leq \sum_j 5r_{x_j}$.

On the other hand, if $\sum_j (ar_{x_j})^\delta \geq 1$, we can use Proposition 5 to find a probability distribution p such that

$$\frac{\sum_j p_j \log\Big(a\big(\phi(x_j+r_{x_j})-\phi(x_j)\big)\Big)}{\sum_j p_j \log(ar_{x_j})} \in F.$$

But the left hand side is a convex combination of terms belonging in E, namely

$$\frac{\sum_j p_j \log\Big(a\big(\phi(x_j+r_{x_j})-\phi(x_j)\big)\Big)}{\sum_j p_j \log(ar_{x_j})} = \sum_j \left(\frac{-p_j\log(ar_{x_j})}{-\sum_\ell p_\ell \log(ar_{x_\ell})}\right) \frac{\log\big(\phi(x_j+r_{x_j})-\phi(x_j)\big)+\log a}{\log(r_{x_j})+\log a}.$$

This implies that $F \cap E \neq \emptyset$, a contradiction. Therefore, we must have $\sum_j (ar_{x_j})^\delta < 1$.

We have obtained

$$m(A) \leq 5 \sum_j r_{x_j} \leq 5 \sum_j r_{x_j}^{\delta} \varepsilon^{1-\delta} \leq 5 a^{-\delta} \varepsilon^{1-\delta}.$$

As $0 < \varepsilon < 1$ was arbitrary, it follows that $m(A) = 0$.

As the complement of B is a countable union of half-planes with positive distance from B, the claim follows.

□

Finally, Corollary 2 is obtained easily from Theorem 1.

Proof of Corollary 2. One simply observes that the slope of any line from the origin to a point in the disk $\overline{B}(1/(1-k^4), k^2/(1-k^4))$ lies on the interval $[-k^2(1-k^4)^{-1/2}, k^2(1-k^4)^{-1/2}]$. □

4. Further comments and discussion

We first discuss the sharpness of our results. In our proof, we used Smirnov's $(1+k^2)$-estimate for the dimension of k-quasicircles [16], but it should be noted that this bound is not sharp. It was shown by Ivrii in [8] that the asymptotical expansion of the upper bound for small k is of the form $1 + \Sigma^2 k^2 + O(k^{2.5})$, with a constant Σ^2 which is strictly smaller than one by the work of Hedenmalm [7]. This sharper version could be used to improve our results slightly since in Lemma 4 we would have a stricter constraint for the holomorphic function. In this way, the sharpness of the rotational results is tied to the sharpness in stretching. We can embed a quasiconformal mapping with (close to) extremal stretching behaviour in a holomorphic motion, and use that to find a mapping with rotational behaviour of similar order. It would be interesting to construct examples in a more direct way.

Let us describe in a somewhat heuristic manner how from a given map with extremal stretching properties, one may produce a map with lot of rotation. For that end, let $\phi : \mathbb{C} \to \mathbb{C}$ be a k-quasiconformal mapping such that $\mathcal{X}_\phi(x) \cap \{\operatorname{Re} z < 1 - ck^2\} \neq \emptyset$ for $x \in A \subset \mathbb{R}$ with $m(A) > 0$, and let it be normalized as $\phi(0) = 0$, $\phi(1) = 1$. Letting μ be the associated Beltrami coefficient, embed this ϕ in a holomorphic motion as in our proof by setting $\mu_\lambda = \lambda \frac{\mu}{k}$ and let ϕ_λ solve the Beltrami equation with this coefficient. For any $x \in A$ and $r > 0$ the function $h_{x,r} : \mathbb{D} \to \mathbb{C}$ defined as $h_{x,r}(\lambda) = \log(\phi_\lambda(x+r) - \phi_\lambda(x))/\log r - 1$ is holomorphic with $h_{x,r}(0) = 0$. Without loss of generality we may assume that for every $x \in A$ the mappings $h_{x,r}$ have the same degree d at origin. Set $\omega = e^{-\frac{\pi}{2d} i}$. Then for sufficiently small r we have $\operatorname{Re} h_{x,r}(k) < -ck^2$ and hence $\operatorname{Im} h_{x,r}(\omega k) > ck^2 + O(k^{d+1})$. This basically implies that $\phi_{\omega k}$ has rotational properties of similar order as the original ϕ has stretching properties. For concrete examples with strong stretching on a large set, we refer to [3, Theorem 5.1] where the considered Julia sets are images of the unit circle under k-quasiconformal mappings and their dimensions behave quadratically in k, which in turn implies the desired stretching behaviour.

We have formulated our results in terms of a distortion on a line for simplicity. Naturally, the same estimates hold on the unit circle. If one instead considers a conformal map in the unit disk then stronger results are available. Namely, according to a well-known theorem of Makarov [9] the stretching exponent $\alpha = 1$ a.e. This has been extended by Binder [6] to cover rotation, accordingly, $\gamma = 0$ a.e.

It is interesting to contrast our results to examples of Julia sets. Due to a classical result of Ruelle [14] the Julia set of $z^2 + \lambda z$ has Hausdorff dimension $1 + \frac{1}{16 \log 2} |\lambda|^2 + O(|\lambda|^3)$. In [6] Binder established an analogous formula for the geometric rotation, which says that the rotation

is $\frac{\sin \arg \lambda}{64 \log 2} |\lambda|^3 + O(|\lambda|^4)$ almost everywhere with respect to the Hausdorff measure on the Julia set. This yields rotation of lower order than what we obtain for quasiconformal mappings in this paper (but with respect to a different measure). However, the dimension ends up being of the same order for both.

The above results concern typical, that is a.e. behaviour. See [12] for the opposite end-point, i.e. for results effective for dimension close to zero. It would be interesting to extend these for the intermediate regime and ask what happens outside of a fixed dimension $s \in (0, 1)$. One could, in principle, follow a similar approach to what we have presented here but in order to get effective estimates one would seem to need detailed multifractal estimates for harmonic measure.

The improved regularity obtained in Theorem 1 compared to the estimate [5] naturally stems from the fact that we are considering subsets of the real axis. More generally, it would be interesting to see what other kind of structural assumptions on the underlying set could replace the real axis in this type of results. One possibility is to consider Jordan curves, or more generally continua, with given dimension or given smoothness.

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