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STABLE COHOMOLOGY OF THE UNIVERSAL DEGREE d HYPERSURFACE IN $\mathbb{P}^n$

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ABSTRACT. Let $U^*_{d,n}$ be the universal degree d hypersurface in $\mathbb{P}^n$. In this paper we compute the stable (with respect to d) cohomology of $U^*_{d,n}$ and give a geometric description of the stable classes. This builds on work of Tommasi [5] and Das [1].

1. INTRODUCTION

Let $U_{d,n}$ be the *parameter space* of smooth degree d hypersurfaces in $\mathbb{P}^n$. We have a natural inclusion $U_{d,n} \subseteq \mathbb{P}^{\binom{n+d}{d}} = \mathbb{P}(V_{d,n})$, where $V_{d,n}$ is the vector space of homogenous degree d complex polynomials in $n+1$ variables. Let

$$U^*_{d,n} := \{(f,p) \in U_{d,n} \times \mathbb{P}^n | f(p) = 0\}.$$

Let $\phi : U^*_{d,n} \to U_{d,n}$ be defined by $\phi(f,p) = f$. The map $\phi : U^*_{d,n} \to U_{d,n}$ is the *universal family* of smooth degree d hypersurfaces in $\mathbb{P}^n$; it satisfies the following property: Given a family $\pi : E \to B$ be a family of smooth degree d hypersurfaces in $\mathbb{P}^n$ there is a unique diagram as follows:

$$\begin{array}{ccc} E & \xrightarrow{\exists!} & U^*_{d,n} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\exists!} & U_{d,n} \end{array}$$

In other words, any family of smooth degree d hypersurfaces is pulled back from this one. Our main result is as follows:

Theorem 1.1. *Let $d, n \geq 1$. Then there is an embedding of graded vecctor spaces:*

$$\phi : H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow H^*(U^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$. Suppose that $d \geq 4n+1$. Then, the map ϕ is an isomorphism in degree $\leq \frac{d-1}{2}$. The class $x \in H^2(U^*_{d,n}, \mathbb{Q})$ is equal to $c_1(\mathscr{L})$ where $\mathscr{L}$ is the fiberwise canonical bundle(defined in Section 2).

We now define some spaces related to $U_{d,n}$, where we will prove similar results. Let $V_{d,n}$ be the vector space of homogenous degree d complex polynomials in $n+1$ variables. Let $X_{d,n} \subseteq V_{d,n}$ be the open subspace of polynomials defining a nonsingular hypersurface. The complement of $X_{d,n}$ in $V_{d,n}$ is known as the *discriminant hypersurface*; it is the zero locus of the classical discriminant polynomial. It is known to be highly singular.

A point of $X_{d,n}$ uniquely determines a projective hypersurface up to a scalar. Quotienting by this action of $\mathbb{C}^*$ we obtain our space $U_{d,n} = X_{d,n}/\mathbb{C}^*$. We can

further quotient to obtain $M_{d,n} := U_{d,n}/PGL_{n+1}\mathbb{C}$, the *moduli space* of degree d smooth hypersurfaces in $\mathbb{P}^n$. Let

$$X^*_{d,n} := \{(f,p)|f \in X_{d,n}, p \in \mathbb{P}^n, f(p) = 0\}.$$

There is a forgetful map $\pi : X^*_{d,n} \to X_{d,n}$ defined by $\pi(f,p) = f$. The fibres of π are

$$Z(f) := \pi^{-1}(f) = \{p \in \mathbb{P}^n | f(p) = 0\} \subseteq \mathbb{P}^n.$$

It is well known that the map π is a fibre bundle.

$X^*_{d,n}$ also has several interesting quotients. The action of GL_{n+1} on $X_{d,n}$ lifts to one on $X^*_{d,n}$. We obtain $U^*_{d,n} = X^*_{d,n}/\mathbb{C}^*$. The map $\pi : X^*_{d,n} \to X_{d,n}$ is $\mathbb{C}^*$-equivariant and descends to the map $\phi : U^*_{d,n} \to U_{d,n}$. We also have $M^*_{d,n} = X^*_{d,n}/GL_{n+1}(\mathbb{C})$.

We can rewrite our result in terms of $X^*_{d,n}$ and $M^*_{d,n}$ as well. This is important to us as our proof will mostly involve understanding the space $X^*_{d,n}$,and $M^*_{d,n}$ is important conceptually.

Theorem 1.2. *Let $d, n \geq 1$.*

(1) There is an embedding of graded vector spaces:

$$\psi : (H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n)) \hookrightarrow H^*(X^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$. Suppose that $d \geq 4n+1$. Then, the map ψ is an isomorphism in degree $\leq \frac{d-1}{2}$.

(2) There is an embedding of graded vector spaces:

$$\phi : (H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n)) \hookrightarrow H^*(U^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$. Suppose that $d \geq 4n+1$. Then, the map ϕ is an isomorphism in degree $\leq \frac{d-1}{2}$.

(3) There is an embedding of graded vector spaces:

$$\varphi : \mathbb{Q}[x]/(x^n)) \hookrightarrow H^*(U^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$. Suppose that $d \geq 4n+1$. Then, the map φ is an isomorphism in degree $\leq \frac{d-1}{2}$.

Nature of stable cohomology: At this point we'd like to comment on Theorem 1.2. Not only does it tell us what the cohomology of $X^*_{d,n}$ is, but as we will see through the course of the proof, it describes the stable classes. In fact the stable classes are tautological in the following sense: There is a line bundle $\mathscr{L}$ on $M^*_{d,n}$ defined by taking the canonical bundle fibrewise (we rigorously define $\mathscr{L}$ in Section 2). We will show that $c_1(\mathscr{L}), \ldots, c_1(\mathscr{L})^{n-1}$ are nonzero in $H^*(M^*_{d,n}; \mathbb{Q})$ and that stably the entire cohomology ring of $M^*_{d,n}$ is just the algebra generated by $c_1(\mathscr{L})$. By [4],

$$H^*(X^*_{d,n}; \mathbb{Q}) \cong H * (GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes H^*(M^*_{d,n}; \mathbb{Q}).$$

In this way we have some qualitative understanding of the stable cohomology of $X^*_{d,n}$.

Both the statement of Theorem 1.2 and our proof of it are heavily influenced by [5], in which Tommasi proves analogous theorems for $X_{d,n}$. Our techniques and approach are also similar to that of Das in [1],where he proves

$$H^*(X^*_{3,3}; \mathbb{Q}) \cong H^*(GL_3(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/x^3$$

with $|x| = 2$.

In some sense, this paper shows that in a stable range, something similar to Das's theorem is true for marked hypersurfaces in general.

Method of Proof: One could attempt to prove Theorem 1.2 by applying the Serre spectral sequence to the fibration $\pi : X^*_{d,n} \to X_{d,n}$. To successfully do this however, one would need to understand $H^p(X_{d,n}; H^q(Z(f); \mathbb{Q}))$ and while Theorem 4.1 tells us what the groups $H^p(X_{d,n}; \mathbb{Q})$ are, it does not give us any understanding of $H^p(X_{d,n}; H^q(Z(f); \mathbb{Q}))$ since $H^q(Z(f); \mathbb{Q})$ is a *nontrivial* local coefficient system. Instead we use an idea of Das and compute $H^*(X^p_{d,n}; \mathbb{Q})$, where $X^p_{d,n} := \{f \in X_{d,n} | f(p) = 0\}$ to avoid any computations with nontrivial coefficient systems. After we have proved Theorem 1.2 we can use it to deduce what these twisted cohomology groups are.

Corollary 1.3. *Let $d, n > 0$. Supposed $\geq 4n + 1$ and $k < \frac{d-1}{2}$. Then,*

$$H^k(X_d; H^{n-1}(Z(f); \mathbb{Q})) = \begin{cases} H^k(X_d; \mathbb{Q}) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{1}$$

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2. A lower bound on $H^k(X^*_{d,n})$

We begin by noting that $H^k(X^*_d)$ contains $H^k(GL_{n+1}(\mathbb{C})) \otimes \mathbb{Q}[x]/(x^n)$ in the stable range. More precisely, we have the following:

Proposition 2.1. *Let $n \geq 0$, and let $d > n + 1$. Then*

$$H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \subseteq H^*(X^*_{d,n}; \mathbb{Q})$$

where $|x| = 2$.

Proof. We first define the *fiberwise canonical bundle* $\mathscr{L}$ as follows:

$$\mathscr{L} = \{(f, p, v) | (f, p) \in X^*_d, v \in \wedge^{n-1} T^*_p(Z(f))\}.$$

By the same argument as in Theorem 1 of [4],

$$H^*(X^*_{d,n}; \mathbb{Q}) \cong H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes H^*(X^*_{d,n}/GL_{n+1}; \mathbb{Q}).$$

Call this quotient $M^*_{d,n}$. Note that the bundle $\mathscr{L}$ is actually pulled back from a bundle on $M^*_{d,n}$.

If we restrict $\mathscr{L}$ to a particular hypersurface Z, the bundle $\mathscr{L}|_Z = \mathscr{O}_Z(d-n-1)$. The chern class

$$c_1(\mathscr{O}_Z(d - n - 1)) = (d - n - 1)c_1(\mathscr{O}_Z(1)) = d(d - n - 1)\omega_Z,$$

where ω_Z is the Kahler class of the variety Z. This implies that for $d > n + 1$, the classes $c_1(\mathscr{L})|Z, \ldots, c_1^{n-1}(\mathscr{L})|Z$ are nonzero. This implies that $H^*(M; \mathbb{Q})$ contains a subalgebra isomorphic to $\mathbb{Q}[x]/x^n$. □

3. The space X_d^p and the Vassiliev method

In order to understand $X_{d,n}$ we will first look at the cohomology of a related space:

$$X_d^p = \{f \in X_d | f(p) = 0\}.$$

Then

$$X_d^p \subseteq V_d^p = \{f \in V_d | f(p) = 0\},$$

which is a vector space. The complement of X_d^p in V_d^p will be called $\Sigma_{d,p}$. We will compute its Borel-Moore homology and use Alexander duality to compute $H^*(X_d^p)$.

In what follows we will often refer to a group G_p defined as follows: if $p \in \mathbb{P}^n$, it is by definition a one-dimensional subspace $p \subseteq \mathbb{C}^{n+1}$. Choose a complementary subspace $W \subseteq \mathbb{C}^{n+1}$ (it is not unique, but we will fix a particular one). We let $G_p = GL(W)$.

Let $x_1, \dots x_n$ be local coordinates in a neighbourhood U containing p. Pick a local trivialisation s of the line bundle $\mathscr{O}(d)$ in U. Then any $f \in V_d^p$ can be restricted to a regular function $U \to \mathbb{C}$. There is an induced map

$$f^* : \mathbb{C} \cong T_0^*(\mathbb{C}) \to T_p^*(U) \cong \mathbb{C}^n.$$

Suppose $f \in X_d^p$. Then the map f^* is nonzero because f has a regular zero locus so $f^*(1) \in \mathbb{C}^n - 0$. This defines a map $\pi : X_d^p \to \mathbb{C}^n - 0$ defined by $\pi(f) = f^*(1)$ which is a fibration. Let $X_v := \pi^{-1}(v)$ and let

$$V_v := \{f \in V_d | f^*(1) = v\}.$$

Clearly, $X_v \subseteq V_v$. Let $\Sigma_v := V_v - X_v$. We will try to understand the Borel-Moore homology of Σ_v.

To accomplish this the Vassiliev method[7] will be applied. The Vassiliev method to compute Borel-Moore homology involves stratifying a space and using the associated spectral sequence to compute its Borel Moore homology. The space Σ_v will be stratified based on the points at which a section f is singular. The techniques used are very similar to that in [5] which contains many of the technical details.

We will now construct a simplicial space X which will be involved in understanding Σ_v. Let $N = \frac{d-1}{2}$. For $k < N$, let

$$X_{0,\dots,k} = \{(f, p_1, \dots p_k) | f \in V_v, p_i \neq p \text{ are singular zeroes of } f\}.$$

If $k \geq N$, let $X_{0\dots k} = \varnothing$. The $X_{0,\dots,k}$ assemble into a simplicial space. Consider the geometric realization $|X|$. Then there is a map $\rho : |X| \to \Sigma_v$, induced by the forgetful maps $X_{0,\dots,k} \to \Sigma_v$.

Cone off the $\Sigma^{\geq N}$ part, i.e. consider the ρ fibres of the map $|X| \to \Sigma_v$ and fiberwise cone them off. More formally, we let

$$Y = |X| \coprod \Sigma^{\geq N} \coprod \rho^{-1}(\Sigma^{\geq N}) \times [0,1]/(x,0) \sim x, (x,1) \sim \rho(x).$$

The map ρ naturally extends to Y.

Y is topologised in a non-standard way. We topologise it as follows: in [5], a space $|\mathscr{X}|$ is constructed with a map $\rho : |\mathscr{X}| \to \Sigma$. Here, $\Sigma = V_d - X_d$. The topology on $|\mathscr{X}|$ is chosen carefully so as to make ρ proper. Roughly speaking, the construction of $|\mathscr{X}|$ as a set is the same as that for Y except that we no longer require $f \in \Sigma_v$ (the only other difference is that [5] uses the cubical space formalism as opposed to simplicial spaces). There is a natural inclusion $Y \to |\mathscr{X}|$. We give Y the subspace topology along this map.

Proposition 3.1. *The map $\rho : Y \to \Sigma_v$ is a proper homotopy equivalence.*

Proof. The properness of $\rho : Y \to \Sigma_v$ follows from the properness of $\rho : |\mathscr{X}| \to \Sigma$ and the properties of the subspace topology. To prove that ρ is a homotopy equivalence, we prove that the fibres are contractible. If $f \notin \bar{\Sigma}_v^{\geq N}$, the fibre is a simplex with vertices the singularities. Now if $f \in \bar{\Sigma}_v^{\geq N}$, the fiber is a cone by construction. □

Now as in any geometric realization, $|X|$ is filtered by

$$F_n = \mathrm{im}(\coprod_{k \leq n} X_{0,\ldots,k} \times \Delta_k).$$

The space Y is also filtered similarly.

Proposition 3.2. *Let $d, n \geq 1$. Let $N = \frac{d-1}{2}$. For $k < N$, the space $F_k - F_{k-1}$ is a Δ_k°- bundle, over a vector bundle over* $\mathrm{UConf}_k(\mathbb{P}^n - p)$.

Proof. The space $F_k - F_{k-1}$ consists of the interiors of k simplices labeled by $\{f, p_0 \ldots p_k\}$. Let

$$B_k = \{(f, \{p_0 \ldots p_k\}) \in \Sigma_v \times \mathrm{UConf}_k(\mathbb{P}^n - p) | p_i \text{ are singular zeroes of } f\}.$$

We have a map $\phi : F_k - F_{k-1} \to B_k$, defined by

$$\phi((f, \{p_0 \ldots p_k\}), s_0, \ldots, s_k) = (f, \{p_0 \ldots p_k\}).$$

The map ϕ expresses $F_k - F_{k-1}$ as a fibre bundle over B_k with Δ_k° fibres, i.e we have a diagram as follows:

$$\begin{array}{ccc} \Delta_k^\circ & \longrightarrow & F_k - F_{k-1} \\ & & \downarrow \\ & & B_k \end{array}$$

We have a map $B_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$ defined by $\{f, p_0 \ldots, p_k\} \mapsto \{p_0, \ldots p_k\}$. This is a vector bundle by Lemma 3.2 in [6]. □

We have a one-dimensional local coefficient system denoted $\pm\mathbb{Q}$ on $\mathrm{UConf}_k(\mathbb{P}^n - p)$ defined in the following way: Let S_k be the symmetric group on k letters. We have a homomorphism $\pi_1 \mathrm{UConf}_k(\mathbb{P}^n - p) \to S_k$ associated to the covering space $\mathrm{PConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. Compose this homomorphism with the sign representation $S_k \to \pm 1 = GL_1(\mathbb{Q})$ to obtain our local system.

Proposition 3.3. *Let $d, n \geq 1$. Let $e_d = \dim_{\mathbb{C}}(V_v)$. For $p < \frac{d-1}{2}$,*

$$\bar{H}_*(F_p - F_{p-1}) \cong H_{*-(p+2e_d-2(n+1)(p+1))}(\mathrm{UConf}_p(\mathbb{P}^n - p), \pm\mathbb{Q}).$$

Proof. This follows immediately from Proposition 3.2. The $\pm\mathbb{Q}$ coefficients follows from the fact that the simplicial bundle $F_p - F_{p-1} \to B_k$ isn't oriented. □

As with any filtered space, we have a spectral sequence with $E_1^{p,q} = \bar{H}_{p+q}(F_p - F_{p-1}; \mathbb{Q})$ converging to $\bar{H}_*(Y; \mathbb{Q})$.

Now for $p < N$ by Proposition 3.3

$$E_1^{p,q} = \bar{H}_{q-(2e_d-2(n+1)(p+1))}(\mathrm{UConf}_p(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

We would like to claim that $E_1^{N,q}$ doesn't matter in the stable range. To that end we have the following:

Lemma 3.4. *Let $d, n \geq 1$. Let $N = \frac{d-1}{2}$. Let $k > 2e_d - N$. Then, $\bar{H}_k(Y - F_N; \mathbb{Q}) \cong \bar{H}_k(Y; \mathbb{Q})$.*

Proof. We first will try to bound the $\bar{H}_*(F_N; \mathbb{Q})$ and then use the long exact sequence of the pair. F_N is the union of locally closed subspaces

$$\phi_k = \{(f, x_1, \ldots, x_k), p | f \in \Sigma^{\geq N}, x_i \text{ are singular zeroes of } f, p \in \Delta_k\}.$$

We have a surjection $\pi : \phi_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. This map π is in fact a fibre bundle with fibres $\Delta^k \times \mathbb{C}^{e_d - N(n+1)}$. The space $\mathrm{UConf}_k(\mathbb{P}^n - p)$ is kn dimensional. Therefore

$$\bar{H}_*(\phi_k; \mathbb{Q}) = 0 \text{ if } * > 2(e_d - (n+1)N) + kn < 2e_d - N.$$

This implies that for all k, $\bar{H}_*(\phi_k; \mathbb{Q}) = 0$, if $* > 2e_d - N$. This implies $\bar{H}_*(F_N; \mathbb{Q}) = 0$, if $* > 2e_d - N$. By the long exact sequence in Borel Moore homology associated to the pair $F_N \hookrightarrow Y$, $\bar{H}_k(Y - F_N; \mathbb{Q}) \cong \bar{H}_k(Y; \mathbb{Q})$ for $k > 2e_d - N$. □

4. Interlude

In [5], Tommasi proves the following result:

Theorem 4.1 ([5])**.** *Let $d, n \geq 1$. Let $f \in X_{d,n}$. Let $\psi : GL_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $\psi(g) = g.f$.*

Then $\psi^* : H^k(X_{d,n}, \mathbb{Q}) \to H^k(GL_{n+1}(\mathbb{C}), \mathbb{Q})$ is an isomorphism for $k < \frac{d+1}{2}$.

In this section we shall look at the proof of Theorem 4.1 in [5] and use it to prove an identity used later on in this paper. One of the ingredients in the proof of Theorem 4.1 is a Vassiliev spectral sequence. We introduce a new convention, by letting h denote the dimension of H. We also define $Gr(p, n)$ to be the Grassmanian of p-planes in $\mathbb{C}^n$. In what follows we shall need a few basic facts about $H_*(Gr(p, n); \mathbb{Q})$ and Schubert symbols. Let

$$0 = E_0 \subsetneq E_1 \cdots \subsetneq E_{n-1} \subsetneq E_n = \mathbb{C}^n$$

be a complete flag. Given $U \in Gr(p, n)$, we can associate to it a sequence of numbers, $a_i = \mathrm{dim} U \cap E_i$. These a_i satisfy the following conditions:

$$0 \leq a_{i+1} - a_i \leq 1, a_0 = 0 \text{ and } a_n = p.$$

Such sequences are called *Schubert symbols*. Let $\mathbf{a} = (a_0 \ldots a_n)$. We call $\mathbf{a}$ a Schubert symbol if $0 \leq a_{i+1} - a_i \leq 1$,$a_0 = 0$ and $a_n = p$. Associated to each Schubert symbol $\mathbf{a}$ we have a subvariety $W_{\mathbf{a}} \subseteq Gr(p, \mathbb{C}^n)$ defined as follows. We define

$$W_{\mathbf{a}} = \overline{\{U \subseteq \mathbb{C}^n | \mathrm{dim} U = a_i\}}.$$

The main result we will be using is the following.

Theorem 4.2. *Let $\mathbf{a}$ be a Schubert symbol. The classes $[W_{\mathbf{a}}] \in H_*(Gr(p, n); \mathbb{Q})$ form a basis.*

For a proof of Theorem 4.2 see page 1071 of [3].

Proposition 4.3. *Let n be a positive integer. Then*

$$\sum_{k,p} h_k(Gr(p, \mathbb{C}^n); \mathbb{Q}) = 2^n.$$

Proof. By Theorem 4.2,

$$\sum_{k,p} h_k(Gr(p,\mathbb{C}^n);\mathbb{Q}) = \sum_p \#\{(a_0,\dots a_n)|0 \le a_{i+1}-a_i \le 1, a_0=0, a_n=p\}$$

$$= \#\{(a_0,\dots a_n)|0 \le a_{i+1}-a_i \le 1, a_0=0\}$$

$$= \#\{(b_1,\dots b_n) \in \{0,1\}\}.$$

The last equality follows because if we are given a sequence of a_i, we can uniquely obtain a sequence of b_i, by letting $b_i = a_i - a_{i-1}$. □

Our main aim of this section is to prove the following technical looking theorem that is necessary for our purposes.

Theorem 4.4. *The Vassiliev spectral sequence in* [5] *immediately collapses in the stable range: if* $p < \frac{d+1}{2}$ *and if* $q > 0$, *then* $E_1^{p,q} \cong E_{p,q}^{\infty}$.

Equivalently, for $k < \frac{d+1}{2}$,

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}) = h_k(GL_{n+1};\mathbb{Q}) \tag{2}$$

(These are the diagonal terms in the spectral sequence).

Proof. We already know that

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) \ge h_k(GL_{n+1};\mathbb{Q})$$

because the left hand side of (2) are the appropriate terms in a spectral sequence converging to the right hand side of (2).

It suffices to prove that

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q})$$

$$= \sum_k h_k(GL_{n+1};\mathbb{Q}) = 2^{n+1}.$$

Lemma 2 in [7] states that:

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n),\pm\mathbb{Q}) = h_{2(p+1)(n+1)-p-k-1-p(p-1)}(Gr(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

Therefore

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k \sum_p h_k(Gr_(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

By Proposition 4.3 this is equal to 2^{n+1}.

□

5. Computation

We would like to know what the groups $\bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$ are. Fortunately this isn't difficult. First note that in [7] Vassiliev computes that :

Proposition 5.1 ([7])**.** *Let $k, n > 0$. Then,*

$$H_*(\mathrm{UConf}_k(\mathbb{P}^n); \pm\mathbb{Q}) \cong H_{*-(k)(k-1)}(Gr_k(\mathbb{C}^{n+1}); \mathbb{Q}).$$

Also note that in light of Theorem 4.2 the homology of Grassmannians is well understood in terms of Schubert cells.

Proposition 5.2. *Let $k, n > 0$. Then*

$$H_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

Proof. We look at the exact sequence in Borel Moore homology associated to

$$\mathrm{UConf}_{k+1}(\mathbb{P}^n - p) \subseteq \mathrm{UConf}_{k+1}(\mathbb{P}^n) \hookleftarrow \mathrm{UConf}_k(\mathbb{P}^n - p).$$

The last inclusion is defined by the map $\phi : \mathrm{UConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_{k+1}(\mathbb{P}^n)$ where $\phi(\{x_1 \dots x_n\}) = \{x_1 \dots x_n, p\}$.

We obtain a long exact sequence as follows:

(3)

$$\cdots \to \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \to \bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \to \bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q}) \to \dots$$

Now we can use the explicit isomorphism in Lemma 2 of [7] to deduce that in fact (3) decomposes into split short exact sequences, i.e.

$$\bar{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \bar{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

□

Remark 5.3. In fact the $H_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q})$ has a basis given by Schubert symbols with $a_1 = 0$.

Proposition 5.4. *If the Vassiliev spectral sequence has no nonzero differentials and $k < \frac{d-1}{2}$, then $H^k(X_v) \cong H^k(G_p)$ as vector spaces.*

Proof. Now in our spectral sequence we had $E_1^{p,q} = \bar{H}_{q-(2e_d-2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$. First collect all terms in the main diagonal, i.e.

$$V := \oplus_{p+q=l} \bar{H}_{q-(2D_n-2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$$

It will suffice to prove that

$$\dim V = \sum_{p \leq 2D_n - k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h^k(GL_n; \mathbb{Q}). \tag{4}$$

Proposition 4.4 implies

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = h_k(GL_{n+1}; \mathbb{Q}). \tag{5}$$

Proposition 5.1 implies,

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = 0 \text{ if } p > n$$

.

So as long as $n < 2(D_n + n + 1) - k$,

$$\sum_{p \leq 2(D_n+n+1)-k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

But the condition $n < 2(D_n + n + 1) - k$ is equivalent to $k < 2(D_n + 1) + n$, which is true if $k < N$. We have another equality from Proposition 5.2,

$$h_k(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) + h_k(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h_k(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

Plugging this into (5) we have

$$h^k(GL_{n+1}; \mathbb{Q}) = \sum h_{2(p+1)(n+1)-p-k}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q})$$

$$= \sum h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}).$$

We have the identity

$$h^k(GL_n; \mathbb{Q}) + h^{k-(2n+1)}(GL_n; \mathbb{Q}) = h^k(GL_{n+1}; \mathbb{Q}).$$

This implies,

(6)
$$h^k(GL_n; \mathbb{Q}) + h^{k-(2n+1)}(GL_n; \mathbb{Q})$$

(7)
$$= \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \mathbb{Q}) + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \mathbb{Q}).$$

Now we will try to prove 4 by induction on k. For $k = 0$, (4) is trivial. By induction

$$h^{k-(2n+1)}(GL_n; \mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}).$$

Putting this into 6 we obtain

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h^k(GL_n; \mathbb{Q}).$$

□

Now we can look at the Serre Spectral sequence associated to the fibration

$$X_v \hookrightarrow X_p \to \mathbb{C}^n - 0.$$

We observe that if there are no nonzero differentials, then

$$H^*(X_p; \mathbb{Q}) \cong H^*(X_v; \mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/e_{2n-1}^2.$$

Proposition 5.5. *Let $d > 0$ and $p \in \mathbb{P}^n$. Then,*

$$H^*(X_{d,p}; \mathbb{Q}) \cong H^*(G_p; \mathbb{Q}) \otimes A$$

, where A is $H^*(X_d^p/G_p; \mathbb{Q})$.

Proof. This follows immediately from Theorem 2 in [4]. □

We will also need the following fact that is a special case of Lemma 2.6 in [1].

Proposition 5.6. *Let $d > 0$, $k < \frac{d-1}{2}$. Let $U_d^* = X_d^*/\mathbb{C}^*$. Then*

$$H^*(X_d^*; \mathbb{Q}) \cong H^*(U_d^*; \mathbb{Q}) \otimes \mathbb{Q}[e_1]/(e_1^2),$$

where $|e_1| = 1$.

Proposition 5.6 implies if there are no nonzero differentials in both our Vassiliev spectral sequence and in the Serre spectral sequence associated to the fibration $X_{d,n}^p \to \mathbb{C}^n - 0$ then

$$H^*(U_{d,p}; \mathbb{Q}) \cong H^*(G_p; \mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]$$

for $* < \frac{d-1}{2}$. In case there are nonzero differentials in either spectral sequence, then $H^*(U_{d,p}; \mathbb{Q}) \cong H^*(G_p; \mathbb{Q})$ for $* < \frac{d-1}{2}$.

6. Comparing fibre bundles

In this section we finish the proof of Theorem 1.2.

Proof of Theorem 1.2. We compare three related fibre bundles and their associated spectral sequences. This is similar to the Proof of Theorem 1.1 in [1].

(8)
$$\begin{array}{ccccc} PG_p := Stab_{PGL(n+1)}p & \longrightarrow & U_{d,p} & \longrightarrow & U_d \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & PGL_{n+1}(\mathbb{C}) & \longrightarrow & U_d^* \longrightarrow U_d \times \mathbb{P}^n \\ & & \downarrow & & \downarrow \quad \downarrow \\ & & \mathbb{P}^n & \xrightarrow{=} & \mathbb{P}^n \xrightarrow{=} \mathbb{P}^n \end{array}$$

By Proposition 5.4 and Theorem 1 of [4] there are two possibilities for $H^*(U_{d,p}; \mathbb{Q})$: either

$$H^*(U_{d,p}; \mathbb{Q}) \cong H^*(PG_p; \mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2) \cong \Lambda < u_1, u_3, \dots u_{2n-1}, e_{2n-1} >$$

(exterior algebra) or

$$H^*(U_{d,p}; \mathbb{Q}) \cong H^*(PG_p; \mathbb{Q}) = \Lambda < u_1, u_3, \dots u_{2n-1} > .$$

Suppose for the sake of contradiction that $H^*(U_{d,p}) = \Lambda < u_3, \dots u_{2n-1} >$ for $* < \frac{d-1}{2}$. In this case $H^*(U_{d,p}; \mathbb{Q}) \cong H^*(PG_p; \mathbb{Q})$ for $* < \frac{d-1}{2}$. Then since the homology of the base and the fibres are isomorphic, $H^*(U_d^*; \mathbb{Q}) \cong H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q})$ for $* < \frac{d-1}{2}$. However by Proposition 2.1,

$$H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes Q[x]/x^n) \subseteq H^*(U_d^*; \mathbb{Q}).$$

But $H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q})$ does not contain a subalgebra isomorphic to $H^*(PGL_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/x^n)$.This is a contradiction.

So we must be in the case,

$$H^*(U_{d,p}; \mathbb{Q}) \cong H^*(PG_p; \mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$. Its E_2 page has terms

$$E_2^{p,q} = H^p(\mathbb{P}^n, H^q(U_d^p; \mathbb{Q})) \cong H^p(\mathbb{P}^n; \mathbb{Q}) \otimes H^q(U_d^p; \mathbb{Q}).$$

Now

$$H^q(U_d^p; \mathbb{Q}) \cong H^q(PG_p; \mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the trivial fibre bundle $U_d \times \mathbb{P}^n \to \mathbb{P}^n$. There is a natural inclusion of fibre bundles as shown in (8). This induces a map of spectral sequences.

Note that any class $\alpha \in H^q(U_d^p; \mathbb{Q})$ that lies in the image of $H^q(U_d; \mathbb{Q})$ is mapped to zero under any differential thanks to the fact that all dfferentials are zero in the spectral sequence associated to a trivial fibration. The only possible nonzero differential in the E_2 page of the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$ is $d(e_{2n-1})$.

Suppose for contradiction that $d(e_{2n-1}) = 0$. This implies that

$$H^k(U_d^*; \mathbb{Q}) \cong (H^*(U_{d,p}; \mathbb{Q}) \otimes H^*(\mathbb{P}^n; \mathbb{Q}))_k = (H^*(PG_p; \mathbb{Q}) \otimes H^*(\mathbb{P}^n, \mathbb{Q}))_k$$

for $k < \frac{d-1}{2}$.

Let $p(t)$ be the Poincare polynomial of U_d^*. We already know that $H^*(U_d^*; \mathbb{Q}) \cong H^*(PGL_{n+1}; \mathbb{Q}) \otimes A$. So $(1+t^3)\dots(1+t^{2n+1})|p(t)$. On the other hand, if $de_{2n-1} = 0$ then

$$p(t) = (1+t^3)\dots(1+t^{2n-1})(1+t^2+t^4\dots t^{2n})(\mod t^{\frac{d-1}{2}}$$

. If $d \geq 4n+1$, then this implies that $(1+t^{2n+1}) \not| p(t)$. This is a contradiction.

So we must have a differential killing the class in $H^{2n}(\mathbb{P}^n, H^0(U_{d,p})); \mathbb{Q})$. The differential must come from from e_{2n-1}, i.e. $d(e_{2n-1}) = ax^n$ for some $a \in \mathbb{Q}^*$. This (along with multiplicativity of differentials) determines all differentials and implies (1). By Proposition 5.6 (1) $\implies$ (2). By Theorem 1 of [4]

$$H^*(X_{d,n}^*; \mathbb{Q}) \cong H^*(M_{d,n}^*; \mathbb{Q}) \otimes (H^*(GL_{n+1})(\mathbb{C}); \mathbb{Q})$$

. In light of this (2) $\implies$ (3).

□

Having finished the proof of Theorem 1.2 we can prove Corollary 1.3.

Proof of Corollary 1.3. Consider the fibration: $\begin{array}{ccc} Z(f) & \longrightarrow & X_d^8 \\ & & \downarrow \\ & & X_d \end{array}$

and its associated Serre spectral sequence whose E_2 page is of the form

$$H^p(X_d; H^q(Z(f); \mathbb{Q})) \implies H^*(X_d^*; \mathbb{Q}).$$

ByTheorem 4.1 for $* < \frac{d+1}{2}$

$$H^*(X_d; \mathbb{Q}) \cong H^*(GL_{n+1}(\mathbb{C}); \mathbb{Q}).$$

By Theorem 1.2, we know that the classes in the E_2 page corresponding to the group $H^p(GL_{n+1}(\mathbb{C}); c_1(\mathscr{L})^q)$ survive till the E^∞ page and in the stable range all other terms are killed by differentials.

Now suppose n is even. Then the only other terms in the spectral sequence are of the form $H^p(X_d; H^{n-1}(Z(f); \mathbb{Q}))$. However it is not possible for any such term to be in the image or in the preimage of a nonzero differential. This is because all other terms survive so any possible nonzero differential must be from $H^{p_1}(X_d; H^{n-1}(Z(f); \mathbb{Q}))$ to $H^{p_2}(X_d; H^{n-1}(Z(f); \mathbb{Q}))$ for some choice of p_1 and p_2. However no differential is of bidegree $(p_2 - p_1, 0)$. This implies that

$$H^p(X_d; H^{n-1}(Z(f); \mathbb{Q})) \cong 0.$$

A similar argument shows that if n is odd, $H^p(X_d; H^{n-1}(Z(f); \mathbb{Q})) \cong H^p(X_d; \mathbb{Q})$. Essentially the only difference between the even case and the odd case is that in

the odd case we have a class $c_1(\mathscr{L})^{(\frac{n-1}{2})} \in H^{n-1}(Z(f);\mathbb{Q})$. By Theorem1.2, we know that $H^p(X_{d,n};\mathbb{Q}c_1(\mathscr{L})^{\frac{n-1}{2}})$ survives till the E^∞ page. An argument similar to that in the even case shows that

$$H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong H^p(X_d;\mathbb{Q}c_1(\mathscr{L})^{\frac{n-1}{2}}).$$

□

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