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7-DIMENSIONAL SIMPLY-CONNECTED SPIN MANIFOLDS WHOSE INTEGRAL COHOMOLOGY RINGS ARE ISOMORPHIC TO THAT OF $\mathbb{C}P^2 \times S^3$ ADMIT ROUND FOLD MAPS

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Abstract. 7-dimensional closed and simply-connected manifolds are important objects in the theory of classical algebraic topology and differential topology. The class has been attractive since the discoveries of 7-dimensional exotic homotopy spheres by Milnor ([11]). Still recently, new understandings via algebraic topological tools such as characteristic classes and bordism relations have been discovered by Kreck, Wang, and so on.

The author has been interested in understanding the class in geometric and constructive ways. Thus the author constructed explicit *fold* maps, which are higher dimensional versions of Morse functions, on some of the manifolds. The studies have been motivated by studies of *special generic* maps, higher dimensional versions of Morse functions on homotopy spheres with exactly two singular points, characterizing them topologically except 4-dimensional cases. The class contains canonical projections of unit spheres for example. This class has been found to be interesting, restricting the topologies and the differentiable structures of the manifolds strictly: Saeki, Sakuma, Wrazidlo, and so on, found explicit phenomena. The present paper concerns fold maps on 7-dimensional closed, simply-connected and spin manifolds whose integral cohomology rings are isomorphic to that of $\mathbb{C}P^2 \times S^3$.

1. Introduction, terminologies and notation.

7-dimensional closed and simply-connected manifolds are important objects in the theory of classical algebraic topology and differential topology (of higher dimensional closed and simply-connected manifolds). The class has been attractive since the discoveries of 7-dimensional so-called *exotic* spheres by Milnor and so on. Still recently, new understandings via algebraic topological tools such as characteristic classes and bordism relations have been performed by Kreck [9] and Wang [18] for example.

Throughout the present paper, manifolds and maps between manifolds are smooth (of class C^∞) unless otherwise stated.

We introduce some terminologies and some of notation on smooth manifolds and maps,

$\mathbb{R}^k$ denotes the k-dimensional Euclidean space, endowed with the standard Euclidean metric for each integer $k \geq 0$. $||x|| \geq 0$ denotes the distance between $x \in \mathbb{R}^k$ and the origin 0. $S^k \subset \mathbb{R}^{k+1}$ denotes the set of all points such that the distances between the points and the origin are 1 for $k \geq 0$. This is the k-dimensional unit sphere. A *homotopy sphere* means a smooth manifold homeomorphic to a unit

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sphere and it is said to be *standard* (resp. *exotic*) if it is (resp. not) diffeomorphic to a (resp. any) unit sphere. $D^k \subset \mathbb{R}^k$ denotes the set of all points such that the distances between the points and the origin are smaller than or equal to 1 for $k \geq 0$. This is the k-dimensional unit disk. A *standard* disk means a smooth manifold diffeomorphic to a unit disk.

The author has been interested in understanding the class of 7-dimensional closed and simply-connected manifolds in geometric and constructive ways and obtained explicit *fold* maps, which are higher dimensional versions of Morse functions, on the manifolds. The studies have been motivated by studies of *special generic* maps, which are higher dimensional versions of Morse functions on homotopy spheres with exactly two singular points. It is well-known a closed manifold is a homotopy sphere if and only if it admits such a function except 4-dimensional cases. In 4-dimensional cases, a closed manifold is diffeomorphic to the standard sphere if and only if it admits such a function.

The class of special generic maps contains canonical projections of unit spheres for example. This class has been found to be interesting, restricting the topologies and the differentiable structures of homotopy spheres and manifolds admitting them strictly. For example 7-dimensional exotic spheres admit no special generic maps into $\mathbb{R}^n$ for $n = 4, 5, 6$. In considerable cases they admit no special generic maps for $n = 3$. See [14], [15], [16], [17] and [19] for example.

It is also an important fact that construction of explicit fold maps on explicit manifolds is difficult even for fundamental manifolds and in general. The present paper concerns fold maps on 7-dimensional simply-connected spin manifolds whose integral cohomology rings are isomorphic to that of $\mathbb{C}P^2 \times S^3$. A recent study on classifications of these manifolds by [18] motivates the author to do this study. Elementary operations on manifolds such as (boundary) connected sums of manifolds are also considered in the smooth category throughout the present paper, unless otherwise stated.

Theorem 1 ([18] with related fundamental and classical results)**.** (1) *For an arbitrary 7-dimensional, closed, simply-connected and spin manifold X whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$, consider an isomorphism ϕ of integral cohomology rings from the integral cohomology ring of $\mathbb{C}P^2 \times S^3$ onto that of X and set a as a generator of $H^2(\mathbb{C}P^2 \times S^3; \mathbb{Z}) \cong \mathbb{Z}$. In this situation, the 1st Pontryagin class of X is represented as $4k\phi(a)$ for some integer k. Moreover, the topology of a 7-dimensional closed, simply-connected and spin manifold whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$ is determined by its 1st Pontryagin class.*

(2) *For any pair (M_1, M_2) of mutually homeomorphic 7-dimensional, closed, simply-connected and spin manifolds whose integral cohomology rings are isomorphic to that of $\mathbb{C}P^2 \times S^3$, M_1 is represented as a connected sum of M_2 and a suitable 7-dimensional homotopy sphere Σ. Moreover, for a 7-dimensional, closed, simply-connected and spin manifold M_0 whose integral cohomology ring is isomorphic to the integral cohomology ring of $\mathbb{C}P^2 \times S^3$, consider a pair (M_1, M_2) of manifolds such that M_i is represented as a connected sum of M_0 and a homotopy sphere Σ_i. They are diffeomorphic if and only if Σ_1 and Σ_2 are mutually diffeomorphic.*

1.1. **Terminologies and notation on smooth maps.** A *singular* point of a smooth map is a point in the domain at which the dimension of the image of the

differential is smaller than both the dimensions of the domain and the target. We call the set of all the singular points the *singular set* of the map. We call the image of the singular set the *singular value set* of the map. The *regular value set* of the map is the complementary set of the singular value set. A *singular* (*regular*) *value* is a point in the singular (resp. regular) value set. For a smooth map c, $S(c)$ denotes the singular set of c.

1.2. **The definition and fundamental properties of a fold map, fold maps on 7-dimensional homotopy spheres, round fold maps and Main Theorem.** Hereafter, let $m > n \geq 1$ be integers. A smooth map from an m-dimensional smooth manifold with no boundary into an n-dimensional smooth manifold with no boundary is said to be a *fold* map if at each singular point p, the map is represented as $(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} {x_k}^2 - \sum_{k=m-i+1}^{m} {x_k}^2)$ for suitable coordinates and an integer $0 \leq i(p) \leq \frac{m-n+1}{2}$. For the singular point p, $i(p)$ is unique and called the *index* of p. The set consisting of all the singular points of a fixed index of the map is a closed submanifold of dimension $n-1$ with no boundary of the m-dimensional manifold. The restriction map to the singular set is an immersion. A *special generic* map, which is explained before, is defined as a fold map such that the index of a singular point is always 0.

Note that proving that fundamental smooth functions and maps such as canonical projections of unit spheres are Morse functions or fold maps is well-known fundamental exercises on smooth manifolds, Morse functions and differential topology of manifolds.

We have obtained the following theorem through challenging construction of explicit fold maps on explicit manifolds. For classical theory on 7-dimensional homotopy spheres, see [1] and [11] for example.

Hereafter, diffeomorphisms on manifolds are always smooth and the *diffeomorphism group* of a manifold is defined as the group of all the smooth diffeomorphisms on the manifold. For bundles whose fibers are manifolds, the structure groups are subgroups of the diffeomorphism groups or the bundles are assumed to be *smooth* unless otherwise stated. A *linear* bundle means a smooth bundle whose fiber is a unit sphere or a unit disk in a Euclidean space with structure group acting linearly in a canonical way on the fiber.

Theorem 2 ([2] and [3] for example.)**.** (1) *Every 7-dimensional homotopy oriented sphere M of 28 types admits a fold map $f : M \to \mathbb{R}^4$ such that $f|_{S(f)}$ is an embedding satisfying $f(S(f)) = \{x \in \mathbb{R}^4 \mid ||x|| = 1, 2, 3\}$, that the index of each singular point is 0 or 1, and the preimage of a regular value in each connected component of f is, empty, diffeomorphic to S^3, diffeomorphic to $S^3 \sqcup S^3$ and diffeomorphic to $S^3 \sqcup S^3 \sqcup S^3$, respectively.*

(2) *7-dimensional homotopy sphere M admits a fold map $f : M \to \mathbb{R}^4$ such that $f|_{S(f)}$ is an embedding and that $f(S(f)) = \{x \in \mathbb{R}^4 \mid ||x|| = 1\}$ if and only if M is a standard sphere. A 7-dimensional homotopy sphere M admits a fold map $f : M \to \mathbb{R}^4$ such that $f|_{S(f)}$ is an embedding and that the index of each singular point is 0 or 1 as before and in addition the following properties hold if and only if M is the total space of a linear bundle whose fiber is S^3 over S^4 (7-dimensional oriented homotopy spheres of 16 types are represented in this way).*

(a) $f(S(f)) = \{x \in \mathbb{R}^4 \mid ||x|| = 1, 2\}$.

(b) *For any connected component $C \subset f(S(f))$ and a small closed tubular neighborhood $N(C)$, the bundle given by the projection represented as the composition of $f|_{f^{-1}(N(C))}$ with the canonical projection to C is trivial.*
(c) *The preimage of a regular value in each connected component is, empty, diffeomorphic to S^3, and diffeomorphic to $S^3 \sqcup S^3$, respectively.*

Especially, f is special generic if and only if $f(S(f)) = \{x \in \mathbb{R}^4 \mid ||x|| = 1\}$. This explicitly presents that the numbers of connected components of the singular value sets affect the differentiable structures of 7-dimensional homotopy spheres for suitable explicit classes of fold maps which may not be the one of special generic maps. In addition, the maps here belong to the class of *round* fold maps.

Definition **1** ([2]–[6].)**.** For an integer $n \geq 2$, a fold map f into $\mathbb{R}^n$ is said to be *round* if for a suitable diffeomorphism ϕ on $\mathbb{R}^n$ and an integer $l > 0$, $(\phi \circ f)(S(f)) = \bigcup_{r=1}^{l} \{||x|| = r \mid x \in \mathbb{R}^n\}$.

This notion is also defined for $n = 1$ in [6] for example. However we do not need this in the present paper.

The following theorem is our main theorem.

Main Theorem. *There exists an infinite family $\{M_k\}_{k \in \mathbb{Z}}$ of closed, simply-connected and spin (oriented) manifolds whose integral cohomology rings are isomorphic to that of $\mathbb{C}P^2 \times S^3$ such that the 1st-Pontryagin class of M_k is $4k$ times a generator of $H^4(M_k; \mathbb{Z}) \cong \mathbb{Z}$ and these manifolds admit round fold maps $\{f_k : M_k \to \mathbb{R}^4\}$. Furthermore, (by virtue of Theorem 1), every 7-dimensional, closed, simply-connected and spin manifold whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$ admits a round fold map into $\mathbb{R}^4$.*

1.3. **The content of the paper.** The next section is devoted to the proof of the main theorem. We construct a round fold map on a 7-dimensional, closed and simply-connected manifold, whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$. The third section is devoted to appendices. In the section, we present related results on explicit fold maps on 7-dimensional closed, and simply-connected and the cohomology rings of the manifolds in [7] and [8]. We see that the integral cohomology rings are not isomorphic to that of $\mathbb{C}P^2 \times S^3$.

2. The proof of Main Theorem.

We will prove the main theorem. We introduce fundamental terminologies and notions.

For a finitely generated and free commutative group G, let $a \in G$ be a non-zero element which we cannot represent as $a = ra'$ for any pair $(r, a') \in (\mathbb{Z} - \{-1, 0, 1\}) \times G$ and let G be the internal direct sum of the group generated by the one element set $\{a\}$ and a subgroup of G. A homomorphism $a^* : G \to \mathbb{Z}$ is said to be the *dual* of a if $a^*(a) = 1$ and for any subgroup G' such that G is the internal direct sum of the group generated by the one element set $\{a\}$ and G', $a^*(G') = \{0\}$ holds.

For a closed and orientable manifold, consider an integral homology class of degree k. It is said to be *represented* by a closed, connected and oriented submanifold of dimension k with no boundary if the class is equal to the value of the homomorphism canonically induced from the inclusion at the *fundamental class* of the submanifold: the *fundamental class* is a generator of the integral cohomology of

degree k (of the k-dimensional closed and oriented manifold) compatible with the given orientation.

2.1. **The 3-dimensional complex projective space and its structure.** The 3-dimensional complex projective space $\mathbb{C}P^3$ is regarded as the total space of a linear bundle over S^4 whose fiber is S^2. It is a spin manifold. The 1st Pontryagin class of the projective space is 4 times a generator of the 4-the integral cohomology group. For this, see [10] for example. For classical systematic theory of Pontryagin classes see [13]. The following theorem presents several classical and important properties.

Theorem 3. *For the projection $\pi_{\mathbb{C}P^3} : \mathbb{C}P^3 \to S^4$ of the linear bundle over S^4 whose fiber is S^2 and the total space or the domain $\mathbb{C}P^3$, the following properties hold.*

1. *There exists a complex projective plane ${\mathbb{C}P^2}_{\mathbb{C}P^3} \subset \mathbb{C}P^3$ being also a complex submanifold. Furthermore, a generator of $H_4(\mathbb{C}P^3;\mathbb{Z})$, isomorphic to $\mathbb{Z}$, is represented by the submanifold with a suitable orientation.*
2. *$\pi_{\mathbb{C}P^3}|_{{\mathbb{C}P^2}_{\mathbb{C}P^3}}$ is a covering on the preimage of a smoothly embedded 4-dimensional standard closed disk in S^4.*
3. *There exists a complex projective line ${\mathbb{C}P^1}_{\mathbb{C}P^2} \subset {\mathbb{C}P^2}_{\mathbb{C}P^3}$ being also a complex submanifold. Furthermore, a generator of $H_2(\mathbb{C}P^3;\mathbb{Z})$, isomorphic to $\mathbb{Z}$, is represented by the submanifold where the submanifold is suitably oriented. The complex projective line can be taken as a fiber of the bundle.*
4. *The square of the dual of the integral homology class represented by ${\mathbb{C}P^1}_{\mathbb{C}P^2}$ is the dual of the integral homology class represented by ${\mathbb{C}P^2}_{\mathbb{C}P^3}$ with the given orientation. The product of the dual of the integral homology class represented by ${\mathbb{C}P^1}_{\mathbb{C}P^2}$ and the dual of the integral homology class represented by ${\mathbb{C}P^2}_{\mathbb{C}P^3}$ is a generator of $H^6(\mathbb{C}P^3;\mathbb{Z})$, isomorphic to $\mathbb{Z}$.*

2.2. **A proof of the main theorem.** We consider a submersion obtained by composing the canonical projection from $\mathbb{C}P^3 \times S^1$ to $\mathbb{C}P^3$ with $\pi_{\mathbb{C}P^3} : \mathbb{C}P^3 \to S^4$ in Theorem 3. We have a 7-dimensional, closed and simply-connected manifold M and a fold map $f_{S^4} : M \to S^4$ by exchanging a restriction to the preimage of a smoothly embedded 4-dimensional standard disk D so that the following properties hold. Let $D' \subset S^4$ be a smoothly embedded 4-dimensional standard closed disk contained in the interior of the disk D.

1. f_{S^4} is a fold map such that the singular set $S(f_{S^4})$ is diffeomorphic to S^3, that $f_{S^4}|_{S(f_{S^4})}$ is an embedding and that $f_{S^4}(S(f_{S^4})) = \partial D'$.
2. The preimage of a regular value of each connected component of $S^4 - f_{S^4}(S(f_{S^4}))$ is S^3 and $S^2 \times S^1$, respectively.

Note that in the second property, (a copy of) S^3 is regarded as a manifold obtained by a so-called handle attachment to (a copy of) $S^2 \times S^1$. In other words, we attach a *3-handle* $D^3 \times D^1$ along $S^2 \times \{*\} \subset S^2 \times S^1$, respecting the structure of a Morse function. See [12] for example for related theory. The following show topological properties of M. We can see them by investigating the two properties of the map before and rigorous proofs are left to readers.

1. There exists a complex projective plane ${\mathbb{C}P^2}_M \subset M$ being also a submanifold such that $f_{S^4}|_{{\mathbb{C}P^2}_M}$ is a covering on the preimage of the disk D in S^4.

(2) The integral cohomology ring of M is isomorphic to that of $\mathbb{C}P^2 \times S^3$.
(3) The integral homology class represented by the submanifold $S^2 \times \{*\} \subset S^2 \times S^1$ of the preimage of a regular value diffeomorphic to $S^2 \times S^1$ is a generator of $H_2(M; \mathbb{Z}) \cong \mathbb{Z}$ where $S^2 \times \{*\}$ is oriented suitably. The integral homology classes represented by the preimages of regular values are a generator of $H_3(M; \mathbb{Z}) \cong \mathbb{Z}$. The square of the dual of the integral homology class represented by $S^2 \times \{*\}$ before with the orientation is the dual of the integral homology class represented by the complex projective plane ${\mathbb{C}P^2}_M \subset M$ with a suitable orientation.
(4) M is spin.
(5) The 1st Pontryagin class of M (, which is oriented,) is 4 times a generator of the 4-the integral cohomology group.

We take the union of two smoothly and disjointly embedded 4-dimensional standard closed disks $D_1 := D'' \subset \mathrm{Int}\ D'$ and D_2 so that they are mutually in distinct connected components of the regular value set of f_{S^4}. By composing the restriction of f_{S^4} to the preimage ${f_{S^4}}^{-1}(D_1 \sqcup D_2)$ with a natural 2-fold covering over a 4-dimensional standard closed disk D_0, we have a trivial bundle over D_0 whose fiber is diffeomorphic to $S^3 \sqcup S^2 \times S^1$. We consider the projection and embed the base space as the space $\{x \in \mathbb{R}^4 \mid |x| \leq r_0\}$ for a positive integer $r_0 > 0$. Let $\tilde{f_{r_0}}$ denote the resulting submersion. The restriction of f_{S^4} to ${f_{S^4}}^{-1}(S^4 - \mathrm{Int}(D_1 \sqcup D_2))$ is regarded as a product map of a Morse function with exactly one singular point and the identity map on id_{S^3}. On the domain of this, we can construct a product map of a Morse function $\tilde{f}$ and the identity map on id_{S^3} such that the Morse function $\tilde{f}$ satisfies the following properties.

(1) The function is a function on a manifold diffeomorphic to a manifold obtained by removing the interior of a 4-dimensional standard closed disk smoothly embedded in the interior of $S^2 \times D^2$.
(2) The minimum is r_0.
(3) The preimage of the minimum r_0 of the function coincides with the boundary and contains no singular point.
(4) The function has exactly three singular points and at distinct singular points, the values are distinct.

We can set the target of the function $\tilde{f}$ as the half-closed interval $[r_0, +\infty) \subset \mathbb{R}$. Thanks to the structures of manifolds and maps, we can glue the maps $\tilde{f_{r_0}}$ and the product map of $\tilde{f}$ and the identity map on the boundary of $\{x \in \mathbb{R}^4 \mid |x| \leq r_0\}$ to obtain a global fold map on the original manifold M into $\mathbb{R}^4$.

Furthermore, the topological properties of M and $f_{S^4} : M \to S^4$ presented before together with fundamental propositions on 1st Pontryagin classes enable us to construct a similar fold map on a manifold whose integral cohomology ring is isomorphic to that of M (, which is oriented suitably,) and whose 1st Pontryagin class is $4k$ times a generator of the 4-th integral homology group, isomorphic to $\mathbb{Z}$, for an arbitrary integer k. We need to change the way we glue the trivial bundle over D_0 whose fiber is diffeomorphic to S^3. For Pontryagin classes see [13] again and see also Theorems 3 and 4 of [7].

This yields the following main theorem.

Theorem 4. *There exist an infinite family $\{M_k\}_{k \in \mathbb{Z}}$ of closed, simply-connected and spin (oriented) manifolds whose integral cohomology rings are isomorphic to*

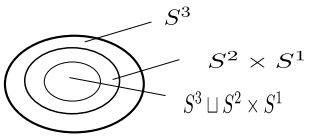

FIGURE 1. The image and the preimages of regular values of a round fold map in Theorem 4: circles represent the singular value set and 3-dimensional spheres.

that of $\mathbb{C}P^2 \times S^3$ such that the 1st-Pontryagin class of M_k is $4k$ times a generator of $H^4(M_k;\mathbb{Z}) \cong \mathbb{Z}$ and a family of round fold maps $\{f_k : M_k \to \mathbb{R}^4\}$ satisfying the following properties.

(1) *The singular set of each map consists of exactly three connected components. More precisely, we can construct the maps so that the restrictions to the singular sets are embeddings and that the images are $\{x \in \mathbb{R}^4 \mid ||x|| = 1, 2, 3\}$.*
(2) *The preimage of a regular value in each connected component of the regular value set in the image is diffeomorphic to S^3, $S^2 \times S^1$ and $S^3 \sqcup S^2 \times S^1$, respectively.*

See also FIGURE 1.
Theorem 1 with [3] yields the following corollary.

Corollary 1. *A 7-dimensional, closed, simply-connected and spin manifold whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$ always admits a round fold map into $\mathbb{R}^4$.*

Proof. Theorem 1 implies that a 7-dimensional, closed, simply-connected and spin manifold whose integral cohomology ring is isomorphic to that of $\mathbb{C}P^2 \times S^3$ is always represented as a connected sum of a manifold in Theorem 4 and a 7-dimensional homotopy sphere. [3] and other papers or preprints such as [2] and [6] show construction of a new round fold map from a round fold map in Theorem 4 and a round fold map in Theorem 2. We take a 4-dimensional, standard and closed disk in the connected component of the regular value set of the former map in the center and remove the interior of a connected component, regarded as the total space of a trivial bundle over the disk whose fiber is S^3 of the preimage. We remove (the preimage of) the union of the interior of a small closed tubular neighborhood of the outermost connected component of the singular value set of the latter map and the complementary set of the image. We glue the two obtained maps in a suitable way to obtain a round fold map on the desired manifold. The manifold is represented as a connected sum of the original two manifolds. See also FIGURE 2, FIGURE 3 and FIGURE 4. This completes the proof. □

Remark **1.** The author believes that we can prove the following fact: 7-dimensional manifolds of Theorem 4 or Main Theorem do not admit special generic maps or fold maps into $\mathbb{R}^n$ (n=1,2,3,4,5,6) such that the following two hold.

(1) The index of each singular point is 0 or 1.
(2) Preimages of regular values are disjoint unions of standard spheres.

For example, fold maps in Theorem 2 and Theorem 5, which is in the next section satisfy these two properties.

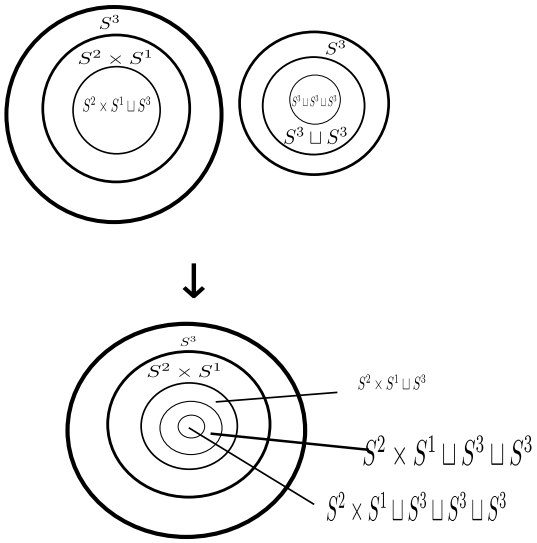

FIGURE 2. Construction of a round fold map on the manifold represented as a connected sum of the two manifolds admitting fold maps in Corollary 1: for example manifolds represent preimages of regular values in the connected components of the regular value sets. More precisely, we attach maps presented in FIGURE 3 and FIGURE 4 to obtain the desired map.

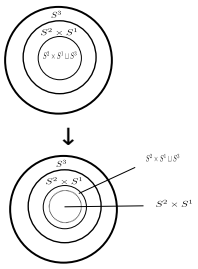

FIGURE 3. The image of a round fold map in Theorem 4. The image of the map obtained by removing a connected component of the preimage of the interior of a 4-dimensional standard closed disk embedded smoothly in the innermost connected component of the regular value set.

3. APPENDICES.

3.1. **Fold maps in Theorem 2 and ones on manifolds represented as connected sums of total spaces of bundles over the standard sphere of a fixed dimension whose fibers are standard spheres.** Note also that fold maps in Theorem 2 have been obtained as specific examples of fold maps on manifolds represented as connected sums of total spaces of bundles over the standard sphere of a fixed dimension whose fibers are standard spheres. See [2], [3] and [6] for example.

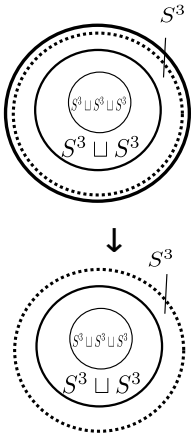

FIGURE 4. The image of a round fold map in Theorem 2. The image of the map obtained by removing the union of the complementary set of the image and the interior of a small closed tubular neighborhood of the outermost connected component of the singular value set (and the preimage).

Related to this fact, manifolds represented as connected sums of manifolds represented as products of standard spheres admit special generic maps into suitable Euclidean spaces whose dimensions are smaller in considerable cases: for example a manifold represented as a connected sum of finite copies of $S^{n-1} \times S^{m-n+1}$ admits a special generic map into $\mathbb{R}^n$. We can construct the map so that the restriction to the singular set is an embedding and that the image is a manifold represented as a boundary connected sum of finite copies of $S^{n-1} \times I$ where I denotes a closed interval.

3.2. **Further examples of fold maps on 7-dimensional closed and simply-connected manifolds.** We present the following recent result, presenting explicit fold maps on infinitely many 7-dimensional closed and simply-connected manifolds. Their integral cohomology rings are mutually non-isomorphic. These manifolds are not represented as connected sums of manifolds represented as products of standard spheres as before in general.

Theorem 5 ([7] and [8])**.** *Let $\{G_j\}_{j=0}^7$ be a sequence of free and finitely generated commutative groups such that $G_0 = G_7 = \mathbb{Z}$, that G_j is zero for $j = 1, 6$, that G_j and G_{7-j} are mutually isomorphic for $j = 2, 3$ and that the rank of G_2 is smaller than or equal to that of G_4.*

In this situation, we have the following properties.

1. *There exist a closed, simply-connected and spin manifold M of dimension 7 such that the homology group is free and that $H_j(M; \mathbb{Z})$ is isomorphic to G_j and a fold map $f : M \to \mathbb{R}^4$ satisfying the following properties.*
   (a) *The index of each singular point is always 0 or 1.*
   (b) *For each connected component of the regular value set of f, the preimage of a regular value in each connected component is, empty, diffeomorphic to S^3, diffeomorphic to $S^3 \sqcup S^3$, or diffeomorphic to $S^3 \sqcup S^3 \sqcup S^3$.*

(2) *Furthermore, if G_2 and G_4 are non-trivial groups, then there exist infinitely many closed and simply-connected manifolds M_λ of dimension 7 admitting the fold maps as before such that $H_j(M_\lambda;\mathbb{Z})$ is isomorphic to G_j and that the cohomology rings of M_{λ_1} and M_{λ_2} are not isomorphic for distinct $\lambda_1, \lambda_2 \in \Lambda$.*

(3) *Furthermore, the integral cohomology rings of the obtained manifolds cannot be isomorphic to those of Theorem 4: the square of every 2nd integral cohomology class must be divisible by 2.*

In very specific cases, they are represented as connected sums of manifolds represented as products of pairs of standard spheres as before.

As Theorem 4, this result is also seen as one capturing the integral cohomology rings of 7-dimensional closed and simply-connected manifolds of suitable families via explicit fold maps on them into $\mathbb{R}^4$.

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