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# ON RECTIFICATION AND ENRICHMENT OF INFINITY PROPERADS

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Abstract. We develop a theory of infinity properads enriched in a general symmetric monoidal infinity category. These are defined as presheaves, satisfying a Segal condition and a Rezk completeness condition, over certain categories of graphs. In particular, we introduce a new category of level graphs which also allow us to give a framework for algebras over an enriched infinity properad. We show that one can vary the category of graphs without changing the underlying theory.

We also show that infinity properads cannot always be rectified, indicating that a conjecture of the second author and Robertson is unlikely to hold. This stands in stark contrast to the situation for infinity operads, and we further demarcate these situations by examining the cases of infinity dioperads and infinity output properads. In both cases, we provide a rectification theorem that says that each up-to-homotopy object is equivalent to a strict one.

Contents

1. Introduction 2
1.1. Notation and conventions 5
1.2. Outline 6
1.3. Further directions 6
Acknowledgments 7
2. Categories of directed graphs 7
2.1. Level graphs 9
2.2. The category **G** of connected, acyclic graphs 22
2.3. From connected level graphs to connected acyclic graphs 29
3. The Segal condition and an algebraic version of enriched $\infty$-properads 31
3.1. Algebraic patterns and homotopy-coherent algebraic structures 31
3.2. Segal presheaves and decorated graph categories 33
3.3. Enrichment in presheaves 44
4. Algebras over enriched $\infty$-properads 46
4.1. Inner horn inclusions and inner anodyne maps 46
4.2. Tensoring with Segal spaces 50
5. Comparison of $\mathbf{L}_c$ and **G** presheaves 56
5.1. The equivalence and its consequences 56

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5.2. Towards the proof of Theorem 5.1.4 59
6. Completeness and enriched $\infty$-properads 66
6.1. Free functors 66
6.2. Fully faithfulness, essential surjectivity and completeness 70
7. Rectification theorems 74
7.1. Operads governing properads 75
7.2. Rectification 83
Appendix A. Equivalence of $\mathbf{G}$ with the properadic graphical category 88
Appendix B. Proof of Proposition 2.3.2 89
References 94

## 1. Introduction

Properads, first introduced by Vallette [Val07] in the context of Koszul duality theory for props, are an intermediate notion between operads and props that are capable of governing certain types of bialgebraic structures. Morphisms in operads take the form $f\colon a_1,\dots,a_n \to b$, that is, they can be interpreted as many-to-one operations. If $g\colon c_1,\dots,c_m \to d$ is some other operation, then we can form

$$g \circ_i f\colon c_1,\dots,c_{i-1},a_1,\dots,a_n,c_{i+1},\dots,c_m \to d$$

whenever $1 \leq i \leq m$ and $b = c_i$. Properads, or more precisely the many-colored variant of them discovered independently by Duncan in his thesis [Dun06, §6.1] under the name "compact symmetric polycategories," are an extension of operads that allows one to consider many-to-many operations

$$f\colon a_1,\dots,a_n \to b_1,\dots,b_p.$$

Composition, rather than connecting one output with one input, is designed to connect several (meaning 'at least one') outputs with inputs. These should be regarded as props (in the sense of [Mac65, §24]) without horizontal composition.[1]

Infinity properads were introduced by the second author, Robertson, and Yau in [HRY15], in part as a potential structural framework for chain-level string topology operations. In the present work we study infinity properads enriched in an arbitrary (presentably symmetric monoidal $\infty$-)category $\mathcal{V}$. The main examples the reader should keep in mind are when $\mathcal{V}$ is the category of chain complexes over a field of characteristic zero (which is the original context for the properads in [Val07]) or the category of spaces (which can be regarded as the 'unenriched' case of infinity properads). The method for $\mathcal{V}$-enrichment presented here works more generally, a story that will be told in forthcoming work of the first author and Haugseng [CH].

The basic idea for $\mathcal{V}$-enrichment is as follows. There is a category of graphs equipped with a suitable amount of structure so that one may identify ordinary properads as certain set-valued presheaves over this graph category, namely those presheaves satisfying a "Segal condition." Infinity properads are then space-valued presheaves satisfying a (homotopical) Segal condition (as well as a discreteness condition for objects). One then builds a new indexing $\infty$-category (Definition 3.2.10) of $\mathcal{V}$-decorated graphs and then $\mathcal{V}$-enriched infinity properads are a subclass of Segal

[1] See also [Dun06, p.80].

presheaves on this new $\infty$-category (Definition 3.2.18, Definition 6.2.7). When $\mathcal{V}$ is just spaces, this recovers the usual unenriched notion of infinity properads.

The present paper concentrates on two major questions in the theory:

(1) Given a $\mathcal{V}$-enriched infinity properad $P$, what should be meant by the $\infty$-category of algebras of $P$? We address this question by showing that the $\infty$-category of $\mathcal{V}$-enriched infinity properads, $\mathrm{Prpd}_\infty^{\mathcal{V}}$, is tensored over $\infty$-categories (Theorem 4.2.2 and Proposition 6.2.12), so by adjunction one can produce the desired $\infty$-category of algebras (Corollary 4.2.8 and Corollary 6.2.13).

(2) Suppose $\mathcal{V}$ is a symmetric monoidal $\infty$-category associated to a symmetric monoidal model category and suppose $P$ is a $\mathcal{V}$-enriched infinity properad, is it possible to rectify it to a strict properad enriched in $\mathcal{V}$? We explain why we expect the answer to this question to be negative in general (even when $\mathcal{V}$ is spaces, see Theorem 7.2.5), but give an affirmative answer in certain special cases such as chain complexes (Theorem 7.2.10) and symmetric spectra (Remark 7.2.11). We also give an affirmative answer in related situations, such as for dioperads (i.e., symmetric polycategories) and for output properads. In particular, we expect that Conjecture 4.14 of [HR18], which occurs in the setting of model categories, is false as stated, but that analogues will be true for both dioperads and for output properads.

The two preceding questions each require us to approach enriched infinity properads using a different base indexing category of graphs. For the second question, it is most appropriate to use the properadic graphical category from [HRY15], which was further developed in [Koc16] and [HRY18]. This is a category of directed graphs with loose ends which are acyclic as directed graphs and connected as undirected graphs. In [HRY15] it was shown that Segal set-valued presheaves on this category are precisely the ordinary properads. In this paper we give a new, conceptual presentation of morphisms of this category as certain homomorphisms on the partially-ordered set of subgraphs. The new description of the properadic graphical category then reveals its tight connection to the operads governing properads which is in turn essential for our rectification procedure.

The first question requires a different approach. Namely, we introduce a new category of directed *level* graphs, which admits a cartesian fibration to the simplex category. This extra structure allows for an alternative description of the Segal condition. This enhanced relationship between the level graph category and the simplex category induces a relationship between infinity properads and infinity categories. Of course level graphs also played a prominent role in [Val07]. Indeed, the 'vertical' structure of our level graph category is already visible in the simplicial bar construction (see the second remark [Val07, p.4920]), though we will need the full structure below.

In each instance, we have utilized a graph category especially suited to the task at hand. In the first question, we used the level graph category which bears a close relationship with the simplex category, while in the second we used the directed graph category which is closely related to operads for properads. A third question arises, which is whether we are really talking about the same kind of enriched infinity properads in both instances. This is indeed the case (Corollary 5.1.6), though the proof is rather involved (§5.2).

In the second question above, we alluded to there being related developments for infinity dioperads and output (or input) properads. For the most part, these developments are entirely parallel to the case of properads, and simply amount to restricting to full subcategories of various graph categories.

Dioperads are like properads, in that operations can have many inputs and outputs, but are also like operads, in that the only compositions we have connect one output with one input. The name 'dioperad' first appeared in work of Gan [Gan03], again in the context of Koszul duality theory, but the many-colored version 'polycategory' had appeared earlier (see [Sza75] for the non-symmetric version and [Hyl02, §5.1] or [Gar08] for the symmetric version). To treat this case we will restrict all of our graph categories to the full subcategory whose objects are graphs which, as topological spaces, have trivial fundamental groups.

Output properads are those properads having the property that if $f\colon a_1,\dots,a_n \to b_1,\dots,b_p$ is an operation, then $p$ is positive. For input properads one instead requires that $n$ is positive. Many interesting properads are, in fact, output (or input) properads. For instance, there are several homological conformal field theories for string topology [CM12, CG04, God, HL15], all of which require at least some type of positive boundary condition (or even noncompactness condition). In the closed part of the theory of [God], this amounts to the structure of an algebra over the output properad given by the homology of the moduli spaces of connected Riemann surfaces with at least one outgoing boundary (see §1 of [Tam09]). It is not possible to relax this condition in string topology situations, that is, to consider a full hcft with both units and counits as the value on a circle would be finite-dimensional, while the (co)homology of a free loop space is usually infinite dimensional (see §1.5 of [HL15] for further details). Further examples of this phenomenon abound, e.g. the topological conformal field theories of [Cos07, §1.1] have a similar restriction.

Though the development of the theory of enriched infinity properads, dioperads and output/input properads follow the same path, we do not have the same rectification theorems in the first case. The main difference is that properads, unlike dioperads or output properads, are not modeled by a $\Sigma$-cofibrant operad. At the very least, this means that one does not have access to standard tools (such as [BM03, Theorem 4.4], which even provides a Quillen equivalence of model categories) for rectifying homotopy properads. We prove slightly more in Theorem 7.2.5, showing that when working over simplicial sets and for a specific model of homotopy properads, that the standard comparison adjunction between strict properads and homotopy properads is not a Quillen equivalence.

We should compare the preceding paragraph to the classical setting of the operads $\mathcal{A}ss$ and $\mathcal{C}om$ in topological spaces: the former is $\Sigma$-free while the latter is not. Connected $\mathcal{A}_\infty$-spaces can be rectified to topological monoids (see [BV73, May72, Sta63]), whereas it can be shown by vanishing of k-invariants or Dyer–Lashof operations for commutative topological monoids that it is not possible to rectify all $\mathcal{E}_\infty$-spaces to commutative monoids (see, e.g., [BV73, p.203]). By analogy, our interpretation of the non-rectification result is that infinity properads are more free than strict properads, and are the correct notion for homotopy theory.

**Remark 1.0.1.** The homotopical setting for this paper is that of $\infty$-categories, and we avoid Quillen model categories until the very end (§7.2). This added flexibility is important when studying enriched properads and their algebras, and much of what we do has no obvious counterpart in the realm of model categories and Quillen

functors. Let us point to two concrete situations where it is clear that the rigidity of model categories and Quillen functors would be an impediment.

- Let $p$ be a prime number and let $\mathbf{Ch}_k$ be the category of chain complexes over the field $k$ with $p$ elements. As was observed in [BM03, 3.3.3], it is not possible to have a model structure on (unreduced, monochrome) operads in $\mathbf{Ch}_k$ so the forgetful functor to symmetric sequences simultaneously creates weak equivalences and is a right Quillen functor. The key step in this argument is that in positive characteristic the free graded commutative algebra functor from chain complexes to the category of CDGAs can take an acyclic chain complex to a CDGA with nontrivial homology. A similar consideration applies in the case of properads; related concerns for props manifest in [Fre10, 4.10].
- If $P$ is a (monochrome) properad in a symmetric monoidal category $\mathbf{V}$, then the forgetful functor from $P$-algebras to $\mathbf{V}$ often does not have a left adjoint. In particular, when $\mathbf{V}$ is a monoidal model category, we cannot expect to have a model structure on $P$-algebras so that the forgetful functor is right Quillen, since this functor is not even a right adjoint for many choices of $\mathbf{V}$ and $P$ (and similarly for left adjoints). To have the desired model structure, there seem to be substantial restrictions on at least one of $\mathbf{V}$ or $P$, such as requiring that $P$ be an operad [BM03], or that $\mathbf{V}$ be cartesian [JY09, Theorem 1.4].

1.1. **Notation and conventions.** We write $\mathbf{\Delta}$ for the usual simplicial indexing category and $[n]$, $n \geq 0$, for its objects. The category of simplicial sets will be denoted by sSet, and the category of small categories by Cat.

The category of all small properads and properad maps will be denoted by Prpd(Set). Objects are unenriched properads, that is properads with sets of morphisms $a_1, \dots, a_n \to b_1, \dots, b_p$. We won't give a formal definition here as it is somewhat involved (see §6.1 [Dun06] or Definition 11.25 or Definition 11.27 of [YJ15]), but one will appear much later in this paper as well, in Definition 7.1.4 (the operad governing $S$-colored properads) and Definition 7.2.6 (the category of all properads).

We let Fin and $\mathrm{Fin}_*$ denote the category of finite sets and pointed finite sets, respectively. An object in $\mathrm{Fin}_*$ is denoted by $A_+ = A \amalg \{*\}$ where $A \in \mathrm{Fin}$ and $*$ is the base point. We write $\mathbf{F}$ for a skeleton of the category Fin, spanned by objects $\mathbf{n} \coloneqq \{1, \dots, n\}$. Similarly, we write $\mathbf{F}_*$ for a skeleton of the category $\mathrm{Fin}_*$, spanned by $\langle n \rangle \coloneqq \{1, \dots, n\} \amalg \{*\}$. We will often implicitly identify an object $K_+ \in \mathrm{Fin}$ such that $|K| = n$ with $\langle n \rangle \in \mathbf{F}_*$.

This paper is mostly written $\infty$-categorically, that is, using the quasicategorical formalism for $(\infty, 1)$-categories presented in [Lur09]. We will regard 1-categories as special $\infty$-categories and all categorical constructions such as taking (co)limits should be interpreted in the $\infty$-categorical setting.

We write $\mathcal{S}$ for the $\infty$-category of spaces (or $\infty$-groupoids) and, for an $\infty$-category $\mathcal{C}$, we write $\mathrm{P}(\mathcal{C})$ for the $\infty$-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ of presheaves of spaces on $\mathcal{C}$. We will use $\mathrm{Cat}_\infty$ to denote the $\infty$-category of $\infty$-categories.

Given a functor $f \colon \mathcal{C} \to \mathcal{D}$ between $\infty$-categories and an object $d \in \mathcal{D}$. We let $\mathcal{C}_{d/}$ denote the pullback $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ whose objects are pairs $(c, \alpha)$ where $c \in \mathcal{C}$ and $\alpha \colon d \to f(c)$. For such an object we will often write $c$ and leave $f$ implicit. We define $\mathcal{C}_{/d}$ analogously.

In this paper, $\mathcal{U}$ will always refer to a small symmetric monoidal $\infty$-category, while $\mathcal{V}$ and $\mathcal{W}$ will denote arbitrary symmetric monoidal $\infty$-categories. That said,

often $\mathcal{V}$ will denote a presentably symmetric monoidal $\infty$-category, (i.e., $\mathcal{V}$ is a presentable $\infty$-category and the tensor product preserves colimits in each variable). The results generally remain true for large symmetric monoidal $\infty$-categories by passage to a larger universe as in [CH20, Remark 3.5.9] and [GH15, Theorem 5.6.6], but we will not comment further on that here.

1.2. **Outline.** Section 2 is devoted to several categories of directed graphs without (directed) cycles. Key among these is the new category $\mathbf{L}$ of level graphs, introduced in §2.1. In §2.2, we give a new, conceptual presentation of the properadic graphical category from [HRY15]; for narrative purposes, a proof of the relevant equivalence is postponed until Appendix A. We also introduce a functor from the full subcategory $\mathbf{L}_{\mathrm{c}}$ of $\mathbf{L}$ on the connected graphs to the properadic graphical category in §2.3, which plays a key role in later comparison results. Several other graph categories appear in §2 as subcategories of the main two, which are useful for studying structures related to properads (trees and forests are for operads, directed graphs without undirected cycles are for dioperads, and so on).

In Section 3, we introduce the algebraic version of $\mathcal{V}$-enriched $\infty$-properads and give the first results. Section 4 restricts attention to categories of level graphs, shows how to tensor by Segal simplicial spaces, and introduces categories of algebras. At this point, we have two competing notions of $\mathcal{V}$-enriched $\infty$-properads, one based on $\mathbf{L}$ and the other based on $\mathbf{G}$. We show in Section 5 that these two approaches coincide.

We leave the algebraic world behind in Section 6, where we introduce a completeness condition for enriched $\infty$-properads. Finally, in Section 7 we compare our notion of enriched $\infty$-properads to enriched ordinary properads.

1.3. **Further directions.** The goal of this paper is to build a foundational framework for enriched $\infty$-properads and their algebras. We now propose several interesting areas of exploration based on the machinery developed here:

- *Enriched $\infty$-properads as monoids*: First of all, one should be able to describe enriched $\infty$-properads as monoids in bicollections in a similar way as Vallette first introduced properads in [Val07]. More precisely, the goal would be to construct a Day convolution double $\infty$-category $\widetilde{\mathbf{L}^{\mathcal{V}}}$ by using the double $\infty$-categorical structure of $\mathbf{L}^{\mathcal{V}}$ (Remark 3.2.11) introduced in Definition 3.2.10. This naturally generalizes the constructions given by [Hau] for the $\infty$-operadic setting. The universal property of $\widetilde{\mathbf{L}^{\mathcal{V}}}$ should then show that the $\infty$-category of monoids in $\widetilde{\mathbf{L}^{\mathcal{V}}}$ is equivalent to the $\infty$-category $\mathrm{Seg}(\mathbf{L}^{\mathcal{V}})$ of Segal objects and by restricting to the $\infty$-category $\mathrm{Prpd}_{\infty}^{\mathcal{V}}$ of $\mathcal{V}$-enriched $\infty$-properads viewed as a full subcategory of $\mathrm{Seg}(\mathbf{L}^{\mathcal{V}})$ we obtain the desired description of enriched $\infty$-properads as monoids.

  One of the reasons for introducing $\mathbf{L}^{\mathcal{V}}$ in this paper is the fact that in contrast to other $\infty$-categories governing enriched $\infty$-properads such as $\mathbf{G}^{\mathcal{V}}$ the $\infty$-category $\mathbf{L}^{\mathcal{V}}$ admits a natural double $\infty$-categorical structure which is essential for the construction of $\widetilde{\mathbf{L}^{\mathcal{V}}}$ mentioned above.
- *Algebras as modules:* As a first application of the previous item, for any enriched $\infty$-properad $P$ we wish to describe $P$-algebras as certain module in bicollections. This would improve on the results about algebras from the present paper by not only proving the existence, but also giving an explicit formula of computing algebras.

- *Koszul duality or bar-cobar construction:* Built on the description of enriched $\infty$-properads as monoids in bicollections Lurie's theory of Koszul duality for associative algebras [Lur, §5.2] then gives an adjunction between enriched $\infty$-properads and enriched $\infty$-coproperads which are coassociative coalgebras in bicollections. We expect that this approach generalizes the bar-cobar construction in the setting of ordinary properads and by restricting our general construction we should then obtain the Koszul duality for enriched $\infty$-operads with the most interesting case being the enrichment over spectra. As was observed by Ching and Harper [Chi12, CH19], the coalgebraic structures in spectra are difficult to work with using model categorical methods, enticing one to work in the $\infty$-categorical setting and to study Koszul duality occurring for any stable symmetric monoidal $\infty$-categories instead of spectra. In [FG12] Francis and Gaitsgory have used the expected properties of enriched $\infty$-operads to obtain Koszul duality equivalences under certain finiteness hypotheses and also conjectured how this should generalize. It would be interesting to compare our $\infty$-categorical construction with the approach suggested by Francis–Gaitsgory.
- *Tensor product for $\infty$-properads:* In Section 4, we use the level graph structure of object in $\mathbf{L}^{\mathcal{V}}$ to define the tensor product of enriched $\infty$-properads with $\infty$-categories. Replacing $\mathbf{L}^{\mathcal{V}}$ with $\mathbf{L}$, one can extend the construction from §4 to give a tensor product of unenriched $\infty$-properads generalizing the tensor product on strict properads from [HRY15, §4.2] and the Boardman–Vogt tensor product of ($\infty$-)operads. Although there is no known tensor product of $\mathcal{V}$-enriched $\infty$-operads unless $\mathcal{V}$ is cartesian, we expect that in presence of a cartesian symmetric monoidal $\infty$-category $\mathcal{V}$ a natural generalization of our construction of the tensor product in Section 4 then gives a closed symmetric monoidal structure on the $\infty$-category of $\mathcal{V}$-enriched $\infty$-properads.
- *Simplicial localization of $P$-algebras:* Let $P$ be a properad in the category of chain complexes in characteristic zero. As mentioned in Remark 1.0.1, we do not expect the category of $P$-algebras to have a meaningful Quillen model structure as the ground category is not cartesian. Nonetheless, one can consider the category of $P$-algebras as a relative category, where the weak equivalences are the quasi-isomorphisms. At this stage, one can use Dwyer–Kan simplicial localization (as in [DK80]) to obtain an $\infty$-category (see [BK12, 6.10]); it would be interesting to see how this compares with the construction of algebras from the present paper. Homotopy-invariance properties of this construction were proved by Yalin in [Yal14] (for props, rather than properads), a remarkable result given the lack of a suitable model structure on $P$-algebras. As our construction of algebras is manifestly homotopy-invariant, it is to be expected that any comparison would be closely related to Yalin's theorem.

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## 2. Categories of directed graphs

In this paper, we are concerned with (finite) graphs which are *directed*, have *loose ends*, and are *acyclic*. Each graph (with loose ends) $G$ is given by two sets $\mathtt{E}(G)$ and $\mathtt{V}(G)$ together with incidence data. Namely, each vertex $v \in \mathtt{V}(G)$ should come equipped with two subsets $\mathtt{in}(v)$ and $\mathtt{out}(v)$ of $\mathtt{E}(G)$ (so that no edge is the input or

output of two different vertices), but a given edge need not be an input (or output) for any vertex of the graph. This last bit is what gives the distinction of 'loose ends.' Let us give a convenient, short formalism for graphs having the first two properties, which we learned from [Koc16, 1.1.1].

**Definition 2.0.1.** Let $\mathscr{G}$ denote the category

$$\begin{array}{ccc} \mathbf{e} & \xleftarrow[s]{} & \mathbf{i} \\ {\scriptstyle t}\uparrow & & \downarrow{\scriptstyle p} \\ \mathbf{o} & \xrightarrow{\quad q \quad} & \mathbf{v}. \end{array}$$

A *graph* $G$ is a functor $\mathscr{G} \to \mathrm{Fin}$ which sends $s$ and $t$ to monomorphisms, that is, a diagram of finite sets of the form

$$\mathtt{E} \xhookleftarrow{s} \mathtt{I} \xrightarrow{p} \mathtt{V} \xleftarrow{q} \mathtt{O} \xhookrightarrow{t} \mathtt{E}.$$

- The image of $\mathbf{e}$, denoted by $\mathtt{E}$ or $\mathtt{E}(G)$, is the set of *edges*.
- The image $\mathbf{v}$, denoted by $\mathtt{V}$ or $\mathtt{V}(G)$, is the set of *vertices*.
- If $v \in \mathtt{V}$, we write $\mathtt{in}(v) = p^{-1}(v)$ and $\mathtt{out}(v) = q^{-1}(v)$.
- We write $\mathtt{in}(G) = \mathtt{E} \setminus t(\mathtt{O})$ and $\mathtt{out}(G) = \mathtt{E} \setminus s(\mathtt{I})$.

We will typically regard $\mathtt{I} \cong \coprod_{v \in \mathtt{V}} \mathtt{in}(v)$ and $\mathtt{O} \cong \coprod_{v \in \mathtt{V}} \mathtt{out}(v)$ as actual subsets of $\mathtt{E}$. Thus the set $\mathtt{E}$ admits two decompositions

$$\mathtt{E} = \mathtt{in}(G) \amalg \coprod_{v \in \mathtt{V}} \mathtt{out}(v) \qquad \mathtt{E} = \mathtt{out}(G) \amalg \coprod_{v \in \mathtt{V}} \mathtt{in}(v).$$

A *naïve morphism* of graphs is simply a natural transformation of functors. In [Koc16, 1.1.7] these were called *morphism of graphs* and defined a full subcategory $\mathbf{Gr}^{+} \subseteq \mathrm{Fun}(\mathscr{G}, \mathrm{Fin})$. We won't have too much use for naïve morphisms as such in the present work (the one exception being as an alternative characterization of 'structured subgraph,' in Definition 2.2.2 – see the proof of Lemma 2.3.1) but they are useful in discussing the concepts of *connectedness* and *acyclicity*, as in [Koc16, §1.2].

**Remark 2.0.2.** There are other possible definitions of directed graph with loose ends, and also of basic notions like connectedness and acyclicity. For instance, in [HRY15, §2.1.2] the definition of 'generalized graph' is given; this formalism was extensively developed in [YJ15]. This Yau–Johnson formalism for directed graphs is nearly equivalent to the one from Definition 2.0.1, the only exception being that in that formalism graphs are allowed to have components that are vertex-free loops. Since we will only be interested in acyclic directed graphs in what follows, this difference would not appear anyway. The equivalence of these approaches can be chained together from [Koc16, 1.1.12], [BB17, Proposition 15.2], and [BB17, Proposition 15.6], noting that the directionality is preserved across all of these bijections.

**Definition 2.0.3** (Étale map)**.** A naïve morphism of graphs $G \to H$ is called *étale* if the middle two squares in the commutative diagram

$$\begin{array}{ccccccccc} \mathtt{E}(G) & \longleftarrow & \mathtt{I}(G) & \longrightarrow & \mathtt{V}(G) & \longleftarrow & \mathtt{O}(G) & \longrightarrow & \mathtt{E}(G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathtt{E}(H) & \longleftarrow & \mathtt{I}(H) & \longrightarrow & \mathtt{V}(H) & \longleftarrow & \mathtt{O}(H) & \longrightarrow & \mathtt{E}(H) \end{array}$$

are pullbacks. We write $\mathbf{K}_{\mathrm{int}}$ for the category whose objects are isomorphism classes of graphs which are both connected and acyclic, and whose morphisms are the étale maps.

The category $\mathbf{K}_{\mathrm{int}}$ was called $\mathbf{Gr}$ in [Koc16]. Kock also had a larger category of graphs $\widetilde{\mathbf{Gr}}$ whose morphisms are more complicated; this category is equivalent to $\mathbf{K}$ from the following definition.

**Definition 2.0.4.** Given a connected, acyclic graph $G$ which is equipped with a total ordering on each of the sets $\mathtt{in}(v)$ and $\mathtt{out}(v)$, an associated $\mathtt{E}(G)$-colored properad is defined in [HRY15, Definition 5.7]. For each object in $\mathbf{K}_{\mathrm{int}}$, make a choice of representative of the isomorphism class and a choice of total ordering on all of the sets $\mathtt{in}(v)$ and $\mathtt{out}(v)$. Let $\mathbf{K}$ denote the full subcategory of the category of colored properads (in Set, see [HRY15, Definition 3.5]) spanned by the objects of $\mathbf{K}_{\mathrm{int}}$, considered as colored properads.

2.1. **Level graphs.** A level graph is a graph whose vertices and edges are arranged in several distinct layers, so that each edge in a middle layer connects vertices in the adjacent layers. More precisely, we have the following, which we will later package as Definition 2.1.8.

**Preliminary Definition 2.1.1.** A *level graph of height* $n$ is a directed graph $G$ together with an assignment of an integer in $[0,n] = \{0,1,\dots,n\}$ to each edge and an assignment of a number in $[1,n] = \{1,\dots,n\}$ to each vertex. The functions $h_E\colon \mathtt{E}(G) \to [0,n]$ and $h_V\colon \mathtt{V}(G) \to [1,n]$ should satisfy

$$h_E(e) = \begin{cases} 0 & \text{if } e \in \mathtt{in}(G), \\ n & \text{if } e \in \mathtt{out}(G), \\ h_V(v) - 1 & \text{if } e \in \mathtt{in}(v), \text{ and} \\ h_V(v) & \text{if } e \in \mathtt{out}(v). \end{cases}$$

See Figure 1 for an example for a level graph of height 4. In general, the extremal layers will be edge layers, whose edges are connected to vertices only at one side, allowing us to glue graphs together. We want to think of the vertices as 'functions' or 'processes' and the edges as 'inputs' or 'outputs' of these, and gluing corresponds to composition of total functions.

Notice that any directed graph which admits this extra structure is automatically acyclic, i.e., wheel-free. On the other hand, there are acyclic graphs that do not admit any level structure, for instance, the graph in Example 2.2.3 below.

**Remark 2.1.2.** Suppose $G$ is a level graph of height $n$ and suppose there exists an edge $e \in \mathtt{E}$ which is not attached to any vertex in $G$. Then the decompositions $\mathtt{E} = \mathtt{in}(G) \amalg \coprod_{v\in\mathtt{V}} \mathtt{out}(v)$ and $\mathtt{E} = \mathtt{out}(G) \amalg \coprod_{v\in\mathtt{V}} \mathtt{in}(v)$ imply that $e \in \mathtt{in}(G) \cap \mathtt{out}(G)$, and the condition on $h_E$ implies that $0 = h_E(e) = n$. Hence, the underlying graph of a level graph containing a loose edge is of height zero, that is, is a finite collection of loose edges.

**Example 2.1.3** (Elementary level graphs)**.** The following level graphs will be called *elementary*:

- If $p, q \geq 0$, the level graph $\mathfrak{c}_{p,q}$ has height 1, $p$ edges in level 0, $q$ edges in level 1, and a single vertex. We call such a level graph a *corolla* and we

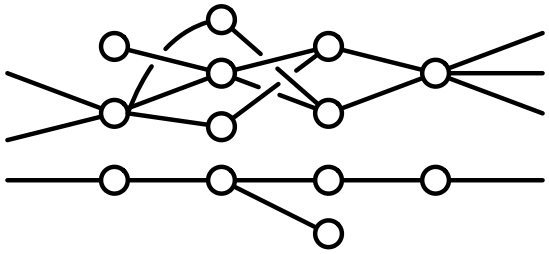


FIGURE 1. A level graph of height 4

write $\mathfrak{c}$ for it if we do not want to emphasize the numbers of input and output edges.
- The level graph $\mathfrak{e}$ which has height 0 (hence no vertices) and a single edge.

**Remark 2.1.4.** If $G$ is a connected graph with $\mathtt{in}(G) \neq \varnothing \neq \mathtt{out}(G)$, then $G$ is a level graph in at most one way. In particular, the height of $G$ is uniquely determined. This is not the case when either $\mathtt{in}(G)$ or $\mathtt{out}(G)$ is empty. For example, consider the graph $G$ with a single vertex and no edges. If $n \geq 1$, then each of the $n$ functions $h_V \colon * \simeq \mathtt{V}(G) \to [1, n]$ exhibits $G$ as a level graph of height $n$. Conversely, if $G$ is a connected level graph of height $n$ that admits no other level graph structures, then both $\mathtt{in}(G)$ and $\mathtt{out}(G)$ are nonempty sets.

In the following we want to give an equivalent definition using certain category $\mathscr{L}_0^n$ (Definition 2.1.6) to the category of finite sets. This will allow us to compare (connected) level graphs with objects in the Hackney–Robertson–Yau category $\mathbf{\Gamma}$ (Theorem A.1 and Corollary 2.3.3). For this purpose we introduce the following definitions.

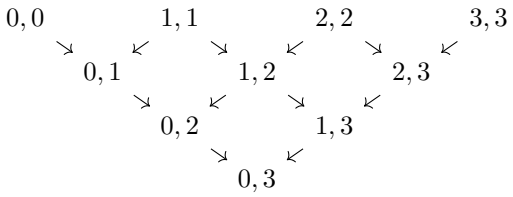
**Definition 2.1.5** (Twisted arrow category)**.** If $\mathcal{C}$ is a category, then the *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$ has as objects the morphisms of $\mathcal{C}$, and morphisms $f \to f'$ in $\mathrm{Tw}(\mathcal{C})$ are given by commutative squares of the form

$$\begin{array}{ccc} \bullet & \longleftarrow & \bullet \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \bullet & \longrightarrow & \bullet \end{array}$$

in $\mathcal{C}$. Let $\epsilon \colon \mathbf{\Delta} \to \mathbf{\Delta}$ denote the functor $[n] \mapsto [n]^{\mathrm{op}} \star [n]$. Then the twisted arrow category is the restriction (to Cat) of the functor $\epsilon^* \colon \mathrm{sSet} \to \mathrm{sSet}$ given by precomposition with $\epsilon$.

Let $n \geq 0$ and consider the twisted arrow category $\mathrm{Tw}(\Delta^n)$. This category is, in fact, a partially-ordered set, and we will give it the alternative name $\mathscr{L}^n \cong \mathrm{Tw}(\Delta^n)$ when we have identified the morphism $i \to j$ of $\Delta^n$ with the pair $(i, j)$. In other words, objects of $\mathscr{L}^n$ are pairs $(i, j)$ with $0 \leq i \leq j \leq n$ and there is a unique morphism $(i, j) \to (i', j')$ exactly when $i' \leq i$ and $j \leq j'$. As an example, here is

the category $\mathscr{L}^3$.




Notice that every square in $\mathscr{L}^n$ is both a pushout and a pullback. The opposite category of Definition 2.1.6(2) appears as [Hau, Definition 2.2.7].

**Definition 2.1.6.** Let $n \geq 0$.

(1) The category $\mathscr{L}^n$ has objects pairs $(i, j)$ with $0 \leq i \leq j \leq n$ and a unique morphism $(i, j) \to (i', j')$ exactly when $0 \leq i' \leq i \leq j \leq j' \leq n$.
(2) Let $\mathscr{L}^n_0$ denote the full subcategory of $\mathscr{L}^n$ spanned by the objects $(i, j)$ where $j - i \leq 1$.

In other words, $\mathscr{L}^n_0$ is the full subcategory of $\mathscr{L}^n$ consisting of all objects of the form $(i, i)$ and $(i, i + 1)$, that is,

$$\mathscr{L}^n_0 = \left( \begin{array}{ccccccc} 0,0 & & 1,1 & \cdots & n{-}1,n{-}1 & & n,n \\ & \searrow \quad \swarrow & & & & \searrow \quad \swarrow & \\ & 0,1 & & \cdots & & n{-}1,n & \end{array} \right).$$

**Notation 2.1.7.** If $G \colon \mathscr{L}^n \to \mathrm{Fin}$ is a functor, we denote its value at an object $(i, j) \in \mathscr{L}^n$ by $G_{i,j}$, and similarly for functors from $\mathscr{L}^n_0$.

**Definition 2.1.8.** A *level graph of height* $n$ is a functor $G \colon \mathscr{L}^n_0 \to \mathrm{Fin}$.

Note that natural transformations of functors correspond to naïve morphisms of graphs. We will cut out a more appropriate class of level-preserving morphisms of height $n$ level graphs in Definition 2.1.16.

**Lemma 2.1.9.** *Preliminary Definition 2.1.1 is equivalent to Definition 2.1.8.*

*Proof.* Given a functor $G \colon \mathscr{L}^n_0 \to \mathrm{Fin}$, the correspondence is realized by $h_E^{-1}(i) = G_{i,i}$ for $0 \leq i \leq n$ and $h_V^{-1}(i) = G_{i-1,i}$ for $1 \leq i \leq n$. That is, the underlying directed graph (as in Definition 2.0.1) is given by

$$\coprod_{i=0}^{n} G_{i,i} \xleftarrow{s} \coprod_{i=0}^{n-1} G_{i,i} \xrightarrow{p} \coprod_{i=1}^{n} G_{i-1,i} \xleftarrow{q} \coprod_{i=1}^{n} G_{i,i} \xhookrightarrow{t} \coprod_{i=0}^{n} G_{i,i}$$

where $p$ restricts to $G_{i,i} \to G_{i,i+1}$ and $q$ restricts to $G_{i,i} \to G_{i-1,i}$. □

We instantly see that level graphs of height 1 are just cospans of finite sets. But cospans assemble into a (weak) double category, as in [GP17, §5], which we could use as a starting point to define a category $\mathbf{L}$ of level graphs. We take an alternative approach, noting that a functor $\mathscr{L}^n_0 \to \mathrm{Fin}$ is essentially the same thing as a *pushout-preserving* functor $\mathscr{L}^n \to \mathrm{Fin}$. This observation is fruitful, as there is a faithful functor $\mathbf{\Delta} \to \mathrm{Cat}$ sending $[n]$ to $\mathscr{L}^n$ (since Tw from Definition 2.1.5 is a functor), but no such functor which sends $[n]$ to $\mathscr{L}^n_0$.

**Definition 2.1.10.** We say a functor $\mathscr{L}^n \to \mathcal{C}$ is *special* if it is a left Kan extension of its restriction to $\mathscr{L}^n_0$. We write $\widetilde{M}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ for the functor which takes $[n]$ to the full subcategory $\widetilde{M}_n$ of $\mathrm{Fun}(\mathscr{L}^n, \mathrm{Fin})$ spanned by the special functors.

Since every functor $\mathscr{L}^n_0 \to \mathrm{Fin}$ admits a left Kan extension $\mathscr{L}^n \to \mathrm{Fin}$, we see that $\widetilde{M}_n$ contains all of the level graphs of height $n$. Notice that a functor $\mathscr{L}^n$ is special if and only if it is pushout-preserving; this is equivalent to saying that *every* square in $\mathscr{L}^n$ is sent to a pushout. Given any $\alpha\colon [n] \to [m]$, the functor $\mathscr{L}^\alpha\colon \mathscr{L}^n \to \mathscr{L}^m$ is automatically pushout-preserving since every square in both categories is a pushout square.

**Remark 2.1.11.** If $F\colon \mathscr{L}^n \to \mathrm{Fin}$ is a special functor, then $F_{i,j}$ is a quotient of

$$\left(\coprod_{k=i}^{j} F_{k,k}\right) \amalg \left(\coprod_{k=i}^{j-1} F_{k,k+1}\right).$$

One can see this via induction on $j-i$. The base cases don't utilize the assumption at all: for $j-i=0$ the statement is clear. For $j-i=1$, elements of $F_{i,i}$ and $F_{j,j} = F_{i+1,i+1}$ are identified with their images in $F_{i,i+1} = F_{i,j}$. For higher values, we have that $F_{i,j}$ is a pushout of $F_{i,j-1} \leftarrow F_{i+1,j-1} \to F_{i+1,j}$, so the result follows.

**Lemma 2.1.12.** *Suppose that $F\colon \mathscr{L}^n \to \mathrm{Fin}$ is a special functor and let $H$ be the directed graph associated to $F|_{\mathscr{L}^n_0}$ as in Lemma 2.1.9. The graph $H$ is connected if and only if $F_{0,n}$ is a one-element set.*

*Proof.* If $F_{0,n} = A_1 \amalg A_2$, then $F = F^1 \amalg F^2$ where $F^k_{i,j} = (F_{i,j} \to F_{0,n})^{-1}(A_k) \subseteq F_{i,j}$ for $k = 1, 2$. Each $F^k$ is a left Kan extension of $F^k|_{\mathscr{L}^n_0}$; further, if $A_k \neq \varnothing$, then $F^k|_{\mathscr{L}^n_0}$ is not the trivial functor (which sends each objects to $\varnothing$) by Remark 2.1.11. Thus if $A_1$ and $A_2$ are both nonempty, we get a nontrivial coproduct decomposition $F|_{\mathscr{L}^n_0} = F^1|_{\mathscr{L}^n_0} \amalg F^2|_{\mathscr{L}^n_0}$. The correspondence from Lemma 2.1.9 preserves coproducts, which tells us that $H$ is not connected.

In the other direction, suppose that $H$ is not connected. As the correspondence preserves coproducts, we obtain a nontrivial decomposition $F|_{\mathscr{L}^n_0} = G^1 \amalg G^2$. Letting $F^k$ be a left Kan extension of $G^k$ $(k = 1, 2)$, we then have $F \cong F^1 \amalg F^2$. As each $G^k$ is nonempty, so is $F^k_{0,n}$ by Remark 2.1.11. Thus we have $F_{0,n} \cong F^1_{0,n} \amalg F^2_{0,n}$ as a decomposition into disjoint nonempty sets. □

We now endeavor (in Remark 2.1.15) to isolate and generalize the construction from the proof of this lemma.

**Definition 2.1.13** (Level subgraphs)**.** Suppose that $F\colon \mathscr{L}^n \to \mathrm{Fin}$ is a special functor. Elements of $F_{i,j}$ will be called $(i,j)$-*level subgraphs* of the level graph $F|_{\mathscr{L}^n_0}$.

Figure 2 provides a graphical representation of this concept, where each element of the set $F_{i,j}$ is depicted as a connected level graph.

**Definition 2.1.14.** If $0 \leq i \leq j \leq n$, write $\mathscr{L}^n_{i,j}$ for the full subcategory consisting of those objects $(k, \ell)$ so that $i \leq k \leq \ell \leq j$. In other words, if $\alpha\colon [j-i] \to [n]$ is given by $\alpha(t) = t + i$, then $\mathscr{L}^n_{i,j}$ is the image of the functor $\mathscr{L}^\alpha\colon \mathscr{L}^{j-i} \to \mathscr{L}^n$.

**Remark 2.1.15.** Each $(i,j)$-level subgraph of $G$ determines a connected, height $(j-i)$-level graph. Specifically, given a special functor $F\colon \mathscr{L}^n \to \mathrm{Fin}$ and an element

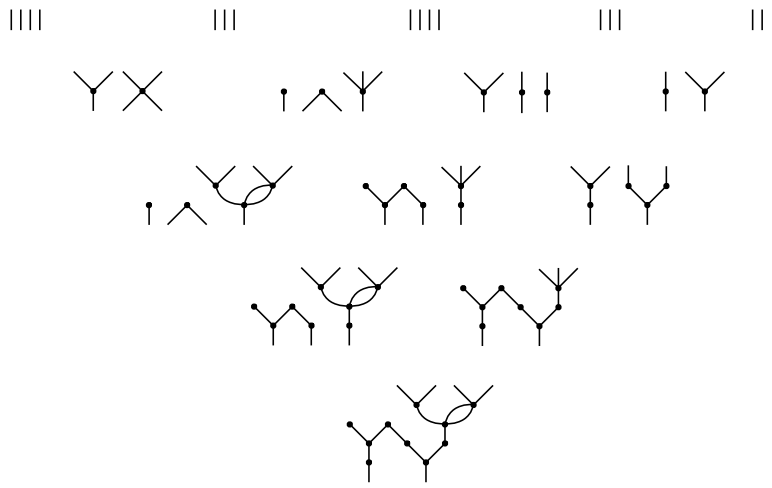


FIGURE 2. Special functor associated to a level graph of height four.

$x \in F_{i,j}$, define for $i \leq k \leq \ell \leq j$ a set $\widetilde{F}_{k,\ell}$ as the pullback

$$\begin{array}{ccc} \widetilde{F}_{k,\ell} & \longrightarrow & F_{k,\ell} \\ \downarrow & & \downarrow \\ \{x\} & \longrightarrow & F_{i,j}. \end{array}$$

This determines a functor $K\colon \mathscr{L}^{j-i} \xrightarrow{\cong} \mathscr{L}^n_{i,j} \to \mathrm{Fin}$ (see Definition 2.1.14) by $K_{a,b} = \widetilde{F}_{a+i,b+i}$. The functor $K$ is special because the pullback functor $\mathrm{Fin}_{/F_{i,j}} \to \mathrm{Fin}_{/\{x\}}$ preserves colimits. The underlying level graph $K|_{\mathscr{L}^{j-i}_0}$ is connected by Lemma 2.1.12 since $K_{0,j-i} = \widetilde{F}_{i,j} \cong \{x\}$.

We now give a crucial definition of this section, which is inspired by the $\Phi$-sequences of [Bar18] (though we are not actually considering a category of $\Phi$-sequences; see Remark 2.1.37 below). It will tell us that any morphism of level graphs which *fixes levels* is a monomorphism on edge and vertex sets (1), and preserves sets of edges incident to a given vertex (2).

**Definition 2.1.16.** Define a functor $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ by declaring that $M_n$ is the wide subcategory of $\widetilde{M}_n$ containing those morphisms $F \to F'$ which satisfy the following two properties:

(1) For every $i \leq j$, the map $F_{i,j} \to F'_{i,j}$ is a monomorphism in Fin.
(2) For every $0 \leq k \leq i \leq j \leq \ell \leq n$, the naturality square

$$\begin{array}{ccc} F_{i,j} & \longrightarrow & F'_{i,j} \\ \downarrow & & \downarrow \\ F_{k,\ell} & \longrightarrow & F'_{k,\ell} \end{array}$$

is cartesian in Fin.

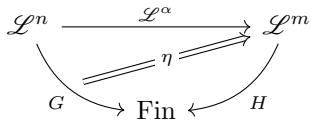
**Definition 2.1.17** (Category of level graphs)**.** Let $\mathcal{E} \to \mathbf{\Delta}$ denote the Grothendieck fibration associated to the functor $M$. We write $\mathbf{L} \to \mathbf{\Delta}$ for the Grothendieck fibration[2] obtained by choosing a skeleton $\mathbf{L}$ of $\mathcal{E}$. We call $\mathbf{L}$ the *category of level graphs*. Write $\mathbf{L}_n$ for the fiber over $[n] \in \mathbf{\Delta}$.

**Remark 2.1.18.** It follows from the previous definition that $\mathbf{L} \to \mathbf{\Delta}$ is associated to a categorical object in Cat, i.e. $\mathbf{L}_n \simeq \mathbf{L}_1 \times_{\mathbf{L}_0} \dots \times_{\mathbf{L}_0} \mathbf{L}_1$. In other words, it induces a double categorical structure.

**Notation 2.1.19.** For each height $n$ level graph $G$, there is a unique special functor $F\colon \mathscr{L}^n \to \mathrm{Fin}$ which is an object of $\mathbf{L}_n$ so that $F|_{\mathscr{L}^n_0}$ is isomorphic to $G$. Further, every object of $\mathbf{L}$ arises in this way. We therefore simplify matters and identify every level graph with its corresponding object in $\mathbf{L}$, and call objects in $\mathbf{L}$ level graphs as well. Henceforth, we generally use $G$ to denote a special functor $\mathscr{L}^n \to \mathrm{Fin}$ since it is essentially the same thing as the level graph $G|_{\mathscr{L}^n_0}$.

Let us unravel Definition 2.1.17 explicitly. Suppose that $G$ and $H$ are two level graphs, of height $n$ and $m$ respectively. Then a morphism from $G$ to $H$ consists of two pieces of data:

- A map $\alpha\colon [n] \to [m]$ in $\mathbf{\Delta}$ and
- A natural transformation




  from $G$ to $H \circ \mathscr{L}^\alpha$.

These should satisfy the following two conditions:

(1) For $0 \le i \le j \le n$, the map $\eta_{i,j}\colon G_{i,j} \to H_{\alpha(i),\alpha(j)}$ is a monomorphism.
(2) For every $0 \le k \le i \le j \le \ell \le n$, the naturality square

$$\begin{array}{ccc} G_{i,j} & \xrightarrow{\eta_{i,j}} & H_{\alpha(i),\alpha(j)} \\ \downarrow & & \downarrow \\ G_{k,\ell} & \xrightarrow{\eta_{k,\ell}} & H_{\alpha(k),\alpha(\ell)} \end{array}$$

  is a pullback.

Given a 1-category $\mathcal{C}$, a *factorization system* (or *orthogonal* factorization system) consists of a pair of subcategories $(\mathcal{C}^L, \mathcal{C}^R)$, each containing all isomorphisms of $\mathcal{C}$, so that each morphism $f$ admits a factorization $f = r \circ \ell$ where $r \in \mathcal{C}^R$ and $\ell \in \mathcal{C}^L$, and this factorization is unique up to *unique* isomorphism (cf., [AHS06, Proposition 14.7]). This notion is subsumed by Definition 3.1.1 in the $\infty$-categorical context.

**Example 2.1.20.** The category $\mathrm{Fin}_*$ of finite pointed sets coincides with the opposite category of Segal's category $\mathbf{\Gamma}$ introduced in [Seg74, Definition 1.1]. In particular, a map $f\colon A_+ \to B_+$ in $\mathrm{Fin}_*$ can be identified with a partial map from $A$ to $B$. The category $\mathrm{Fin}_*$ has an inert-active factorization system (see, for instance, [Lur, Remark 2.1.2.2]), where a map $f\colon A_+ \to B_+$ is *inert* if $|f^{-1}(b)| = 1$ for every $b \in B$, and *active* if $f^{-1}(*) = \{*\}$. This factorization system restricts to one on the skeleton, $\mathbf{F}_*$, of $\mathrm{Fin}_*$.

[2]Since $\mathbf{\Delta}$ is a skeletal category, the composite $\mathbf{L} \to \mathcal{E} \to \mathbf{\Delta}$ is again a Grothendieck fibration.

**Example 2.1.21.** Let $\alpha\colon [m] \to [n]$ be a morphism in $\mathbf{\Delta}$. We call it *active* if it is boundary preserving, i.e. $\alpha(0) = 0$ and $\alpha(m) = n$, and *inert* if there is a constant $c_\alpha$ so that $\alpha(t) = c_\alpha + t$ for all $t$. It is standard that this constitutes an active-inert factorization system on $\mathbf{\Delta}$.

**Definition 2.1.22.** Let $(\alpha, \eta)\colon G \to H$ be a morphism in $\mathbf{L}$ where $\alpha\colon [m] \to [n]$.

- The map is called *inert* if $\alpha$ is inert in $\mathbf{\Delta}$.
- The map is called *active* if $\alpha$ is active in $\mathbf{\Delta}$ and $\eta_{i,j}\colon G_{i,j} \to H_{\alpha(i),\alpha(j)}$ is an isomorphism for every $0 \leq i \leq j \leq m$.

We have the following three subcategories of $\mathbf{L}$:

- Write $\mathbf{L}_{\text{int}}$ for the wide subcategory of $\mathbf{L}$ consisting of the inert maps.
- Write $\mathbf{L}_{\text{el}}$ for the full subcategory of $\mathbf{L}_{\text{int}}$ spanned by the elementary graphs from Example 2.1.3.
- Write $\mathbf{L}_{\text{act}}$ for the wide subcategory of $\mathbf{L}$ containing only active morphisms.

**Remark 2.1.23.** Inert maps are, in particular, monomorphisms. Suppose $(\alpha, \eta)\colon G \to H$ is an inert map in $\mathbf{L}$. As $\alpha$ is inert, it is of the form $\alpha(i) = i + t$. By Definition 2.1.16(1), $G_{i,j} \to H_{\alpha(i),\alpha(j)} = H_{i+t,j+t}$ is a monomorphism. It follows that

$$\mathtt{E}(G) = \coprod G_{i,i} \to \coprod H_{k,k} = \mathtt{E}(H) \qquad \text{and}$$
$$\mathtt{V}(G) = \coprod G_{i-1,i} \to \coprod H_{k-1,k} = \mathtt{V}(H)$$

are monomorphisms. Together with Lemma 2.1.9, we see that every inert morphism in $\mathbf{L}$ determines an inclusion of a subgraph.

**Remark 2.1.24** (Weaker condition for active maps)**.** To show that a map as in Definition 2.1.22 is active, it is enough that $\alpha$ be active and $\eta_{0,m}\colon G_{0,m} \to H_{0,n}$ be a bijection. Indeed, for every $0 \leq i \leq j \leq m$, the diagram

$$\begin{array}{ccc} G_{i,j} & \xrightarrow{\eta_{i,j}} & H_{\alpha(i),\alpha(j)} \\ \downarrow & & \downarrow \\ G_{0,m} & \xrightarrow[\cong]{\eta_{0,m}} & H_{\alpha(0),\alpha(m)} \end{array}$$

is a pullback by (2) of Definition 2.1.16, which implies that $\eta_{i,j}$ is an isomorphism.

**Lemma 2.1.25.** *The pair of subcategories* $(\mathbf{L}_{\text{act}}, \mathbf{L}_{\text{int}})$ *constitute an orthogonal factorization system on* $\mathbf{L}$.

*Proof.* Let us construct a factorization of a morphism $(\alpha, \eta)\colon (n, G) \to (m, H)$ in $\mathbf{L}$. For $0 \leq i \leq j \leq \alpha(n) - \alpha(0) = p$, let $K_{i,j}$ be the pullback

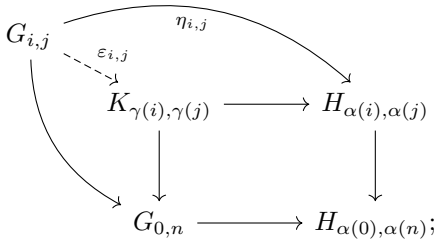
$$\begin{array}{ccc} K_{i,j} & \longrightarrow & H_{i+\alpha(0),j+\alpha(0)} \\ \downarrow & & \downarrow \\ G_{0,n} & \xhookrightarrow{\eta_{0,n}} & H_{\alpha(0),\alpha(n)}. \end{array}$$

As in Remark 2.1.15, the functor $K\colon \mathscr{L}^p \to \mathrm{Fin}$ is a special functor. Letting $\beta\colon [p] \to [m]$ be $\beta(i) = i + \alpha(0)$ and $\gamma\colon [n] \to [p]$ be $\gamma(i) = \alpha(i) - \alpha(0)$, we have a

factorization

$$\begin{array}{ccccccc} G & \longrightarrow & K & \longrightarrow & \beta^* H & \longrightarrow & H \\ \vdots & & \vdots & & \vdots & & \vdots \\ [n] & \xrightarrow{\gamma} & [p] & \xrightarrow{\mathrm{id}} & [p] & \xrightarrow{\beta} & [m] \end{array}$$

lying above the usual active-inert factorization in $\mathbf{\Delta}$. The morphism $(\gamma, \varepsilon)\colon G \to K$ is defined via the diagram

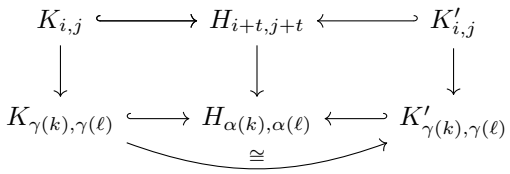



since both the inner and outer squares are pullbacks, $\varepsilon_{i,j}$ is a bijection. It follows that $(\gamma, \varepsilon)$ is active. The map $K \to H$ is inert since $\beta$ is.

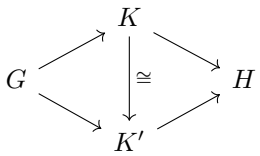
Suppose we have some other factorization

$$G \xrightarrow{(\gamma', \varepsilon')} K' \xrightarrow{(\beta', \mu')} H$$

into an active map followed by an inert map. By uniqueness of factorizations in $\mathbf{\Delta}$, we know that $\gamma' = \gamma$ and $\beta' = \beta$. Since $\gamma$ is active, for each $0 \leq i \leq j \leq p$, there exists $0 \leq k \leq \ell \leq n$ with $0 \leq \gamma(k) \leq i \leq j \leq \gamma(j) \leq p$. Since $(\gamma, \varepsilon')$ is active, we know that $\varepsilon'_{k,\ell}\colon G_{k,\ell} \to K'_{\gamma(k),\gamma(\ell)}$ is a bijection, hence we have an isomorphism $\varepsilon'_{k,\ell}\varepsilon_{k,\ell}^{-1}\colon K_{\gamma(k),\gamma(\ell)} \cong K'_{\gamma(k),\gamma(\ell)}$. If $t = \beta(0)$, then (2) of Definition 2.1.16 gives that both squares in




are pullbacks, exhibiting an isomorphism so that the following diagram commutes.




As inert maps are monomorphisms (Remark 2.1.23), there is at most one such isomorphism $K \to K'$ making the right triangle commute. Thus every morphism in $\mathbf{L}$ factors as an active map followed by an inert map, and this factorization is unique up to unique isomorphism. □

We now want to define a vertex functor $\mathtt{V}_{\mathbf{L}}\colon \mathbf{L} \to \mathrm{Fin}_*^{\mathrm{op}}$ which takes every level graph to the set of its vertices with added base point. Recall that if $H$ is a height $m$ level graph, then we have a decomposition $\mathtt{V}(H) = \coprod_{k=0}^{m-1} H_{k,k+1}$.

**Definition 2.1.26** (The functor $\mathbf{L} \to \mathrm{Fin}_*^{\mathrm{op}}$)**.** Suppose that $G\colon \mathscr{L}^n \to \mathrm{Fin}$ and $H\colon \mathscr{L}^m \to \mathrm{Fin}$ are level graphs and $f\colon G \to H$ is a morphism of $\mathbf{L}$ lying over $\alpha\colon [n] \to [m]$. Define

$$\mathtt{V}_{\mathbf{L}}(f)\colon \mathtt{V}(H)_+ \to \mathtt{V}(G)_+$$

by specifying that $\mathtt{V}_{\mathbf{L}}(f)(v) = w$ just when there is a commutative square

$$\begin{array}{ccc} * & \xrightarrow{v} & H_{k,k+1} \\ \downarrow{\scriptstyle w} & & \downarrow \\ G_{i,i+1} & \xrightarrow{f} & H_{\alpha(i),\alpha(i+1)}, \end{array}$$

and otherwise $\mathtt{V}_{\mathbf{L}}(f)(v) = *$.

In other words, $\mathtt{V}_{\mathbf{L}}(f)(v) = w$ if $v$ is in the level subgraph associated to $f(w)$ and $\mathtt{V}_{\mathbf{L}}(f)(v)$ is the base point otherwise. The following proposition shows that this rule defines a functor.

**Proposition 2.1.27.** *The function $\mathtt{V}_{\mathbf{L}}(f)$ from the previous definition is well-defined. Further, this assignment is compatible with composition.*

*Proof.* First, note that for any $k$ there is at most one $i$ so that

$$(k, k+1) \to (\alpha(i), \alpha(i+1)) \tag{1}$$

exists in $\mathscr{L}^m$. Suppose that there are $i < i'$ so that (1) exists. The existence of (1) for $i$ means that $k+1 \leq \alpha(i+1)$, while (1) for $i'$ implies $\alpha(i') \leq k$. But $i+1 \leq i'$ and so we have $k+1 \leq \alpha(i+1) \leq \alpha(i') \leq k$, a contradiction.

If $k$ is such that (1) does not exist for any $i$, then $\mathtt{V}_{\mathbf{L}}(f)$ takes every vertex in $H_{k,k+1}$ to the base point. Otherwise, form the pullback

$$\begin{array}{ccc} P & \hookrightarrow & H_{k,k+1} \\ \downarrow & & \downarrow \\ G_{i,i+1} & \hookrightarrow & H_{\alpha(i),\alpha(i+1)}; \end{array}$$

since the bottom map is a monomorphism, so is the top. Hence it gives us a morphism $\mathtt{V}(H_{k,k+1})_+ \to \mathtt{V}(P)_+$ which takes every vertex in $\mathtt{V}(H_{k,k+1})$ that does not lie in the image of the monomorphism to the base point. Composing with $\mathtt{V}(P)_+ \to \mathtt{V}(G_{i,i+1})_+ \hookrightarrow \mathtt{V}(G)_+$, we attain our desired map $\mathtt{V}_{\mathbf{L}}(f)\colon \mathtt{V}(H)_+ \to \mathtt{V}(G)_+$.

We now show that $\mathtt{V}_{\mathbf{L}}$ is a functor. It clearly sends identities to identities.

Consider maps $f\colon G \to H$ and $f'\colon H \to K$ (lying over $\alpha$ and $\beta$, respectively). We wish to show that $\mathtt{V}_{\mathbf{L}}(f'f) = \mathtt{V}_{\mathbf{L}}(f)\mathtt{V}_{\mathbf{L}}(f')$. Let $S \subseteq \mathtt{V}(K)$ be the preimage of $\mathtt{V}(G)$ under $\mathtt{V}_{\mathbf{L}}(f'f)$, and let $T \subseteq \mathtt{V}(K)$ be the preimage of $\mathtt{V}(G)$ under $\mathtt{V}_{\mathbf{L}}(f)\mathtt{V}_{\mathbf{L}}(f')$. Our goal is to show that $S = T$ and that the two functions agree on this subset. To that end, let $x \in K_{\ell,\ell+1}$ and consider the following diagram.

$$\begin{array}{ccccc} * & & \xrightarrow{x} & & K_{\ell,\ell+1} \\ & \searrow^{w} & & & \downarrow \\ \downarrow{\scriptstyle u} & & H_{k,k+1} & \xrightarrow{f'} & K_{\beta(k),\beta(k+1)} \\ & & \downarrow & & \downarrow \\ G_{i,i+1} & \xrightarrow{f} & H_{\alpha(i),\alpha(i+1)} & \xrightarrow{f'} & K_{\beta\alpha(i),\beta\alpha(i+1)} \end{array}$$

It instantly shows that if $\mathtt{V}_{\mathbf{L}}(f')(x) = w$ and $\mathtt{V}_{\mathbf{L}}(f)(w) = u$, then $\mathtt{V}_{\mathbf{L}}(f'f)(x) = u = \mathtt{V}_{\mathbf{L}}(f)(\mathtt{V}_{\mathbf{L}}(f')(x))$. Hence $T \subseteq S$, and the two functions agree on $T$.

To see that $\mathtt{V}_{\mathbf{L}}(f'f) = \mathtt{V}_{\mathbf{L}}(f)\mathtt{V}_{\mathbf{L}}(f')$, it remains to show that $S \subseteq T$. Suppose that $x \in K_{\ell,\ell+1}$ is such that $\mathtt{V}_{\mathbf{L}}(f'f)(x) = u \in \mathtt{V}(G)$, that is, so that the outer rectangle exists. Since $\beta\alpha(i) \leq \ell < \ell + 1 \leq \beta\alpha(i+1)$, there exits a unique $k$ so that $\alpha(i) \leq k < \alpha(i+1)$, $\beta(k) \leq \ell$, and $\beta(k+1) > \ell$. It follows that the lower-right rectangle exists in the diagram; it is a pullback since $f'$ is a morphism of $\mathbf{L}$. Thus the indicated $w$ exists and we have that $S \subseteq T$. □

**Proposition 2.1.28.** *The functor* $\mathtt{V}_{\mathbf{L}} \colon \mathbf{L} \to \mathrm{Fin}_*^{\mathrm{op}}$ *preserves the active-inert factorization systems.*

*Proof.* If $(\alpha, \eta) \colon G \to H$ is an active map in $\mathbf{L}$, then $\alpha \colon [n] \to [m]$ is boundary preserving. Therefore, if a vertex $v \in \mathtt{V}(H)$ lies in $H_{k,k+1}$ for some $0 \leq k \leq m$, then by the previous proposition there exists a unique $0 \leq i \leq n$ such that $\alpha(i) \leq k$ and $k + 1 \leq \alpha(i+1)$. Since $\eta_i \colon G_{i,i+1} \to H_{\alpha(i),\alpha(i+1)}$ is an isomorphism, the composite $H_{k,k+1} \to H_{\alpha(i),\alpha(i+1)} \to G_{i,i+1}$ exists. It induces a map $\mathtt{V}_{\mathbf{L}}(f) \colon \mathtt{V}(H)_+ \to \mathtt{V}(G)_+$ by the construction of Definition 2.1.26 which takes only the base point to the base point, hence, $\mathtt{V}_{\mathbf{L}}$ preserves active morphisms.

Now suppose $(\alpha, \eta) \colon G \to H$ is inert and $v \in \mathtt{V}(H)$ lies in $H_{k,k+1}$ for some $k$. Since $\alpha$ is inert, i.e. an interval inclusion, the construction of $\mathtt{V}_{\mathbf{L}}(f) \colon \mathtt{V}(H)_+ \to \mathtt{V}(G)_+$ shows that the preimage of every element in $\mathtt{V}(G) \subseteq \mathtt{V}(G)_+$ has cardinality 1. This proves that the functor $\mathtt{V}_{\mathbf{L}}$ also preserves inert maps. □

**Notation 2.1.29.** We will also write $\mathtt{V}_{\mathbf{L}}$ for a composite $\mathbf{L} \to \mathrm{Fin}_*^{\mathrm{op}} \simeq \mathbf{F}_*^{\mathrm{op}}$.

We now discuss several important subcategories of $\mathbf{L}$. Recall that the any level graph has an underlying directed graph (see Lemma 2.1.9). See Figure 3 for a schematic.

**Definition 2.1.30** (Other categories of level graphs)**.** We define several full subcategories of $\mathbf{L}$.

- The most important is $\mathbf{L}_{\mathrm{c}}$, which is the full subcategory on the *connected* level graphs. By Lemma 2.1.12, these are characterized by the property that the associated special functor $F \colon \mathscr{L}^n \to \mathrm{Fin}$ preserves terminal objects.
- The full subcategory $\mathbf{L}_{\mathrm{sc}}$ consists of the *simply-connected* level graphs.
- There is also the larger subcategory $\mathbf{L}_{0\text{-type}}$, consisting of possibly disconnected graphs where each connected component is simply-connected.
- The full subcategory $\mathbf{L}_{\mathrm{out}}$ consists of those level graphs $G$ so that each vertex has at least one output. That is, the functions $G_{k,k} \to G_{k-1,k}$ are surjective.
- The full subcategory $\mathbf{L}_{\mathrm{out,c}}$ consists of those level graphs $G$ which are both connected and so that each vertex has at least one output.
- Likewise there is a subcategory consisting of level graphs $G$ which so that each $G_{k,k} \to G_{k,k+1}$ surjective, and a subcategory of that where additionally the graphs are connected. By symmetry, all theorems concerning $\mathbf{L}_{\mathrm{out}}$ and $\mathbf{L}_{\mathrm{out,c}}$ also hold for these nonempty input versions, so we will not mention them again.

- There is a subcategory of $\mathbf{L}$ consisting of the level *forests*, which are those graphs level $G$ so that $G_{k,k} \to G_{k-1,k}$ is bijective. Following Proposition 2.1.35 below we identify this category with the category $\mathbf{\Delta_F}$ of Fin-sequences defined in [Bar18]. Likewise, there is subcategory of level *trees*, which are the level forests $G$ so that $G_{n,n}$ is a point. Again by Proposition 2.1.35 this category is equivalent to the category $\mathbf{\Delta}^1_{\mathbf{F}}$ defined in [CHH18].
- The simplex category $\mathbf{\Delta}$ can be identified with a full subcategory of all of the above. It is spanned by the linear graphs, i.e. those graphs $G$ such that $G_{k,k}$ is a point for all $k$.

**Remark 2.1.31.** Restricting the Grothendieck fibration $\mathbf{L} \to \mathbf{\Delta}$ to any of the subcategories $\mathbf{L}_{0\text{-type}}$, $\mathbf{L}_{\text{out}}$, $\mathbf{\Delta_F}$, or $\mathbf{\Delta}$ again gives a Grothendieck fibration. The same is not true for $\mathbf{L}_{\text{c}}$, $\mathbf{L}_{\text{sc}}$, $\mathbf{L}_{\text{out,c}}$, and $\mathbf{\Delta}^1_{\mathbf{F}}$, as the domain of a cartesian lift in $\mathbf{L}$ over an inert morphism is sometimes disconnected, even when the codomain is connected.

**Remark 2.1.32** (Simply-connected graphs)**.** Most of the types of level graphs above are characterized by clear properties of $G\colon \mathscr{L}_0^n \to \text{Fin}$ or of the associated special functor – certain maps are surjections or bijections, or certain sets like $G_{0,n}$ or $G_{n,n}$ are terminal. The exceptions are $\mathbf{L}_{\text{sc}}$ and $\mathbf{L}_{0\text{-type}}$. Though it is possible to give a criterion for a level graph $G$ to be in one of these subcategories strictly in terms of the associated special functor, the only criterion we know is somewhat unsatisfactory, as it simply reformulates the notion of 'undirected path' in the underlying directed graph. It does not seem worthwhile to do so here, as we have developed other tools in this paper that do the job just as well. Namely, in §2.3 we define a functor $\tau\colon \mathbf{L}_{\text{c}} \to \mathbf{G}$ which takes a level graph to its underlying directed graph. The codomain of $\tau$ is the properadic graphical category from [HRY15], so we may use theorems therein. For example, we have the following proposition, which says that $\mathbf{L}_{\text{sc}} \subseteq \mathbf{L}_{\text{c}}$ and $\mathbf{L}_{0\text{-type}} \subseteq \mathbf{L}$ are both sieves. We emphasize that there is no logical circularity here: nothing in §2.2 and §2.3 (including Appendix A and Appendix B) depends on this result.

**Lemma 2.1.33.** *Suppose that $G \to H$ is a map in $\mathbf{L}$ with $H \in \mathbf{L}_{0\text{-type}}$. Then $G$ is also an object of $\mathbf{L}_{0\text{-type}}$. In particular, if $G$ is connected then $G \in \mathbf{L}_{\text{sc}}$.*

*Proof.* Let $G'$ be an arbitrary connected component of $G$, and let $H'$ be the unique connected component of $H$ so that the following factorization exists

$$\begin{array}{ccc} G' & \dashrightarrow & H' \\ \downarrow & & \downarrow \\ G & \longrightarrow & H. \end{array}$$

Apply $\tau$ from §2.3 to the morphism $G' \to H'$ of $\mathbf{L}_{\text{c}}$. By assumption $\tau H'$ is simply-connected, so [HRY15, Proposition 5.2.8] gives that the graph $\tau G'$ is also simply-connected. Each component of $G$ is simply-connected, hence $G \in \mathbf{L}_{0\text{-type}}$. □

We will show that several other of these subcategory inclusions are sieves in Lemma 2.1.38 below.

**Lemma 2.1.34.** *The active-inert factorization system on $\mathbf{L}$ restricts to one on each of the subcategories from Definition 2.1.30.*

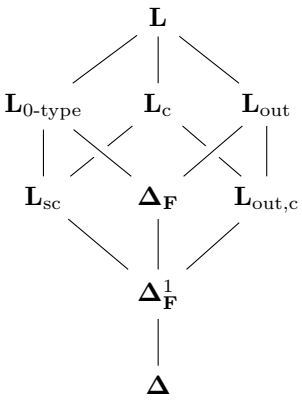



FIGURE 3. The lattice of subcategories from Definition 2.1.30

*Proof.* Let $G \to K \to H$ be the active-inert factorization of a morphism in $\mathbf{L}$, where $G$ has height $m$ and $K$ has height $n$. If $G$ is connected, then Lemma 2.1.12 implies that $G_{0,m} \cong \{*\}$. By definition of active morphisms we have $G_{0,m} \xrightarrow{\sim} K_{0,n}$ where $n$ is the height of $K$, hence, $K$ is connected as well. It follows that the active-inert factorization system on $\mathbf{L}$ restricts to one on $\mathbf{L}_{\mathrm{c}}$.

Now suppose that $G$ is connected and $H$ is simply-connected. By the previous paragraph, we know that $K$ is connected, and by Lemma 2.1.33, $K$ is simply-connected as well. Thus the active-inert factorization system on $\mathbf{L}_{\mathrm{c}}$ further restricts to one on $\mathbf{L}_{\mathrm{sc}}$.

The remaining cases are proved in a similar manner, with $K$ inheriting properties from $G$ or $H$. □

**Proposition 2.1.35.** *There is an equivalence between the full subcategory of* $\mathbf{L}$ *consisting of the forests and the category* $\mathbf{\Delta_F}$ *defined in* [Bar18].

*Proof.* It follows from [Lur09, Proposition 4.3.2.15] or classical results about Kan extension that $\widetilde{M}_n \simeq \mathrm{Fun}(\mathscr{L}_0^n, \mathrm{Fin})$ for every $n \geq 0$ (see Definition 2.1.10). This equivalence shows that the full subcategory in $\widetilde{M}_n$ spanned by forests can be identified with $\mathrm{Fun}([n], \mathrm{Fin})$. The two conditions in Definition 2.1.16 implies that $M_n$ is equivalent to the subcategory of Fin-sequences in $\mathrm{Fun}([n], \mathrm{Fin})$ in the sense of [Bar18, Definition 2.4]. Hence, we have an equivalence of the corresponding cartesian fibrations, which implies the result. □

**Remark 2.1.36.** In [Bar18], Barwick uses the full subcategory $\mathbf{\Delta_F}$ of $\mathbf{L}$ to construct 'complete Segal operads' and proves that they are equivalent to Lurie's $\infty$-operads. In [CHH18] the authors further show that Barwick's approach to $\infty$-operads is equivalent to dendroidal Segal spaces, which are in turn equivalent to simplicial operads by [CM13].

**Remark 2.1.37.** In contrast to the level forest situation, the category $\mathbf{L}$ does not arise from an operator category $\Phi$ in the sense of [Bar18, Definition 1.2]. Indeed, if $\mathbf{L}$ was the category of $\Phi$-sequences ([Bar18, Definition 2.4]) for some operator category $\Phi$, then

- height 0 level graphs would correspond to functors $[0] \to \Phi$, and
- height 1 level graphs would correspond to functors $[1] \to \Phi$.

It would then follow that the objects of $\Phi$ are finite sets and the morphisms of $\Phi$ are cospans of finite sets. In particular, each $\hom(X, Y)$ is infinite, so neither (1.2.1) nor (1.2.3) from [Bar18] is satisfied.

**Lemma 2.1.38.** *Suppose that $\mathbf{A} \subseteq \mathbf{B}$ is one of the fully-faithful inclusions appearing in the following diagram*

$$\begin{array}{ccccc} \boldsymbol{\Delta} & \longrightarrow & \boldsymbol{\Delta}^1_{\mathbf{F}} & \longrightarrow & \mathbf{L}_{\mathrm{out,c}} \\ & & \downarrow & & \downarrow \\ & & \mathbf{L}_{\mathrm{sc}} & \dashrightarrow & \mathbf{L}_{\mathrm{c}} \end{array} \qquad \begin{array}{ccc} \boldsymbol{\Delta}_{\mathbf{F}} & \longrightarrow & \mathbf{L}_{\mathrm{out}} \\ \downarrow & & \downarrow \\ \mathbf{L}_{0\text{-type}} & \dashrightarrow & \mathbf{L} \end{array}$$

*including the two red dashed maps.*

*(1) If $f\colon G \to H$ is a morphism in $\mathbf{B}$ and $H \in \mathbf{A}$, then $G \in \mathbf{A}$.*

*Now suppose that $\mathbf{A} \subseteq \mathbf{B}$ is one of the solid black arrows.*

*(2) If $G \in \mathbf{B}$ is such that for each corolla $\mathfrak{c}$ of $\mathbf{B}$ and each inert map $\mathfrak{c} \to G$, we have that $\mathfrak{c} \in \mathbf{A}$, then $G \in \mathbf{A}$.*

*Proof.* Item (1) for the two red dashed maps is exactly the content of Lemma 2.1.33. We thus concentrate on the cases where $\mathbf{A} \subseteq \mathbf{B}$ is one of the solid black arrows. In each case, we can characterize objects $F$ in $\mathbf{A}$ among those in $\mathbf{B}$ by the following conditions on the functions comprising $F\colon \mathscr{L}^n_0 \to \mathrm{Fin}$

$$\begin{array}{ccccc} \boldsymbol{\Delta} & \xrightarrow{F_{k,k} \cong F_{k+1,k}} & \boldsymbol{\Delta}^1_{\mathbf{F}} & \xrightarrow{F_{k,k} \cong F_{k-1,k}} & \mathbf{L}_{\mathrm{out,c}} \\ & & \downarrow{\scriptstyle F_{k,k} \cong F_{k-1,k}} & & \downarrow{\scriptstyle F_{k,k} \twoheadrightarrow F_{k-1,k}} \\ & & \mathbf{L}_{\mathrm{sc}} & & \mathbf{L}_{\mathrm{c}} \end{array}$$

and likewise for the other solid black arrow maps. As bijections and surjections are preserved by pushouts, we get extended conditions when working with special functors $F\colon \mathscr{L}^n \to \mathrm{Fin}$. For example, a special functor $F\colon \mathscr{L}^n \to \mathrm{Fin}$ with $F_{0,n} = *$ (that is, a connected graph) is in $\mathbf{L}_{\mathrm{out,c}}$ if and only if $F_{k,\ell} \to F_{k-1,\ell}$ is a surjection whenever $0 < k \leq \ell \leq n$. It follows that if $H \in \mathbf{L}_{\mathrm{out,c}}$ and $G \to H$ is a morphism, then we have a pullback square

$$\begin{array}{ccc} G_{i,i} & \longrightarrow & H_{\alpha(i),\alpha(i)} \\ \downarrow & \lrcorner & \downarrow \\ G_{i-1,i} & \longrightarrow & H_{\alpha(i-1),\alpha(i)} \end{array}$$

which implies $G \in \mathbf{L}_{\mathrm{out,c}}$ as well. The other six black arrow cases of (1) follow similarly.

For (2) again consider the case $\mathbf{L}_{\mathrm{out,c}} \to \mathbf{L}_{\mathrm{c}}$, as the others are similar. Fixing $k$, for each $x \in G_{k-1,k}$ there is a corresponding inert map $\mathfrak{c}_x \to G$, which assembles into an inert map $\coprod_x \mathfrak{c}_x \to G$ in $\mathbf{L}$. This map is a bijection on the level $k-1$ and $k$ edges and the level $k$ vertices, and we see that $G_{k,k} \to G_{k-1,k}$ is surjective if and only if each $\mathfrak{c}_x$ is in $\mathbf{L}_{\mathrm{out,c}}$. □

2.2. **The category G of connected, acyclic graphs.** In [HRY15] the authors introduced $\infty$-properads as presheaves on an indexing category $\mathbf{\Gamma}$, called the 'graphical category' or 'properadic graphical category.' In this section, we give a new presentation for a category $\mathbf{G}$ (see Definition 2.2.11 and Definition 2.2.14), and in Appendix A we show that our $\mathbf{G}$ is indeed equivalent to the category $\mathbf{\Gamma}$. Accordingly, going forward we will use whichever description of the properadic graphical category that is most convenient at the time, and to avoid duplication we will frequently use results from [HRY15].

Let $G$ and $H$ be directed graphs, and suppose that we are given a vertex $v \in G$ as well as bijections $b_i \colon \mathtt{in}(v) \cong \mathtt{in}(H)$ and $b_o \colon \mathtt{out}(v) \cong \mathtt{out}(H)$. Then we can define a new graph $G(H)$, the *graph substitution* of $H$ into $G$, where the vertex $v$ has been replaced by the graph $H$. We call the quintuple $(G, H, v, b_i, b_o)$ *graph substitution data*. See Figure 4 for an example (where all edges are oriented by gravity and the bijections are left implicit). Basic facts about graph substitution (including associativity, unitality, and so on) may be found in [YJ15].

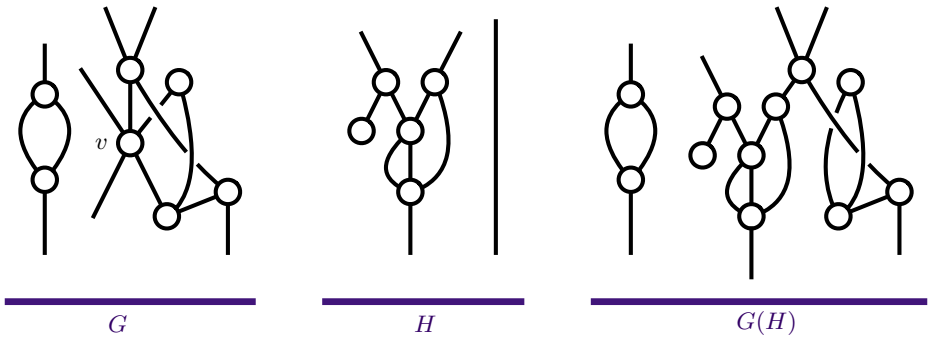



FIGURE 4. Substitution of the graph $H$ into the vertex $v$ of $G$.

**Definition 2.2.1.** Suppose that $K$ is a directed graph.

- An *ordinary subgraph* $H$ of $K$ consists of a pair of subsets $\mathtt{E}(H) \subseteq \mathtt{E}(K)$ and $\mathtt{V}(H) \subseteq \mathtt{V}(K)$, with incidence data inherited from $K$. That is, if $v \in \mathtt{V}(H)$, then we define $\mathtt{in}^H(v)$ to be $\mathtt{in}^K(v) \cap \mathtt{E}(H)$, and likewise for $\mathtt{out}(v)$.
- An *open* subgraph $H$ of $K$ is an ordinary subgraph so that $\mathtt{in}^H(v) = \mathtt{in}^K(v)$ and $\mathtt{out}^H(v) = \mathtt{out}^K(v)$ for all $v \in \mathtt{V}(H)$.
- If $G$ and $H$ are both open subgraphs of $K$, then write $G \pitchfork H$ and $G \cup H$ for the open subgraphs with

$$\mathtt{E}(G \pitchfork H) = \mathtt{E}(G) \cap \mathtt{E}(H) \qquad \mathtt{V}(G \pitchfork H) = \mathtt{V}(G) \cap \mathtt{V}(H)$$
$$\mathtt{E}(G \cup H) = \mathtt{E}(G) \cup \mathtt{E}(H) \qquad \mathtt{V}(G \cup H) = \mathtt{V}(G) \cup \mathtt{V}(H).$$

Suppose that we are given graph substitution data so that $G(H)$ is well-defined. Then $H$ is an open subgraph of $G(H)$. We are primarily interested in acyclic graphs (that is, graphs without directed cycles), and in that case the reverse implication does not hold.

**Definition 2.2.2.** Suppose that $K$ is a connected acyclic directed graph.

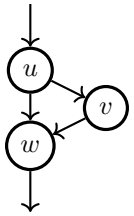
- An ordinary subgraph $H$ of $K$ is said to be a *structured subgraph* if
  - $H$ is connected, and

  - there exists a connected acyclic directed graph $G$, graph substitution data $(G, H, v, b_i, b_o)$, and an isomorphism $G(H) \cong K$ which is the identity on $H$.
- If $H$ is a structured subgraph of $K$, we write $H \sqsubset K$.
- Write $\mathtt{Sb}(K)$ for the set of structured subgraphs of $K$.

Note that the data which guarantees that $H$ is a structured subgraph of $K$ is unique up to isomorphism, and we generally disregard it. Also note that structured subgraphs are automatically open subgraphs.

We note that any vertex $v$ of $K$ determines a subgraph $C_v$, which consists of $v$ and all edges incident to $v$. Indeed, $K(C_v)$ is canonically isomorphic to $K$ (unitality of graph substitution). Likewise, every edge $e$ of $K$ determines a subgraph $\downarrow_e$.

**Example 2.2.3.** Consider the following three-vertex graph.




The open subgraph with vertices $u$ and $w$ is *not* a structured subgraph. The open subgraph with vertices $u$ and $v$ *is* a structured subgraph. Informally, collapsing the first subgraph down to a vertex yields a graph that contains a directed cycle, whereas collapsing the second graph down to a vertex yields a graph with no directed cycles.

**Remark 2.2.4.** In [HRY15], structured subgraphs were simply called *subgraphs*. In [Koc16], structured subgraphs are called *convex open subgraphs* and by reformulating [Koc16, 1.6.5] and [Koc16, 1.6.10] we obtain the following characterization of structured subgraphs:

Suppose $G$ is an open subgraph of $K$. Then $G$ is a structured subgraph of $K$ if and only if the associated inclusion $i\colon G \to K$ has the right lifting property with respect to all naïve morphisms of graphs $\downarrow_0 \amalg \downarrow_m \to L$, where $L$ is a linear graph (see Definition 2.1.30) of height $m \geq 0$ and which take $\downarrow_0$ and $\downarrow_m$ to the unique edges in $\mathtt{in}(L)$ and $\mathtt{out}(L)$, respectively. In other words, every commutative square

$$\begin{array}{ccc} \downarrow_0 \amalg \downarrow_m & \longrightarrow & G \\ \downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle i} \\ L & \longrightarrow & K \end{array}$$

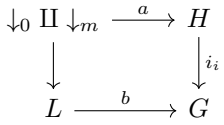
of naïve morphisms has a lift $k$.

Notice in the case when $m = 0$, the map $\downarrow_0 \amalg \downarrow_m \to L$ from the remark is just the 2-fold cover of $\downarrow$.

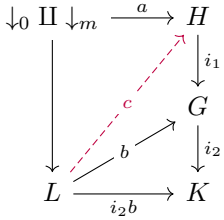
**Proposition 2.2.5.** *Suppose that $G \sqsubset K$ is a structured subgraph of $K$ and $H$ is an ordinary subgraph of $G$. Then $H \sqsubset K$ if and only if $H \sqsubset G$.*

*Proof.* According to the previous remark being a structured subgraph is equivalent to being right orthogonal to all naïve maps of inclusions of endpoints into linear

graphs $L$. If $H$ is a structured subgraph of $K$, then we can extend any commutative square




to a commutative diagram




The assumption $H \sqsubset K$ gives the existence of the map $c$ such that $cf = a$ and $i_2 i_1 c = i_2 b$. It follows from the definition of ordinary subgraphs that the morphism $i_2$ is an monomorphism of graphs, hence, $i_1 c = b$ and the 'only if' direction is established. The 'if' direction is just closure under composition of maps in a right orthogonality class; alternatively, one can use associativity of graph substitution $K \cong G'(G) \cong G'(H'(H)) \cong ((G'(H'))(H)$. □

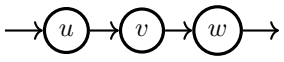
**Remark 2.2.6.** The set $\mathtt{Sb}(K)$ is a partially-ordered set, ordered by inclusion of subgraphs. In light of the previous proposition, it does not matter whether we take $H \leq G$ to mean '$H$ is an ordinary subgraph of $G$,' the stronger '$H$ is an open subgraph of $G$,' or the strongest '$H$ is a structured subgraph of $G$.' This partially-ordered set has a unique maximal element, $K$. Each edge of $K$ is a minimal element.

**Definition 2.2.7.** Suppose that $G, H \in \mathtt{Sb}(K)$. If $G \cup H$ is a structured subgraph of $K$, we write $G \tilde{\cup} H \coloneqq G \cup H \in \mathtt{Sb}(K)$. If not, we say that $G \tilde{\cup} H$ does not exist.

The following is immediate.

**Proposition 2.2.8.** *Suppose that $G, H \in \mathtt{Sb}(K)$. If $G \cup H$ is a structured subgraph of $K$, then it is the least upper bound of $G$ and $H$.* □

It is possible to have least upper bounds in $\mathtt{Sb}(K)$ which are not of this form, and we are entirely uninterested in those. For example, in the graph




the structured subgraphs $C_u$ and $C_w$ have a least upper bound, namely the entire graph.

**Example 2.2.9.** It may be that $H_1 \cup H_2$ is connected, but still is not a structured subgraph. This should be immediate since $\mathtt{V}(K) \hookrightarrow \mathtt{Sb}(K)$ is a monomorphism, but there can exist open connected subgraphs that are not structured subgraphs. For example, in the graph from Example 2.2.3, the open subgraph $C_u \cup C_w$ is connected, but is not a structured subgraph.

**Definition 2.2.10.** If $K$ is a connected acyclic directed graph, then there are two functions $\mathtt{in}$ and $\mathtt{out}$ that go from $\mathtt{Sb}(K)$ to the powerset $\wp(\mathtt{E}(K))$ of $\mathtt{E}(K)$. The

first takes a subgraph $H$ to its set of input edges, while the second takes $H$ to its set of output edges.

The following definition was inspired both by a question from Steve Lack and by Definition 1.12 of [HRY19], which concerned morphisms in a category of undirected trees.

**Definition 2.2.11.** Let $G$ and $K$ be connected acyclic directed graphs. A *morphism* $f\colon G \to K$ consists of two functions $f_0\colon \mathtt{E}(G) \to \mathtt{E}(K)$ and $f_1\colon \mathtt{Sb}(G) \to \mathtt{Sb}(K)$. These should satisfy the following:

(1) The diagram

$$\begin{array}{ccccc} \wp(\mathtt{E}(G)) & \xleftarrow{\mathtt{in}} & \mathtt{Sb}(G) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(G)) \\ \downarrow{\scriptstyle \wp(f_0)} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \wp(f_0)} \\ \wp(\mathtt{E}(K)) & \xleftarrow{\mathtt{in}} & \mathtt{Sb}(K) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(K)) \end{array}$$

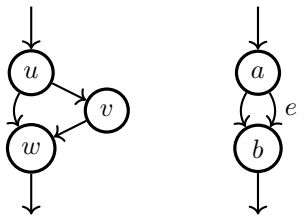
commutes.

(2) Suppose that $H_1, H_2 \in \mathtt{Sb}(G)$ and $H_1 \tilde{\cup} H_2$ exists, then $f_1(H_1 \tilde{\cup} H_2) = f_1(H_1) \tilde{\cup} f_1(H_2)$.

This definition of morphism turns out to be equivalent to the (properadic) graphical maps from [HRY15, Definition 6.46]. We show this in Theorem A.1. This implies that a morphism $f\colon G \to K$ is completely determined by $f_0$ by [HRY15, Corollary 6.62].

The reader familiar with [HRY19, Definition 1.12] may wonder why Definition 2.2.11 does not mention intersections of subgraphs. This may be understood with an example.

**Example 2.2.12.** Properadic graphical maps in the sense of [HRY15] need not preserve intersections of subgraphs, hence this is not part of the definition of morphism. Indeed, consider the following situation.




There is a properadic graphical map $f$ which sends $C_u$ to $C_a$, $C_w$ to $C_b$, and $C_v$ to $\downarrow_e$. But $C_u \cap C_w$ consists of a single edge. On the other hand, $f(C_u) \cap f(C_w) = C_a \cap C_b$ consists of *two* edges, and is not a structured subgraph. Morphisms of this type are essential in establishing the nerve theorem (see [HRY15, Theorem 7.42]) in this setting.

**Example 2.2.13.** Consider the following two graphs $G$ and $K$.

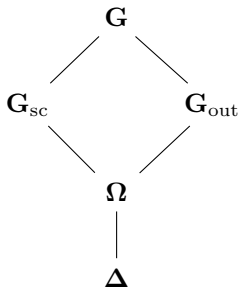
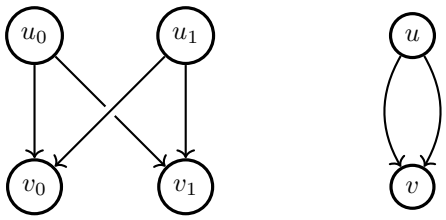



FIGURE 5. Lattice of subcategories (see Definition 2.2.14)




There is no morphism $f\colon G \to K$. Such a morphism would necessarily have $f_1(C_{u_0}) = f_1(C_{u_1}) = C_u$ as there is exactly one structured subgraph of $K$ with two outputs; likewise $f_1(C_{v_0}) = f_1(C_{v_1}) = C_v$. But $H = C_{u_1} \tilde{\cup} C_{v_1} \in \mathtt{Sb}(G)$, so is required that $f_1(H) = f_1(C_{u_1}) \tilde{\cup} f_1(C_{v_1}) = C_u \tilde{\cup} C_v = K$. But then $H$ has one input and one output, whereas $K$ has an empty set of inputs and empty set of outputs. Likewise, there is no morphism $g\colon K \to G$. If one existed, we would have $g_1(C_u) = C_{u_i}$ and $g_1(C_v) = C_{v_j}$ for some $i, j$. But then $C_{u_i} \tilde{\cup} C_{v_j}$ is a structured subgraph with one input and one output, whereas $C_u \tilde{\cup} C_v = K$ has no inputs or outputs.

**Definition 2.2.14** (Properadic Graphical Category)**.** We let $\mathbf{G}$ denote the category whose objects are (isomorphism classes of) connected acyclic directed graphs together with the morphisms of Definition 2.2.11. Composition is given by composition of pairs of functions. Similar to Definition 2.1.30 we write

- $\mathbf{G}_{\mathrm{out}}$ for the full subcategory of $\mathbf{G}$ spanned by graphs whose vertices have at least one output and
- $\mathbf{G}_{\mathrm{sc}}$ for the full subcategory of $\mathbf{G}$ spanned by simply-connected graphs.

The dendroidal category, $\boldsymbol{\Omega}$, from [MW07] is the full subcategory of $\mathbf{G}_{\mathrm{out}}$ so that each vertex has precisely one output.

Under the equivalence of Theorem A.1, the subcategory $\mathbf{G}_{\mathrm{sc}}$ was called '$\Theta$' in [HRY15]. The dendroidal category $\boldsymbol{\Omega}$ is a full subcategory of both $\mathbf{G}_{\mathrm{out}}$ and $\mathbf{G}_{\mathrm{sc}}$. See Figure 5.

**Example 2.2.15.** If $H \in \mathtt{Sb}(K)$, then there is a canonical morphism $H \to K$ given by inclusion of subsets $\mathtt{E}(H) \subseteq \mathtt{E}(K)$ and $\mathtt{Sb}(H) \subseteq \mathtt{Sb}(K)$.

**Remark 2.2.16** (Alternative characterization of morphisms)**.** Definition 2.2.11 is not the original definition of graphical map that appeared in [HRY15]. Let us give an alternative definition more closely aligned with the original. Suppose that

$f\colon G \to K$ is a morphism. We can identify each vertex with a corolla in $\mathtt{Sb}(G)$, which induces a function $\mathtt{V}(G) \to \mathtt{Sb}(K)$ so that the diagram

$$\begin{array}{ccccc} \wp(\mathtt{E}(G)) & \xleftarrow{\mathtt{in}} & \mathtt{V}(G) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(G)) \\ \downarrow{\scriptstyle \wp(f_0)} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \wp(f_0)} \\ \wp(\mathtt{E}(K)) & \xleftarrow{\mathtt{in}} & \mathtt{Sb}(K) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(K)) \end{array}$$

commutes. Condition (2) no longer makes sense, as taking the union of two corollas will yield a graph that is not a corolla. Instead, we have the following two potential conditions (essentially borrowed from [HRY15]), where we write $H_v \in \mathtt{Sb}(K)$ for the image of $C_v$ under $f_1$.

(3) For each $J \in \mathtt{Sb}(G)$, the induced étale map $J\{H_v\}_{v\in\mathtt{V}(J)} \to K$ is convex open.
(4) The induced étale map $G\{H_v\}_{v\in\mathtt{V}(J)} \to K$ is convex open.

Combining one of these two conditions with (1) yields a definition equivalent to the one from Definition 2.2.11. This equivalence follows from Theorem A.1.

**Definition 2.2.17.** Let $f\colon G \to K$ be a morphism.

- If $f_1(G) = K$, then $f$ is said to be *active*.
- If $f$ is isomorphic to a morphism of the form of Example 2.2.15, then $f$ is said to be *inert*.

Notice that active maps are automatically bijective on boundaries. The reverse implication also holds.

**Lemma 2.2.18.** *If $f\colon G \to K$ is a morphism of $\mathbf{G}$ inducing a bijection between the inputs / outputs of $G$ and $K$, then $f$ is active.*

*Proof.* There is at most one subgraph possessing a given boundary by [HRY15, Lemma 6.39]. Our assumption is that the inputs / outputs of $f_1(G)$ are the same as those of $K$, so it follows that $f_1(G) = K$. □

Kock described a weak factorization system on $\mathbf{K}$ (see Definition 2.0.4) with $\mathbf{K}_{\mathrm{int}}$ as the right class. As observed in [Koc16, 2.4.14], this restricts to an orthogonal factorization system $(\mathbf{G}_{\mathrm{act}}, \mathbf{G}_{\mathrm{int}})$ on $\mathbf{G}$.

**Theorem 2.2.19** (Kock)**.** *The pair $(\mathbf{G}_{\mathrm{act}}, \mathbf{G}_{\mathrm{int}})$ is an orthogonal factorization system on $\mathbf{G}$.* □

**Remark 2.2.20.** The equivalence $\mathbf{G} \simeq \mathbf{\Gamma}$ of Theorem A.1 and [Koc16, 2.4.14] show that $G \to H$ is inert in $\mathbf{G}$ if and only if $G$ is a convex open subgraph of $H$. Therefore, if $H$ lies in $\mathbf{G}_{\mathrm{out}}$, $\mathbf{G}_{\mathrm{sc}}$, $\mathbf{\Omega}$, or $\mathbf{\Delta}$, so does $G$. In particular, the active-inert factorization system on $\mathbf{G}$ restricts to each of these subcategories. This restricted factorization system on $\mathbf{\Delta}$ coincides with the one described in Example 2.1.21 and with the restriction of the factorization system on $\mathbf{L}$ from Lemma 2.1.34.

We also have the following analogue of Lemma 2.1.38(1). The corresponding statement for item (2) of Lemma 2.1.38 is immediate from Definition 2.2.14.

**Lemma 2.2.21.** *Suppose that* $\mathbf{A} \subseteq \mathbf{B}$ *is one of the fully-faithful inclusions appearing in the following diagram*

$$\begin{array}{ccccc} \mathbf{\Delta} & \longrightarrow & \mathbf{\Omega} & \longrightarrow & \mathbf{G}_{\mathrm{out}} \\ & & \downarrow & & \downarrow \\ & & \mathbf{G}_{\mathrm{sc}} & \dashrightarrow & \mathbf{G}. \end{array}$$

*If* $f\colon G \to H$ *is a morphism in* $\mathbf{B}$ *and* $H \in \mathbf{A}$*, then* $G \in \mathbf{A}$*.*

*Proof.* The statement for the dashed map is [HRY15, Proposition 5.2.8]. We thus concentrate on the cases where $\mathbf{A} \subseteq \mathbf{B}$ is one of the solid black arrows. In each case, we may distinguish graphs in $\mathbf{A}$ among those in $\mathbf{B}$ by a corresponding property not just for vertices, but for subgraphs. For instance,

- a graph $G \in \mathbf{G}$ is in $\mathbf{G}_{\mathrm{out}}$ if and only if each structured subgraph of $G$ has at least one output,
- a graph $G \in \mathbf{G}_{\mathrm{sc}}$ is in $\mathbf{\Omega}$ if and only if each structured subgraph has exactly one output,

and so on. We have the active-inert factorization $G \to G\{K_v\} \to H$ from [HRY15, Lemma 6.42]. As each $K_v$ is a structured subgraph of $H$, we have $K_v \in \mathbf{A}$. But $K_v$ has the same number of inputs and outputs as the vertex $v \in G$, hence $G \in \mathbf{A}$ as well. □

We now introduce the functor $\mathtt{V}_{\mathbf{G}}\colon \mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}}$.

**Definition 2.2.22.** We define the functor $\mathtt{V}_{\mathbf{G}}\colon \mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}}$ by requiring that it takes

- an object $G \in \mathbf{G}$ to the set $\mathtt{V}(G)_+$ of its vertices together with a base point,
- a morphism $f\colon G \to K$ to the based map

$$\mathtt{V}_{\mathbf{G}}(f)\colon \mathtt{V}(K)_+ \to \mathtt{V}(G)_+$$

defined by the rule

$$\mathtt{V}_{\mathbf{G}}(f)(v) = w \qquad \text{if and only if} \qquad v \in \mathtt{V}(f_1(w)) \tag{2}$$

and otherwise $\mathtt{V}_{\mathbf{G}}(f)(v) = *$.

We will also write $\mathtt{V}_{\mathbf{G}}$ for the composite $\mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}} \simeq \mathbf{F}_*^{\mathrm{op}}$.

We now want to check that $\mathtt{V}_{\mathbf{G}}$ is indeed a functor. First note that since $f$ is a graphical map, there is *at most one* $w$ so that $v \in \mathtt{V}(f_1(w))$, hence this is a well-defined map.

Suppose that $g\colon H \to G$ is another graphical map; let us verify that $\mathtt{V}_{\mathbf{G}}(fg) = \mathtt{V}_{\mathbf{G}}(g)\mathtt{V}_{\mathbf{G}}(f)$. We have $\mathtt{V}_{\mathbf{G}}(fg)(v) = w$ if and only if $v \in \mathtt{V}((fg)_1(w))$. By definition of composition in $\mathbf{G}$,

$$\mathtt{V}((fg)_1(w)) = \coprod_{x \in \mathtt{V}(g_1(w))} \mathtt{V}(f_1(x));$$

it follows that $v \in \mathtt{V}((fg)_1(w))$ if and only if $v \in \mathtt{V}(f_1(x))$ for some (unique) $x \in \mathtt{V}(g_1(w))$. This of course happens if and only if $\mathtt{V}_{\mathbf{G}}(f)(v) = x$ and $\mathtt{V}_{\mathbf{G}}(g)(x) = w$. Thus if $\mathtt{V}_{\mathbf{G}}(fg)(v)$ is in $\mathtt{V}(H)$, then so is $\mathtt{V}_{\mathbf{G}}(g)(\mathtt{V}_{\mathbf{G}}(f)(v))$, and we have the equality

$$\mathtt{V}_{\mathbf{G}}(fg)(v) = \mathtt{V}_{\mathbf{G}}(g)(\mathtt{V}_{\mathbf{G}}(f)(v)).$$

To finish showing that $\mathtt{V}_{\mathbf{G}}(fg) = \mathtt{V}_{\mathbf{G}}(g)\mathtt{V}_{\mathbf{G}}(f)$, simply observe that $\mathtt{V}_{\mathbf{G}}(g)(\mathtt{V}_{\mathbf{G}}(f)(v))$ is in $\mathtt{V}(H)$ if and only if $\mathtt{V}_{\mathbf{G}}(f)(v)$ is in $\mathtt{V}(G)$ and $\mathtt{V}_{\mathbf{G}}(g)(\mathtt{V}_{\mathbf{G}}(f)(v))$ is in $\mathtt{V}(H)$, which in turn implies that $\mathtt{V}_{\mathbf{G}}(fg)(v)$ is in $\mathtt{V}(H)$.

**Proposition 2.2.23.** *The functor $\mathtt{V}_{\mathbf{G}}\colon \mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}}$ preserves the active-inert factorization systems (see Example 2.1.20).*

*Proof.* If $f\colon G \to K$ is an active map in $\mathbf{G}$ then $f(G) = K$. Therefore, we have

$$\mathtt{V}(K) = \mathtt{V}(f(G)) = \coprod_{w \in \mathtt{V}(G)} \mathtt{V}(f_1(w))$$

and $\mathtt{V}_{\mathbf{G}}(f)$ is active by construction. If $f$ is inert then it easily follows from the definition of $\mathtt{V}_{\mathbf{G}}(f)$ that for a subgraph we have $|\mathtt{V}(f_1(w))| = 1$ for every $w \in \mathtt{V}(G)$. In other words, the functor $\mathtt{V}_{\mathbf{G}}\colon \mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}}$ also preserves inert morphisms. □

2.3. **From connected level graphs to connected acyclic graphs.** Let $\mathbf{L}_{\mathrm{c}}$ denote the full subcategory of $\mathbf{L}$ spanned by the connected level graphs. The present goal is to define a functor $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$. On objects, $\tau$ simply forgets the level structure, that is, $\tau(G)$ is the directed graph from Lemma 2.1.9.

**Lemma 2.3.1.** *If $G \in \mathbf{L}_{\mathrm{c}}$, then any level subgraph (Definition 2.1.13) is a structured subgraph (Definition 2.2.2) of $\tau(G)$.*

*Proof.* Suppose that $x \in G_{i,j}$ is a level subgraph. As in Remark 2.1.15, let $\widetilde{H}\colon \mathscr{L}^n_{i,j} \to \mathrm{Fin}$ be given by pullbacks

$$\begin{array}{ccc} \widetilde{H}_{k,\ell} & \longrightarrow & G_{k,\ell} \\ \downarrow & & \downarrow \\ \{x\} & \longrightarrow & G_{i,j} \end{array} \tag{3}$$

and let $H$ be the composite $\mathscr{L}^{j-i} \xrightarrow{\cong} \mathscr{L}^n_{i,j} \to \mathrm{Fin}$ with $\widetilde{H}_{k,\ell} = H_{k-i,\ell-i}$. As this is defined by pullbacks, $\tau(H)$ is automatically an open subgraph of $\tau(G)$. Remark 2.2.4 implies that it suffices to construct a lift $k$ for any commutative square

$$\begin{array}{ccc} \downarrow_0 \amalg \downarrow_m & \xrightarrow{a} & \tau(H) \\ \downarrow & \overset{k}{\nearrow} & \downarrow \\ L & \xrightarrow{f} & \tau(G) \end{array}$$

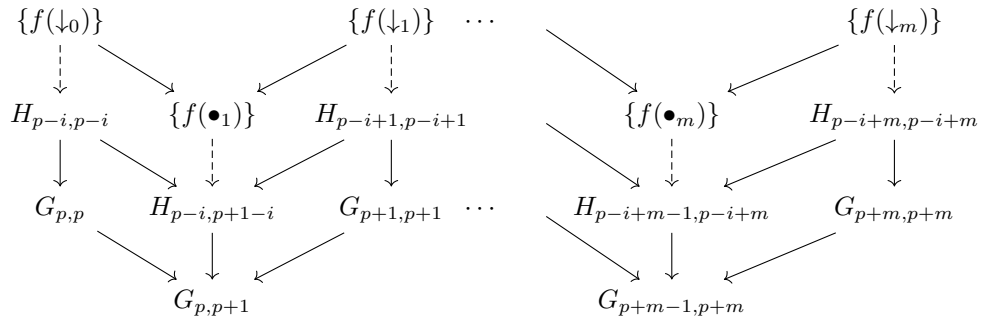
of naïve morphisms of graphs (that is, in $\mathrm{Fun}(\mathscr{G}, \mathrm{Fin})$), where $L$ is a linear graph of height $m \geq 0$ with $\mathtt{E}(L) = \{\downarrow_0, \ldots, \downarrow_m\}$ and $\mathtt{V}(L) = \{\bullet_1, \ldots, \bullet_m\}$. When $m = 0$ this is automatic, as $\tau(H) \to \tau(G)$ is a monomorphism.

There is a $p$ with $f(\downarrow_w) \in G_{p+w,p+w}$, $f(\bullet_w) \in G_{p+w-1,p+w}$, and $i \leq p$, $p+m \leq j$. In $G$, we have commutative diagrams

$$\begin{array}{ccccccc} \{f(\downarrow_w)\} & \longrightarrow & \{f(\bullet_{w+1})\} & & \{f(\bullet_u)\} & \longleftarrow & \{f(\downarrow_u)\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_{p+w,p+w} & \longrightarrow & G_{p+w,p+w+1} & \longrightarrow G_{i,j} \longleftarrow & G_{p+u-1,p+u} & \longleftarrow & G_{p+u,p+u} \end{array}$$

for $0 \leq w \leq m-1$ and $1 \leq u \leq m$. Since $f(\downarrow_0)$ maps to $x \in G_{i,j}$ by assumption, it follows that all of $f(\downarrow_w)$ and $f(\bullet_w)$ map to $x \in G_{i,j}$. The vertical maps thus factor

through the pullbacks from (3), yielding the dashed maps in the following diagram.




This determines the naïve morphism $k\colon L \to \tau(H)$. By uniqueness of the map to the pullback, we have $k(\downarrow_0) = a(\downarrow_0)$ and $k(\downarrow_m) = a(\downarrow_m)$, hence $k$ is the desired lift. □

We now want to construct a functor $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ which is on objects is taking the underlying directed graph (see Lemma 2.1.9). To define $\tau$ on morphisms, note that a morphism in $\mathbf{L}_{\mathrm{c}}$ lying over $\alpha : [n] \to [m]$ consists of the following data:

(1) For each $0 \leq i \leq n$, a monomorphism $G_{i,i} \to H_{\alpha(i),\alpha(i)}$; these assemble into a function $\mathtt{E}(\tau(G)) \to \mathtt{E}(\tau(H))$.
(2) If $1 \leq i \leq n$, a monomorphism $G_{i-1,i} \to H_{\alpha(i-1),\alpha(i)}$. In light of the previous lemma, these assemble into a function $\mathtt{V}(\tau(G)) \to \mathtt{Sb}(\tau(H))$.

As a provisional definition, we would like $\tau(G \to H)$ to be specified by the above data. Let us first check that this is plausible. From the definition of morphism in $\mathbf{L}$, the following diagrams are pullbacks

$$\begin{array}{ccc} G_{i,i} & \hookrightarrow & H_{\alpha(i),\alpha(i)} \\ \downarrow & & \downarrow \\ G_{i,i+1} & \hookrightarrow & H_{\alpha(i),\alpha(i+1)} \end{array} \qquad \begin{array}{ccc} G_{i,i} & \hookrightarrow & H_{\alpha(i),\alpha(i)} \\ \downarrow & & \downarrow \\ G_{i-1,i} & \hookrightarrow & H_{\alpha(i-1),\alpha(i)} \end{array}$$

whenever $i$ is in the appropriate range. In particular, for every $v \in \mathtt{V}(G)$, there exists $i$ such that $\mathtt{in}(v) \simeq \{v\} \times_{G_{i,i+1}} G_{i,i} \simeq \{v\} \times_{H_{\alpha(i),\alpha(i+1)}} H_{\alpha(i),\alpha(i)}$ and similarly for $\mathtt{out}(v)$. Hence, the diagram

$$\begin{array}{ccccc} \wp(\mathtt{E}(G)) & \xleftarrow{\mathtt{in}} & \mathtt{V}(G) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(G)) \\ \downarrow{\scriptstyle \wp(-)} & & \downarrow & & \downarrow{\scriptstyle \wp(-)} \\ \wp(\mathtt{E}(H)) & \xleftarrow{\mathtt{in}} & \mathtt{Sb}(H) & \xrightarrow{\mathtt{out}} & \wp(\mathtt{E}(H)) \end{array}$$

from Remark 2.2.16 commutes.

**Proposition 2.3.2.** *If $f\colon G \to H$ is a morphism in $\mathbf{L}_{\mathrm{c}}$ lying over $\alpha\colon [n] \to [m]$, then the pair $\mathtt{E}(\tau(G)) \to \mathtt{E}(\tau(H))$, $\mathtt{V}(\tau(G)) \to \mathtt{Sb}(\tau(H))$ from above indeed constitutes a morphism in $\mathbf{G}$.*

The proof this proposition is rather involved, utilizing the equivalence of $\mathbf{G}$ with the graphical category $\mathbf{\Gamma}$ from [HRY15] (see Theorem A.1). As the methods used are rather different than what we are dealing with otherwise, we have separated this proof out into Appendix B.

**Corollary 2.3.3.** *The assignment $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ is a functor.*

*Proof.* We wish to show that $\tau(f)\tau(g) = \tau(fg)$ whenever $f$ and $g$ are composable morphisms in $\mathbf{L}_{\mathrm{c}}$. By Corollary 6.62 of [HRY15], the functor $\mathtt{E}\colon \mathbf{G} \to \mathrm{Set}$ is faithful, so it is enough to show that $\mathtt{E}(\tau(fg))$ is equal to $\mathtt{E}(\tau(f)\tau(g)) = \mathtt{E}(\tau(f))\mathtt{E}(\tau(g))$. This follows because the assignment on objects $G \mapsto \mathtt{E}(\tau(G))$ constitutes a functor $\mathbf{L}_{\mathrm{c}} \to \mathrm{Set}$, so we have $\mathtt{E}\tau(fg) = \mathtt{E}\tau(f)\mathtt{E}\tau(g)$. □

**Lemma 2.3.4.** *The functor $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ is compatible with the active-inert factorization systems. Further, $\tau$ restricts to an isomorphism $\mathbf{L}_{\mathrm{el}} \cong \mathbf{G}_{\mathrm{el}}$.*

*Proof.* We first show that $\tau$ preserves active-inert factorization systems. Let $f\colon G \to H$ be an active morphism in $\mathbf{L}_{\mathrm{c}}$. We need to show that $\tau(f)_1(G) = H$. This is clear if $H$ consists of just one edge. For a non-trivial graph $H$ the connectivity implies that $\tau(f)_1(G) \neq H$ can only happen if there is a vertex $w \in \mathtt{V}(H)$ such that there is no $v \in \mathtt{V}(G)$ with $w \in \mathtt{V}(\tau(f)_1(v))$. But this case cannot occur due to the fact that $\mathtt{V}_{\mathbf{L}}(f)$ is active in $\mathrm{Fin}_*^{\mathrm{op}}$ by Lemma 2.1.28.

Suppose $f\colon G \to H$ is inert in $\mathbf{L}_{\mathrm{c}}$ lying over an interval inclusion $\alpha\colon [m] \to [n]$. Then the monomorphisms $G_{i,j} \hookrightarrow H_{\alpha(i),\alpha(i)-i+j} = H_{\alpha(i),\alpha(j)}$ for every $0 \leq i \leq j \leq m$ and the connectivity of $G$ show that $G$ is a level subgraph of $H$. By Lemma 2.3.1, $\tau(f)\colon \tau(G) \to \tau(H)$ is the inclusion of a structured subgraph and inert by definition.

The definition of $\tau$ implies that the restriction $\mathbf{L}_{\mathrm{el}} = \mathbf{L}_{\mathrm{c,el}} \xrightarrow{\sim} \mathbf{G}_{\mathrm{el}}$ is an equivalence. It is an isomorphism as both $\mathbf{L}$ and $\mathbf{G}$ are skeletal categories. □

## 3. The Segal condition and an algebraic version of enriched ∞-properads

In this section, we give a preliminary version of the notion of enriched ∞-properad. We first recall in §3.1 a general framework for Segal objects. This is applied in §3.2 to give and compare definitions for 'algebraic' $\mathcal{V}$-enriched ∞-properads. For many purposes (in particular, for existence of certain adjoints), it is important to work with small symmetric monoidal ∞-categories $\mathcal{U}$ rather than general symmetric monoidal ∞-categories $\mathcal{V}$. In §3.3 we make precise how one can work in the small, rather than presentable, setting.

There is some overlap between the material in this section and that which will appear in the forthcoming [CH], but many of the constructions and results below depend on special properties of the category of level graphs $\mathbf{L}$. These will be important in the next section, where we take up the question of algebras over ∞-properads.

### 3.1. Algebraic patterns and homotopy-coherent algebraic structures.

We have already encountered (Example 2.1.20, Example 2.1.21, Lemma 2.1.25, Theorem 2.2.19) inert-active factorization systems on several 1-categories

$$\mathrm{Fin}_* \simeq \mathbf{F}_*, \boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{L}^{\mathrm{op}}, \mathbf{G}^{\mathrm{op}}$$

as well as several restrictions of the latter two (Lemma 2.1.34 and Remark 2.2.20). By declaring certain objects to be 'elementary objects,' these categories and factorization systems allow us to define Segal objects. As the remainder of the paper deals with ∞-categories, we recall the definition of a factorization system in that context.

**Definition 3.1.1.** Let $\mathcal{C}$ be an ∞-category and let $(\mathcal{C}^L, \mathcal{C}^R)$ be a pair of wide subcategories of $\mathcal{C}$. Suppose $\mathrm{Fun}_{L,R}(\Delta^2, \mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$

spanned by those diagrams $\sigma$ such that $\sigma|_{\Delta^{\{0,1\}}}$ is in $\mathcal{C}^L$ and $\sigma|_{\Delta^{\{1,2\}}}$ is in $\mathcal{C}^R$. Then we say $\mathcal{C}$ has a *factorization system* if the functor

$$\mathrm{Fun}_{L,R}(\Delta^2,\mathcal{C}) \to \mathrm{Fun}(\Delta^1,\mathcal{C})$$

given by restriction to $\Delta^{\{0,2\}}$ is an equivalence.

The following is our first important example of factorization system that is not 1-categorical in nature.

**Example 3.1.2** (Symmetric monoidal $\infty$-categories)**.** Recall that a symmetric monoidal $\infty$-category is a cocartesian fibration $\mathcal{V}^{\otimes} \to \mathbf{F}_*$ so that $\prod \rho^i_! \colon \mathcal{V}^{\otimes}_n \to (\mathcal{V}^{\otimes}_1)^{\times n}$ is an equivalence, where $\rho^i_! \colon \mathcal{V}^{\otimes}_n \to \mathcal{V}^{\otimes}_1$ denotes the cocartesian pushout along the inert map $\rho^i \colon \langle n\rangle \to \langle 1\rangle$ determined by $\rho^i(i) = 1$. Any symmetric monoidal $\infty$-category $\mathcal{V}^{\otimes} \to \mathbf{F}_*$ has a canonical inert-active factorization system by [Lur, Proposition 2.1.2.4]. Here a map in $\mathcal{V}^{\otimes}$ is called inert if it is cocartesian and lies over an inert map in $\mathbf{F}_*$, and active if it lies over an active map in $\mathbf{F}_*$.

The categories from Section 2, along with several others, fit into a general framework developed by the first author with Haugseng in [CH21].

**Definition 3.1.3.** Let $\mathcal{Q}$ be an $\infty$-category with an active-inert factorization system, and let $\mathcal{Q}^{\mathrm{op}}$ be its opposite, which comes with an inert-active factorization system. Fix a full subcategory $\mathcal{Q}^{\mathrm{op}}_{\mathrm{el}}$ of $\mathcal{Q}^{\mathrm{op}}_{\mathrm{int}}$ whose objects are called elementary objects. In [CH21, Definition 2.1] this data is called an *algebraic pattern*. Following [CH21, Definition 2.7] we can define *Segal $\mathcal{Q}$-spaces* to be functors $F \colon \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}$ such that the canonical map

$$F(X) \to \lim_{E \in (\mathcal{Q}^{\mathrm{op}}_{\mathrm{el}})_{X/}} F(E)$$

is an equivalence for each $X \in \mathcal{Q}^{\mathrm{op}}$. We write $\mathrm{Seg}(\mathcal{Q})$ for the full subcategory of $\mathrm{P}(\mathcal{Q}) = \mathrm{Fun}(\mathcal{Q}^{\mathrm{op}}, \mathcal{S})$ spanned by Segal $\mathcal{Q}$-spaces (this is denoted by $\mathrm{Seg}_{\mathcal{Q}^{\mathrm{op}}}(\mathcal{S})$ in [CH21]).

In other words, $F$ is a Segal $\mathcal{Q}$-space if and only if the restriction $F|_{\mathcal{Q}^{\mathrm{op}}_{\mathrm{int}}}$ is the right Kan extension of $F|_{\mathcal{Q}^{\mathrm{op}}_{\mathrm{el}}}$ along the inclusion $\mathcal{Q}^{\mathrm{op}}_{\mathrm{el}} \hookrightarrow \mathcal{Q}^{\mathrm{op}}_{\mathrm{int}}$ (see [CH21, Lemma 2.9]). We write $\mathrm{Seg}(\mathcal{Q})$ for the full subcategory of $\mathrm{P}(\mathcal{Q}) = \mathrm{Fun}(\mathcal{Q}^{\mathrm{op}}, \mathcal{S})$ spanned by Segal $\mathcal{Q}$-spaces (this is denoted by $\mathrm{Seg}_{\mathcal{Q}^{\mathrm{op}}}(\mathcal{S})$ in [CH21]). The description of Segal $\mathcal{Q}$-spaces as right Kan extensions and [Lur09, Proposition 4.3.2.15] imply that $\mathrm{Seg}(\mathcal{Q}_{\mathrm{int}}) \simeq \mathrm{P}(\mathcal{Q}_{\mathrm{el}})$.

The most prominent example of an algebraic pattern is $\mathcal{Q} = \mathbf{\Delta}$ which has an active-inert factorization system where the inert morphisms are interval inclusions and active morphisms are boundary preserving maps (see Example 2.1.21). By choosing the elementary objects to be [1] and [0], we see that $\mathbf{\Delta}^{\mathrm{op}}$ admits the structure of an algebraic pattern and Segal $\mathbf{\Delta}$-spaces are exactly Segal spaces in the sense of [Rez01], which turn out to model $\infty$-categories. The basic idea of the construction of Segal $\mathcal{Q}$-presheaves is that the elementary objects play the role of building blocks of an algebraic structure while the Segal condition, i.e. the requirement that the canonical maps $F(X) \to \lim_{E \in (\mathcal{Q}^{\mathrm{op}}_{\mathrm{el}})_{X/}} F(E)$ are equivalences, says that every space $F(X)$ is given by gluing these building blocks along inert morphisms. The algebraic operations such as compositions are induced by active morphisms. In general, we observe that every $\infty$-category with an inert-active factorization system and elementary objects defines a kind of homotopy-coherent

algebra. As we will see in the next section this construction recovers the notion of $\infty$-properads.

Following the idea that the algebraic structure of objects in $\mathrm{Seg}(\mathcal{Q})$ are controlled by inert/active morphisms in $\mathcal{Q}^{\mathrm{op}}$, it is natural that a functor $f\colon \mathcal{Q} \to \mathcal{R}$ induces a functor $\mathrm{Seg}(\mathcal{R}) \to \mathrm{Seg}(\mathcal{Q})$ (by precomposition) when $f$ is compatible with the additional data. A precise characterization can be found in [CH21, Lemma 4.5], which includes that $f$ must preserve both the factorization system and the elementary objects.

3.2. **Segal presheaves and decorated graph categories.** In this section, we introduce '$\mathcal{V}$-decorated graph categories' and use the framework from §3.1 to give a first model for enriched $\infty$-properads (Definition 3.2.18). We also provide another model in terms of algebras over categories of graphs whose edges are decorated by elements of a space (Definition 3.2.32); these two approaches are equivalent (Theorem 3.2.33).

It is a theorem of the first author, Robertson, and Yau that the full subcategory of $\mathrm{Fun}(\mathbf{G}^{\mathrm{op}}, \mathrm{Set})$ on the Segal objects is equivalent to the category $\mathrm{Prpd}(\mathrm{Set})$ (see [HRY15]). We thus regard a Segal object in $\mathrm{P}(\mathbf{G}) = \mathrm{Fun}(\mathbf{G}^{\mathrm{op}}, \mathcal{S})$ to be, at least as a first approximation, a reasonable notion of an $\infty$-properad.

We will shortly unravel the following definition (see Remark 3.2.6).

**Definition 3.2.1.** We now introduce a few algebraic structures given by the construction described in Definition 3.1.3.

(1) We write $\mathbf{G}_{\mathrm{el}}^{\mathrm{op}}$ for the full subcategory of $\mathbf{G}_{\mathrm{int}}^{\mathrm{op}}$ of Theorem 2.2.19 spanned by the corollas and the edge $\downarrow$. This yields the $\infty$-category $\mathrm{Seg}(\mathbf{G}) \subseteq \mathrm{Fun}(\mathbf{G}^{\mathrm{op}}, \mathcal{S})$ of Segal $\mathbf{G}$-spaces.

(2) We write $\mathbf{L}_{\mathrm{el}}^{\mathrm{op}}$ (resp. $\mathbf{L}_{\mathrm{c,el}}^{\mathrm{op}}$) for the full subcategory of $\mathbf{L}_{\mathrm{int}}^{\mathrm{op}}$ (resp. $\mathbf{L}_{\mathrm{c,int}}^{\mathrm{op}}$) spanned by the elementary level graphs (Example 2.1.3). Note that $\mathbf{L}_{\mathrm{el}}^{\mathrm{op}} = \mathbf{L}_{\mathrm{c,el}}^{\mathrm{op}}$. We let $\mathrm{Seg}(\mathbf{L})$ and $\mathrm{Seg}(\mathbf{L}_{\mathrm{c}})$ denote the $\infty$-category of Segal $\mathbf{L}$-spaces and Segal $\mathbf{L}_{\mathrm{c}}$-spaces.

As we will see in the next section, the $\infty$-category of Segal $\mathbf{G}$-spaces is equivalent to that of Segal $\mathbf{L}$-spaces.

**Notation 3.2.2** (Graph categories $\boldsymbol{\Xi}$)**.** So far, we have seen several examples of categories of directed graphs which we will return to again and again. In what follows, we will use a generic symbol $\boldsymbol{\Xi}$ for $\mathbf{L}$, $\mathbf{G}$, or any of the variations from Definition 2.1.30 and Definition 2.2.14. We refer generically to these types of categories as *graph categories*. The common features that we will utilize are the following:

- A factorization system $(\boldsymbol{\Xi}_{\mathrm{act}}, \boldsymbol{\Xi}_{\mathrm{int}})$. See Lemma 2.1.25, Lemma 2.1.34, and Theorem 2.2.19.
- A full subcategory $\boldsymbol{\Xi}_{\mathrm{el}} \subseteq \boldsymbol{\Xi}_{\mathrm{int}}$ of elementary graphs, whose objects are the corollas and the edge. We generically denote these by $\mathfrak{c}_{p,q}$ and $\mathfrak{e}$, respectively. As an example, when $\boldsymbol{\Xi} = \mathbf{G}$, we take $\mathfrak{c}_{p,q} \coloneqq C_{p,q}$ and $\mathfrak{e} \coloneqq \downarrow$.
- A vertex functor $\mathtt{V}_{\boldsymbol{\Xi}}\colon \boldsymbol{\Xi} \to \mathrm{Fin}_*^{\mathrm{op}} \simeq \mathbf{F}_*^{\mathrm{op}}$ preserving the factorization systems. See Proposition 2.1.28 and Proposition 2.2.23.
- A canonical inclusion $\boldsymbol{\Delta} \to \boldsymbol{\Xi}$ as the linear level graphs. (This will become relevant only in Section 6.)

The subcategories $\Xi_{\mathrm{el}}$ are all isomorphic for $\Xi \in \{\mathbf{L}, \mathbf{L}_{\mathrm{c}}, \mathbf{L}_{0\text{-type}}, \mathbf{L}_{\mathrm{sc}}, \mathbf{G}, \mathbf{G}_{\mathrm{sc}}\}$; likewise, the subcategories $\Xi_{\mathrm{el}}$ are all isomorphic for $\Xi \in \{\mathbf{L}_{\mathrm{out}}, \mathbf{L}_{\mathrm{out,c}}, \mathbf{G}_{\mathrm{out}}\}$. Note that $\mathbf{G}_{\mathrm{out,el}}$ is missing the objects $\mathfrak{c}_{p,0}$ which are present in $\mathbf{G}_{\mathrm{el}}$.

**Remark 3.2.3.**

(i) All the opposites of these graph categories are in particular algebraic patterns, but they come equipped with more structure. In particular, for each $\Xi$ we have an $\infty$-category $\mathrm{Seg}(\Xi) \subseteq \mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$.

(ii) The category $\mathbf{K}$ from Definition 2.0.4 is unfortunately not a graph category in this sense: it has only a weak factorization system, rather than an orthogonal one, and it does not admit a vertex functor (see Remark 7.1.10 below).

(iii) Corollas of the form $C_{0,n}$ or $C_{m,0}$ in $\mathbf{G}$ will admit many level graph structures (as in Remark 2.1.4), though only one of those will be elementary in the sense of Example 2.1.3. It thus seems harmless to use $\mathfrak{c}_{p,q}$ as a common notation for corollas.

We do not pursue an abstract version of Notation 3.2.2 here.

**Remark 3.2.4** (Spans of graph categories)**.** In [CH20], the graph categories $\mathbf{\Delta}^1_{\mathbf{F}}$, $\mathbf{\Delta}_{\mathbf{F}}$, and $\mathbf{\Omega}$ (all of whose objects are trees or forests) played a primary role. Each of these three indexing categories is suitable to define (enriched) $\infty$-operads. Indeed, these fit into a span $\mathbf{\Delta}_{\mathbf{F}} \leftarrow \mathbf{\Delta}^1_{\mathbf{F}} \rightarrow \mathbf{\Omega}$ of algebraic patterns respecting the extra structure specified in Notation 3.2.2. Both of these functors induce equivalences at the level of Segal objects.

At a high level, much of the present paper is about extending constructions of [CH20] to the corresponding span $\mathbf{L} \leftarrow \mathbf{L}_{\mathrm{c}} \rightarrow \mathbf{G}$ of graph categories. The graph categories $\mathbf{L}$ and $\mathbf{G}$ each give a different approach to (enriched) $\infty$-properads, and this zig-zag allows us to compare them. Likewise, we will utilize the spans $\mathbf{L}_{0\text{-type}} \leftarrow \mathbf{L}_{\mathrm{sc}} \rightarrow \mathbf{G}_{\mathrm{sc}}$ (for $\infty$-dioperads) and $\mathbf{L}_{\mathrm{out}} \leftarrow \mathbf{L}_{\mathrm{out,c}} \rightarrow \mathbf{G}_{\mathrm{out}}$ (for $\infty$-output-properads), as well as the inclusions among these four spans. For notational reasons we generally deemphasize the final two contexts.

**Definition 3.2.5** (Segal cores)**.** Let $\Xi$ be a graph category. We will not distinguish between a graph $G \in \Xi$ and its image in $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ under the Yoneda embedding.

- If $G \in \Xi$ is a directed graph, define the *Segal core* to be
$$G_{\mathrm{Seg}} \coloneqq \operatorname*{colim}_{H \in (\Xi^{\mathrm{op}}_{\mathrm{el}})_{G/}} H$$
in $\mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$. The map $G_{\mathrm{Seg}} \to G$ will be called a *Segal core inclusion*.
- A level graph $L \in \mathbf{L}$ will be called *short* if the height of $L$ is 0 or 1. A *short Segal core inclusion* is just a Segal core inclusion $L_{\mathrm{Seg}} \to L$ where $L$ is a short level graph.

**Remark 3.2.6.** It follows directly from the previous definition and the Yoneda embedding that an object $F \in \mathrm{Fun}(\Xi^{\mathrm{op}}, \mathcal{S})$ lies in $\mathrm{Seg}(\Xi)$ if and only if $F$ is local with respect to the Segal core inclusions. The main advantage of working with $\mathbf{L}^{\mathrm{op}}$ instead of $\mathbf{G}^{\mathrm{op}}$ is that its "rigid" structure allows us to reformulate this description of Segal $\mathbf{L}$-spaces by rewriting the colimits $L_{\mathrm{Seg}}$ in various ways. In Section 4.2 we will use an alternative characterization of Segal $\mathbf{L}$-spaces, given in Proposition 3.2.9, to construct the tensor product between $\infty$-properads and Segal spaces.

**Example 3.2.7.** We emphasize that representable **G**-presheaves do not necessarily possess the Segal property. We saw, in Example 2.2.13, a pair of graphs $G$ and $K$ so that $\hom(G, K)$ is empty. But in this same example, the set $\hom(G_{\mathrm{Seg}}, K)$ is inhabited. Indeed, there is a map $C_{u_0} \amalg C_{u_1} \amalg C_{v_0} \amalg C_{v_1} \to K$ which factors through $G_{\mathrm{Seg}}$. Hence $K$ is not local with respect to all Segal core inclusions. A similar phenomenon occurs in other settings where there is a distinction between the representable presheaf on a graph and the nerve of the free object generated by the graph (see, for example, [HRY19, Remark 5.10]).

**Definition 3.2.8** (Segmentation map)**.** Suppose $L\colon \mathscr{L}^n \to \mathrm{Fin}$ is a height $n$ level graph. For $1 \leq i \leq n$, let $L^{\{i-1,i\}}$ be the composite

$$\mathscr{L}^1 \simeq \mathscr{L}^n_{i-1,i} \xrightarrow{L} \mathrm{Fin},$$

(see Definition 2.1.14) that is, the restriction of $L$ to height $i$ vertices and their adjacent edges, and similarly for $L^{\{i\}}\colon \mathscr{L}^0 \simeq \mathscr{L}^n_{i,i} \xrightarrow{L} \mathrm{Fin}$ $(0 \leq i \leq n)$. Each of these objects admits an evident inert map in $\mathbf{L}$ with codomain $L$, and we define the *segmentation map* associated to $L$ to be the morphism

$$L|_{\Delta^n_{\mathrm{Seg}}} \coloneqq L^{\{0,1\}} \amalg_{L^{\{1\}}} L^{\{1,2\}} \amalg_{L^{\{2\}}} \cdots \amalg_{L^{\{n-1\}}} L^{\{n-1,n\}} \to L$$

in $\mathrm{Fun}(\mathbf{L}^{\mathrm{op}}, \mathcal{S})$. In particular, when $n \leq 1$, the segmentation map is the identity.

Observe that we do not define segmentation maps in the setting of $\mathbf{L}_{\mathrm{c}}$-presheaves. Indeed, if $L \in \mathbf{L}_{\mathrm{c}}$, then the restricted graphs $L^{\{i-1,i\}}$ and $L^{\{i\}}$ are usually not connected (they will all be connected if and only if $L$ is a linear graph). This problem disappears when working with the categories of disconnected level graphs $\mathbf{L}_{0\text{-type}}$ and $\mathbf{L}_{\mathrm{out}}$ (Definition 2.1.30). In particular, a similar statement to the following holds when $\mathbf{L}$ is replaced by $\mathbf{L}_{0\text{-type}}$ or $\mathbf{L}_{\mathrm{out}}$.

**Proposition 3.2.9.** *Suppose $F \in \mathrm{Fun}(\mathbf{L}^{\mathrm{op}}, \mathcal{S})$. The following are equivalent:*

1. *$F$ is a Segal $\mathbf{L}$-space.*
2. *$F$ is local with respect to all Segal core inclusions $L_{\mathrm{Seg}} \to L$ (Definition 3.2.5).*
3. *$F$ is local with respect to the short Segal core inclusions (Definition 3.2.5) and the segmentation maps (Definition 3.2.8).*

*Proof.* The first two are equivalent by Remark 3.2.6. If $L$ is a height $n$ level graph, then the Segal core inclusion factors as the following composite.

$$\begin{array}{ccc} L_{\mathrm{Seg}} & \xrightarrow{\simeq} & (L^{\{0,1\}})_{\mathrm{Seg}} \amalg_{(L^{\{1\}})_{\mathrm{Seg}}} \cdots \amalg_{(L^{\{n-1\}})_{\mathrm{Seg}}} (L^{\{n-1,n\}})_{\mathrm{Seg}} \\ \downarrow & & \downarrow \\ L & \longleftarrow & L^{\{0,1\}} \amalg_{L^{\{1\}}} \cdots \amalg_{L^{\{n-1\}}} L^{\{n-1,n\}} \end{array}$$

If (2) is satisfied, then $F$ is local with respect to the vertical maps in the diagram. Hence, the 2-of-3 property implies that $F$ is local with respect to the bottom horizontal segmentation map. Further, it is automatic that if $F$ satisfies (2), then $F$ is local with respect to the short Segal core inclusions. Hence (2) implies (3).

On the other hand, the right vertical map of the diagram above is a pushout of short Segal core inclusions, so if $F$ is local with respect to short Segal core inclusions then it is local with respect to this map. As the bottom map is a segmentation map, we see that (3) implies (2). □

In the 1-categorical setting, the concept of properads is a generalization of that of operads, which in turn is a generalization of categories. Following Rezk [Rez01], [CM13] and the previous definition in the $\infty$-categorical setting these algebraic structures can be described as presheaves satisfying Segal conditions. Using the terminology of Definition 3.1.3, $\infty$-categories, $\infty$-operads, and $\infty$-properads are modeled by objects in $\mathrm{Seg}(\mathbf{\Delta})$, $\mathrm{Seg}(\mathbf{\Omega})$, and $\mathrm{Seg}(\mathbf{G})$, respectively, and the two generalization steps are induced by embeddings $\mathbf{\Delta}^{\mathrm{op}} \overset{i}{\hookrightarrow} \mathbf{\Omega}^{\mathrm{op}} \overset{j}{\hookrightarrow} \mathbf{G}^{\mathrm{op}}$ of the corresponding indexing categories which respect both the inert-active factorization systems and the elementary objects. More precisely, the precompositions with $i$ and $j$ induce functors (see [CH21, Lemma 4.5])

$$i^* \colon \mathrm{Seg}(\mathbf{\Omega}) \to \mathrm{Seg}(\mathbf{\Delta}) \text{ and } j^* \colon \mathrm{Seg}(\mathbf{G}) \to \mathrm{Seg}(\mathbf{\Omega}),$$

where $j^*$ takes any $\infty$-properad to its underlying $\infty$-operad while $i^*$ associates to any $\infty$-operad its underlying $\infty$-category (see Proposition 3.2.27 below).

By writing $\mathbf{\Delta}_{\mathbf{F}}^{1,\mathrm{op}}$ for the full subcategory of $\mathbf{\Delta}_{\mathbf{F}}^{\mathrm{op}}$ (see Proposition 2.1.35) spanned by connected objects, that is, trees instead of forests, we then obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{\Delta}_{\mathbf{F}}^{1,\mathrm{op}} & \longrightarrow & \mathbf{\Omega}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{L}_{\mathrm{c}}^{\mathrm{op}} & \xrightarrow{\tau} & \mathbf{G}^{\mathrm{op}}, \end{array}$$

where the bottom horizontal map is the morphism $\tau$ of Lemma 2.3.4. In [CH20], the upper horizontal map was first extended to a functor $\mathbf{\Delta}_{\mathbf{F}}^{1,\mathrm{op},\mathcal{V}} \to \mathbf{\Omega}^{\mathrm{op},\mathcal{V}}$ for any symmetric monoidal $\infty$-category $\mathcal{V}$, where the objects of $\mathbf{\Delta}_{\mathbf{F}}^{1,\mathrm{op},\mathcal{V}}$ and $\mathbf{\Omega}^{\mathrm{op},\mathcal{V}}$ are trees with each vertex decorated by an object of $\mathcal{V}$. Then it is was shown that this functor induces an equivalence of two corresponding models for $\mathcal{V}$-enriched $\infty$-operads. We want to generalize this idea to the $\infty$-properadic setting and extend $\tau$ to $\overline{\tau} \colon \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}} \to \mathbf{G}^{\mathrm{op},\mathcal{V}}$ (these $\infty$-categories are defined just below). Our focus lies on studying the various properties of the associated $\infty$-categories of Segal spaces. Finally, by a careful examination of the map $\overline{\tau}$ we will prove in Section 5 that $\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}}$ and $\mathbf{G}^{\mathrm{op},\mathcal{V}}$ describe $\mathcal{V}$-enriched $\infty$-properads. The reason for introducing $\mathbf{L}^{\mathrm{op}}$ and $\mathbf{L}_{\mathrm{c}}^{\mathrm{op}}$ will be clear in Section 4.2, where we use the particular structure of $\mathbf{L}_{\mathrm{c}}^{\mathrm{op}}$ to prove that $\infty$-properads are tensored over Segal spaces, which then gives us the notion of algebras by adjunction.

We now introduce the $\infty$-categories used to define enriched $\infty$-properads:

**Definition 3.2.10.** Let $\mathbf{\Xi}$ be a graph category (Notation 3.2.2) and let $q \colon \mathcal{V}^{\otimes} \to \mathbf{F}_*$ be a symmetric monoidal $\infty$-category. Write $q' \colon \mathcal{V}^{\mathrm{op},\otimes} \to \mathbf{F}_*$ for the opposite symmetric monoidal $\infty$-category. More precisely, $q'$ is the cocartesian fibration associated to the composite

$$\mathbf{F}_* \to \mathrm{Cat}_\infty \xrightarrow{\mathrm{op}} \mathrm{Cat}_\infty$$

whose first map classifies the cocartesian fibration $q$.[3] We let $\Xi^{\mathrm{op},\mathcal{V}}$ be given by the pullback

$$\begin{array}{ccc} \Xi^{\mathrm{op},\mathcal{V}} & \longrightarrow & \mathcal{V}^{\mathrm{op},\otimes} \\ \downarrow & & \downarrow{\scriptstyle q'} \\ \Xi^{\mathrm{op}} & \xrightarrow{\mathtt{V}_{\Xi}^{\mathrm{op}}} & \mathbf{F}_*. \end{array}$$

**Remark 3.2.11.** Similar to Remark 2.1.18 it is easy to see that $\mathbf{L}^{\mathcal{V}}$ admits a double $\infty$-categorical structure.

**Notation 3.2.12.** By definition an object in $\Xi^{\mathrm{op},\mathcal{V}}$ is given by a pair $(G,(v_c)_{c\in \mathtt{V}_{\Xi}(G)})$, consisting of an object of $\Xi^{\mathrm{op}}$ and an object of $\mathcal{V}^{\mathrm{op},\otimes}$. We think of this as a graph $G\in\Xi^{\mathrm{op}}$ whose vertices are labeled by objects of $\mathcal{V}$. We will write $\overline{G}$ for the object $(G,(v_c)_{c\in \mathtt{V}_{\Xi}(G)})$ when we do not wish to emphasize the labeling. When we *do* wish to emphasize the labeling, we will write such an object as $G(v_c)_{c\in \mathtt{V}_{\Xi}(G)}$. Since $\Xi^{\mathrm{op},\mathcal{V}}_{\mathfrak{e}}\simeq\{*\}$ we also use $\mathfrak{e}$ to denote the edge in $\Xi^{\mathrm{op},\mathcal{V}}_{\mathfrak{e}}\subseteq\Xi^{\mathrm{op},\mathcal{V}}$.

**Remark 3.2.13.** Every morphism in $\Xi^{\mathrm{op},\mathcal{V}}$ is a pair $(f,g)$ where $f$ and $g$ are morphisms in $\Xi^{\mathrm{op}}$ and $\mathcal{V}^{\mathrm{op},\otimes}$, respectively. Hence, the inert-active factorization systems on $\Xi^{\mathrm{op}}$ and $\mathcal{V}^{\mathrm{op},\otimes}$ induce one on $\Xi^{\mathrm{op},\mathcal{V}}$. More explicitly, a morphism in $\Xi^{\mathrm{op},\mathcal{V}}$ is inert if and only if its images in $\Xi^{\mathrm{op}}$ and $\mathcal{V}^{\mathrm{op},\otimes}$ are both inert. Active maps and elementary objects are defined similarly, but we can be more specific in these cases.

- A morphism $(f,g)$ of $\Xi^{\mathrm{op},\mathcal{V}}$ is active if and only if $f$ is active in $\Xi^{\mathrm{op}}$. To see this, suppose that $f$ is active. Then its image in $\mathbf{F}_*$ is active, so by Example 3.1.2, $g$ is active as well.
- Each elementary object of $\Xi^{\mathrm{op}}$ maps to $\langle 0\rangle$ or $\langle 1\rangle$ in $\mathbf{F}_*$. It follows that elementary objects in $\Xi^{\mathrm{op},\mathcal{V}}$ are those of the form $\mathfrak{c}(v)$ where $\mathfrak{c}$ is a corolla and $v\in\mathcal{V}$, and also $\mathfrak{e}$.

Following Definition 3.1.3 we can define the $\infty$-category $\mathrm{Seg}(\Xi^{\mathcal{V}})$ of Segal $\Xi^{\mathcal{V}}$-spaces.

**Definition 3.2.14.** Let $\overline{G}$ denote an object of $\Xi^{\mathrm{op},\mathcal{V}}$ lying over the graph $G$.

- The *Segal core inclusion* of $\overline{G}$ is
$$\overline{G}_{\mathrm{Seg}} \coloneqq \operatorname*{colim}_{\overline{E}\in(\Xi^{\mathrm{op},\mathcal{V}}_{\mathrm{el}})_{\overline{G}/}} \overline{E}\to\overline{G}$$
in $\mathrm{Fun}(\Xi^{\mathrm{op},\mathcal{V}},\mathcal{S})$, where $\overline{G}_{\mathrm{Seg}}$ is called the *Segal core* of $\overline{G}$.
- If $L$ is a level graph, a Segal core inclusion $\overline{L}_{\mathrm{Seg}}\to\overline{L}$ is called a *short* Segal core inclusion just when the underlying level graph $L$ is short (Definition 3.2.5).
- Suppose that $L\in\mathbf{L}$ has height $n$ and $\overline{L}\in\mathbf{L}^{\mathcal{V}}$. Defining $\overline{L}^{\{i\}}$ and $\overline{L}^{\{i-1,i\}}$ analogously to Definition 3.2.8, the *segmentation map* associated to $\overline{L}$ is
$$\overline{L}|_{\Delta^n_{\mathrm{Seg}}} \coloneqq \overline{L}^{\{0,1\}} \amalg_{\overline{L}^{\{1\}}} \cdots \amalg_{\overline{L}^{\{n-1\}}} \overline{L}^{\{n-1,n\}} \to \overline{L}$$
in $\mathrm{Fun}(\mathbf{L}^{\mathrm{op},\mathcal{V}},\mathcal{S})$.

[3] The reader wishing an explicit description of these dualities is encouraged to consult [BGN18].

Notice that if $p\colon \mathbf{L}^{\mathcal{V}} \to \mathbf{L} \to \mathbf{\Delta}$ denotes the canonical cartesian fibration, then $\overline{L}|_{\Delta^n_{\mathrm{Seg}}}$ fits into a pullback

$$\begin{array}{ccc} \overline{L}|_{\Delta^n_{\mathrm{Seg}}} & \longrightarrow & \overline{L} \\ \downarrow & & \downarrow \\ p^*(\Delta^n_{\mathrm{Seg}}) & \longrightarrow & p^*(\Delta^n) \end{array}$$

in the $\infty$-topos $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$.

As in Remark 3.2.6 the next proposition easily follows from the Yoneda embedding.

**Proposition 3.2.15.** *An object $F \in \mathrm{Fun}(\mathbf{\Xi}^{\mathrm{op},\mathcal{V}}, \mathcal{S})$ is a Segal $\mathbf{\Xi}^{\mathcal{V}}$-space if and only if it is local with respect to all Segal core inclusions $\overline{G}_{\mathrm{Seg}} \to \overline{G}$.* □

**Remark 3.2.16.** It follows from the definition that the cocartesian fibration $\mathbf{\Xi}^{\mathrm{op},\mathcal{V}} \to \mathbf{\Xi}^{\mathrm{op}}$ restricts to a cocartesian fibration $\mathbf{\Xi}^{\mathrm{op},\mathcal{V}}_{\mathrm{int}} \to \mathbf{\Xi}^{\mathrm{op}}_{\mathrm{int}}$. By applying (the dual of) [CH20, Lemma 2.3.13] to this cocartesian fibration and the full subcategory $\mathbf{\Xi}^{\mathrm{op}}_{\mathrm{el}}$, we see that the $\infty$-categories $(\mathbf{\Xi}^{\mathrm{op},\mathcal{V}}_{\mathrm{el}})_{\overline{G}/}$ and $(\mathbf{\Xi}^{\mathrm{op}}_{\mathrm{el}})_{G/}$ are equivalent. In particular, $\overline{G}_{\mathrm{Seg}} = \mathrm{colim}_{(\mathbf{\Xi}^{\mathrm{op},\mathcal{V}}_{\mathrm{el}})_{\overline{G}/}} \overline{E}$ can be identified with $\mathrm{colim}_{E \in (\mathbf{\Xi}^{\mathrm{op}}_{\mathrm{el}})_{G/}} \overline{E}$ where $\overline{G} \to \overline{E}$ are cocartesian lifts.

In the special case $\mathbf{\Xi} = \mathbf{L}$, we can give the following improvement, which can be proven in the same manner as Proposition 3.2.9. A similar statement holds when $\mathbf{L}$ is replaced by $\mathbf{L}_{0\text{-type}}$ or $\mathbf{L}_{\mathrm{out}}$.

**Proposition 3.2.17.** *Suppose $F \in \mathrm{Fun}(\mathbf{L}^{\mathrm{op},\mathcal{V}}, \mathcal{S})$. The following are equivalent:*

1. *$F$ is a Segal $\mathbf{L}^{\mathcal{V}}$-space.*
2. *$F$ is local with respect to all Segal core inclusions $\overline{L}_{\mathrm{Seg}} \to \overline{L}$.*
3. *$F$ is local with respect to the short Segal core inclusions and the segmentation maps from Definition 3.2.14.* □

**Definition 3.2.18** (Fibrewise representability)**.** Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category and $p\colon \mathbf{\Xi}^{\mathrm{op},\mathcal{V}} \to \mathbf{\Xi}^{\mathrm{op}}$ be the cocartesian fibration constructed in Definition 3.2.10.

- If $F \in \mathrm{Seg}(\mathbf{\Xi}^{\mathcal{V}})$ and $\mathfrak{c}_{m,n}$ is a corolla in $\mathbf{\Xi}_{\mathrm{el}}$, write
$$F(\mathfrak{c}_{m,n}(-))\colon \mathcal{V}^{\mathrm{op}} \simeq (\mathbf{\Xi}^{\mathrm{op},\mathcal{V}})_{\mathfrak{c}_{m,n}} \to \mathcal{S}_{/F(\mathfrak{e})^{m+n}} \simeq \mathrm{Fun}(F(\mathfrak{e})^{m+n}, \mathcal{S})$$
for the functor induced by the $p$-cocartesian lifts of the $m+n$ morphisms $\mathfrak{c}_{m,n} \to \mathfrak{e}$ in $\mathbf{\Xi}^{\mathrm{op}}_{\mathrm{el}}$.
- We say that $F \in \mathrm{Seg}(\mathbf{\Xi}^{\mathcal{V}})$ is *fibrewise representable* if for each corolla $\mathfrak{c}_{m,n} \in \mathbf{\Xi}_{\mathrm{el}}$ and each object $\underline{xy} = (x_1, \dots, x_m; y_1, \dots, y_n) \in F(\mathfrak{e})^{m+n}$, the composite functor
$$F(\mathfrak{c}_{m,n}(-, \underline{xy}))\colon \mathcal{V}^{\mathrm{op}} \xrightarrow{F(\mathfrak{c}_{m,n}(-))} \mathrm{Fun}(F(\mathfrak{e})^{m+n}, \mathcal{S}) \xrightarrow{\mathrm{ev}} \mathcal{S}$$
(with 'ev' being evaluation at $\underline{xy}$) is representable. In this case we let $\mathrm{Map}_F(x_1, \dots, x_m; y_1, \dots, y_n)$ denote the object in $\mathcal{V}$ representing the composite $F(\mathfrak{c}_{m,n}(-, \underline{xy})) = \mathrm{ev} \circ F(\mathfrak{c}_{m,n}(-))$.
- We write $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{\Xi}^{\mathcal{V}})$ for the full subcategory of $\mathrm{Seg}(\mathbf{\Xi}^{\mathcal{V}})$ spanned by the fibrewise representable Segal $\mathbf{\Xi}^{\mathcal{V}}$-spaces.

Suppose that $F \in \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$. Unraveling the definition, we have, for each $v \in \mathcal{V}$ and each $\underline{xy} = x_1, \dots, x_m; y_1, \dots, y_n$ a pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{V}}(v, \mathrm{Map}_F(x_1, \dots, x_m; y_1, \dots, y_n)) & \longrightarrow & F(\mathfrak{c}_{m,n}(v)) \\ \downarrow & & \downarrow \\ \{x_1, \dots, x_m; y_1, \dots, y_n\} & \longrightarrow & F(\mathfrak{e})^{m+n}. \end{array}$$

**Remark 3.2.19.** Informally, we occasionally refer to objects of the $\infty$-categories $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$, $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$, and $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ as *algebraic $\mathcal{V}$-enriched $\infty$-properads*. We later will see that the choice among the three graph categories $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{L}_{\mathrm{c}}$ give equivalent notions for this term. This is a preliminary definition, and $\mathcal{V}$-enriched $\infty$-properads will appear below in Definition 6.2.7. Likewise, we refer to fibrewise representable $\mathbf{G}_{\mathrm{out}}^{\mathcal{V}}$-Segal spaces and $\mathbf{L}_{\mathrm{out}}^{\mathcal{V}}$-Segal spaces as algebraic $\mathcal{V}$-enriched $\infty$-output-properads, and fibrewise representable $\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}}$-Segal spaces and $\mathbf{L}_{0\text{-type}}^{\mathcal{V}}$-Segal spaces as algebraic $\mathcal{V}$-enriched $\infty$-dioperads.

**Remark 3.2.20.** As we will see in Corollary 5.1.6, the $\infty$-categories $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ and $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ are equivalent, while the interpretation of Corollary 7.1.24 says that the objects in these $\infty$-categories can be interpreted as enriched $\mathcal{V}$-properads. In this picture the object $\mathrm{Map}_F(x_1, \dots, x_m; y_1, \dots, y_n) \in \mathcal{V}$ in Definition 3.2.18 should be thought of as the mapping object of the $\mathcal{V}$-enriched $\infty$-properad $F$.

**Remark 3.2.21.** In [CH20] the authors apply the construction of Definition 3.2.10 to the subcategory $\boldsymbol{\Delta}_{\mathbf{F}}$ of $\mathbf{L}$ mentioned in Proposition 2.1.35. The resulting $\infty$-category $\boldsymbol{\Delta}_{\mathbf{F}}^{\mathcal{V}}$ is then used in [CH20, Definition 2.3.9] to define continuous Segal presheaves, which model enriched $\infty$-operads. Although the definition of fibrewise representable Segal $\mathbf{L}^{\mathcal{V}}$-spaces from above naturally generalizes the continuous definition [CH20, Definition 2.3.9] we prefer to use "fibrewise representable" instead of "continuous" as we believe that it describes the phenomenon better.

We now give alternative characterizations of fibrewise representable Segal $\boldsymbol{\Xi}^{\mathcal{V}}$-spaces:

**Proposition 3.2.22.** *Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category and let $\varnothing$ denote the initial object in $\mathcal{V}$. For $F \in \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{V}})$ the following are equivalent:*

*(1) $F$ is fibrewise representable.*

*(2) For every corolla $\mathfrak{c}_{m,n} \in \boldsymbol{\Xi}_{\mathrm{el}}$, the restriction*

$$\mathcal{V}^{\mathrm{op}} \simeq (\boldsymbol{\Xi}^{\mathrm{op},\mathcal{V}})_{\mathfrak{c}_{m,n}} \xrightarrow{F(\mathfrak{c}_{m,n}(-))} \mathcal{S}_{/F(\mathfrak{e})^{m+n}} \to \mathcal{S}$$

*preserves weakly contractible limits, and the natural map $F(\mathfrak{c}_{m,n}(\varnothing)) \to F(\mathfrak{e})^{m+n}$ is an equivalence.*

*(3) $F$ is local with respect to the maps $\coprod_{m+n} \mathfrak{e} \to \mathfrak{c}_{m,n}(\varnothing)$ and the maps $\mathrm{colim}_{\mathcal{I}} \mathfrak{c}_{m,n}(\phi) \to \mathfrak{c}_{m,n}(\mathrm{colim}_{\mathcal{I}} \phi)$ in $\mathrm{Fun}(\boldsymbol{\Xi}^{\mathrm{op},\mathcal{V}}, \mathcal{S})$, where $\phi$ is a weakly contractible diagram in $\mathcal{V}$.*

*Proof.* By definition $F$ is fibrewise representable if for every corolla $\mathfrak{c}_{m,n}$ and every $\underline{xy} \in F(\mathfrak{e})^{m+n}$ the functor $\mathcal{V}^{\mathrm{op}} \to \mathcal{S}$ given by composite of $F(\mathfrak{c}_{m,n}(-))$ and the evaluation map $\mathrm{ev}_{\underline{xy}}$ is representable. The assumption that $\mathcal{V}$ is presentable implies that a functor $\mathcal{V}^{\mathrm{op}} \to \mathcal{S}$ is representable if and only if it preserves all limits. Since limits in $\mathrm{Fun}(F(\mathfrak{e})^{m+n}, \mathcal{S})$ are computed objectwise, (1) is equivalent to saying that

the functor $F(\mathfrak{c}_{m,n}(-))\colon \mathcal{V}^{\mathrm{op}} \to \mathcal{S}_{/F(\mathfrak{e})^{m+n}} \simeq \mathrm{Fun}(F(\mathfrak{e})^{m+n}, \mathcal{S})$ preserves all limits. This is equivalent to saying that $F(\mathfrak{c}_{m,n}(-))$ preserves terminal objects and weakly contractible limits by [GHK21, Lemma 2.2.7]. The terminal object in $\mathcal{S}_{/F(\mathfrak{e})^{m+n}}$ is $F(\mathfrak{e})^{m+n}$ and the forgetful functor $\mathcal{S}_{/F(\mathfrak{e})^{m+n}} \to \mathcal{S}$ preserves and detects weakly contractible limits. Hence, (1) holds if and only $F(\mathfrak{c}_{m,n}(-))$ takes the terminal object $\varnothing$ to $F(\mathfrak{e})^{m+n}$ and preserves weakly contractible limits, which is equivalent to (2). The equivalence between (2) and (3) follows from the Yoneda lemma. □

**Definition 3.2.23.** Let $\boldsymbol{\Xi}$ be a graph category (Notation 3.2.2).

(1) A map $X \to X'$ in $\mathrm{P}(\boldsymbol{\Xi}) = \mathrm{Fun}(\boldsymbol{\Xi}^{\mathrm{op}}, \mathcal{S})$ (resp. $\mathrm{P}(\boldsymbol{\Xi}^{\mathcal{V}})$) is called a *Segal equivalence* if $\mathrm{Map}(X', F) \to \mathrm{Map}(X, F)$ is an equivalence for every $F \in \mathrm{Seg}(\boldsymbol{\Xi})$ (resp. $F \in \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{V}})$),
(2) A map $X \to X'$ in $\mathrm{P}(\boldsymbol{\Xi}^{\mathcal{V}})$ is called a *$\mathcal{V}$-Segal equivalence* if $\mathrm{Map}(X', F) \to \mathrm{Map}(X, F)$ is an equivalence for every $F \in \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$.

Note that each Segal equivalence in $\mathrm{P}(\boldsymbol{\Xi}^{\mathcal{V}})$ is also a $\mathcal{V}$-Segal equivalence.

**Definition 3.2.24.** We call a class $S$ of morphisms in a cocomplete $\infty$-category $\mathcal{C}$ *strongly saturated* if

(i) it satisfies the 2-of-3 property,
(ii) it is stable under pushouts along any morphism in $\mathcal{C}$,
(iii) the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by $S$ is stable under small colimits.

We say that the strongly saturated class $S$ is *strongly generated* by $\mathbb{S}$ if $S$ is the smallest strongly saturated class containing $\mathbb{S}$. In this case we call elements in $\mathbb{S}$ the *generators* of $S$.

**Remark 3.2.25.** If $\mathbb{S}$ is a proper set of morphisms in a presentable $\infty$-category $\mathcal{C}$, then [Lur09, Proposition 5.5.4.15] implies that the strongly saturated class generated by $\mathbb{S}$ coincides with the class of morphisms $f$ such that $\mathrm{Map}_{\mathcal{C}}(f, X)$ is an equivalence for every $\mathbb{S}$-local object $X$ in $\mathcal{C}$.

**Example 3.2.26.** By [Lur09, Lemma 5.5.4.11] the class of Segal equivalences is strongly saturated, and likewise for $\mathcal{V}$-Segal equivalences. The previous remark and Remark 3.2.6 show that the class of Segal equivalences in $\mathrm{P}(\boldsymbol{\Xi})$ is generated by Segal core inclusions. If $\mathcal{U}$ is a small symmetric monoidal $\infty$-category, then by Proposition 3.2.15 the class of Segal equivalences in $\mathrm{P}(\boldsymbol{\Xi}^{\mathcal{U}})$ is generated by Segal core inclusions. In addition, Proposition 3.2.9 (resp. Proposition 3.2.17) gives that the class of Segal equivalences in $\mathrm{P}(\mathbf{L})$ (resp. $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$) can instead be generated by the set of short Segal core inclusions together with the segmentation maps.

Even in the case of a small $\mathcal{U}$, one cannot immediately produce generators for $\mathcal{U}$-Segal equivalences since, by Proposition 3.2.22, one would want to index some of these generators by the proper class of all weakly contractible diagrams in $\mathcal{U}$. In the present paper we will never need explicit generators for $\mathcal{U}$ or $\mathcal{V}$-Segal equivalences.

One could ask about compatibility between the above constructions for various graph categories. Along these lines, we have the following.

**Proposition 3.2.27.** *Suppose that $i\colon \boldsymbol{\Xi} \to \boldsymbol{\Upsilon}$ is one of the fully-faithful inclusions of graph categories appearing in Figure 3 or Figure 5. Then $i^*\colon \mathrm{P}(\boldsymbol{\Upsilon}) \to \mathrm{P}(\boldsymbol{\Xi})$ restricts to $i^*\colon \mathrm{Seg}(\boldsymbol{\Upsilon}) \to \mathrm{Seg}(\boldsymbol{\Xi})$. Likewise, we have restrictions $\bar{i}^*\colon \mathrm{Seg}(\boldsymbol{\Upsilon}^{\mathcal{V}}) \to \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{V}})$ and $\bar{i}^*\colon \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Upsilon}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$.*

*Proof.* By declaring the edge and each corolla to be elementary objects, by Remark 3.2.3(i) we have the opposites of all categories appearing in Figure 3 and Figure 5 are algebraic patterns. According to [CH21, Lemma 4.5] and [CH21, Remark 4.4] the functor $i^*\colon \mathrm{P}(\boldsymbol{\Upsilon}) \to \mathrm{P}(\boldsymbol{\Xi})$ given by the precomposition with $i^{\mathrm{op}}\colon \boldsymbol{\Xi}^{\mathrm{op}} \to \boldsymbol{\Upsilon}^{\mathrm{op}}$ restricts to $\mathrm{Seg}(\boldsymbol{\Upsilon}) \to \mathrm{Seg}(\boldsymbol{\Xi})$ if for every object $G \in \boldsymbol{\Xi}^{\mathrm{op}}$ the functor $i$ induces an equivalence $\boldsymbol{\Xi}^{\mathrm{op}}_{\mathrm{el},G/} \xrightarrow{\sim} \boldsymbol{\Upsilon}^{\mathrm{op}}_{\mathrm{el},i(G)/}$. Note that the existence of an inert morphism $i(G) \to E$ with $E \in \boldsymbol{\Upsilon}^{\mathrm{op}}_{\mathrm{el}}$ implies that $E \simeq i(E')$ for some $E' \in \boldsymbol{\Xi}^{\mathrm{op}}_{\mathrm{el}}$. The fully faithfulness of $i$ then implies that $\boldsymbol{\Xi}^{\mathrm{op}}_{\mathrm{el},G/} \to \boldsymbol{\Upsilon}^{\mathrm{op}}_{\mathrm{el},i(G)/}$ is essentially surjective as well as fully faithful.

Suppose $\overline{G} = G(v_c)_{c \in \mathtt{V}_{\boldsymbol{\Xi}}(G)}$, then it follows from Remark 3.2.13 that $\boldsymbol{\Xi}^{\mathcal{V},\mathrm{op}}_{\mathrm{el},\overline{G}/} \simeq \boldsymbol{\Xi}^{\mathrm{op}}_{\mathrm{el},G/} \times_{\mathbf{F}_*,\mathtt{V}_{\boldsymbol{\Xi}}(G)/} \mathcal{V}^{\mathrm{op},\otimes}_{(v_c)_c/}$. Hence the equivalence $\boldsymbol{\Xi}^{\mathrm{op}}_{\mathrm{el},G/} \xrightarrow{\sim} \boldsymbol{\Upsilon}^{\mathrm{op}}_{\mathrm{el},i(G)/}$ induces an equivalence $\boldsymbol{\Xi}^{\mathcal{V},\mathrm{op}}_{\mathrm{el},\overline{G}/} \xrightarrow{\sim} \boldsymbol{\Upsilon}^{\mathcal{V},\mathrm{op}}_{\mathrm{el},\overline{G}/}$ which shows that $\mathrm{P}(\boldsymbol{\Upsilon}^{\mathcal{V}}) \to \mathrm{P}(\boldsymbol{\Xi}^{\mathcal{V}})$ restricts to $\mathrm{Seg}(\boldsymbol{\Upsilon}^{\mathcal{V}}) \to \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{V}})$ by [CH21, Lemma 4.5]. Finally, it further restricts to $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Upsilon}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$ as being fibrewise representable is a property that only concerns elementary objects and the map $\boldsymbol{\Xi}^{\mathcal{V}} \to \boldsymbol{\Upsilon}^{\mathcal{V}}$ preserves elementary objects. □

**Lemma 3.2.28.** *Let $i\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{L}$ be the inclusion. The functor $i_*\colon \mathrm{P}(\mathbf{L}_{\mathrm{c}}) \to \mathrm{P}(\mathbf{L})$ given by the right Kan extension along $i^{\mathrm{op}}$ restricts to a functor $\mathrm{Seg}(\mathbf{L}_{\mathrm{c}}) \to \mathrm{Seg}(\mathbf{L})$. Similarly, if $\mathcal{U}$ is a small symmetric monoidal $\infty$-category, then the right Kan extension restricts to Segal objects:*

$$\begin{array}{ccc} \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}) \\ \downarrow & & \downarrow \\ \mathrm{Seg}(\mathbf{L}^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\mathbf{L}^{\mathcal{U}}). \end{array}$$

*In both cases, the restricted functors are right adjoints to the precomposition functors appearing in Proposition 3.2.27. Similar statements hold when the pair $(\mathbf{L}_{\mathrm{c}}, \mathbf{L})$ is replaced by $(\mathbf{L}_{\mathrm{sc}}, \mathbf{L}_{0\text{-type}})$ or $(\mathbf{L}_{\mathrm{out,c}}, \mathbf{L}_{\mathrm{out}})$.*

*Proof.* We write $i\colon \boldsymbol{\Xi}_{\mathrm{c}} \to \boldsymbol{\Xi}$ for any of the three inclusions under consideration. For the first part, it suffices to show that both adjoint functors $i^*\colon \mathrm{P}(\boldsymbol{\Xi}) \rightleftarrows \mathrm{P}(\boldsymbol{\Xi}_{\mathrm{c}}) : i_*$ restrict to functors between the corresponding $\infty$-category of Segal objects. By the previous proposition this is true for $i^*$. By [CH21, Proposition 6.3] the right adjoint $i_*$ restricts to $\mathrm{Seg}(\boldsymbol{\Xi}_{\mathrm{c}}) \to \mathrm{Seg}(\boldsymbol{\Xi})$ if $i^{\mathrm{op}}$ has unique lifting of active morphisms. Suppose we have an active map $f\colon H \to i(G)$ in $\boldsymbol{\Xi}^{\mathrm{op}}$ lying over some active morphism $[n] \to [m]$ in $\boldsymbol{\Delta}^{\mathrm{op}}$, then we have $H_{0,n} \xrightarrow{\sim} G_{0,m}$. By Lemma 2.1.12, $G$ is an object of $\boldsymbol{\Xi}_{\mathrm{c}}$ if and only if $G_{0,m} \simeq \{*\}$. The same lemma then implies that $H \in \boldsymbol{\Xi}_{\mathrm{c}}$. Hence $f$ lifts to an active map of $\boldsymbol{\Xi}_{\mathrm{c}}^{\mathrm{op}}$, and since $i$ is fully-faithful the unique lifting property holds.

Since a morphism in $\boldsymbol{\Xi}^{\mathcal{U}}$ is active if and only if its projection to $\boldsymbol{\Xi}$ is active (see Remark 3.2.13), the same argument shows that the inclusion $\boldsymbol{\Xi}_{\mathrm{c}}^{\mathcal{U}} \to \boldsymbol{\Xi}^{\mathcal{U}}$ induces a right adjoint $\mathrm{Seg}(\boldsymbol{\Xi}_{\mathrm{c}}^{\mathcal{U}}) \to \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{U}})$ given by restriction. □

**Proposition 3.2.29.** *The inclusion $i\colon \mathbf{L}_{\mathrm{c}} \hookrightarrow \mathbf{L}$ induces an equivalence $i^*\colon \mathrm{Seg}(\mathbf{L}) \xrightarrow{\sim} \mathrm{Seg}(\mathbf{L}_{\mathrm{c}})$. For every small symmetric monoidal $\infty$-category $\mathcal{U}$, the inclusion $\mathbf{L}_{\mathrm{c}}^{\mathcal{U}} \hookrightarrow \mathbf{L}^{\mathcal{U}}$ induces an equivalence $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$. Similar statements hold when the pair $(\mathbf{L}_{\mathrm{c}}, \mathbf{L})$ is replaced by $(\mathbf{L}_{\mathrm{sc}}, \mathbf{L}_{0\text{-type}})$ or $(\mathbf{L}_{\mathrm{out,c}}, \mathbf{L}_{\mathrm{out}})$.*

*Proof.* We only give a proof for $(\mathbf{L}_{\mathrm{c}}, \mathbf{L})$, as the other two situations are entirely analogous. By Proposition 3.2.27 and Lemma 3.2.28 the adjunction $i^*\colon \mathrm{P}(\mathbf{L}) \rightleftarrows \mathrm{P}(\mathbf{L}_{\mathrm{c}})\colon i_*$, where the right adjoint $i_*$ is given by right Kan extension, restricts to an adjunction

$$i^*\colon \mathrm{Seg}(\mathbf{L}) \rightleftarrows \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}) : i_*$$

Since $i$ is fully faithful, the right adjoint $i_*$ is fully faithful and the counit is an equivalence. It only remains to show that the unit $\mathrm{id} \to i_* i^*$ is an equivalence after evaluating at any object $F \in \mathrm{Seg}(\mathbf{L})$ and $I \in \mathbf{L}^{\mathrm{op}}$. The description of right Kan extension give $i_* i^* F(I) \simeq \lim_{J \in (\mathbf{L}_{\mathrm{c}}^{\mathrm{op}})_{I/}} F(iJ)$. Let us regard the set $\{I_j\}$ of objects $I \to I_j$ in $(\mathbf{L}_{\mathrm{c}}^{\mathrm{op}})_{I/}$ such that $I_j$ is a connected component of $I$. We now want to see that the inclusion of $\{I_j\}$ into $(\mathbf{L}_{\mathrm{c}}^{\mathrm{op}})_{I/}$ is final. According to [Lur09, Theorem 4.1.3.1] this is can be proven by verifying that for every $\phi \in (\mathbf{L}_{\mathrm{c}}^{\mathrm{op}})_{I/}$ the category $\{I_j\}_{/\phi}$ is weakly contractible. As the codomain of $\phi\colon I \to J$ in $\mathbf{L}^{\mathrm{op}}$ is connected, the morphism $\phi$ necessarily factors through a unique connected components $I_j$, and thus, the category $\{I_j\}_{/\phi}$ contains only the object $(I \to I_j) \to (I \to J)$.

This implies that $\lim_{J \in (\mathbf{L}_{\mathrm{c}}^{\mathrm{op}})_{I/}} F(iJ) \simeq \prod_j F(I_j)$ and the assumption that $F \in \mathrm{Seg}(\mathbf{L})$ implies that $\prod_j F(I_j) \simeq \prod_j \lim_{E \in (\mathbf{L}_{\mathrm{el}}^{\mathrm{op}})_{I_j/}} F(E) \simeq \lim_{E \in (\mathbf{L}_{\mathrm{el}}^{\mathrm{op}})_{I/}} F(E)$ where the last equivalence is given by writing $(\mathbf{L}_{\mathrm{el}}^{\mathrm{op}})_{I/}$ as a coproduct of $(\mathbf{L}_{\mathrm{el}}^{\mathrm{op}})_{I_j/}$.

The equivalence $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}}) \simeq \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$ is proved in essentially the same way. □

In the 1-categorical setting, taking the underlying set of colors gives a forgetful functor from the category of properads to the category of sets such that each fibre can be constructed as algebras over an operad (see Lemma 7.1.5 below). Theorem 3.2.33 below can be interpreted as an $\infty$-categorical version of this classical fact where taking the underlying set corresponds to evaluation at the edge $\mathfrak{e}$ in $\Xi^{\mathrm{op},\mathcal{V}}$. As in the classical case the fibres are certain algebras.

**Definition 3.2.30** (Edge decorations)**.** Given a space $X \in \mathcal{S}$, we can construct pictured right Kan extension

$$\begin{array}{ccc} \{\mathfrak{e}\} & \hookrightarrow & \Xi^{\mathrm{op}} \\ {\scriptstyle X}\downarrow & \swarrow & \\ \mathcal{S} & & \end{array}$$

and we write $\Xi_X^{\mathrm{op}} \to \Xi^{\mathrm{op}}$ for the left fibration associated to $\Xi^{\mathrm{op}} \to \mathcal{S}$. We call a morphism in $\Xi_X^{\mathrm{op}}$

- *inert* if it is cocartesian and lies over an inert morphism in $\Xi^{\mathrm{op}}$, or
- *active* if it lies over an active morphism in $\Xi^{\mathrm{op}}$.

Likewise, we say that an object is elementary if its image in $\Xi^{\mathrm{op}}$ is elementary. This gives $\Xi_X^{\mathrm{op}}$ the structure of an algebraic pattern. We let $\Xi_X^{\mathrm{op},\mathcal{V}}$ denote the pullback $\Xi_X^{\mathrm{op}} \times_{\Xi^{\mathrm{op}}} \Xi^{\mathrm{op},\mathcal{V}} \simeq \Xi_X^{\mathrm{op}} \times_{\mathbf{F}_*} \mathcal{V}^{\mathrm{op},\otimes}$ whose inert/active morphisms as well as elementary objects are defined by the components.

**Notation 3.2.31.** By unwinding the previous definition the fibre of $\Xi_X^{\mathrm{op}} \to \Xi^{\mathrm{op}}$ over a graph $G$ is given by the evaluation of the associated functor $\Xi^{\mathrm{op}} \to \mathcal{S}$ at $G$. If $G$ has $n$ edges, then this functor takes $G$ to $X^n$. Therefore, we can view an object in $\Xi_X^{\mathrm{op}}$ as an object in $\Xi^{\mathrm{op}}$ together with a labeling of its edges by elements of the space $X$. We write $G(x_e)_{e \in \mathbb{E}(G)}$ for an object in $\Xi_X^{\mathrm{op}}$ and $G(v_c, x_e)_{c,e}$ for an object in $\Xi_X^{\mathrm{op},\mathcal{V}}$.

**Definition 3.2.32.** Let $\mathcal{V}^{\otimes} \to \mathbf{F}_*$ be a symmetric monoidal $\infty$-category. We define a $\Xi^{\mathrm{op}}_X$-*algebra* in $\mathcal{V}$ to be a functor $\Xi^{\mathrm{op}}_X \to \mathcal{V}^{\otimes}$ which renders the diagram

$$\begin{array}{ccc} \Xi^{\mathrm{op}}_X & \longrightarrow & \mathcal{V}^{\otimes} \\ \downarrow & & \downarrow \\ \Xi^{\mathrm{op}} & \xrightarrow[\mathtt{v}^{\mathrm{op}}_{\Xi}]{} & \mathbf{F}_* \end{array}$$

commutative and takes the inert morphisms lying over $\rho_i$ to inert morphisms in $\mathcal{V}^{\otimes}$. We write $\mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}_{\mathbf{F}_*}(\Xi^{\mathrm{op}}_X, \mathcal{V}^{\otimes})$ spanned by the $\Xi^{\mathrm{op}}_X$-algebras. This construction is (contravariantly) functorial in $X$, and we write $\mathrm{Alg}_{\Xi^{\mathrm{op}}/\mathcal{S}}(\mathcal{V}) \to \mathcal{S}$ for the cartesian fibration associated to the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ which sends $X$ to $\mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{V})$.

In particular, when $X$ is a point, the $\infty$-category $\mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}_X}(\mathcal{S})$ coincides with the $\infty$-category presented by the model structure for Segal properads given in Theorem 5.3 of [HRY18].

The following analogue of Theorem 2.4.4 of [CH20] can be proved in a similar manner, after replacing $\mathbf{\Delta}^{\mathrm{op},\mathcal{V}}_{\mathbf{F}}$ with $\Xi^{\mathrm{op},\mathcal{V}}$. Recall from Remark 3.2.21 that the continuous condition from [CH20] is called fibrewise representability here.

**Theorem 3.2.33.** *Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category. There is an equivalence*

$$\begin{array}{ccc} \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}}) & \xrightarrow{\sim} & \mathrm{Alg}_{\Xi^{\mathrm{op}}/\mathcal{S}}(\mathcal{V}) \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

*of cartesian fibrations where the left diagonal functor is given by the evaluation at the edge $\mathfrak{e}$.* □

The next two items, regarding base change along lax monoidal functors, constitute analogues of Proposition 2.4.11 and Corollary 2.4.12 of [CH20] and are proved similarly.

**Proposition 3.2.34.** *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably symmetric monoidal $\infty$-categories.*

*(i) If $F\colon \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor, then $F$ induces a functor*

$$F_*\colon \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{V}) \to \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{W}).$$

*(ii) If $F\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal left adjoint, with (lax monoidal) right adjoint $G$, then there is an adjunction*

$$F_*\colon \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{W}) :G_*.$$

*(iii) If $L\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal localization with (lax monoidal) fully faithful right adjoint $i$, then the right adjoint $i_*\colon \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{W}) \to \mathrm{Alg}_{\Xi^{\mathrm{op}}_X}(\mathcal{V})$ is fully faithful. The image consists of those algebras $A\colon \Xi^{\mathrm{op}}_X \to \mathcal{V}^{\otimes}$ so that $A$ takes each object of $\Xi^{\mathrm{op}}_X$ lying over a corolla in $\Xi^{\mathrm{op}}$ to an object in $i(\mathcal{W})$.* □

**Corollary 3.2.35.** *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably symmetric monoidal $\infty$-categories.*

(i) *If $F\colon \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor, then $F$ induces a functor $F_*\colon \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{W}})$.*

(ii) *If $F\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal left adjoint, with (lax monoidal) right adjoint $G$, then there is an adjunction*

$$F_*\colon \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}}) \rightleftarrows \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{W}}) : G_*.$$

*Moreover, the functor $G_*$ can be identified with $F^*$.*

(iii) *If $L\colon \mathcal{V} \to \mathcal{W}$ is a symmetric monoidal localization with (lax monoidal) fully faithful right adjoint $i$, then the right adjoint $i_* \simeq L^*\colon \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{W}}) \to \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}})$ is fully faithful. The image consists of those $F \in \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}})$ so that the functors*

$$F(\mathfrak{c}_{m,n}(-, \underline{xy}))\colon \mathcal{V}^{\mathrm{op}} \to \mathcal{S}$$

*from Definition 3.2.18 are representable by objects in $\mathcal{W}$ for every corolla $\mathfrak{c}_{m,n}$ in $\Xi$ and every $\underline{xy} = (x_1, \dots, x_m; y_1, \dots, y_n) \in F(\mathfrak{e})^{m+n}$.* □

3.3. **Enrichment in presheaves.** In this subsection we recall some results which can be proven in a similar manner to their $\infty$-operadic counterparts in [CH20, §2.5 and §2.6], by replacing $\mathbf{\Delta_F}$ with a graph category $\Xi$.

Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category, then according to [Lur, Corollary 4.8.1.12] the presheaf $\infty$-category $\mathrm{P}(\mathcal{U})$ has a unique symmetric monoidal structure (called 'Day convolution') such that the tensor product preserves colimits in each variable and the Yoneda embedding $y\colon \mathcal{U} \to \mathrm{P}(\mathcal{U})$ is symmetric monoidal.

**Theorem 3.3.1.** *For every small symmetric monoidal $\infty$-category $\mathcal{U}$, the fully faithful functor $\Xi^{\mathcal{U}} \to \Xi^{\mathrm{P}(\mathcal{U})}$, induced by the symmetric monoidal Yoneda embedding $y\colon \mathcal{U} \to \mathrm{P}(\mathcal{U})$, gives a fully faithful functor $y_*\colon \mathrm{Seg}(\Xi^{\mathcal{U}}) \hookrightarrow \mathrm{Seg}(\Xi^{\mathrm{P}(\mathcal{U})})$. The functor $y_*$ restricts to an equivalence*

$$y_*\colon \mathrm{Seg}(\Xi^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}(\mathcal{U})}).$$

*Proof.* As in [CH20, Theorem 2.5.2]. □

Applying this theorem to the case $\Xi = \mathbf{G}$ and $\mathcal{U} = *$ yields the following.

**Corollary 3.3.2.** *Segal $\mathbf{G}$-spaces are equivalent to fibrewise representable $\mathbf{G}^{\mathcal{S}}$-spaces.* □

By taking the full subcategory of $\mathrm{Seg}(\mathbf{G})$ for those functors which send the edge to a point, we obtain the $\infty$-category presented by the model structure for Segal properads from [HRY18, Theorem 5.3]. Thus we may regard fibrewise representable $\mathbf{G}^{\mathcal{S}}$-spaces as a good algebraic version of $\infty$-properads.

**Definition 3.3.3.** Suppose that $\mathcal{U}$ is a small symmetric monoidal $\infty$-category $\mathcal{U}$ and $\mathbb{S}$ is a set of morphisms in $\mathrm{P}(\mathcal{U})$ such that the strongly saturated class generated by $\mathbb{S}$ is closed under tensor products. Let $y\colon \mathcal{U} \to \mathrm{P}(\mathcal{U})$ be the Yoneda embedding.

(i) We write $\mathrm{P}_{\mathbb{S}}(\mathcal{U})$ for the full subcategory of $\mathrm{P}(\mathcal{U})$ spanned by the $\mathbb{S}$-local objects. By [Lur, Proposition 2.2.1.9], it inherits a symmetric monoidal structure such that the localization $\mathrm{P}(\mathcal{U}) \to \mathrm{P}_{\mathbb{S}}(\mathcal{U})$ is symmetric monoidal.

(ii) We let $\mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\Xi^{\mathrm{P}(\mathcal{U})})$ denote the full subcategory of $\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}(\mathcal{U})})$ spanned by functors which are local with respect to the maps $\mathfrak{c}(s)$ where $\mathfrak{c}$ is a corolla in $\Xi$ and $s$ is in $\mathbb{S}$.

(iii) We write $\mathrm{Seg}_{\mathbb{S}}(\Xi^{\mathcal{U}})$ for the full subcategory of $\mathrm{Seg}(\Xi^{\mathcal{U}})$ spanned by functors which are local with respect to the maps $y^*\mathfrak{c}(s)$ where $\mathfrak{c}$ is a corolla in $\Xi$ and $s$ is in $\mathbb{S}$.

**Remark 3.3.4.** It follows from the definition that an object $F \in \mathrm{Seg}(\Xi^{\mathcal{U}})$ lies in $\mathrm{Seg}_{\mathbb{S}}(\Xi^{\mathcal{U}})$ if and only if for every corolla $\mathfrak{c}_{m,n}$ the induced functor

$$\mathcal{U}^{\mathrm{op}} \xrightarrow{F(\mathfrak{c}_{m,n}(-))} \mathcal{S}_{/F(\mathfrak{c})^{m+n}} \longrightarrow \mathcal{S}$$

is local with respect to all maps in $\mathbb{S}$. Analogously, $F \in \mathrm{Seg}(\Xi^{\mathrm{P}(\mathcal{U})})$ lies in $\mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\Xi^{\mathrm{P}(\mathcal{U})})$ if and only if for every corolla $\mathfrak{c}$, the induced functor $F(\mathfrak{c}(-, \underline{xy}))\colon \mathrm{P}(\mathcal{U})^{\mathrm{op}} \to \mathcal{S}$ is representable, and the representing object in $\mathrm{P}(\mathcal{U})$ is local with respect to all maps in $\mathbb{S}$.

Since $\mathbf{L}_{\mathrm{c}}$ and $\mathbf{L}$ have the same set of corollas, Proposition 3.2.29 and the first part of the previous remark imply the following.

**Proposition 3.3.5.** *If $\mathcal{U}$ is a small symmetric monoidal $\infty$-category and $\mathbb{S}$ is a set of morphisms in $\mathrm{P}(\mathcal{U})$ so that the strongly saturated class generated by $\mathbb{S}$ is closed under tensor products, then $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}) \to \mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{c}})$ is an equivalence.* □

Under the same hypotheses, and by the same reasoning, we also have $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{0\text{-type}}) \simeq \mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{sc}})$ and $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{out}}) \simeq \mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{out,c}})$.

After replacing $\mathbf{\Delta_F}$ by $\Xi$ in the proof of [CH20, Corollary 2.6.3], we obtain the following result:

**Proposition 3.3.6.** *Let $\mathcal{U}$ and $\mathbb{S}$ be as in Definition 3.3.3.*

*(i) The equivalence $\mathrm{Seg}(\Xi^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}(\mathcal{U})})$ of Theorem 3.3.1 restricts to an equivalence*

$$\mathrm{Seg}_{\mathbb{S}}(\Xi^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\Xi^{\mathrm{P}(\mathcal{U})}).$$

*(ii) Let $\Xi^{\mathrm{P}(\mathcal{U})} \to \Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}$ be the functor induced by the symmetric monoidal localization $\mathrm{P}(\mathcal{U}) \to \mathrm{P}_{\mathbb{S}}(\mathcal{U})$. Precomposition induces an equivalence*

$$\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\Xi^{\mathrm{P}(\mathcal{U})}).$$

*Combined, these imply that $\mathrm{Seg}_{\mathbb{S}}(\Xi^{\mathcal{U}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$.* □

As an immediate consequence we have:

**Corollary 3.3.7.** *The $\infty$-category $\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$ is given by an accessible localization of $\mathrm{Seg}(\Xi^{\mathcal{U}})$. In particular, it is presentable.*

*Proof.* Lemma 2.11 of [CH21] shows that $\mathrm{Seg}(\Xi^{\mathcal{U}})$ is presentable. By the previous proposition we can identify $\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$ with $\mathrm{Seg}_{\mathbb{S}}(\Xi^{\mathcal{U}})$. This latter $\infty$-category is given by an accessible localization of the presentable $\infty$-category $\mathrm{Seg}(\Xi^{\mathcal{U}})$ with respect to the set of maps $y^*\mathfrak{c}(s)$, where $\mathfrak{c}$ is a corolla and $s \in \mathbb{S}$ (see [Lur09, Proposition 5.5.4.2]). It follows that $\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$ is presentable. □

This result implies the following.

**Corollary 3.3.8.** *If $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, then $\mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{V}})$ is a presentable $\infty$-category.*

*Proof.* According to [CH20, Proposition 2.6.9] and [Lur09, Theorem 5.5.1.1], there is an equivalence $\mathcal{V} \simeq \mathrm{P}_{\mathbb{S}}(\mathcal{U})$ of symmetric monoidal $\infty$-categories, where $\mathcal{U}$ is a small symmetric monoidal $\infty$-category admitting all $\kappa$-small colimits for some regular cardinal $\kappa$ and $\mathbb{S}$ is the set of maps of the form $\operatorname{colim} y \circ \phi \to y(\operatorname{colim} \phi)$ and $\phi$ runs over a set of representatives for equivalence classes of $\kappa$-small colimits in $\mathcal{U}$. The previous corollary shows that this $\infty$-category $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$ is presentable. □

**Theorem 3.3.9.** *If $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, then the map*

$$\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{c}})$$

*from Proposition 3.2.27 is an equivalence.*

A similar proof to the following will also give equivalences $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{0\text{-type}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{sc}})$ and $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{out}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{out,c}})$.

*Proof.* Let $\mathbb{S}$ and $\mathcal{U}$ be as in the proof of the previous corollary, with $\mathcal{V} \simeq \mathrm{P}_{\mathbb{S}}(\mathcal{U})$. We have a diagram

$$\begin{array}{ccccccc}
\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}) & \xrightarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\mathbf{L}^{\mathrm{P}(\mathcal{U})}) & \xleftarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}) & \xrightarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \\
\simeq\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{c}}) & \xrightarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}_{\mathbb{S}}(\mathbf{L}_{\mathrm{c}}^{\mathrm{P}(\mathcal{U})}) & \xleftarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}) & \xrightarrow{\simeq} & \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{c}})
\end{array}$$

where the first two equivalences on each row are from Proposition 3.3.6 and the vertical equivalence is from Proposition 3.3.5. □

## 4. Algebras over enriched $\infty$-properads

The purpose of this section is to introduce, for two $\mathcal{V}$-enriched $\infty$-properads $\mathcal{Q}$ and $\mathcal{R}$, the $\infty$-category $\mathrm{Alg}^{\mathcal{V}}_{\mathcal{Q}}(\mathcal{R})$ of $\mathcal{Q}$-algebras in $\mathcal{R}$. Often we have that $\mathcal{V}$ is self-enriched, and we take $\mathcal{R}$ to be $\mathcal{V}$, appropriately regarded as a $\mathcal{V}$-enriched $\infty$-properad.

We begin with an extension of the notion of 'inner anodyne map' to $\mathbf{L}$ and $\mathbf{L}^{\mathcal{U}}$-presheaves; this is a key technical tool which has no analogue for $\mathbf{G}$-presheaves. The core of the section is in §4.2, where we show that $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ is tensored over $\mathrm{Seg}(\boldsymbol{\Delta})$. This tensor product is a partial extension of a Boardman–Vogt-style tensor product of properads (see [HRY15, §4]) in the unenriched case. The adjoint functor theorem then provides us with our notion of algebras (and also cotensors).

Everything in this section is in the setting of algebraic $\mathcal{V}$-enriched $\infty$-properads. We will return to the complete case in §6.2.

### 4.1. Inner horn inclusions and inner anodyne maps.

In this subsection we generalize the definition of inner anodyne maps in $\mathrm{P}(\boldsymbol{\Delta})$ to the setting of $\mathrm{P}(\mathbf{L})$ and $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$. After some formal observations we then finally prove the main result is Proposition 4.1.6 which says that inner anodyne maps in $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ are Segal equivalences. Its proof is a properadic generalization of the operadic case treated in [CH20, §2.7].

**Definition 4.1.1.** Let $p\colon \mathbf{L} \to \mathbf{\Delta}$ denote the cartesian fibration from Definition 2.1.17 and let $I \in \mathbf{L}$ be of height $n$. For $0 \leq k \leq n$, define $\Lambda^n_k I$ by the pullback

$$\begin{array}{ccc} \Lambda^n_k I & \longrightarrow & I \\ \downarrow & & \downarrow \\ p^*\Lambda^n_k & \longrightarrow & p^*(\Delta^n). \end{array}$$

Likewise, suppose that $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, and $q\colon \mathbf{L}^{\mathcal{V}} \to \mathbf{L}$ denote the cartesian fibration from Definition 3.2.10. If $\overline{I} \in \mathbf{L}^{\mathcal{V}}$ is of height $n$, we define $\Lambda^n_k \overline{I} \coloneqq (pq)^*\Lambda^n_k \times_{(pq)^*(\Delta^n)} \overline{I}$. When $0 < k < n$, we call the inclusions $\Lambda^n_k I \hookrightarrow I$ and $\Lambda^n_k \overline{I} \hookrightarrow \overline{I}$ *inner horn inclusions*. The class of *inner anodyne maps* in $\mathrm{P}(\mathbf{L})$ or $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ is the *weakly saturated class* generated by the inner horn inclusions, that is, the smallest class which both contains the inner horn inclusions and is closed under pushouts, transfinite compositions, and retracts.

We make use of the following notion of 'simple' morphism from [CH20, Definition 2.7.11].

**Definition 4.1.2.** Let $p\colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration between small $\infty$-categories. For $F \in \mathrm{P}(\mathcal{E})$ and $Y \in \mathrm{P}(\mathcal{C})$, we say a morphism $F \to p^*(Y)$ is *simple* if for every map $\sigma\colon X \to Y$ from a representable object $X$, in the pullback

$$\begin{array}{ccc} F' & \longrightarrow & F \\ \downarrow & & \downarrow \\ p^*(X) & \longrightarrow & p^*(Y) \end{array}$$

the presheaf $F'$ is representable and the adjunct $p_!F' \to X$ is an equivalence.

**Remark 4.1.3.** It is immediate from the definition, and was pointed out in Remark 2.7.12 of [CH20], that

(1) if $F \in \mathcal{E}$, then the counit map $F \to p^*p_!F \simeq p^*(pF)$ is simple, and
(2) the pullback of a simple map is simple.

We record a relative version of (1), whose conception and proof are joint with Rune Haugseng.

**Lemma 4.1.4.** *Let*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\beta} & \mathcal{E} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathcal{B} & \xrightarrow{\alpha} & \mathcal{C} \end{array} \tag{4}$$

*be a pullback of small $\infty$-categories with $p$ and $q$ cartesian fibrations. For each $e \in \mathcal{E}$, the unit map $\beta^*e \to q^*q_!\beta^*e$ is simple with respect to $q$.*

Since the square is a pullback and $\mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cocartesian fibration, a Beck–Chevalley condition gives $\alpha^*p_! \simeq q_!\beta^*\colon \mathrm{P}(\mathcal{E}) \to \mathrm{P}(\mathcal{B})$; see Lemma 9.2.7 and Lemma 12.3.11 of [RV]. We use this fact freely in the following proof. We note also that $\beta^*e \to q^*q_!\beta^*e$ is equivalent to $\beta^*$ applied to the unit $e \to p^*p_!e$.

*Proof.* Fix $e \in \mathcal{E}$ as in the statement, and let $f\colon b \to q_!\beta^* e \simeq \alpha^* p_! e$ be an arbitrary map of $\mathrm{P}(\mathcal{B})$ whose domain is representable. We need to show that the pullback of the unit map $\beta^* e \to q^* q_! \beta^* e$ along $q^*(f)$ is the unit map at a representable object $d \in \mathcal{D}$. Let $f'\colon \alpha(b) \to p(e)$ be adjunct to $f$. By [CH20, Lemma 2.7.10], there is a pullback

$$\begin{array}{ccc} e' & \xrightarrow{\overline{f}'} & e \\ \downarrow & & \downarrow \\ p^*(\alpha b) & \xrightarrow{p^*(f')} & p^*(pe) \end{array} \tag{5}$$

where $\overline{f}'$ in $\mathcal{E}$ is a $p$-cartesian lift of $f'$ and $p(e') \simeq \alpha b$. Since (4) is a pullback, there exists an object $d \in \mathcal{D}$ given by the pair $(b, e')$. Suppose we define the morphism $\overline{f}\colon d \to \beta^* e$ to be adjunct to $\overline{f}'\colon e' \simeq \beta d \to e$ then we have a commutative diagram

$$\begin{array}{ccc} d & \xrightarrow{\overline{f}} & \beta^* e \\ \downarrow & & \downarrow \\ q^* b & \xrightarrow{q^*(f)} & q^* q_! \beta^* e \end{array}$$

where the vertical maps are units. The statement of the lemma holds if we can show that this square is cartesian. In other words, it suffices to show that the image of this diagram under the functor $\mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, -)$ is a pullback of spaces for every $x \in \mathcal{D}$.

We first note that since the left adjoint $p_!$ preserves representable objects, for every $x \in \mathcal{D}$ the functor $\mathrm{Map}_{\mathrm{P}(\mathcal{E})}(\beta x, -)$ takes the pullback square (5) to the pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(\beta x, \beta d) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(\beta x, e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}}(p\beta x, \alpha b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(p\beta x, pe). \end{array}$$

On the other hand, since (4) is a pullback, we also have a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(x, d) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(\beta x, \beta d) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(qx, b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(\alpha q x, \alpha b). \end{array}$$

Pasting these together, we have the left-displayed pullback

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}}(x, d) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(\beta x, e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(qx, b) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(p\beta x, pe) \end{array} \qquad \begin{array}{ccc} \mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, d) & \longrightarrow & \mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, \beta^* e) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, q^* b) & \longrightarrow & \mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, q^* q_! \beta^* e) \end{array}$$

which is equivalent to the right-hand square. Hence, the functor $\mathrm{Map}_{\mathrm{P}(\mathcal{D})}(x, -)$ indeed takes the commutative square (5) to a pullback. □

**Lemma 4.1.5.** *Let $p^*\colon \mathrm{P}(\mathcal{C}) \to \mathrm{P}(\mathcal{E})$ be the functor induced by the composition with a cartesian fibration $p\colon \mathcal{E} \to \mathcal{C}$. Suppose that $A, L \in \mathcal{C}$, and let $f\colon A \to L \times B$*

*and* $g\colon L \times B \to L$ *be two morphisms in* $\mathrm{P}(\mathcal{C})$ *where* $g$ *is the projection map. Let*

$$\begin{array}{ccccc} F & \longrightarrow & \overline{L} \times p^*B & \longrightarrow & \overline{L} \\ \downarrow & & \downarrow & & \downarrow \\ p^*A & \xrightarrow{p^*f} & p^*(L \times B) & \xrightarrow{p^*g} & p^*L. \end{array}$$

*be a commutative diagram in* $\mathrm{P}(\mathcal{E})$ *where the left-hand and the right-hand squares are pullbacks and the right vertical map is the adjunction unit. Then the presheaf* $F$ *is represented by the object* $(gf)^*\overline{L} \in \mathcal{E}$ *given by the p-cartesian lift* $(gf)^*\overline{L} \to \overline{L}$ *of* $gf\colon A \to L \times B \to L$.

*Proof.* The outer rectangle is a pullback, so [CH20, Lemma 2.7.10] shows that $F \simeq (gf)^*\overline{L}$. □

**Proposition 4.1.6.** *The inner anodyne maps in* $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ *are Segal equivalences.*

*Proof.* As the class of Segal equivalences in $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ is strongly saturated (Example 3.2.26), and inner anodyne maps are the weakly saturated class generated by the inner horn inclusions, it suffices to show that every inner horn inclusion $\bar{\jmath}\colon \Lambda^n_k\overline{I} \to \overline{I}$ in $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ is a Segal equivalence. If $\mathbb{T}$ is a class of morphisms in a cocomplete $\infty$-category, write $\langle\mathbb{T}\rangle^r$ for the right-cancellative class generated by $\mathbb{T}$, that is, the smallest class of morphisms containing $\mathbb{T}$ and closed under finite compositions, pushouts, and right cancellations (i.e. $fg \in \langle\mathbb{T}\rangle^r$ and $g \in \langle\mathbb{T}\rangle^r$ implies $f \in \langle\mathbb{T}\rangle^r$).

Let $\mathbb{S}_1$ denote the set of spine inclusions $\Delta^m_{\mathrm{Seg}} \to \Delta^m$ in $\mathrm{P}(\mathbf{\Delta})$; by the proof of Lemma 3.5 of [JT07], each inner horn inclusion $\Lambda^n_k \to \Delta^n$ is contained in $\langle\mathbb{S}_1\rangle^r$. Suppose we are given an inner horn inclusion

$$\begin{array}{ccc} \Lambda^n_k\overline{I} & \xrightarrow{\bar{\jmath}} & \overline{I} \\ \downarrow & & \downarrow \\ (pq)^*(\Lambda^n_k) & \xrightarrow{j} & (pq)^*(\Delta^n). \end{array}$$

Let $\mathbb{S}_2$ denote the class (depending on $\overline{I}$) of maps $s_2$ in $\mathrm{P}(\mathbf{L}^{\mathcal{V}})$ appearing in a diagram of the form

$$\begin{array}{ccccc} A & \xrightarrow{s_2} & B & \longrightarrow & \overline{I} \\ \downarrow & & \downarrow & & \downarrow \\ (pq)^*(\Delta^m_{\mathrm{Seg}}) & \xrightarrow{(pq)^*s_1} & (pq)^*\Delta^m & \longrightarrow & (pq)^*\Delta^n \end{array} \tag{6}$$

where both squares are pullbacks, $s_1 \in \mathbb{S}_1$, and the bottom right map is arbitrary. Since inner horn inclusions $\Lambda^n_k \to \Delta^n$ are in $\langle\mathbb{S}_1\rangle^r$, by Proposition 2.7.8 of [CH20], we have that $\bar{\jmath}\colon \Lambda^n_k\overline{I} \to \overline{I}$ is contained in $\langle\mathbb{S}_2\rangle^r$. In particular, $\bar{\jmath}$ is contained in the strongly saturated class generated by $\mathbb{S}_2$.

As the right-hand map of (6) is a unit map it is simple, hence so too is the pullback $B \to (pq)^*\Delta^m$. In particular, $B$ is representable and has height $m$. It follows that $s_2$ is equivalent to a segmentation map, thus is also a Segal equivalence. Since $\mathbb{S}_2$ is contained in the strongly saturated class of Segal equivalences, so too is $\bar{\jmath}$. □

**Corollary 4.1.7.** *Let* $\mathcal{U}$ *be a small symmetric monoidal* $\infty$*-category, and let* $\mathbf{L}^{\mathcal{U}} \xrightarrow{q} \mathbf{L} \xrightarrow{p} \mathbf{\Delta}$ *be the usual cartesian fibrations.*

(1) *If* $f\colon F \to (pq)^* K'$ *is a simple map in* $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ *and* $K \to K'$ *is an inner anodyne map in* $\mathrm{P}(\mathbf{\Delta})$*, then* $(pq)^* K \times_{(pq)^* K'} F \to F$ *in* $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ *is inner anodyne.*
(2) *If* $f\colon F \to q^* B$ *is a simple map in* $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ *and* $A \to B$ *is an inner anodyne map in* $\mathrm{P}(\mathbf{L})$*, then* $q^* A \times_{q^* B} F \to F$ *in* $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ *is inner anodyne.*

*In both cases, the indicated map is also a Segal equivalence.*

*Proof.* For the first item, let $\mathbb{S}$ denote the set of inner horn inclusions $\{\Lambda^n_k \to \Delta^n \mid 0 < k < n\}$ in $\mathrm{P}(\mathbf{\Delta})$. Applying [CH20, Lemma 2.7.14], we have that the map under consideration is in the weakly saturated class generated by the inner horn inclusions $\Lambda^n_k \overline{I} \to \overline{I}$ in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$. This is the definition of being inner anodyne. The proof of the second item is similar, except that $\mathbb{S}$ should be taken to be the set of inner horn inclusions $\{\Lambda^n_k I \to I\}$ in $\mathrm{P}(\mathbf{L})$. □

4.2. **Tensoring with Segal spaces.** In this subsection we prove that for a presentably symmetric monoidal $\infty$-category $\mathcal{V}$, the $\infty$-category $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ is a module over the symmetric monoidal $\infty$-category $\mathrm{Seg}(\mathbf{\Delta})$ of Segal spaces. At the end we will see that by applying the adjoint functor theorem to this tensoring functor we get an algebra functor which takes two objects in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ to the corresponding $\infty$-category of algebras between them. Notice that we are no longer working with an arbitrary graph category $\mathbf{\Xi}$ (Notation 3.2.2), but rather just[4] with $\mathbf{L}$. This shift is important for our proofs, though eventually we shall see (Corollary 5.1.6) that $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$, so our main results extend to that context (e.g., Proposition 5.1.7).

**Definition 4.2.1.** Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category, and let $p^*\colon \mathrm{P}(\mathbf{\Delta}) \to \mathrm{P}(\mathbf{L}^{\mathcal{U}})$ be the functor induced by $p\colon \mathbf{L}^{\mathcal{U}} \to \mathbf{\Delta}$ given by the composition of the two cartesian fibrations $\mathbf{L}^{\mathcal{U}} \to \mathbf{L}$ and $\mathbf{L} \to \mathbf{\Delta}$. The two presheaf categories are symmetric monoidal $\infty$-categories with respect to the cartesian product. Since $p^*$ is right adjoint to the functor $p_!$ given by left Kan extension, $p^*$ preserves products and hence, it is a morphism of commutative algebra objects in $\mathrm{Cat}_\infty$ (or even in the $\infty$-category $\mathrm{Pr}^{\mathrm{L}}$ of presentable $\infty$-categories). Hence, by [Lur, Corollary 3.4.1.7], the functor

$$- \times p^*(-)\colon \mathrm{P}(\mathbf{L}^{\mathcal{U}}) \times \mathrm{P}(\mathbf{\Delta}) \to \mathrm{P}(\mathbf{L}^{\mathcal{U}}).$$

exhibits $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ as a module over $\mathrm{P}(\mathbf{\Delta})$. This functor then preserves colimits in each variable, because the cartesian product in $\mathcal{S}$ does and $p^*$ is left adjoint to $p_*$ (given by right Kan extension).

The main result of this section is the following theorem.

**Theorem 4.2.2.** *Let* $\mathcal{U}$ *be a small symmetric monoidal* $\infty$*-category and let* $L$ *denote the localization* $\mathrm{P}(\mathbf{L}^{\mathcal{U}}) \to \mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$*. The* $\mathrm{P}(\mathbf{\Delta})$*-module structure on* $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ *induces a* $\mathrm{Seg}(\mathbf{\Delta})$*-module structure on* $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$ *and the tensor product*

$$\otimes\colon \mathrm{Seg}(\mathbf{L}^{\mathcal{U}}) \times \mathrm{Seg}(\mathbf{\Delta}) \to \mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$$

*is given by* $F \otimes K = L(F \times p^* K)$*. In particular, the tensor product preserves colimits in each variable.*

[4] Our techniques are slightly more general than this, and apply to $\mathbf{L}_{\mathrm{out}}$ and $\mathbf{L}_{0\text{-type}}$ as mentioned in Remark 4.2.9.

If follows from [Lur, Proposition 2.2.1.9] that for the proof of Theorem 4.2.2 it suffices to show that the module structure on $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ is compatible with the Segal equivalences in the following sense:

**Proposition 4.2.3.** *Suppose $f\colon F \to F'$ is a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ and $g\colon K \to K'$ is a Segal equivalence in $\mathrm{P}(\mathbf{\Delta})$. Then $f \times p^*(g)\colon F \times p^*(K) \to F' \times p^*(K')$ is a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$.*

As already mentioned in Definition 4.2.1 the tensor functor $- \times p^*(-)$ preserves colimits in each variable. This allows us to prove the proposition by reducing it to a few key special cases of Segal equivalences. We start with two easy cases.

**Proposition 4.2.4.** *Given an object $\overline{I} \in \mathbf{L}^{\mathcal{U}}$ of height $n$ and a Segal equivalence $Z \to \Delta^n$, we write $f\colon \overline{I}|_Z \to \overline{I}$ for the map $\overline{I} \times_{p^*(\Delta^n)} p^*(Z) \to \overline{I}$. For $g\colon K \to K'$ in $\mathrm{P}(\mathbf{\Delta})$, consider the map*

$$f \times p^*(g)\colon \overline{I}|_Z \times p^*(K) \to \overline{I} \times p^*(K').$$

*(1) If $K \in \mathrm{P}(\mathbf{\Delta})$ and $f\colon \overline{I}|_{\Delta^n_{\mathrm{Seg}}} \to \overline{I}$ is a segmentation map, then $f \times p^*(K)$ is a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$.*

*(2) If $g\colon K \to K'$ is a Segal equivalence in $\mathrm{P}(\mathbf{\Delta})$, then $\overline{I} \times p^*(g)$ is a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$.*

*Proof.* Suppose $Z \to \Delta^n$ and $K \to K'$ are inner anodyne maps of simplicial sets. Two applications (first to $(Z \to \Delta^n, \varnothing \to K)$ and then to $(\varnothing \to \Delta^n, K \to K')$) of [Lur09, Corollary 2.3.2.4] gives that the product $Z \times K \to \Delta^n \times K'$ of inner horn inclusions is inner anodyne. We have a commutative diagram

$$\begin{array}{ccccc}
\overline{I}|_Z \times p^*(K) & \xrightarrow{f \times p^*(g)} & \overline{I} \times p^*(K') & \longrightarrow & \overline{I} \\
\downarrow & & \downarrow & & \downarrow \\
p^*(Z) \times p^*(K) & \longrightarrow & p^*(\Delta^n) \times p^*(K') & \longrightarrow & p^*(\Delta^n)
\end{array}$$

consisting of two pullback squares. Remark 4.1.3 implies that the right two vertical maps are simple. Corollary 4.1.7 gives that the left upper horizontal map $f \times p^*(g)$ is inner anodyne, hence a Segal equivalence.

Since spine inclusions are inner anodyne by [Joy08, Proposition 2.13], the first item (1) follows immediately from the previous paragraph by taking $Z = \Delta^n_{\mathrm{Seg}}$ and $K \to K'$ an identity.

For the second statement, first observe that the class of maps $g \in \mathrm{P}(\mathbf{\Delta})$ so that $\overline{I} \times p^*(g)$ is a Segal equivalence is strongly saturated by [Lur09, Remark 5.5.4.10]. In the first paragraph we showed that if $g$ is an inner horn inclusion then $\overline{I} \times p^*(g)$ is inner anodyne, hence a Segal equivalence. On the other hand, inner horn inclusions in $\mathrm{P}(\mathbf{\Delta})$ generate the strongly saturated class of Segal equivalences (see [CH20, Proposition 2.7.7]), so (2) holds. □

For the proof of Proposition 4.2.3 we need to understand explicitly the tensor product of a corolla with $\Delta^1$. For this purpose, it is convenient to introduce some notation:

**Definition 4.2.5.** Given an object $\overline{I} \in \mathbf{L}^{\mathcal{U}}$ of height 1, write $\overline{I}^+, \overline{I}^- \to \overline{I}$ for the cartesian lifts of $s_0, s_1\colon [2] \to [1]$, respectively. If $\overline{I}$ lives over $I \in \mathbf{L}$ of the form

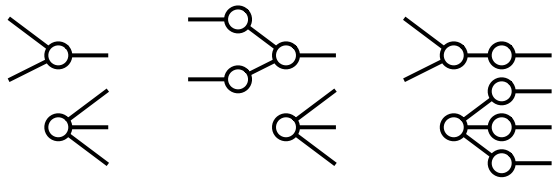


FIGURE 6. An example $I$ (of the form $\mathbf{2} \to \mathbf{2} \leftarrow \mathbf{4}$), $I^+$, and $I^-$.

$\mathbf{m} \xrightarrow{f} \mathbf{k} \xleftarrow{g} \mathbf{n}$, then the objects $\overline{I}^+, \overline{I}^-$ lie over the height 2 level graphs (see Figure 6)

$$I^+ = \left( \begin{array}{ccccc} \mathbf{m} & & \mathbf{m} & & \mathbf{n} \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}} & & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & \\ & \mathbf{m} & & \mathbf{k} & \end{array} \right), \qquad I^- = \left( \begin{array}{ccccc} \mathbf{m} & & \mathbf{n} & & \mathbf{n} \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle g} & & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}} & \\ & \mathbf{k} & & \mathbf{n} & \end{array} \right).$$

If $\overline{I}$ is connected, that is, if $\overline{I} = \mathfrak{c}_{m,n}(v)$ then the corolla in $\overline{I}^+, \overline{I}^-$ corresponding to $\mathfrak{c}_{m,n}$ is labeled by $v$ while all other corollas are of the form $\mathfrak{c}_{1,1}$ and labeled by the unit $\mathbb{1} \in \mathcal{U}$; see Figure 7 for an illustration in the disconnected case. We write $\mathfrak{c}^{\pm}_{m,n}(v)$ for $\overline{I}^{\pm}$ when $I$ is connected.

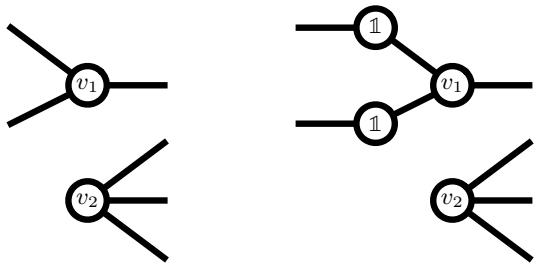



FIGURE 7. An example of $\overline{I}$ and $\overline{I}^+$

**Lemma 4.2.6.** *For a height 1 object $\overline{I} \in \mathbf{L}^{\mathcal{U}}$, there is an equivalence*

$$\overline{I}^+ \amalg_{\overline{I}} \overline{I}^- \to \overline{I} \times p^*(\Delta^1).$$

*In particular, in the connected case where $\overline{I} = \mathfrak{c}_{m,n}(v)$, we have an equivalence $\mathfrak{c}^+_{m,n}(v) \amalg_{\mathfrak{c}_{m,n}(v)} \mathfrak{c}^-_{m,n}(v) \to \mathfrak{c}_{m,n}(v) \times p^*(\Delta^1)$, natural in $v$.*

*Proof.* First, let us introduce some notation for a simplicial subdivision of the square. Namely, let $\sigma^+, \sigma^- \colon \Delta^2 \to \Delta^1 \times \Delta^1$ denote the two non-degenerate 2-simplices of $\Delta^1 \times \Delta^1$, taking $(0,1,2)$ to $((0,0),(0,1),(1,1))$ and $((0,0),(1,0),(1,1))$, respectively, and let $\delta \colon \Delta^1 \to \Delta^1 \times \Delta^1$ denote the diagonal map. Then the maps $\sigma^{\pm}$ and $\delta$ induce an equivalence $\Delta^2 \amalg_{\Delta^1} \Delta^2 \xrightarrow{\sim} \Delta^1 \times \Delta^1$.

Let $A^+$ and $A$ denote the pullbacks

$$\begin{array}{ccc} A^+ & \longrightarrow & \overline{I} \times p^*(\Delta^1) \\ \downarrow & & \downarrow \\ p^*(\Delta^2) & \xrightarrow{p^*(\sigma^+)} & p^*(\Delta^1) \times p^*(\Delta^1) \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & \overline{I} \times p^*(\Delta^1) \\ \downarrow & & \downarrow \\ p^*(\Delta^1) & \xrightarrow{p^*(\delta)} & p^*(\Delta^1) \times p^*(\Delta^1) \end{array}$$

and similarly for $A^-$. We can extend these diagrams by projection onto the first factor, yielding in the first case

$$\begin{array}{ccccc} A^+ & \longrightarrow & \overline{I} \times p^*(\Delta^1) & \longrightarrow & \overline{I} \\ \downarrow & & \downarrow & & \downarrow \\ p^*(\Delta^2) & \xrightarrow{p^*(\sigma^+)} & p^*(\Delta^1) \times p^*(\Delta^1) & \longrightarrow & p^*(\Delta^1). \end{array}$$

$$p^*(s^0)$$

It follows from this diagram and Lemma 4.1.5 that $A^+ \simeq \overline{I}^+$. Using the corresponding diagrams for $A^-$ and $A$, one concludes also that $A^- \simeq \overline{I}^-$ and $A \simeq \overline{I}$. Since pullbacks in $\mathrm{P}(\mathbf{L}^{\mathcal{U}})$ preserve colimits, we have a natural pullback square

$$\begin{array}{ccc} \overline{I}^+ \amalg_{\overline{I}} \overline{I}^- & \xrightarrow{\sim} & \overline{I} \times p^*(\Delta^1) \\ \downarrow & & \downarrow \\ p^*(\Delta^2) \amalg_{p^*(\Delta^1)} p^*(\Delta^2) & \xrightarrow{\sim} & p^*(\Delta^1) \times p^*(\Delta^1), \end{array}$$

which completes the proof. □

*Proof of Proposition 4.2.3.* The map $f \times p^*(g) \colon F \times p^*(K) \to F' \times p^*(K')$ in the statement factors as $F \times p^*(K) \to F' \times p^*(K) \to F' \times p^*(K')$ and below we prove separately that both morphisms are Segal equivalences.

By Proposition 4.2.4(2) the map $\mathrm{id} \times p^*(g) \colon \overline{L} \times p^*(K) \to \overline{L} \times p^*(K')$ is a Segal equivalence for every object $\overline{L} \in \mathbf{L}^{\mathcal{U}}$ and every Segal equivalence $g \colon K \to K'$ in $\mathrm{P}(\mathbf{\Delta})$. Since $- \times p^*(-)$ preserves colimits in each variable and Segal equivalences are closed under colimits, the map $F \times p^*(K) \to F \times p^*(K')$ is a Segal equivalence for every $F \in \mathrm{P}(\mathbf{L}^{\mathcal{U}})$ and every Segal equivalence $K \to K'$ in $\mathrm{P}(\mathbf{\Delta})$.

In the remaining part of this proof we show that the first map $F \times p^*(K) \to F' \times p^*(K)$ is a Segal equivalence as well. As $- \times p^*(-)$ preserves colimits the map $F \times p^*(K) \to F' \times p^*(K)$ is a Segal equivalence if it is true in the case where $K$ is a simplex $\Delta^n$. Therefore, it suffices to show that the bottom map of the commutative diagram

$$\begin{array}{ccc} F \times p^*(\Delta^n_{\mathrm{Seg}}) & \longrightarrow & F' \times p^*(\Delta^n_{\mathrm{Seg}}) \\ \downarrow & & \downarrow \\ F \times p^*(\Delta^n) & \longrightarrow & F' \times p^*(\Delta^n) \end{array}$$

is a Segal equivalence. The previous paragraph shows that the vertical morphisms are Segal equivalences and the definition of $\Delta^n_{\mathrm{Seg}}$ implies that the upper horizontal map is a colimit of maps of the form $f \times p^*(\Delta^1)$. Hence, we only need to prove that these maps are a Segal equivalences. By Proposition 3.2.17, we can further reduce to the case where $f$ is either a segmentation map or a short Segal core inclusion (Definition 3.2.14).

For the segmentation maps, the claim follows from Proposition 4.2.4(1). We now consider the case where $f \colon \coprod \overline{L}_i \to \overline{L}$ a short Segal core inclusion, $\overline{L}$ is of height 1 and each $\overline{L}_i$ is connected. Using that $- \times p^*(\Delta^1)$ commutes with colimits,

Lemma 4.2.6 tells us that to show $f \times p^*\Delta^1$ is a Segal equivalence, it suffices to show that

$$\left(\coprod_i \overline{L}_i^+\right) \amalg_{(\coprod_i \overline{L}_i)} \left(\coprod_i \overline{L}_i^-\right) \simeq \coprod_i (\overline{L}_i^+ \amalg_{\overline{L}_i} \overline{L}_i^-) \to \overline{L}^+ \amalg_{\overline{L}} \overline{L}^-$$

is a Segal equivalence. We know that $f\colon \coprod_i \overline{L}_i \to \overline{L}$ is a short Segal core inclusion, hence a generating Segal equivalence. It remains to show that $\coprod_i \overline{L}_i^+ \to \overline{L}^+$ and $\coprod_i \overline{L}_i^- \to \overline{L}^-$ are Segal equivalences; we consider only the first case (the other is similar). There is a commutative square

$$\begin{array}{ccc} \coprod_i \overline{L}_i^+|_{\Delta^2_{\mathrm{Seg}}} & \longrightarrow & \overline{L}^+|_{\Delta^2_{\mathrm{Seg}}} \\ \downarrow & & \downarrow \\ \coprod_i \overline{L}_i^+ & \longrightarrow & \overline{L}^+, \end{array}$$

where the vertical maps are Segal equivalences and the upper horizontal map is of the form

$$\coprod_i \left(\overline{L}_i^{+,\{0,1\}} \amalg_{\overline{L}_i^{+,\{1\}}} \overline{L}_i^{+,\{1,2\}}\right) \to \overline{L}^{+,\{0,1\}} \amalg_{\overline{L}^{+,\{1\}}} \overline{L}^{+,\{1,2\}}.$$

To show that the bottom map in the square is a Segal equivalence, it suffices to show that the top map is such. This top map is a pushout of the following:

$$\coprod_i \overline{L}_i^{+,\{0,1\}} \to \overline{L}^{+,\{0,1\}} \tag{7}$$

$$\coprod_i \overline{L}_i^{+,\{1,2\}} \to \overline{L}^{+,\{1,2\}} \tag{8}$$

$$\coprod_i \overline{L}_i^{+,\{1\}} \to \overline{L}^{+,\{1\}} \tag{9}$$

Since we generally have $\overline{I}^{+,\{1,2\}} = \overline{I}$, map (8) is a short Segal core inclusion. If $m$ is the number of input edges of $L$, then there are commutative triangles

$$\begin{array}{ccc} \coprod_{j\in\mathbf{m}} \mathfrak{c}_{1,1}(\mathbb{1}) & \longrightarrow & \overline{L}^{+,\{0,1\}} \\ & \searrow \quad \nearrow_{(7)} & \\ & \coprod_i \overline{L}_i^{+,\{0,1\}} & \end{array} \qquad \begin{array}{ccc} \coprod_{j\in\mathbf{m}} \mathfrak{e} & \longrightarrow & \overline{L}^{+,\{1\}} \\ & \searrow \quad \nearrow_{(9)} & \\ & \coprod_i \overline{L}_i^{+,\{1\}} & \end{array}$$

where the top maps are short Segal core inclusions, and the downward arrows are coproducts of such. Hence (7) and (9) are Segal equivalences as well. It follows that $f \times p^*\Delta^1$ is a Segal equivalence when $f$ is the Segal core inclusion into a height 1 graph.

It remains to consider the case where $f\colon \coprod_{i\in\mathbf{n}} \mathfrak{e} \to \overline{L}$ is a short Segal core inclusion and $\overline{L}$ is of height 0. As above, we must understand $\overline{L} \times p^*(\Delta^1)$, but unlike the height 1 case (Lemma 4.2.6), in the height 0 case this object is representable. We

first note that canonical equivalence $\sigma\colon \Delta^1 \xrightarrow{\sim} \Delta^0 \times \Delta^1$ induces a pullback square

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \overline{L} \times p^*(\Delta^1) \\ \downarrow & & \downarrow \\ p^*(\Delta^1) & \xrightarrow[p^*\sigma]{} & p^*(\Delta^0) \times p^*(\Delta^1) \end{array}$$

where the horizontal maps are equivalences. Lemma 4.1.5 implies that the presheaf $X$ is represented by $s_0\overline{L}$ determined by the cartesian lift $s_0\overline{L} \to \overline{L}$ of the projection $s^0\colon \Delta^1 \to \Delta^0$. Hence, the presheaf $X$ is represented by $L'(\mathbb{1}_c)_{c\in \mathbf{V}_{\mathbf{L}}(L')}$, where $L'$ is the graph $\mathbf{n} \xrightarrow{\mathrm{id}} \mathbf{n} \xleftarrow{\mathrm{id}} \mathbf{n}$ and $\mathbb{1}$ denotes the unit in $\mathcal{U}$. In particular, we have $\mathfrak{e} \times p^*(\Delta^1) \simeq (\mathfrak{c}_{1,1}, \mathbb{1})$. Therefore, the map $f \times p^*(\Delta^1)$ is given by

$$\left(\coprod_{i\in\mathbf{n}} \mathfrak{e}\right) \times p^*(\Delta^1) \simeq \coprod_{i\in\mathbf{n}} \left(\mathfrak{e} \times p^*(\Delta^1)\right) \simeq \coprod_{i\in\mathbf{n}} \mathfrak{c}_{1,1}(\mathbb{1}) \to L'(\mathbb{1}_c)_{c\in \mathbf{V}_{\mathbf{L}}(L')} \simeq \overline{L} \times p^*(\Delta^1),$$

which is a short Segal core inclusion. □

This completes the proof of Theorem 4.2.2. As a consequence, we get:

**Corollary 4.2.7.** *Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category and let $\mathbb{S}$ be a set of morphisms in $\mathrm{P}(\mathcal{U})$ compatible with the symmetric monoidal structure. Then the $\mathrm{Seg}(\mathbf{\Delta})$-module structure on $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$ induces a $\mathrm{Seg}(\mathbf{\Delta})$-module structure on $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}})$. Moreover, this tensor product preserves colimits in each variable.*

*Proof.* By definition $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}})$ is a localization of $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$ with respect to maps of the form $y^*\mathfrak{c}(s)$, $s \in \mathbb{S}$, according to [Lur, Proposition 2.2.1.9] the $\mathrm{Seg}(\mathbf{\Delta})$-module structure on $\mathrm{Seg}(\mathbf{L}^{\mathcal{U}})$ induces one on $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}})$ if for every map $s\colon X \to Y$ in $\mathbb{S}$ and every $K \in \mathrm{Seg}(\mathbf{\Delta})$, the map

$$y^*\mathfrak{c}(s) \times p^*K\colon y^*\mathfrak{c}(X) \times p^*K \to y^*\mathfrak{c}(Y) \times p^*K$$

is an $\mathbb{S}$-Segal equivalence. The proof of Proposition 4.2.3 shows that it suffices to verify the this for the case $K = \Delta^1$. By Lemma 4.2.6, there is an equivalence $\overline{I}^+ \amalg_{\overline{I}} \overline{I}^- \to \overline{I} \times p^*(\Delta^1)$ for every $\overline{I} \in \mathbf{L}^{\mathcal{U}}$ of height 1. Since $y^*\mathfrak{c}(X)$ is a colimit of these objects, the map $y^*\mathfrak{c}(s) \times p^*(\Delta^1)$ is equivalent to

$$y^*\mathfrak{c}^+(X) \amalg_{y^*\mathfrak{c}(X)} y^*\mathfrak{c}^-(X) \to y^*\mathfrak{c}^+(Y) \amalg_{y^*\mathfrak{c}(Y)} y^*\mathfrak{c}^-(Y).$$

It then suffices to show that the morphisms $y^*\mathfrak{c}^{\pm}(X) \to y^*\mathfrak{c}^{\pm}(Y)$ are both $\mathbb{S}$-Segal equivalences. We consider the case of $\mathfrak{c}^+$; the proof for $\mathfrak{c}^-$ is the same. If $\mathfrak{c} = \mathfrak{c}_{m,n}$ then the definition of $\mathfrak{c}_{m,n}$ implies that the upper horizontal map of the commutative diagram

$$\begin{array}{ccc} y^*\mathfrak{c}^+(X)_{\mathrm{Seg}} & \longrightarrow & y^*\mathfrak{c}^+(Y)_{\mathrm{Seg}} \\ \downarrow & & \downarrow \\ y^*\mathfrak{c}^+(X) & \longrightarrow & y^*\mathfrak{c}^+(Y) \end{array}$$

is given by $(\coprod_m \mathfrak{c}_{1,1}(\mathbb{1})) \amalg_{(\coprod_m \mathfrak{e})} y^*\mathfrak{c}(X) \to (\coprod_m \mathfrak{c}_{1,1}(\mathbb{1})) \amalg_{(\coprod_m \mathfrak{e})} y^*\mathfrak{c}(Y)$. As a pushout of $y^*\mathfrak{c}(X) \to y^*\mathfrak{c}(Y)$, this map is an $\mathbb{S}$-Segal equivalence. The vertical maps are

Segal equivalences by definition, hence, the bottom horizontal map is also an $\mathbb{S}$-Segal equivalence. □

**Corollary 4.2.8.** *Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category. There exists a tensor product $\otimes\colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \times \mathrm{Seg}(\mathbf{\Delta}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ and it induces*

$$\mathrm{Alg}^{\mathcal{V}}_{(-)}(-)\colon (\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}))^{\mathrm{op}} \times \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \to \mathrm{Seg}(\mathbf{\Delta})$$

*such that*

$$\mathrm{Map}_{\mathrm{Seg}(\mathbf{\Delta})}(\mathcal{C}, \mathrm{Alg}^{\mathcal{V}}_{\mathcal{Q}}(\mathcal{R})) \simeq \mathrm{Map}_{\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})}(\mathcal{Q} \otimes \mathcal{C}, \mathcal{R})$$

*and a cotensor product*

$$(-)^{(-)}\colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \times (\mathrm{Seg}(\mathbf{\Delta}))^{\mathrm{op}} \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$$

*such that*

$$\mathrm{Map}_{\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})}(\mathcal{Q}, \mathcal{R}^{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})}(\mathcal{Q} \otimes \mathcal{C}, \mathcal{R}).$$

*Moreover, both of these functors preserve limits in each variable.*

*Proof.* Since $\mathcal{V}$ is presentable there exists a small $\infty$-category $\mathcal{U}$ and a set of morphisms $\mathbb{S}$ in $\mathrm{P}(\mathcal{U})$ such that $\mathcal{V} \simeq \mathrm{P}_{\mathbb{S}}(\mathcal{U})$. The existence of the tensor product then follows from Corollary 4.2.7 and the equivalence $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ of Proposition 3.3.6. The remaining statements follow from the adjoint functor theorem. □

**Remark 4.2.9.** All proofs and statements in §4.2 hold equally well if $\mathbf{L}$ is replaced by $\mathbf{L}_{\mathrm{out}}$ or $\mathbf{L}_{0\text{-type}}$ from Definition 2.1.30. This leads to the natural question about the compatibility of the various tensor products. We will address this in Theorem 6.1.3 below, where we show that they are related via left Kan extension.

## 5. Comparison of $\mathbf{L}_{\mathrm{c}}$ and $\mathbf{G}$ presheaves

In Theorem 3.3.9 we have shown the canonical inclusion $\mathbf{L}^{\mathcal{V}}_{\mathrm{c}} \hookrightarrow \mathbf{L}^{\mathcal{V}}$ induces an equivalence $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{c}})$. In this section we introduce a functor $\overline{\tau}\colon \mathbf{L}^{\mathcal{V}}_{\mathrm{c}} \to \mathbf{G}^{\mathcal{V}}$ lying over $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ from Lemma 2.3.4. The main result is Theorem 5.1.4, which shows that $\overline{\tau}$ induces an equivalence $\overline{\tau}^*\colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{c}})$. The key step is to show that these $\infty$-categories are the $\infty$-categories of algebras of the same monad. Once this is proven an $\infty$-categorical version of the Barr–Beck Theorem gives the desired equivalence.

### 5.1. The equivalence and its consequences.

Let $\mathcal{V}$ be a symmetric monoidal $\infty$-category. Recall that the category of connected level graphs admits a vertex functor $\mathbf{L}_{\mathrm{c}} \hookrightarrow \mathbf{L} \xrightarrow{\mathtt{V}_{\mathbf{L}}} \mathrm{Fin}^{\mathrm{op}}_*$ (see Definition 2.1.26), which was used in Definition 3.2.10 to define $\mathbf{L}^{\mathrm{op},\mathcal{V}}_{\mathrm{c}}$ as the pullback

$$\begin{array}{ccc} \mathbf{L}^{\mathrm{op},\mathcal{V}}_{\mathrm{c}} & \longrightarrow & \mathcal{V}^{\mathrm{op},\otimes} \\ \downarrow & & \downarrow \\ \mathbf{L}^{\mathrm{op}}_{\mathrm{c}} & \longrightarrow & \mathbf{F}_*. \end{array}$$

The functor $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ from Lemma 2.3.4 fits into the commutative diagram

$$\begin{array}{ccccc} \mathbf{L} & \hookleftarrow & \mathbf{L}_{\mathrm{c}} & \longrightarrow & \mathbf{G} \\ & {\scriptstyle \mathtt{V}_{\mathbf{L}}} \searrow & & \swarrow {\scriptstyle \mathtt{V}_{\mathbf{G}}} & \\ & & \mathrm{Fin}_*^{\mathrm{op}} & & \end{array}$$

(where $\mathtt{V}_{\mathbf{G}}$ is from Definition 2.2.22). It follows that there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{L}^{\mathrm{op},\mathcal{V}} & \hookleftarrow & \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}} & \longrightarrow & \mathbf{G}^{\mathrm{op},\mathcal{V}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{L}^{\mathrm{op}} & \hookleftarrow & \mathbf{L}_{\mathrm{c}}^{\mathrm{op}} & \xrightarrow{\tau^{\mathrm{op}}} & \mathbf{G}^{\mathrm{op}} \end{array}$$

in which both squares are pullbacks.

**Definition 5.1.1.** For a symmetric monoidal $\infty$-category $\mathcal{V}$, write

$$\overline{\tau}^{\mathrm{op}}\colon \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}} \to \mathbf{G}^{\mathrm{op},\mathcal{V}}$$

for the arrow appearing in the preceding pullback diagram.

**Remark 5.1.2.** The functor $\tau$ restricts to an isomorphism of categories $\mathbf{L}_{\mathrm{el}} \cong \mathbf{G}_{\mathrm{el}}$ (Lemma 2.3.4), hence the restriction $\mathbf{L}_{\mathrm{c,el}}^{\mathcal{V}} \to \mathbf{G}_{\mathrm{el}}^{\mathcal{V}}$ of $\overline{\tau}$ is an equivalence.

**Lemma 5.1.3.** *For $\overline{L} \in \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}}$ and $\overline{E} \in \mathbf{L}_{\mathrm{c,el}}^{\mathrm{op},\mathcal{V}}$, we have equivalences $(\mathbf{L}_{\mathrm{c,el}}^{\mathrm{op},\mathcal{V}})_{\overline{L}/} \simeq (\mathbf{G}_{\mathrm{el}}^{\mathrm{op},\mathcal{V}})_{\overline{\tau}\overline{L}/}$ and $(\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}})_{\overline{E}/} \simeq \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}} \times_{\mathbf{G}^{\mathrm{op},\mathcal{V}}} (\mathbf{G}^{\mathrm{op},\mathcal{V}})_{\overline{\tau}\overline{E}/}$.*

*Proof.* The definitions of inert maps in $\mathbf{L}$ and $\mathbf{G}$ imply that $\tau$ induces an equivalence $\mathrm{Map}_{\mathbf{L}_{\mathrm{c,int}}}(E,L) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{G}_{\mathrm{int}}}(\tau E, \tau L)$. Since the images $L$ and $\tau L$ in $\mathrm{Fin}_*$ coincide, the constructions of $\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}}$ and $\mathbf{G}^{\mathrm{op},\mathcal{V}}$ show that the map

$$\mathrm{Map}_{\mathbf{L}_{\mathrm{c,int}}^{\mathrm{op},\mathcal{V}}}(\overline{L},\overline{E}) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{G}_{\mathrm{int}}^{\mathrm{op},\mathcal{V}}}(\overline{\tau}\overline{L},\overline{\tau}\overline{E})$$

is an equivalence. This shows that the fibres of the vertical maps of the commutative square

$$\begin{array}{ccc} (\mathbf{L}_{\mathrm{c,el}}^{\mathrm{op},\mathcal{V}})_{\overline{L}/} & \longrightarrow & (\mathbf{G}_{\mathrm{el}}^{\mathrm{op},\mathcal{V}})_{\overline{\tau}\overline{L}/} \\ \downarrow & & \downarrow \\ \mathbf{L}_{\mathrm{c,el}}^{\mathrm{op},\mathcal{V}} & \xrightarrow[\sim]{} & \mathbf{G}_{\mathrm{el}}^{\mathrm{op},\mathcal{V}} \end{array}$$

are equivalent and therefore, the upper horizontal map is an equivalence.

For the second equivalence we first observe that a map $\tau L \to \tau E$ in $\mathbf{G}$ is either active or factors through $\tau\mathfrak{e} = \downarrow$. If $\tau L \to \tau E$ is active, then it is given by contracting all inner edges, i.e. edges bounded by two vertices. It follows that in both cases $\tau L \to \tau E$ lies in the image of $\tau$ and hence $\mathrm{Map}_{\mathbf{L}_{\mathrm{c}}}(L,E) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{G}}(\tau L, \tau E)$ is an equivalence. Once again this gives this equivalence

$$\mathrm{Map}_{\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}}}(\overline{E},\overline{L}) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{G}^{\mathrm{op},\mathcal{V}}}(\overline{\tau}\overline{E},\overline{\tau}\overline{L})$$

which proves that the fibres of the vertical maps of the commutative square

$$\begin{array}{ccc} (\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}})_{\overline{E}/} & \longrightarrow & (\mathbf{G}^{\mathrm{op},\mathcal{V}})_{\overline{\tau}\overline{E}/} \\ \downarrow & & \downarrow \\ \mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{V}} & \longrightarrow & \mathbf{G}^{\mathrm{op},\mathcal{V}} \end{array}$$

are equivalent. It follows that the this square is cartesian, giving the second desired equivalence. □

Our goal is to prove:

**Theorem 5.1.4.** *Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category. The functor $\overline{\tau}^* \colon \mathrm{P}(\mathbf{G}^{\mathcal{U}}) \to \mathrm{P}(\mathbf{L}^{\mathcal{U}})$ given by composition with $\overline{\tau}^{\mathrm{op}}$ restricts to an equivalence*

$$\mathrm{Seg}(\mathbf{G}^{\mathcal{U}}) \to \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}).$$

Before we give the proof, we want to derive some consequences from this theorem.

**Corollary 5.1.5.** *Given a small symmetric monoidal $\infty$-category $\mathcal{U}$ and a small set $\mathbb{S}$ of morphisms in $\mathrm{P}(\mathcal{U})$ which is compatible with the symmetric monoidal structure. Then the following hold:*

*(i) The functor $\overline{\tau} \colon \mathbf{L}_{\mathrm{c}}^{\mathcal{U}} \to \mathbf{G}^{\mathcal{U}}$ induces an equivalence*

$$\overline{\tau}^* \colon \mathrm{Seg}_{\mathbb{S}}(\mathbf{G}^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}_{\mathbb{S}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}).$$

*(ii) The functor $\overline{\tau} \colon \mathbf{L}_{\mathrm{c}}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})} \to \mathbf{G}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}$ induces an equivalence*

$$\overline{\tau}^* \colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}).$$

*Proof.* By Definition 3.3.3, $\mathrm{Seg}_{\mathbb{S}}(\mathbf{G}^{\mathcal{U}})$ and $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$ are the full subcategories of the respective $\infty$-categories $\mathrm{Seg}(\mathbf{G}^{\mathcal{U}})$ and $\mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$, spanned by objects which are local with respect to the same set of morphisms under the equivalence $\mathbf{L}_{\mathrm{c,el}}^{\mathcal{U}} \xrightarrow{\sim} \mathbf{G}_{\mathrm{el}}^{\mathcal{U}}$ of Remark 5.1.2. Hence, the equivalence $\overline{\tau}^* \colon \mathrm{Seg}(\mathbf{G}^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$ of Theorem 5.1.4 restricts to an equivalence $\overline{\tau}^* \colon \mathrm{Seg}_{\mathbb{S}}(\mathbf{G}^{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}})$ of (i).

The equivalence (ii) follows by combining Proposition 3.3.6 with (i). □

**Corollary 5.1.6.** *If $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, then the following $\infty$-categories are equivalent*

$$\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}).$$

*Proof.* The first equivalence can be identified with $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})}) \xrightarrow{\sim} \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathrm{P}_{\mathbb{S}}(\mathcal{U})})$ of Corollary 5.1.5(ii) for some small symmetric monoidal $\infty$-category $\mathcal{U}$ and some set $\mathbb{S}$ and the second equivalence is given by Theorem 3.3.9. □

**Proposition 5.1.7.** *For every presentably symmetric monoidal $\infty$-category $\mathcal{V}$, the $\infty$-category $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ has a $\mathrm{Seg}(\mathbf{\Delta})$-module structure where the tensor product preserves colimits in each variable.*

*Proof.* The previous corollary shows that $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ which by Proposition 3.3.6 can be identified with $\mathrm{Seg}_{\mathbb{S}}(\mathbf{L}^{\mathcal{U}})$. This $\infty$-category has a $\mathrm{Seg}(\mathbf{\Delta})$-module structure by Corollary 4.2.7. □

**Remark 5.1.8.** The functor $\tau\colon \mathbf{L}_{\mathrm{c}} \to \mathbf{G}$ restricts to functors $\mathbf{L}_{\mathrm{out,c}} \to \mathbf{G}_{\mathrm{out}}$ and $\mathbf{L}_{\mathrm{sc}} \to \mathbf{G}_{\mathrm{sc}}$ (from Definition 2.1.30 and Definition 2.2.14), and the proofs of all of the above statements remain valid when applied to these restrictions. For instance, the analogue of Theorem 5.1.4 tells us that

$$\mathrm{Seg}(\mathbf{G}^{\mathcal{U}}_{\mathrm{out}}) \to \mathrm{Seg}(\mathbf{L}^{\mathcal{U}}_{\mathrm{out,c}}) \quad \& \quad \mathrm{Seg}(\mathbf{G}^{\mathcal{U}}_{\mathrm{sc}}) \to \mathrm{Seg}(\mathbf{L}^{\mathcal{U}}_{\mathrm{sc}})$$

are equivalences. The proof of Corollary 5.1.6 gives equivalences

$$\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}_{\mathrm{sc}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{sc}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{0\text{-type}})$$
$$\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}_{\mathrm{out}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{out,c}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}_{\mathrm{out}}).$$

Finally, the proof of Proposition 5.1.7 along with Remark 4.2.9 tells us that both $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}_{\mathrm{sc}})$ and $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}_{\mathrm{out}})$ have appropriate $\mathrm{Seg}(\mathbf{\Delta})$-module structures.

5.2. **Towards the proof of Theorem 5.1.4.** The proof of Theorem 5.1.4 relies on certain intricate filtrations. We build these decompositions in this subsection, and conclude with a proof of the theorem.

**Definition 5.2.1.** Suppose that $G \in \mathbf{G}$, $I$ is a connected, height $n$ level graph, and $\varphi\colon I \to \tau^* G$ is a morphism in $\mathrm{P}(\mathbf{L}_{\mathrm{c}})$.

- We say that $\varphi$ is *non-degenerate* if, whenever $I \to J \to \tau^* G$ is a factorization of $\varphi$ with $I \to J$ in $\mathbf{L}$ lying over a surjective morphism in $\mathbf{\Delta}$, we have that $I \to J$ is an isomorphism.
- We let $\varphi$ also denote the adjoint map $\tau I \to G$ in $\mathbf{G}$ and we say that $\varphi$ is *admissible* if the following conditions hold:
  (1) If $v \in I_{n-1,n}$ is a bottom vertex, then $\varphi(v)$ is either an edge or a corolla.
  (2) There is a unique bottom vertex $v \in I_{n-1,n}$ so that $\varphi(v)$ is a corolla.

**Proposition 5.2.2.** *Let $I, J \in \mathbf{L}_{\mathrm{c}}$ be of height $m$ and $n$, respectively, and $G \in \mathbf{G}$ a graph with at least one vertex. Suppose*

$$\begin{array}{ccc} I & \longrightarrow & J \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \tau^* G & \end{array} \qquad \begin{array}{ccc} \tau I & \xrightarrow{h} & \tau J \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & G & \end{array}$$

*are adjoint commutative diagrams in $\mathrm{P}(\mathbf{L}_{\mathrm{c}})$ and $\mathbf{G}$, where $I \to J$ lies over $\alpha\colon [m] \to [n]$. If $g$ is active, $\varphi$ is admissible, and $\psi$ is non-degenerate and not admissible, then $n > m$.*

*Proof.* Since $\psi$ is non-degenerate, the map $\alpha$ is injective, hence $n \geq m$. Since $g$ is active and $G$ has at least one vertex, we have $m > 0$.

We first show that $\alpha(m) = n$. Let $v \in J_{n-1,n}$ be the unique vertex so that $f(C_v)$ is a corolla, and let $w \in \mathtt{V}(G)$ be the vertex in this corolla. The diagram

$$\begin{array}{ccccc} \mathtt{V}(\tau I)_+ & \xleftarrow{\mathtt{V}(h)} & \mathtt{V}(\tau J)_+ & & \ni v \\ & {\scriptstyle \mathtt{V}(g)}\nwarrow \quad \nearrow{\scriptstyle \mathtt{V}(f)} & & \nearrow & \\ & \mathtt{V}(G)_+ & & \ni w & \end{array}$$

in $\mathrm{Fin}_*$ commutes and $\mathtt{V}(g)$ is active (Example 2.1.20), so $\mathtt{V}(h)(v)$ is not the base point. In other words, since $g$ is active, there exists a vertex $u \in \mathtt{V}(\tau I)$ with $w \in g(C_u)$. Since $u \in I_{k-1,k}$ and $v \in J_{n-1,n}$, then $\alpha(k-1) \leq n-1 < n \leq \alpha(k)$, which implies that $n = \alpha(k)$. This implies that $\alpha(m) = n$.

By injectivity, we know that $\alpha(m-1) < \alpha(m) = n$; now suppose that $\alpha(m-1) = n-1$. Then $h(v)$ is a corolla for each $v \in I_{m-1,m} \subset \mathtt{V}(\tau I)$. In particular, if (1) fails for $\psi$, then there exists a $v \in I_{m-1,m}$ with $g(v)$ contains more than one vertex. But this implies that $\varphi$ also fails (1) for the vertex $h(v) \in J_{n-1,n}$. On the other hand, if (1) holds for $\psi$, then non-degeneracy and the fact that (2) fails for $\psi$ implies that there are distinct vertices $v, v' \in I_{m-1,m}$ which are both sent to corollas by $\psi$. But this can't happen, otherwise $\varphi$ would fail (2) for $h(v) \neq h(v') \in J_{n-1,n}$.

We conclude that $\alpha(m-1)$ cannot be $n-1$, hence $n-1$ is not in the image of the injective map $\alpha$ and the conclusion follows. □

**Lemma 5.2.3.** *Suppose that $\varphi\colon I \to \tau^* G$ is non-degenerate and admissible. If $I \in \mathbf{L}_{\mathrm{c}}$ is of height $n > 1$, then the composite $d_{n-1}I \to I \to \tau^* G$ is not admissible.*

*Proof.* Write $\varphi'\colon d_{n-1}I \to I \to \tau^* G$ for the induced map. As $\varphi$ is admissible, there is a unique bottom vertex $v \in I_{n-1,n}$ with $\varphi(v)$ a corolla. Consider the commutative square

$$\begin{array}{ccc} [1] & \xrightarrow{d^1} & [2] \\ \downarrow{\scriptstyle \beta} & \searrow^{\gamma} & \downarrow{\scriptstyle \alpha} \\ [n-1] & \xrightarrow[d^{n-1}]{} & [n] \end{array}$$

where $\alpha(t) = t + n - 2$ and $\beta(t) = t + n - 2$. We have that $\alpha^* I \to I$ is a (possibly disconnected) height two subgraph, whose connected components are in bijection with the vertices of $\gamma^* I$. Since $\varphi$ is non-degenerate, there exists at least one vertex $w \in I_{n-2,n-1}$ so that $\varphi(w)$ is not an edge.

If $w$ and $v$ are in the same component of $\alpha^* I$, then there is a bottom vertex $x$ of $d_{n-1}I$ so that both $w$ and $v$ are in the image of $x$. In this case, $\varphi'$ does not satisfy (1) since $\varphi'(x)$ contains $\varphi(w)$ and $\varphi(v)$, hence contains more than one vertex.

If $w$ and $v$ are in different components of $\alpha^* I$, then there are distinct bottom vertices $y$ and $x$ of $d_{n-1}I$ with $w \mapsto y$ and $v \mapsto x$ under $\mathtt{V}(I)_+ \to \mathtt{V}(d_{n-1}I)_+$. We have $\varphi(w) \subseteq \varphi'(y)$ and $\varphi(v) \subseteq \varphi'(x)$, so $\varphi'(y)$ and $\varphi'(x)$ are not edges. If $\varphi'(y)$ and $\varphi'(x)$ are both corollas, then $\varphi'$ does not satisfy (2), while if one of them is not a corolla, then $\varphi'$ does not satisfy (1). □

**Lemma 5.2.4.** *Suppose that $\varphi\colon I \to \tau^* G$ is admissible and $I \in \mathbf{L}_{\mathrm{c}}$ is of height $n$. If $0 < k < n-1$, then the composite $d_k I \to I \to \tau^* G$ is admissible.*

*Proof.* The map $d_k I \to I$ in $\mathbf{L}$ identifies the sets of bottom vertices. □

**Lemma 5.2.5.** *Suppose that $\varphi\colon I \to \tau^* G$ is non-degenerate and $I \in \mathbf{L}_{\mathrm{c}}$ is of height $n$. If $0 < k < n$, then the composite $d_k I \to I \to \tau^* G$ is non-degenerate.*

*Proof.* Write $\psi'$ for this composite. Since $d_k I \to I$ restricts to an identity for vertices $v \in d_k I$ which are not in $(d_k I)_{k-1,k}$, we only need to exclude the possibility that $\psi'(v)$ is an edge for each vertex $v$ in $(d_k I)_{k-1,k} \cong I_{k-1,k+1} = I_{k-1,k} \amalg_{I_{k,k}} I_{k,k+1}$. If this were the case, then every vertex $w \in I_{k-1,k} \amalg I_{k,k+1}$ maps to some $v \in (d_k I)_{k-1,k}$ under this isomorphism and $\psi(w) \subset \psi'(v)$. Thus each vertex of $I$ in level $k$ or $k+1$ must go to an edge, a contradiction. □

We now endeavor to show that there is a sufficient supply of non-degenerate, admissible maps $I \to \tau^* G$. The next construction, which proves the existence of certain factorizations, is essential for the proof of Proposition 5.2.8. In the

construction, we say that a vertex $v$ of a graph $G$ is a *bottom vertex* if all of its outgoing edges are also outgoing edges for the graph, $\mathtt{out}(v) \subseteq \mathtt{out}(G)$. Given a bottom vertex $v$, we can form a (possibly disconnected) graph $G \setminus v$ with $\mathtt{V}(G \setminus v) = \mathtt{V}(G) \setminus \{v\}$, $\mathtt{E}(G \setminus v) = \mathtt{E}(G) \setminus \mathtt{out}(v)$ with the rest of the structure evident. We have $C_v \amalg_{\mathtt{in}(v)} G \setminus v \cong C_v \cup G \setminus v = G$, where the second expression is a union as subobjects of $G$. If $H$ is a connected component of $G \setminus v$, then $H \in \mathtt{Sb}(G)$ is a structured subgraph.

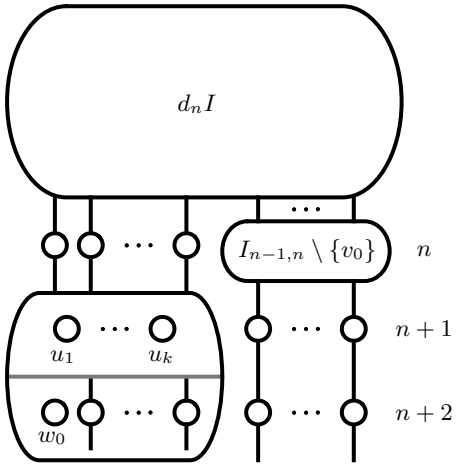



FIGURE 8. Schematic of the level graph $J$

**Construction 5.2.6.** Let $I \in \mathbf{L}_{\mathrm{c}}$ be of height $n > 0$, $G \in \mathbf{G}$, and suppose that $\psi\colon \tau I \to G$ is an active map whose adjoint $I \to \tau^* G$ is non-degenerate. We assume that the adjoint of $\psi$ is not admissible, and construct a factorization $\tau I \to \tau J \to G$ with $I \to J$ in $\mathbf{L}$ so that $J \to \tau^* G$ is admissible and non-degenerate. The level graph $J$ will have height $n+1$ or $n+2$, depending on $\psi$. A schematic picture is given in Figure 8, where the $u_t$ and $w_0$ will be explained below. The map $\tau J \to G$ will send all of the $1,1$ vertices in the picture to edges, and will behave as $\psi$ on $d_n I$ and on $I_{n-1,n} \setminus \{w_0\}$. The schematic tells us that we should have $J_{k,k} = I_{k,k}$ and $J_{k-1,k} = I_{k-1,k}$ for $k \leq n-1$, so that the top of $J$ coincides with the (possibly disconnected) level graph $d_n I$ of height $n-1$.

Since the adjoint of $\psi$ is non-degenerate, there exists a vertex $v_0$ in level $n$ of $I$ (that is, $v_0 \in I_{n-1,n}$) with $\psi(v_0) \in \mathtt{Sb}(G)$ not an edge. Let $w_0$ be a bottom vertex of the graph $\psi(v_0)$, and let $H_1, \ldots, H_k \in \mathtt{Sb}(\psi(v_0)) \subseteq \mathtt{Sb}(G)$ be the connected components of the (possibly disconnected) graph $\psi(v_0) \setminus w_0$. For each $t = 1, \ldots, k$, the set $\mathtt{in}(w_0) \cap \mathtt{out}(H_t)$ is nonempty, otherwise $\psi(v_0)$ would be disconnected. Notice that

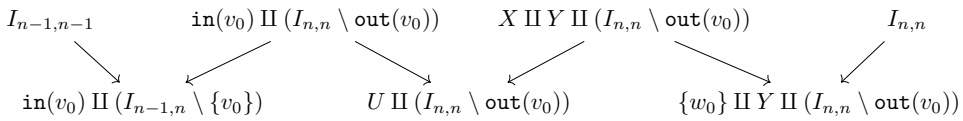
$$\mathtt{in}(\psi(v_0) \setminus w_0) = \mathtt{in}(H_1 \amalg \cdots \amalg H_k) \cong \mathtt{in}(v_0) \subseteq I_{n-1,n-1}, \tag{10}$$

and let

$$\mathtt{out}(\psi(v_0) \setminus w_0) \cong X \amalg Y \tag{11}$$

where $X = \mathtt{in}(w_0)$ and $Y \subseteq \mathtt{out}(v_0) \subseteq I_{n,n}$ is the subset so that $\psi(Y) \amalg \mathtt{out}(w_0) = \mathtt{out}(\psi(v_0)) \cong \mathtt{out}(v_0)$. Let $U = \{u_1, \ldots, u_k\}$ be a set of size $k$ (equal to the number

of components of $\psi(v_0) \setminus w_0$). Define a level graph $K \colon \mathscr{L}_0^3 \to \mathrm{Set}$ of height 3 which is of the form




We must of course define these maps. All of these functions are defined to be the identity when possible (for example, the two maps on the left are the identity on $\mathtt{in}(v_0) \subseteq I_{n-1,n-1}$), and otherwise the two maps on the left are induced from the maps in $I$. The third function involves a map $\mathtt{in}(v_0) \to U$, which sends $a \in \mathtt{in}(v_0)$ to $u_t$ if $a$ corresponds to an input of $H_t$ under the isomorphism (10). Likewise, the map $X \amalg Y \to U$ in the fourth map comes from sending $a$ to $u_t$ if $a$ corresponds to an output of $H_t$ under the isomorphism (11). The fifth map sends $X$ to $w_0$ and the final map sends those elements of $I_{n,n}$ which map to $\mathtt{out}(w_0)$ under $\psi$ to $w_0$.

Let us now define $f \colon \tau K \to G$ on vertices. At level one, $f$ is given by

$$\mathtt{in}(v_0) \amalg (I_{n-1,n} \setminus \{v_0\}) \hookrightarrow \mathtt{E}(\tau I) \amalg \mathtt{V}(\tau I) \xrightarrow{\psi} \mathtt{Sb}(G).$$

The binary vertices labeled by $I_{n,n} \setminus \mathtt{out}(v_0)$ in levels two and three are sent to the appropriate elements of $\mathtt{E}(G) \subseteq \mathtt{Sb}(G)$ using $\psi$. The remainder of level two is the set $U$, and we declare that $f(u_t) = H_t$. At level three, we define $f(w_0) = C_{w_0}$ and $f(y) = \psi(y) \in \mathtt{E}(G)$ for $y \in Y$.

As constructed, it is possible that the adjoint of $f \colon \tau K \to G$ is degenerate. This will happen either when $\psi(v_0) = C_{w_0}$, in which case $f$ sends every vertex in level two to edges, or when there is a *unique* vertex $v_0 \in I_{n-1,n}$ with $\psi(v_0)$ not an edge, in which case $f$ sends all vertices in level one to edges. Notice that we cannot have both of these situations occur simultaneously, otherwise the adjoint of $\psi$ would have been admissible already. We thus let $f' \colon \tau K' \to G$ be a non-degenerate factorization, where $K'$ is of height two or three and $K \to K'$ lives over $s^0 \colon [3] \to [2]$, $s^1 \colon [3] \to [2]$, or $\mathrm{id} \colon [3] \to [3]$. We do not say that the adjoint of $f'$ is admissible and non-degenerate, but only because those terms were only defined when the domain is a *connected* level graph. Let

$$J \coloneqq d_n I \coprod_{I_{n-1,n-1}} K'$$

be the graph of height $n+1$ or $n+2$, and $\psi' \colon \tau J \to G$ be induced by $d_n \psi$ and $f'$. Note that $J$ is connected, $\psi'$ is active, and the adjoint of $\psi'$ is admissible and non-degenerate. Further, we either have $I = d_n J$ or $I = d_n d_{n+1} J$, and $\tau I \to \tau J \to G$ is just $\psi$.

**Definition 5.2.7** (External boundary)**.**

- Suppose that $G \in \mathbf{G}$ is a graph, and let $\mathrm{Sub}(G)$ be the full subcategory of $(\mathbf{G}_{\mathrm{int}})_{/G}$ spanned by the non-invertible morphisms. We let $\partial_{\mathrm{ext}} G$ be the colimit of the composition

  $$\mathrm{Sub}(G) \to \mathbf{G} \to \mathrm{P}(\mathbf{G}),$$

  which we call the *external boundary* of $G$.

- There is an analogous functor $\mathrm{Sub}(G) \to \mathrm{P}(\mathbf{G})$ which takes $H$ to $H_{\mathrm{Seg}}$. We write $(\partial_{\mathrm{ext}} G)_{\mathrm{Seg}}$ for the colimit of this functor, which is the same as

$$\operatorname*{colim}_{H \in \mathrm{Sub}(G)} H_{\mathrm{Seg}} \simeq \operatorname*{colim}_{H \in \mathrm{Sub}(G)} (H \times_G G_{\mathrm{Seg}}) \simeq (\partial_{\mathrm{ext}} G) \times_G G_{\mathrm{Seg}}.$$

- Analogously, suppose we are given an object $\overline{G} \in \mathbf{G}^{\mathcal{V}}$. We write $\mathrm{Sub}(\overline{G})$ for the full subcategory of $(\mathbf{G}^{\mathcal{V}}_{\mathrm{int}})_{/\overline{G}}$ spanned by non-equivalence morphisms and we define $\partial_{\mathrm{ext}} \overline{G}$ to be the colimit of the composition $\mathrm{Sub}(\overline{G}) \to \mathbf{G}^{\mathcal{V}} \to \mathrm{P}(\mathbf{G}^{\mathcal{V}})$. We call $\partial_{\mathrm{ext}} \overline{G}$ the *external boundary* of $\overline{G}$.

**Proposition 5.2.8.** *For every object $G$ in $\mathbf{G}$ with at least two vertices, the map $\tau^*(\partial_{\mathrm{ext}} G) \to \tau^* G$ is an inner anodyne map in $\mathrm{P}(\mathbf{L}_{\mathrm{c}})$.*

*Proof.* The presheaf $\tau^*(\partial_{\mathrm{ext}} G)$ is a subpresheaf of $\tau^* G$, which is a presheaf of sets. For $n \geq 0$, we define the presheaf $F_n$ by declaring that $F_n(I)$ is the union of $\tau^*(\partial_{\mathrm{ext}} G)(I)$ with the set of maps $I \to \tau^* G$ which factor through an admissible and non-degenerate morphism $J \to \tau^* G$ with $J$ of height $\leq n$ and whose adjoint $\tau I \to G$ is active. Note that then $\tau J \to G$ has to be active too. To see this let $\tau J \to K \to G$ be the active-inert factorization of $\tau J \to G$ and let $\tau I \to L \to K$ the active-inert factorization of $\tau I \to \tau J \to K$. Since $\tau I \to L$ is active, $L \to K \to G$ is inert and their composition equals the active morphism $\tau I \to G$, the uniqueness of the factorization implies that $L \to G$ is an equivalence. Now, the equivalence $\mathbf{G} \simeq \mathbf{\Gamma}$ of Theorem A.1 shows that $\mathbf{G}$ is a generalized Reedy category by [HRY15, Theorem 6.70]. According to [HRY15, Lemma 6.65] inert morphisms such as $G \simeq L \to K$ and $K \to G$ preserve or raise degrees. Since their composite is an equivalence, it preserves the degree, hence, $G \to K$ and $K \to G$ also preserve the degree and are isomorphisms by the definition of generalized Reedy categories [BM11, Definition 1.1]. In particular, we see that $\tau J \to G$ is active if $\tau I \to G$ is.

Each morphism $I \to \tau^* G$ whose adjoint is not active is automatically contained in $\tau^*(\partial_{\mathrm{ext}} G)$. Thus each map $\mathfrak{e} \to \tau^* G$ (where $\mathfrak{e}$ is the unique height zero object of $\mathbf{L}_{\mathrm{c}}$) is in $\tau^*(\partial_{\mathrm{ext}} G)$. On the other hand, $\mathfrak{e} \to G$ is never active when $G$ has at least one vertex, hence $F_0 = \tau^*(\partial_{\mathrm{ext}} G)$. By Construction 5.2.6, every active map $\tau J \to G$ with $J \in \mathbf{L}_{\mathrm{c}}$ necessarily factors through a non-degenerate, admissible map, hence the filtration

$$F_0 = \tau^*(\partial_{\mathrm{ext}} G) \subseteq F_1 \subseteq \cdots \subseteq \operatorname*{colim}_{n \to \infty} F_n = \tau^* G$$

is exhaustive.

It suffices to show that each inclusion $F_{n-1} \hookrightarrow F_n$ is inner anodyne. Let $S_n$ denote the set of isomorphism classes of non-degenerate, admissible, active maps $\varphi \colon \tau(I) \to G$ where $I \in \mathbf{L}_{\mathrm{c}}$ is has height $n$. Since $G$ has at least two vertices, there is no map $\varphi \colon \mathfrak{e} \to \tau^* G$ which is admissible and has active adjoint, so $S_1 = \varnothing$. It follows that $F_0 = F_1$. In general, for $\varphi \in F_n$ $(n \geq 2)$ the following hold:

- For $i \in \{0, n\}$, the assumption that $\varphi$ is non-degenerate and the definition of $\tau^*(\partial_{\mathrm{ext}} G)$ imply that the faces $d_i I \to I \to \tau^* G$ factor through $\tau^*(\partial_{\mathrm{ext}} G)$, and thus through $F_{n-1}$.
- For $0 < i < n-1$, admissibility and non-degeneracy of $\varphi$ implies the same about the faces $d_i I \to I \to \tau^* G$ (by Lemma 5.2.4 and Lemma 5.2.5) which thus factor through $F_{n-1}$.

- The face $d_{n-1}I \to I \to \tau^* G$ is *not* admissible by Lemma 5.2.3. Further, it is non-degenerate (by Lemma 5.2.5) and its adjoint is active, so Proposition 5.2.2 applies that it can only factor through an admissible map whose domain has at least height $n$. In particular, $d_{n-1}I \to I \to \tau^* G$ cannot lie in $F_{n-1}$.

This shows that there exists a pushout diagram

$$\begin{array}{ccc} \coprod_{S_n} \Lambda^n_{n-1} I & \longrightarrow & F_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{S_n} I & \longrightarrow & F_n. \end{array}$$

Since the left vertical morphism is inner anodyne by definition, so is the right vertical map. □

**Proposition 5.2.9.** *Let $\mathcal{U}$ be a symmetric monoidal $\infty$-category. The functor $\overline{\tau}^* \colon \mathrm{P}(\mathbf{G}^{\mathcal{U}}) \to \mathrm{P}(\mathbf{L}^{\mathcal{U}}_{\mathrm{c}})$ preserves Segal equivalences.*

*Proof.* It suffices to show that $\overline{\tau}^*$ preserves Segal core inclusions, which generate Segal equivalences. In other words, for every $\overline{G} \in \mathbf{G}^{\mathcal{U}}$, the map $\overline{\tau}^* \overline{G}_{\mathrm{Seg}} \to \overline{\tau}^* \overline{G}$ needs to be a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}}_{\mathrm{c}})$. We prove this by inducting on the number of vertices of $\overline{G}$. Since the statement is vacuous if $\overline{G}$ has zero or one vertices, we assume that $\overline{G}$ has at least two vertices and lies over $G \in \mathbf{G}$. Let $\pi^* \colon \mathrm{P}(\mathbf{G}) \to \mathrm{P}(\mathbf{G}^{\mathcal{U}})$ be given by composition with the opposite of the natural projection $\pi \colon \mathbf{G}^{\mathcal{U}} \to \mathbf{G}$. We then define $\partial_{\mathrm{ext}} \overline{G}$ to be the presheaf given by the pullback square

$$\begin{array}{ccc} \partial_{\mathrm{ext}} \overline{G} & \longrightarrow & \overline{G} \\ \downarrow & & \downarrow \\ \pi^* \partial_{\mathrm{ext}} G & \longrightarrow & \pi^* G. \end{array}$$

We let $(\partial_{\mathrm{ext}} \overline{G})_{\mathrm{Seg}}$ be defined analogously (see Definition 5.2.7). By applying the idea of the proof of [CHH18, Lemma 5.8] to our case, we see that the canonical map $(\partial_{\mathrm{ext}} G)_{\mathrm{Seg}} \to G_{\mathrm{Seg}}$ is an equivalence.

Since $\mathrm{P}(\mathbf{G}^{\mathcal{U}})$ is an $\infty$-topos, pullbacks commute with colimits, and the map $(\partial_{\mathrm{ext}} \overline{G})_{\mathrm{Seg}} \xrightarrow{\sim} \overline{G}_{\mathrm{Seg}}$, given by the pullback of $\pi^*(\partial_{\mathrm{ext}} G)_{\mathrm{Seg}} \xrightarrow{\sim} \pi^* G_{\mathrm{Seg}}$, is an equivalence. We now have a commutative square

$$\begin{array}{ccc} (\partial_{\mathrm{ext}} \overline{G})_{\mathrm{Seg}} & \longrightarrow & \partial_{\mathrm{ext}} \overline{G} \\ \wr \downarrow & & \downarrow \\ \overline{G}_{\mathrm{Seg}} & \longrightarrow & \overline{G}. \end{array}$$

The upper horizontal morphism is a colimit indexed by $\mathrm{Sub}(G)$ of generating Segal equivalences for graphs with fewer vertices than $\overline{G}$, and is therefore mapped to a Segal equivalence in $\mathrm{P}(\mathbf{L}^{\mathcal{U}}_{\mathrm{c}})$ by the inductive hypothesis. By the 2-of-3 property the claim follows if $\overline{\tau}^* \partial_{\mathrm{ext}} \overline{G} \to \overline{\tau}^* \overline{G}$ is a Segal equivalence. Let $p \colon \mathbf{L}^{\mathcal{U}}_{\mathrm{c}} \to \mathbf{L}_{\mathrm{c}}$ be the

projection. Since $\tau p \simeq \pi\overline{\tau}$ and the right adjoint $\overline{\tau}^*$ preserves pullbacks we have a cartesian square

$$\begin{array}{ccc} \overline{\tau}^*\partial_{\mathrm{ext}}\overline{G} & \longrightarrow & \overline{\tau}^*\overline{G} \\ \downarrow & & \downarrow \\ p^*\tau^*\partial_{\mathrm{ext}}G & \longrightarrow & p^*\tau^*G. \end{array}$$

The previous proposition shows that $\tau^*\partial_{\mathrm{ext}}G \to \tau^*G$ is an inner anodyne map in $\mathrm{P}(\mathbf{L})$. The right-hand map $\overline{\tau}^*\overline{G} \to p^*\tau^*G$ is equivalent to the unit $\overline{\tau}^*\overline{G} \to p^*p_!\overline{\tau}^*\overline{G}$, so by Lemma 4.1.4 is simple. By Corollary 4.1.7, the top map is inner anodyne, hence a Segal equivalence. □

**Proposition 5.2.10.** *Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category. The adjunction $\overline{\tau}^*\colon \mathrm{Fun}(\mathbf{G}^{\mathrm{op},\mathcal{U}},\mathcal{S}) \rightleftarrows \mathrm{Fun}(\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{U}},\mathcal{S}) :\overline{\tau}_*$ restricts to an adjunction*

$$\overline{\tau}^*\colon \mathrm{Seg}(\mathbf{G}^{\mathcal{U}}) \rightleftarrows \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}) :\overline{\tau}_*.$$

*Proof.* The claim is saying that the functors $\overline{\tau}_*$ and $\overline{\tau}^*$ preserve local objects which is equivalent to requiring that the corresponding left adjoints $\overline{\tau}^*$ and $\overline{\tau}_!$ preserve Segal equivalences. The functor $\overline{\tau}^*$ preserves Segal equivalences by the previous proposition and it remains to prove that the same is true for $\overline{\tau}_!$. Since the left adjoint $\overline{\tau}_!$ preserves colimits and $\overline{\tau}_!\overline{K}$ is represented by $\overline{\tau}\overline{K}$ we have

$$\overline{\tau}_!(\overline{L}_{\mathrm{Seg}}) \simeq \operatorname*{colim}_{\overline{E}\in(\mathbf{L}_{\mathrm{el}}^{\mathrm{op},\mathcal{U}})_{\overline{L}/}} \overline{\tau}\overline{E}.$$

The equivalence $\mathbf{L}_{\mathrm{c,el}}^{\mathcal{U}} \to \mathbf{G}_{\mathrm{el}}^{\mathcal{U}}$ and $(\mathbf{L}_{\mathrm{c,el}}^{\mathrm{op},\mathcal{U}})_{\overline{L}/} \simeq (\mathbf{G}_{\mathrm{el}}^{\mathrm{op},\mathcal{U}})_{\overline{\tau}\overline{L}/}$ of Remark 5.1.2 and Lemma 5.1.3 implies that

$$\overline{\tau}_!(\overline{L}_{\mathrm{Seg}}) \simeq \operatorname*{colim}_{\overline{G}\in(\mathbf{G}_{\mathrm{el}}^{\mathrm{op},\mathcal{U}})_{\overline{\tau}\overline{L}/}} \overline{G}.$$

Hence, $\overline{\tau}_!$ takes a generating Segal equivalence $\overline{L}_{\mathrm{Seg}} \to \overline{L}$ to a generating Segal equivalence $(\overline{\tau}\overline{L})_{\mathrm{Seg}} \to \overline{\tau}\overline{L}$. □

*Proof of Theorem 5.1.4.* We have a commutative square

$$\begin{array}{ccc} \mathrm{Seg}(\mathbf{G}^{\mathcal{U}}) & \xrightarrow{\overline{\tau}^*} & \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}}) \\ \downarrow{\scriptstyle \overline{\imath}_{\mathbf{G}}^*} & & \downarrow{\scriptstyle \overline{\imath}_{\mathbf{L}}^*} \\ \mathrm{Seg}(\mathbf{G}_{\mathrm{int}}^{\mathcal{U}}) & \xrightarrow{\overline{\tau}_{\mathrm{int}}^*} & \mathrm{Seg}(\mathbf{L}_{\mathrm{c,int}}^{\mathcal{U}}) \end{array}$$

of right adjoints (where $\overline{\tau}_{\mathrm{int}}\colon \mathbf{L}_{\mathrm{c,int}}^{\mathcal{U}} \to \mathbf{G}_{\mathrm{int}}^{\mathcal{U}}$ is the restriction of $\overline{\tau}$), where the vertical morphisms induced by precompositions of the canonical inclusions $\overline{\imath}_{\mathbf{G}}\colon \mathbf{G}_{\mathrm{int}}^{\mathcal{U}} \to \mathbf{G}^{\mathcal{U}}$, $\overline{\imath}_{\mathbf{L}}\colon \mathbf{L}_{\mathrm{c,int}}^{\mathcal{U}} \to \mathbf{L}_{\mathrm{c}}^{\mathcal{U}}$ are monadic according to [CH21, Corollary 8.2]. Let $F_{\mathbf{G}}$ and $F_{\mathbf{L}}$ denote the corresponding left adjoints. By the discussion in the paragraph following Definition 3.1.3, the bottom horizontal map $\overline{\tau}_{\mathrm{int}}^*$ can be identified with $\mathrm{P}(\mathbf{G}_{\mathrm{el}}^{\mathcal{U}}) \to \mathrm{P}(\mathbf{L}_{\mathrm{c,el}}^{\mathcal{U}})$ which is an equivalence by Remark 5.1.2. Then [Lur, Corollary 4.7.3.16] implies that $\overline{\tau}^*$ is an equivalence if the canonical natural transformation $F_{\mathbf{L}} \to \overline{\tau}^*F_{\mathbf{G}}(\overline{\tau}_{\mathrm{int}}^*)^{-1}$ is an equivalence, which is the same as requiring that the corresponding transformation of right adjoints $\overline{\tau}_{\mathrm{int}}^*\overline{\imath}_{\mathbf{G}}^*\overline{\tau}_* \to \overline{\imath}_{\mathbf{L}}^*$ is an equivalence. In

other words it suffices to show that for every $F \in \mathrm{Seg}(\mathbf{L}_{\mathrm{c}}^{\mathcal{U}})$ and $\overline{E} \in \mathbf{L}_{\mathrm{c,int}}^{\mathcal{U},\mathrm{el}}$, the canonical map $\overline{\tau}_{\mathrm{int}}^{*}\overline{\imath}_{\mathbf{G}}^{*}\overline{\tau}_{*}F(\overline{E}) \simeq \overline{\imath}_{\mathbf{L}}^{*}\overline{\tau}^{*}\overline{\tau}_{*}F(\overline{E}) \to \overline{\imath}_{\mathbf{L}}^{*}F(\overline{E})$ is an equivalence.

The description of right Kan extension allows us to identify the domain of this map with

$$(\overline{\tau}_{*}F)(\overline{\tau}\overline{E}) \simeq \lim F(\overline{L}) \tag{12}$$

where the limit is over $\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{U}} \times_{\mathbf{G}^{\mathrm{op},\mathcal{U}}} (\mathbf{G}^{\mathrm{op},\mathcal{U}})_{\overline{\tau}\overline{E}/}$. The equivalence

$$\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{U}} \times_{\mathbf{G}^{\mathrm{op},\mathcal{U}}} (\mathbf{G}^{\mathrm{op},\mathcal{U}})_{\overline{\tau}\overline{E}/} \simeq (\mathbf{L}_{\mathrm{c}}^{\mathrm{op},\mathcal{U}})_{\overline{E}/}$$

of Lemma 5.1.3 implies that this $\infty$-category has an initial object $\mathrm{id}_{\overline{E}}$, hence the limit in (12) is equivalent to $F(\overline{E})$. □

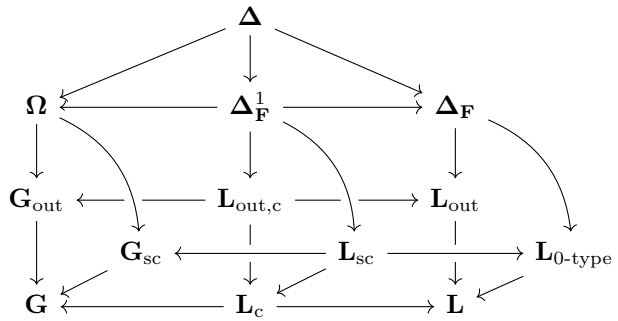
## 6. Completeness and enriched $\infty$-properads

Two ordinary properads are equivalent if there is a fully faithful and essentially surjective functor between them. These two notions have natural generalizations in the $\infty$-categorical setting, and the $\infty$-category of $\mathcal{V}$-enriched $\infty$-properads should be given by a localization of the algebraic model $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ for $\mathcal{V}$-enriched $\infty$-properads with respect to fully faithful and essentially surjective functors. In [Rez01], Rezk introduced the notion of completeness and proved that complete Segal spaces model $\infty$-categories. This idea was generalized to enriched $\infty$-operads in [CH20]. After defining completeness for objects in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ we show that the full subcategory of complete objects is the correct $\infty$-category of $\mathcal{V}$-enriched $\infty$-properads, in the sense that it is given by localizing $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ with respect to fully faithful and essentially surjective functors.

There are analogues of the statements in this section to dioperads and output properads, which will be discussed in Remark 6.2.14. We begin with a necessary detour comparing Segal objects for presheaves over various graph categories.

6.1. **Free functors.** Consider any of the functors appearing in the following diagram.

(13)




Write $i\colon \Xi \to \Upsilon$ for such a functor, and, for a small symmetric monoidal $\infty$-category $\mathcal{U}$, write $\bar{\imath}\colon \Xi^{\mathcal{U}} \to \Upsilon^{\mathcal{U}}$. We have previously shown that we have restrictions

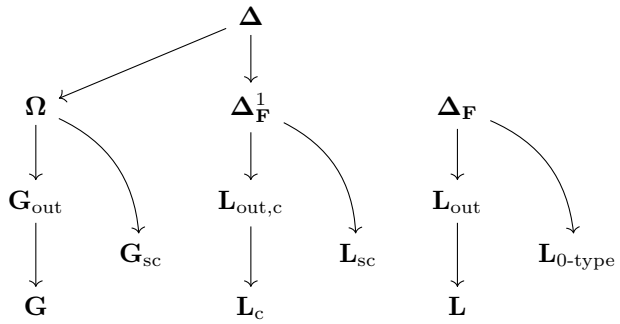
$$\begin{array}{ccc} \mathrm{Seg}(\Upsilon) & \longrightarrow & \mathrm{P}(\Upsilon) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\ \mathrm{Seg}(\Xi) & \longrightarrow & \mathrm{P}(\Xi) \end{array} \qquad \begin{array}{ccccc} \mathrm{Seg}^{\mathrm{rep}}(\Upsilon^{\mathcal{U}}) & \longrightarrow & \mathrm{Seg}(\Upsilon^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\Upsilon^{\mathcal{U}}) \\ \downarrow{\scriptstyle \bar{\imath}^*} & & \downarrow{\scriptstyle \bar{\imath}^*} & & \downarrow{\scriptstyle \bar{\imath}^*} \\ \mathrm{Seg}^{\mathrm{rep}}(\Xi^{\mathcal{U}}) & \longrightarrow & \mathrm{Seg}(\Xi^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\Xi^{\mathcal{U}}) \end{array} \tag{14}$$

of $i^*$ and $\bar{\imath}^*$ (see Proposition 3.2.27 and Proposition 5.2.10 along with Remark 5.1.8). These should generally be considered as forgetful functors. For example, the map $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{U}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{U}}_{\mathrm{sc}})$ takes an enriched $\infty$-properad to its underlying enriched $\infty$-dioperad.

By [CH21, Proposition 4.7], each dashed arrow in (14) admits a left adjoint, given by first taking the left Kan extension and then localizing.[5] We generically denote this by $Li_! \dashv i^*$ (resp. $L\bar{\imath}_! \dashv \bar{\imath}^*$). Our main goal is to prove Theorem 6.1.3, which says that the various $\mathrm{Seg}(\mathbf{\Delta})$-module structures are compatible via these left adjoints.

For several of the functors in (13), we can actually get away with just using the raw left Kan extension, rather than composing with localization. For example, the free $\infty$-properad generated by an $\infty$-operad is just given by left Kan extension at the presheaf level. This will be convenient in the next section.

**Lemma 6.1.1.** *Suppose $i\colon \mathbf{\Xi} \to \mathbf{\Upsilon}$ is one of the fully-faithful functors appearing in the diagram*




*and let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category. Then we have restrictions*

$$\begin{array}{ccc} \mathrm{Seg}(\mathbf{\Xi}) & \longrightarrow & \mathrm{P}(\mathbf{\Xi}) \\ \downarrow & & \downarrow{\scriptstyle i_!} \\ \mathrm{Seg}(\mathbf{\Upsilon}) & \longrightarrow & \mathrm{P}(\mathbf{\Upsilon}) \end{array} \qquad \begin{array}{ccccc} \mathrm{Seg}^{\mathrm{rep}}(\mathbf{\Xi}^{\mathcal{U}}) & \longrightarrow & \mathrm{Seg}(\mathbf{\Xi}^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\mathbf{\Xi}^{\mathcal{U}}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{\imath}_!} \\ \mathrm{Seg}^{\mathrm{rep}}(\mathbf{\Upsilon}^{\mathcal{U}}) & \longrightarrow & \mathrm{Seg}(\mathbf{\Upsilon}^{\mathcal{U}}) & \longrightarrow & \mathrm{P}(\mathbf{\Upsilon}^{\mathcal{U}}) \end{array}$$

*which we also call $i_!$ and $\bar{\imath}_!$. These restrictions are left adjoint to the restrictions $i^*$ and $\bar{\imath}^*$.*

*Proof.* We give the proof for the version without '$\mathcal{U}$,' the other is similar. We know that $i_!$ takes representable presheaves to representable presheaves, and, since $i_!$ preserves colimits, that $i_!(K_{\mathrm{Seg}} \to K) = K_{\mathrm{Seg}} \to K$ for any $K \in \mathbf{\Xi}$. Further, since $i$ is fully-faithful, the unit of the adjunction $i_! \dashv i^*$ is an equivalence. Suppose that $A$ is an object of $\mathrm{Seg}(\mathbf{\Xi})$. As $A \simeq i^* i_! A$ is again Segal, we have, for each $K \in \mathbf{\Xi}$, the right equivalence in the following natural diagram.

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{P}(\mathbf{\Upsilon})}(K, i_!A) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathrm{P}(\mathbf{\Upsilon})}(i_!K, i_!A) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathrm{P}(\mathbf{\Xi})}(K, i^*i_!A) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Map}_{\mathrm{P}(\mathbf{\Upsilon})}(K_{\mathrm{Seg}}, i_!A) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathrm{P}(\mathbf{\Upsilon})}(i_!(K_{\mathrm{Seg}}), i_!A) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathrm{P}(\mathbf{\Xi})}(K_{\mathrm{Seg}}, i^*i_!A) \end{array}$$

It follows that the left-hand map is also an equivalence.

[5] In the case of the horizontal functors, which induce equivalences, this left adjoint can instead be described as the restriction of the *right* Kan extension.

It remains to show that the map $\mathrm{Map}_{\mathrm{P}(\boldsymbol{\Upsilon})}(K, i_! A) \to \mathrm{Map}_{\mathrm{P}(\boldsymbol{\Upsilon})}(K_{\mathrm{Seg}}, i_! A)$ is an equivalence for $A \in \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{U}})$ and $K \notin \boldsymbol{\Xi}$. If $G \to H$ is a morphism in $\boldsymbol{\Upsilon}^{\mathrm{op}}$ with $G \in \boldsymbol{\Xi}$, then $H$ is also in $\boldsymbol{\Xi}$ (Lemma 2.1.38(1) and Lemma 2.2.21). We thus have $(\boldsymbol{\Xi}^{\mathrm{op}})_{/K}$ is empty whenever $K \notin \boldsymbol{\Xi}$, hence $\mathrm{Map}_{\mathrm{P}(\boldsymbol{\Upsilon})}(K, i_! A) = (i_! A)(K) = \varnothing$. But by Lemma 2.1.38(2) or Definition 2.2.14 we see that $K \notin \boldsymbol{\Xi}$ implies that there exists an inert map $K \to \mathfrak{c}$ of $\boldsymbol{\Upsilon}^{\mathrm{op}}$ with $\mathfrak{c}$ a corolla in $\boldsymbol{\Upsilon}^{\mathrm{op}}$. As $(i_! A)(\mathfrak{c}) = \varnothing$, it follows that $\mathrm{Map}_{\mathrm{P}(\boldsymbol{\Upsilon})}(K_{\mathrm{Seg}}, i_! A) = \varnothing$. Thus $i_! A$ is a Segal presheaf. □

By adjointness, we have that $i_!$ commutes with the localization functors

$$\begin{array}{ccc} \mathrm{P}(\boldsymbol{\Xi}) & \longrightarrow & \mathrm{Seg}(\boldsymbol{\Xi}) \\ \downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle i_!} \\ \mathrm{P}(\boldsymbol{\Upsilon}) & \longrightarrow & \mathrm{Seg}(\boldsymbol{\Upsilon}) \end{array}$$

(and likewise for $\bar{\imath}_!$), hence is a retract of $i_! \colon \mathrm{P}(\boldsymbol{\Xi}) \to \mathrm{P}(\boldsymbol{\Upsilon})$.

**Example 6.1.2.** The proof Lemma 6.1.1 fails for $i \colon \mathbf{L}_{\mathrm{sc}} \to \mathbf{L}_{\mathrm{c}}$, since even if $G$ is not in $\mathbf{L}_{\mathrm{sc}}$, all its elementary subgraphs will be. But the result also does not hold. Indeed, if $F \in \mathrm{Seg}(\mathbf{L}_{\mathrm{sc}})$ and $L$ is the graph  then, as in the proof of Lemma 6.1.1, $(i_! F)(L) = \varnothing$. On the other hand,

$$\mathrm{Map}(L_{\mathrm{Seg}}, i_! F) \simeq F(\mathfrak{c}_{1,2}) \times_{F(\mathfrak{c})^2} F(\mathfrak{c}_{2,1}),$$

which is often not equivalent to $\varnothing$.

The distinction between the two situations should not be too surprising. To pass from an operad $O$ to the free properad $P$ it generates, we have

$$P(a_1, \dots, a_m; b_1, \dots, b_n) = \begin{cases} \varnothing & n \neq 1 \\ O(a_1, \dots, a_m; b_1) & n = 1. \end{cases}$$

On the other hand, to go from a dioperad $D$ to the free properad $P$ it generates, we likely have that $P(a_1, \dots, a_m; b_1, \dots, b_n)$ and $D(a_1, \dots, a_m; b_1, \dots, b_n)$ are both inhabited, but very different. The properad $P$ has many operations formally generated from those in $D$.

**Theorem 6.1.3.** *Let $\mathcal{U}$ be a small symmetric monoidal $\infty$-category and let $i \colon \boldsymbol{\Xi} \to \boldsymbol{\Upsilon}$ be a composite of fully-faithful functors appearing in Lemma 6.1.1. Then for $A \in \mathrm{Seg}(\boldsymbol{\Xi}^{\mathcal{U}})$ and $\mathcal{C} \in \mathrm{Seg}(\boldsymbol{\Delta})$, we have*

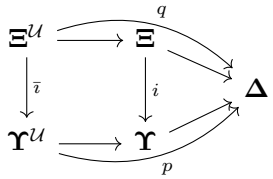
$$\bar{\imath}_!(A \otimes \mathcal{C}) \simeq (\bar{\imath}_! A) \otimes \mathcal{C}$$

*in $\mathrm{Seg}(\boldsymbol{\Upsilon}^{\mathcal{U}})$. If instead $i$ appears in the diagram (13), then $L\bar{\imath}_!(A \otimes \mathcal{C}) \simeq (L\bar{\imath}_! A) \otimes \mathcal{C}$.*

The tensorings here are those from Theorem 4.2.2, Remark 4.2.9, and [CH20].

*Proof.* It is enough to prove the result for functors between categories of possibly disconnected level graphs (that is, in the rightmost part of (13)), as the other tensorings are defined along the compatible equivalences induced by the horizontal functors. We prove the statement for $\bar{\imath}_!$, as the proof for $L\bar{\imath}_!$ is nearly identical.

Write $q$ and $p$ for the relevant composite cartesian fibrations from $\mathbf{\Xi}^{\mathcal{U}}$ and $\mathbf{\Upsilon}^{\mathcal{U}}$ to $\mathbf{\Delta}$, that is, the following diagram commutes:



Write $L_1\colon \mathrm{P}(\mathbf{\Xi}^{\mathcal{U}}) \to \mathrm{Seg}(\mathbf{\Xi}^{\mathcal{U}})$ and $L_2\colon \mathrm{P}(\mathbf{\Upsilon}^{\mathcal{U}}) \to \mathrm{Seg}(\mathbf{\Upsilon}^{\mathcal{U}})$ for the localization functors.

For $B \in \mathrm{Seg}(\mathbf{\Upsilon}^{\mathcal{U}})$ and $\mathcal{C} \in \mathrm{Seg}(\mathbf{\Delta})$, we have that $B \otimes \mathcal{C}$ is defined to be $L_2(B \times p^*\mathcal{C})$. We apply $\bar{\imath}^*$ to the localization map $B \times p^*\mathcal{C} \to B \otimes \mathcal{C}$ in $\mathrm{P}(\mathbf{\Upsilon}^{\mathcal{U}})$. As $\bar{\imath}^*$ preserves Segal objects, $\bar{\imath}^*(B \otimes \mathcal{C})$ is Segal and we have the indicated factorization

$$\begin{array}{ccc} \bar{\imath}^*(B \times p^*\mathcal{C}) & \longrightarrow & \bar{\imath}^*(B \otimes \mathcal{C}) \\ \downarrow \simeq & & \uparrow \\ (\bar{\imath}^*B) \times q^*\mathcal{C} & \longrightarrow & (\bar{\imath}^*B) \otimes \mathcal{C} \end{array}$$

in $\mathrm{P}(\mathbf{\Xi}^{\mathcal{U}})$. We thus have constructed a map

$$(\bar{\imath}^*B) \otimes \mathcal{C} \to \bar{\imath}^*(B \otimes \mathcal{C})$$

natural in $B \in \mathrm{Seg}(\mathbf{\Upsilon}^{\mathcal{U}})$ and $\mathcal{C} \in \mathrm{Seg}(\mathbf{\Delta})$, hence we have

$$A \otimes \mathcal{C} \to (\bar{\imath}^*\bar{\imath}_!A) \otimes \mathcal{C} \to \bar{\imath}^*((\bar{\imath}_!A) \otimes \mathcal{C})$$

natural in $A \in \mathrm{Seg}(\mathbf{\Xi}^{\mathcal{U}})$ and $\mathcal{C} \in \mathrm{Seg}(\mathbf{\Delta})$. By adjointness, this amounts to

$$\bar{\imath}_!(A \otimes \mathcal{C}) \to (\bar{\imath}_!A) \otimes \mathcal{C} \tag{15}$$

where both sides commute with colimits in each variable.

First note that objects $A$ in $\mathrm{Seg}(\mathbf{\Xi}^{\mathcal{U}})$ are colimits over corollas $\mathfrak{c}_{m,n}(v)$ where $\mathfrak{c}_{m,n}$ ranges over corollas of $\mathbf{\Xi}$. As usual it is true that any presheaf is a colimit of representables $\overline{G}$, and applying $L_1$ each of these representables splits into a colimit of elementary representables. Moreover, $\mathfrak{e}$ is a retract of $\mathfrak{c}_{1,1}(\mathbb{1})$. Finally, $A$ was already assumed to be Segal, so the result follows for $A$ and not just $L_1(A)$. Similarly, every object of $\mathrm{Seg}(\mathbf{\Delta})$ is a colimit of $\Delta^1$.

By cocontinuity, to see that (15) is an equivalence it suffices to show that $\bar{\imath}_!(\mathfrak{c}(v) \otimes \Delta^1) \to (\bar{\imath}_!\mathfrak{c}(v)) \otimes \Delta^1$ is an equivalence for each corolla $\mathfrak{c} \in \mathbf{\Xi}$. This follows more or less by Lemma 4.2.6. Specifically, the lemma gives that the left hand-side is

$$\begin{aligned} \bar{\imath}_!L_1(\mathfrak{c}(v) \times q^*\Delta^1) &\simeq \bar{\imath}_!L_1(\mathfrak{c}^+(v) \amalg_{\mathfrak{c}(v)} \mathfrak{c}^-(v)) \\ &\simeq (\bar{\imath}_!L_1\mathfrak{c}^+(v) \amalg_{\bar{\imath}_!L_1\mathfrak{c}(v)} \bar{\imath}_!L_1\mathfrak{c}^-(v)) \\ &\simeq \mathfrak{c}^+(v) \amalg_{\mathfrak{c}(v)} \mathfrak{c}^-(v), \end{aligned}$$

using that representables lying over simply-connected graphs are already local, and that $\bar{\imath}_!$ sends representables to representables. Likewise, the right-hand side is

$$\begin{aligned}(\bar{\imath}_!\mathfrak{c}(v)) \otimes \Delta^1 &\simeq \mathfrak{c}(v) \otimes \Delta^1 \\ &\simeq L_2(\mathfrak{c}(v) \times p^*\Delta^1) \\ &\simeq L_2(\mathfrak{c}^+(v) \amalg_{\mathfrak{c}(v)} \mathfrak{c}^-(v)) \\ &\simeq L_2\mathfrak{c}^+(v) \amalg_{L_2\mathfrak{c}(v)} L_2\mathfrak{c}^-(v) \\ &\simeq \mathfrak{c}^+(v) \amalg_{\mathfrak{c}(v)} \mathfrak{c}^-(v). \qquad \square\end{aligned}$$

We can extend these results to the presentable case in the usual way, by choosing a suitable subcategory of $\kappa$-compact objects. This gives the following result.

**Corollary 6.1.4.** *Suppose that $i\colon \boldsymbol{\Xi} \to \boldsymbol{\Upsilon}$ is one of the maps in (13). Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category and let $\bar{\imath}\colon \boldsymbol{\Xi}^{\mathcal{V}} \to \boldsymbol{\Upsilon}^{\mathcal{V}}$. Then*

*(i) There is an adjunction*

$$\bar{\imath}_\Delta : \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}}) \rightleftarrows \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Upsilon}^{\mathcal{V}}) : \bar{\imath}^*.$$

*(ii) The left adjoint $\bar{\imath}_\Delta$ is compatible with the tensoring with $\mathrm{Seg}(\boldsymbol{\Delta})$ in the sense that there is a natural equivalence $\bar{\imath}_\Delta(X \otimes \mathcal{C}) \simeq (\bar{\imath}_\Delta X) \otimes \mathcal{C}$ for $X \in \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$ and $\mathcal{C} \in \mathrm{Seg}(\boldsymbol{\Delta})$.*

*(iii) There is a natural equivalence $\bar{\imath}^*(A^{\mathcal{C}}) \simeq (\bar{\imath}^* A)^{\mathcal{C}}$ for $A \in \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Upsilon}^{\mathcal{V}})$ and $\mathcal{C} \in \mathrm{Seg}(\boldsymbol{\Delta})$.*

*Proof.* Note that there exists a small symmetric monoidal $\infty$-category $\mathcal{U}$ and a set $\mathbb{S}$ such that $\mathcal{V} \simeq \mathrm{P}_{\mathbb{S}}(\mathcal{U})$. The claims (i) and (ii) then follow from $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}}) \simeq \mathrm{Seg}_{\mathbb{S}}(\boldsymbol{\Xi}^{\mathcal{U}})$ and the previous theorem. In particular, $\bar{\imath}_\Delta$ is induced along equivalences from either $\bar{\imath}_!$ or $L\bar{\imath}_!$ from $\mathrm{Seg}_{\mathbb{S}}(\boldsymbol{\Xi}^{\mathcal{U}})$ to $\mathrm{Seg}_{\mathbb{S}}(\boldsymbol{\Upsilon}^{\mathcal{U}})$ (depending on the domain and codomain of $i$).

We show only (iii); let $Q \in \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$ be arbitrary. On the one hand we have (see Proposition 4.2.8)

$$\mathrm{Map}(Q, (\bar{\imath}^* A)^{\mathcal{C}}) \simeq \mathrm{Map}(Q \otimes \mathcal{C}, \bar{\imath}^* A) \simeq \mathrm{Map}(\bar{\imath}_\Delta(Q \otimes \mathcal{C}), A),$$

while on the other we have

$$\mathrm{Map}(Q, \bar{\imath}^*(A^{\mathcal{C}})) \simeq \mathrm{Map}(\bar{\imath}_\Delta Q, A^{\mathcal{C}}) \simeq \mathrm{Map}((\bar{\imath}_\Delta Q) \otimes \mathcal{C}, A).$$

These coincide by (ii). $\square$

Notice that we have written $\bar{\imath}_\Delta$ for the left adjoint, rather than $\bar{\imath}_!$, as the functor is not literally given by left Kan extension even in the situation of Lemma 6.1.1. This contrasts with the conventions of [CH20], see Warning 2.9.7 there. This notation will only be used again at the end of §6.2.

6.2. **Fully faithfulness, essential surjectivity and completeness.** In this section, we generalize the definition of the $\infty$-category $\mathrm{Opd}_\infty^{\mathcal{V}}$ of $\mathcal{V}$-enriched $\infty$-operads from [CH20, §3.2] to give $\infty$-categories of $\mathcal{V}$-enriched $\infty$-properads, $\infty$-output-properads, and $\infty$-dioperads. In each case, these are full subcategories of representable presheaves on the objects whose underlying simplicial presheaf is complete.

Given a graph category $\boldsymbol{\Xi}$, there is a functor $\boldsymbol{\Xi} \to \boldsymbol{\Xi}^{\mathcal{V}}$ which on objects sends an object $G$ to $G(\mathbb{1}_c)_{c \in \mathsf{V}(G)}$. We thus have functors, both denoted by $u$,

$$u\colon \boldsymbol{\Delta} \to \mathbf{G}^{\mathcal{V}} \qquad u\colon \boldsymbol{\Delta} \to \mathbf{L}_{\mathrm{c}}^{\mathcal{V}}.$$

By similar techniques to above, one can show that $u^*\colon \mathrm{P}(\mathbf{G}^{\mathcal{V}}) \to \mathrm{P}(\mathbf{\Delta})$ and $u^*\colon \mathrm{P}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \to \mathrm{P}(\mathbf{\Delta})$ restrict to Segal objects, and we have:

**Definition 6.2.1** (Underlying $\infty$-category)**.** Given $F$ in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ or $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$, we write call

$$u^*F \in \mathrm{Seg}(\mathbf{\Delta})$$

the *underlying $\infty$-category of $F$*.

**Remark 6.2.2** (Another description of the underlying $\infty$-category)**.** Let $\varphi \coloneqq \mathrm{Map}_{\mathcal{V}}(\mathbb{1}, -)\colon \mathcal{V} \to \mathcal{S}$ denote the lax monoidal functor which is right adjoint to the unique colimit-preserving functor $\mathcal{S} \to \mathcal{V}$ taking $*$ to the unit $\mathbb{1}$. Let $\varphi_*\colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \leftrightarrows \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{S}})$ be the right adjoint of Corollary 3.2.35 induced by $\varphi$. Given an object $F \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$, we can use the equivalence $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{S}}) \simeq \mathrm{Seg}(\mathbf{G})$ of Corollary 3.3.2 to consider $\varphi_*F \in \mathrm{Seg}(\mathbf{G})$. This object restricts further, along $i\colon \mathbf{\Delta} \to \mathbf{G}$, and we have $i^*\varphi_*F \simeq u^*F$ is the underlying enriched $\infty$-properad associated to $F$.

**Definition 6.2.3.** Let $\mathrm{Map}_F(x_1, \dots, x_m; y_1, \dots, y_n)$ be as defined in Definition 3.2.18. We say a morphism $f\colon F \to F'$ in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ is

- *fully faithful* if it induces an equivalence

  $$\mathrm{Map}_F(x_1, \dots, x_m; y_1, \dots, y_n) \xrightarrow{\sim} \mathrm{Map}_{F'}(fx_1, \dots, fx_m; fy_1, \dots, fy_n)$$

  in $\mathcal{V}$ for every corolla $\mathfrak{c}_{m,n}$ and every $\underline{xy} \in F(\mathfrak{c})^{m+n}$, and
- *essentially surjective* if the induced functor $u^*(f)\colon u^*(F) \to u^*(F')$ of the underlying Segal space is essentially surjective.

**Definition 6.2.4.** We write $E^n$ for the indiscrete category (viewed as a Segal space) with $n+1$ objects, i.e. it has a unique morphism between any pair of objects.

**Remark 6.2.5.** The category $E^1$ is the "generic isomorphism," so giving a morphism of Segal spaces $E^1 \to X$ corresponds to giving two objects of $X$ and an equivalence between them. Similarly, giving a map $E^n \to X$ is equivalent to specifying $n+1$ equivalent objects in $X$.

**Remark 6.2.6.** Following [Rez01], the correct space of objects of a Segal space $\mathcal{C}$ should be given by $\iota\mathcal{C}$ defined as the colimit of the simplicial space $\mathrm{Map}_{\mathrm{Seg}(\mathbf{\Delta})}(E^{(-)}, \mathcal{C})$. Using this notation, the functor $f$ of Definition 6.2.3 is essentially surjective if and only if $\pi_0(\iota u^*(f))\colon \pi_0(\iota u^*(F)) \to \pi_0(\iota u^*(F'))$ is a surjection of sets.

The following extends Definition 3.2.1 and Definition 4.4.12 of [CH20] to the setting of properads.

**Definition 6.2.7.** Let $\mathcal{V}$ be a presentably symmetric monoidal $\infty$-category.

(1) We say a Segal space $F$ is *complete* if the map

$$F([0]) \simeq \mathrm{Map}(E^0, F) \to \mathrm{Map}(E^1, F)$$

induced by the map $s^0\colon E^1 \to E^0 \simeq \{[0]\}$, is an equivalence of spaces. We write $\mathrm{Cat}_\infty$ for the full subcategory of $\mathrm{Seg}(\mathbf{\Delta})$ spanned by the complete Segal spaces.

(2) We say an object $F \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ is *complete* if its underlying Segal space $u^*F$ is complete and we let $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ denote the full subcategory of $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ spanned by the complete objects.

(3) A similar definition holds for objects of $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$, and we write $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ for the full subcategory of $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ spanned by complete objects. The equivalence $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ of Corollary 5.1.6 induces an equivalence $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$.
(4) A *$\mathcal{V}$-enriched $\infty$-properad* is a complete, fibrewise representable $\mathbf{G}^{\mathcal{V}}$-Segal space or $\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}$-Segal space.

**Remark 6.2.8.** It follows from the previous definition and [CH20, Definition 3.2.1] that an object in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ is complete if and only if the underlying object in $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Delta}_{\mathbf{F}}^{1,\mathcal{V}}) \simeq \mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Omega}^{\mathcal{V}})$ is complete.

By replacing $\boldsymbol{\Delta}_{\mathbf{F}}^{\mathcal{V}}$ with $\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}$ the proofs of [CH20, §3.5] give the following result.

**Theorem 6.2.9.** *There is a completion functor*

$$\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \to \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$$

*which takes every object in $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ to a complete one and is left adjoint to the inclusion $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \hookrightarrow \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ of the full subcategory. Moreover, it exhibits $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ as a localization of $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ with respect to the class of fully faithful and essentially surjective functors.* □

Of course a similar theorem holds if $\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}$ is replaced by $\mathbf{G}^{\mathcal{V}}$. This theorem says that the $\infty$-categories $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ are the correct $\infty$-category of $\mathcal{V}$-enriched $\infty$-properads. We now introduce a new notation, similar to $\mathrm{Opd}_{\infty}^{\mathcal{V}}$ for $\mathcal{V}$-enriched $\infty$-operads from Notation 3.2.2 of [CH20].

**Notation 6.2.10.** We write $\mathrm{Prpd}_{\infty}^{\mathcal{V}}$ for $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}})$ or $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ when we do not want to emphasize the specific implementation of $\infty$-category of $\mathcal{V}$-enriched $\infty$-properads.

It follows from the definition that a lax symmetric monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ induces a functor $\mathrm{Alg}_{\mathbf{L}_{\mathrm{c}}^{\otimes,\mathrm{op}}/\mathcal{S}}(\mathcal{V}) \to \mathrm{Alg}_{\mathbf{L}_{\mathrm{c}}^{\otimes,\mathrm{op}}/\mathcal{S}}(\mathcal{W})$ of $\infty$-categories. This functor can be identified with $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{W}})$ under the equivalence $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbf{L}_{\mathrm{c}}^{\otimes,\mathrm{op}}/\mathcal{S}}(\mathcal{V})$ of Theorem 3.2.33. We then obtain the next proposition by localizing $\mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{c}}^{\mathcal{W}})$ with respect to fully faithful and essential surjective functors.

**Proposition 6.2.11.** *The $\infty$-category $\mathrm{Prpd}_{\infty}^{\mathcal{V}}$ is functorial in $\mathcal{V}$ with respect to lax symmetric monoidal functors. Moreover, if $F\colon \mathcal{V} \to \mathcal{W}$ is a colimit-preserving symmetric monoidal functor then $F_*\colon \mathrm{Prpd}_{\infty}^{\mathcal{V}} \to \mathrm{Prpd}_{\infty}^{\mathcal{W}}$ preserves colimits; thus $\mathrm{Prpd}_{\infty}^{(-)}$ defines a functor $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{Pr}^{\mathrm{L}}$ where $\mathrm{Pr}^{\mathrm{L}}$ is the $\infty$-category of presentable $\infty$-categories.* □

The proof of [CH20, Proposition 3.4.9] gives the following proposition.

**Proposition 6.2.12.** *The tensor product $\otimes\colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}}) \times \mathrm{Seg}(\boldsymbol{\Delta}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{L}^{\mathcal{V}})$ of Proposition 5.1.7 restricts to a tensor product functor*

$$\otimes\colon \mathrm{Prpd}_{\infty}^{\mathcal{V}} \times \mathrm{Cat}_{\infty} \to \mathrm{Prpd}_{\infty}^{\mathcal{V}}$$

*of presentable $\infty$-categories which preserves colimits in each variable.* □

The adjoint functor theorem gives the following corollary.

**Corollary 6.2.13.** *Let* $\mathcal{C} \in \mathrm{Cat}_\infty$ *and* $\mathcal{Q}, \mathcal{R} \in \mathrm{Prpd}_\infty^{\mathcal{V}}$. *The tensor product* $\otimes \colon \mathrm{Prpd}_\infty^{\mathcal{V}} \times \mathrm{Cat}_\infty \to \mathrm{Prpd}_\infty^{\mathcal{V}}$ *induces*

$$\mathrm{Alg}^{\mathcal{V}}_{(-)}(-) \colon (\mathrm{Prpd}_\infty^{\mathcal{V}})^{\mathrm{op}} \times \mathrm{Prpd}_\infty^{\mathcal{V}} \to \mathrm{Cat}_\infty$$

*such that*

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}^{\mathcal{V}}_{\mathcal{Q}}(\mathcal{R})) \simeq \mathrm{Map}_{\mathrm{Prpd}_\infty^{\mathcal{V}}}(\mathcal{Q} \otimes \mathcal{C}, \mathcal{R})$$

*and a cotensor product*

$$(-)^{(-)} \colon \mathrm{Prpd}_\infty^{\mathcal{V}} \times \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Prpd}_\infty^{\mathcal{V}}$$

*such that*

$$\mathrm{Map}_{\mathrm{Prpd}_\infty^{\mathcal{V}}}(\mathcal{Q}, \mathcal{R}^{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Prpd}_\infty^{\mathcal{V}}}(\mathcal{Q} \otimes \mathcal{C}, \mathcal{R}).$$

*Moreover, both of these functors preserve limits in each variable.* □

Though we are using the same notation for algebras and cotensors as we did in Corollary 4.2.8, we do not know that these two notions coincide. Specifically, we do not know that the objects produced by Corollary 4.2.8 are complete even when the inputs are.

**Remark 6.2.14.** As is usual, the theorems and definitions of this section can be carried out using other graph categories. We write

$$\mathrm{DOpd}_\infty^{\mathcal{V}} \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{sc}}^{\mathcal{V}}) \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}})$$

for the $\infty$-category of $\mathcal{V}$-enriched $\infty$-dioperads and

$$\mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}} \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{L}_{\mathrm{out,c}}^{\mathcal{V}}) \simeq \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}})$$

for the $\infty$-category of $\mathcal{V}$-enriched $\infty$-output-properads. These are defined as full subcategories of the appropriate $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}})$ on the complete objects (where completeness is created by $\mathrm{Seg}^{\mathrm{rep}}(\boldsymbol{\Xi}^{\mathcal{V}}) \to \mathrm{Seg}(\boldsymbol{\Delta})$), and then shown to be localizations at the fully-faithful and essentially surjective functors (as in Theorem 6.2.9). Similar statements also hold for $\mathrm{Opd}_\infty^{\mathcal{V}}$, and previously appeared in [CH20].

**Proposition 6.2.15.** *There are adjunctions*

$$\mathrm{DOpd}_\infty^{\mathcal{V}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathrm{Opd}_\infty^{\mathcal{V}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Prpd}_\infty^{\mathcal{V}}$$

*restricted from those in Corollary 6.1.4*(i). *The left adjoints preserve tensors.*

*Proof.* For concreteness, we only prove the statement about $\mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}} \rightleftarrows \mathrm{Prpd}_\infty^{\mathcal{V}}$ and we do so in terms of $\mathbf{G}_{\mathrm{out}}^{\mathcal{V}}$ and $\mathbf{G}^{\mathcal{V}}$-presheaves. Write $\bar{\imath} \colon \mathbf{G}_{\mathrm{out}}^{\mathcal{V}} \to \mathbf{G}^{\mathcal{V}}$. The key fact that we use about this situation, which falls under the setting of Lemma 6.1.1, is that the unit $1 \to \bar{\imath}^* \bar{\imath}_\Delta$ of the adjunction $\bar{\imath}_\Delta \colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}}) \rightleftarrows \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) : \bar{\imath}^*$ of Corollary 6.1.4(i) is an equivalence. Indeed, this follows from the corresponding fact for the equivalent adjunction $\mathrm{Seg}_{\mathbb{S}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{U}}) \rightleftarrows \mathrm{Seg}_{\mathbb{S}}(\mathbf{G}^{\mathcal{U}})$ (for an appropriately chosen $\mathcal{U}$ and $\mathbb{S}$), using that $\mathbf{G}_{\mathrm{out}}^{\mathcal{U}} \to \mathbf{G}^{\mathcal{U}}$ is fully-faithful.

We name two additional functors

$$\begin{array}{ccc} \boldsymbol{\Delta} & \xrightarrow{u'} & \mathbf{G}_{\mathrm{out}}^{\mathcal{V}} \\ & \searrow^{u} & \downarrow{\scriptstyle \bar{\imath}} \\ & & \mathbf{G}^{\mathcal{V}}. \end{array}$$

An object $F \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ is complete if and only if $u^*F = (u')^*\bar{\imath}^*F \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{\Delta})$ is complete if and only if $\bar{\imath}^*F \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}})$ is complete. Thus $\bar{\imath}^*$ restricts to $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \to \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}})$. Now suppose that $A \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}})$ is complete. As

$$u^*\bar{\imath}_{\vartriangle}A \simeq (u')^*\bar{\imath}^*\bar{\imath}_{\vartriangle}A \simeq (u')^*A, \tag{16}$$

it follows that $\bar{\imath}_{\vartriangle}A \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ is also complete. Thus $\bar{\imath}_{\vartriangle} \dashv \bar{\imath}^*$ restricts to an adjunction $\bar{\imath}_{\vartriangle} \colon \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{out}}^{\mathcal{V}}) \rightleftarrows \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) : \bar{\imath}^*$. Using the two tensorings (from Proposition 6.2.12 and Remark 6.2.14) we know that $\bar{\imath}_{\vartriangle}(A \otimes \mathcal{C}) \simeq (\bar{\imath}_{\vartriangle}A) \otimes \mathcal{C}$ by Corollary 6.1.4(ii). □

**Remark 6.2.16.** The proof of the preceding proposition can be modified to show that if $i \colon \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ is the inclusion, then $\bar{\imath}^* \colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}})$ restricts to $\mathrm{Prpd}_{\infty}^{\mathcal{V}} \to \mathrm{DOpd}_{\infty}^{\mathcal{V}}$. By Theorem 6.2.9 this functor will have a left adjoint, though it is not clear whether or not it is restriction of the functor $\bar{\imath}_{\vartriangle} \colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}}) \to \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ from Corollary 6.1.4. Indeed, $\bar{\imath}_{\vartriangle}$ will typically significantly enlarge the underlying $\mathrm{Seg}(\mathbf{\Delta})$ object, meaning that the equivalence corresponding to (16) will not hold. This is in the same spirit as Example 6.1.2, since $\bar{\imath}_{\vartriangle}$ requires a localization.

That said, [HRY17, §4] provides evidence that the adjunction

$$\bar{\imath}_{\vartriangle} \colon \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}}) \rightleftarrows \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}}) : \bar{\imath}^*$$

restricts to $\mathrm{DOpd}_{\infty}^{\mathcal{V}} \rightleftarrows \mathrm{Prpd}_{\infty}^{\mathcal{V}}$. Suppose that $F$ is the functor from simplicially-enriched dioperads to simplicially-enriched properads, left adjoint to the forgetful functor. One consequence of Proposition 4.4 of [HRY17] is that if $D$ is a nice-enough dioperad, then the underlying simplicially-enriched categories of $D$ and $FD$ will have the same classes of equivalences. The main idea is that the new operations in $FD$ are obtained by decorating graphs having nonzero first betti number by operations in $D$. Composing operations in $FD$ is additive on first betti numbers of said graphs, so in particular the newly added operations are never invertible. It would be interesting to see if a similar argument can be carried out in the present situation, but such a detailed argument is beyond what we are trying to achieve here. Thus we leave this as Conjecture 6.2.17 below. Notice that if $A$ and $\bar{\imath}_{\vartriangle}A$ have the same objects and same equivalences, then $A$ will be complete if and only if $\bar{\imath}_{\vartriangle}A$ is complete. This tells us that the rest of the proof of Proposition 6.2.15 carries through without change.

**Conjecture 6.2.17.** If $A \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}})$, then the equivalences in $\bar{\imath}_{\vartriangle}A \in \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ coincide with the equivalences in $A$. Consequently, $\bar{\imath}_{\vartriangle}$ restricts to $\widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}_{\mathrm{sc}}^{\mathcal{V}}) \to \widehat{\mathrm{Seg}}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})$ which preserves tensors.

## 7. Rectification theorems

The aim of this section is to understand whether the homotopy theory of enriched $\infty$-properads is equivalent to a Dwyer–Kan-type homotopy theory for ordinary enriched properads. In §7.1 we show that $\mathbf{G}$ (resp. $\mathbf{G}_{\mathrm{sc}}$, $\mathbf{G}_{\mathrm{out}}$) and the operads governing properads (resp. dioperads, output properads) induce the same $\infty$-category of algebras. Then, in §7.2 we turn to the question of rectifying enriched $\infty$-properads. This we can do only over very particular bases (see Theorem 7.2.5), though for enriched $\infty$-dioperads and enriched $\infty$-output-properads rectification holds quite generally (Theorem 7.2.9).

7.1. **Operads governing properads.** In this subsection we first recall, for a set $S$, the operad whose algebras in a symmetric monoidal $\infty$-category are enriched $S$-colored properads. The main result of this subsection is that the $\infty$-category $\mathbf{G}_S^{\mathrm{op}}$ is an "approximation" to this operad in the sense of [Lur, §2.3.3]. This observation immediately implies that an enriched $\infty$-properad in our sense is indeed equivalent to an enriched $\infty$-properad defined as an algebra over the operad for properads.

**Definition 7.1.1.** Let $Z$ be a finite set. For our purposes, a $Z$-graph will consist of a connected, acyclic graph $G$ together with total orderings on each of the sets $\mathtt{in}(G)$, $\mathtt{out}(G)$, $\mathtt{in}(v)$ (for each $v \in \mathtt{V}(G)$), and $\mathtt{out}(v)$, as well as a bijection $Z \xrightarrow{\sim} \mathtt{V}(G)$. Likewise, an *$S$-colored $Z$-graph* will additionally come with a function $\mathtt{E}(G) \to S$. We say that two $Z$-graphs are *strictly isomorphic* if there is a graph isomorphism preserving all of the structure.

Note that there is at most one strict isomorphism between any two $Z$-graphs.

We now recall, from [YJ15, §14.1], a colored operad $\mathbf{Prpd}_S$ which controls $S$-colored properads. Before we define a colored operad $\mathbf{Prpd}_S$, we first introduce the special case where $S$ is the terminal set.

**Definition 7.1.2.** Let $Z$ be a finite set. Suppose that $\vec{k}$ and $\vec{k}_z$ (indexed over $z \in Z$) are elements of $\mathbb{N} \times \mathbb{N}$. Define

$$\mathbf{Prpd}(\{\vec{k}_z\}_{z\in Z}; \vec{k})$$

to be the set of strict isomorphism classes of $Z$-graphs $G$ with $(|\mathtt{in}(v_z)|, |\mathtt{out}(v_z)|) = \vec{k}_z$ for each $z \in Z$ and $(|\mathtt{in}(G)|, |\mathtt{out}(G)|) = \vec{k}$. This forms an $\mathbb{N} \times \mathbb{N}$-colored operad $\mathbf{Prpd}$ with operadic composition given by graph substitution. As all of the sets $\mathtt{in}(v)$, $\mathtt{in}(G)$ and so on are totally ordered, we use the unique order-preserving isomorphisms as our graph substitution data.

- For $Z$ a one-element set, the identity element in $\mathbf{Prpd}(\{\vec{k}\}; \vec{k})$ is a corolla $C$ with vertex $v$ so that the two orderings on $\mathtt{in}(C) = \mathtt{in}(v)$ agree, and likewise for $\mathtt{out}(C) = \mathtt{out}(v)$.
- If $\sigma\colon Z \to Z'$ is a bijection, there is an isomorphism

  $$\mathbf{Prpd}(\{\vec{k}_{z'}\}_{z'\in Z'}; \vec{k}) \to \mathbf{Prpd}(\{\vec{k}_{\sigma(z)}\}_{z\in Z}; \vec{k})$$

  given on a $Z'$-graph $G$ by precomposing the bijection $Z' \to \mathtt{V}(G)$ with $\sigma$.

A special case of this definition is when $Z$ is the empty set. As there is a unique graph $G$ in $\mathbf{G}$ which does not have any vertices, we have

$$\mathbf{Prpd}(\{\,\}; \vec{k}) = \begin{cases} \{\mathfrak{e}\} & \text{if } \vec{k} = (1,1), \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

**Remark 7.1.3.** Note that the operad from [YJ15, §14.1] is actually the 'skeletal' version of this one, that is, is only indexed on the finite sets $Z = \mathbf{n}$. As is customary, we will write the corresponding set of operations as

$$\mathbf{Prpd}(\vec{k}_1, \ldots, \vec{k}_n; \vec{k})$$

since $\mathbf{n}$ has a natural total order.

We now want to extend $\mathbf{Prpd}$ to a colored operad $\mathbf{Prpd}_S$.

**Definition 7.1.4.** Let $S$ be a nonempty set. The operations in $\mathbf{Prpd}_S$ as are strict isomorphism classes of $S$-colored $Z$-graphs (where strict isomorphism is interpreted to mean strict isomorphism preserving the coloring functions). The set of colors is $(\coprod_{n\geq 0} S^{\times n})^{\times 2}$, the set of pairs of ordered lists of elements of $S$. Given an $S$-colored $Z$-graph, the vertex $v_z$ has an associated pair of lists of elements of $S$ using the two functions $\mathtt{in}(v_z) \to S$ and $\mathtt{out}(v_z) \to S$, as does the whole graph using $\mathtt{in}(G) \to S$ and $\mathtt{out}(G) \to S$. This determines the profile where this operation lives. Otherwise, the structure is very similar to that of $\mathbf{Prpd}$.

**Lemma 7.1.5** (Section 14.1 of [YJ15])**.** *For a set $S$, the $\mathbf{Prpd}_S$-algebras in a symmetric monoidal category $\mathcal{C}$ are $\mathcal{C}$-enriched properads with $S$ as the set of colors.*

Let us now look at some special cases of $\mathbf{Prpd}_S$. Notice that there is an operad map $\mathbf{Prpd}_S \to \mathbf{Prpd}$ which forgets the edge colorings on operations, and whose color map is induced from $S \to *$:

$$\left(\coprod_{n\geq 0} S^{\times n}\right) \times \left(\coprod_{n\geq 0} S^{\times n}\right) \to \left(\coprod_{n\geq 0} *\right) \times \left(\coprod_{n\geq 0} *\right) \cong \mathbb{N} \times \mathbb{N}.$$

**Example 7.1.6.**

(1) For $S = *$, $\mathbf{Prpd}_*$ coincides with $\mathbf{Prpd}$ of Definition 7.1.2.
(2) Let $\mathbf{Cat}_S$ denote the full suboperad of $\mathbf{Prpd}_S$ with color set the pullback of $\{1\} \times \{1\} \hookrightarrow \mathbb{N} \times \mathbb{N}$. Algebras over $\mathbf{Cat}_S$ are categories with $S$ as the set of objects (see [GH15, 2.1]). Likewise, letting $\mathbf{Opd}_S$ denote the full suboperad of $\mathbf{Prpd}_S$ with color set the pullback of $\mathbb{N} \times \{1\} \hookrightarrow \mathbb{N} \times \mathbb{N}$, we recover the operad whose algebras are $S$-colored operads (see [CH20, Definition 5.1.5]).
(3) Let $\mathbf{Prpd}_S^{\mathtt{out}}$ denote the full suboperad of $\mathbf{Prpd}_S$ with color set the pullback of $\mathbb{N} \times \{1, 2, 3, \dots\} = \mathbb{N} \times \mathbb{N}_+ \hookrightarrow \mathbb{N} \times \mathbb{N}$. Algebras over $\mathbf{Prpd}_S^{\mathtt{out}}$ are $S$-colored properads in which every operation has at least one output color. Likewise, there is a colored operad $\mathbf{Prpd}_S^{\mathtt{in}}$ with color set living over $\mathbb{N}_+ \times \mathbb{N}$.
(4) Let $\mathbf{DOpd}_S$ be the suboperad of $\mathbf{Prpd}_S$ with the same color set, with the requirement that the underlying graph of any operation is simply-connected. Algebras over $\mathbf{DOpd}_S$ are $S$-colored dioperads (see [YJ15, §11.5]).

**Remark 7.1.7.** The operad $\mathbf{Prpd}_S$ is not $\Sigma$-free. Indeed, consider the left graph from Example 2.2.13 with the output orderings at $u_i$ and the input orderings at $v_i$ are given from left to right, and with vertex ordering $u_0, u_1, v_0, v_1$. With this convention, the graph represents an element of $\mathbf{Prpd}((0,2),(0,2),(2,0),(2,0);(0,0))$. This element is fixed by the group element $(12)(34) \in \Sigma_4$. This issue is intrinsic, that is, any other operad governing properads will also have such a fixed point. In particular, this means that our conception of properads is different from that of [BB17, 10.4] which arises as algebras over a finitary polynomial monad in Set, as a polynomial monad always describes a $\Sigma$-free colored operad (see Section 6 of [BB17]).

The other colored operads from Example 7.1.6 are $\Sigma$-free. Indeed, for each of the other types of graphs, the only strict automorphisms are identities. For $\mathbf{DOpd}_S$, this fact is [YJ15, Proposition 4.14], while for the others ($\mathbf{Prpd}_S^{\mathtt{out}}$, $\mathbf{Opd}_S$, and so on) this is [YJ15, Lemma 4.8].

For the reader's convenience we now recall the definition of the $\infty$-operad associated to a symmetric operad introduced in [Lur, Construction 2.1.1.7].

**Definition 7.1.8.** For a symmetric operad $\mathbf{O}$, we define its associated $\infty$-operad $\mathcal{O} \to \mathbf{F}_*$ to be the functor determined by the following:

(1) The objects in $\mathcal{O}$ are finite sequences $(x_1, \dots, x_m)$ of colors in $\mathbf{O}$.
(2) For two objects $(x_1, \dots, x_m), (y_1, \dots, y_n)$ in $\mathcal{O}$, we define the set of morphisms
$$\mathcal{O}((x_1, \dots, x_m), (y_1, \dots, y_n)) \coloneqq \coprod_{\alpha \in \mathrm{Hom}(\langle m \rangle, \langle n \rangle)} \prod_{1 \leq j \leq n} \mathbf{O}(\{x_i\}_{\alpha(i)=j}, y_j).$$
(3) The composition in $\mathcal{O}$ is induced by that of $\mathbf{F}_*$ and $\mathbf{O}$.
(4) The map $\mathcal{O} \to \mathbf{F}_*$ is the obvious projection map.

**Notation 7.1.9.** We let $\mathcal{P}rpd_S \to \mathbf{F}_*$ denote the $\infty$-operad associated to the simplicial operad $\mathbf{Prpd}_S$.

If $S$ is a set and $\mathcal{V}$ is a symmetric monoidal $\infty$-category, then we can consider the $\infty$-category $\mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{V})$ of algebras over $\mathcal{P}rpd_S$. We should consider objects of this $\infty$-category as $S$-colored, $\mathcal{V}$-enriched $\infty$-properads. Our next main goal is to show this is reasonable, by proving in Corollary 7.1.20 that this $\infty$-category is equivalent to the $\infty$-category $\mathrm{Alg}_{\mathbf{G}_S^{\mathrm{op}}}(\mathcal{V})$ from Definition 3.2.32. The stated equivalence utilizes a relationship between $\mathcal{P}rpd_S$ and $\mathbf{G}_S^{\mathrm{op}}$. Our initial task is to define a family of functors $\Theta_S \colon \mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$. Before doing so, we explain in the next remark why we do not use the category $\mathbf{K}$, and then we replace $\mathbf{G}$ with a convenient, equivalent category.

**Remark 7.1.10.** In Proposition 2.2.23, we studied a functor $\mathtt{V}_{\mathbf{G}} \colon \mathbf{G} \to \mathrm{Fin}_*^{\mathrm{op}}$ which took a graph to its set of vertices. There is no corresponding functor from $\mathbf{K}$ (Definition 2.0.4), or even from $\mathbf{K}_{\mathrm{int}}$. Indeed, consider the graphs from Example 2.2.13. In $\mathbf{K}_{\mathrm{int}}$ there is an étale map from $G$ to $K$ which takes each $u_i$ to $u$ and each $v_i$ to $v$, so it is unclear how one should construct a meaningful base point preserving function
$$\mathtt{V}(K)_+ = \{u, v, *\} \to \{u_0, u_1, v_0, v_1, *\} = \mathtt{V}(G)_+$$
in the same manner. Certainly the rule (2) from Definition 2.2.22 is not single-valued. As there is no meaningful vertex functor $\mathbf{K} \to \mathrm{Fin}_*^{\mathrm{op}}$, there will not be a meaningful functor $\mathbf{K}^{\mathrm{op}} \to \mathcal{P}rpd$.

A variant of the following notion appeared in Appendix A.

**Notation 7.1.11** (Ordered variant of $\mathbf{G}$)**.** For the remainder of this section, we will work with the category whose objects are graphs together with orderings on the sets $\mathtt{in}(v)$, $\mathtt{out}(v)$, and also on $\mathtt{V}(G)$ (but not on $\mathtt{in}(G)$ or $\mathtt{out}(G)$). Morphisms are just morphisms in $\mathbf{G}$, that is, they ignore this extra structure. This category is equivalent to the usual $\mathbf{G}$. To avoid clutter, we will simply write $\mathbf{G}$ for this category until the end of §7.1.

**Definition 7.1.12.** Define a functor $\Theta \colon \mathbf{G}^{\mathrm{op}} \to \mathcal{P}rpd$ as follows. Each graph in $\mathbf{G}$ comes equipped with a chosen ordering of the vertices. On objects, send a graph $G$ to $(\vec{k}_1, \dots, \vec{k}_n)$ where $\vec{k}_a = (|\mathtt{in}(v_a)|, |\mathtt{out}(v_a)|)$. Now suppose that $f \colon H \to G$ is a morphism in $\mathbf{G}$. We already know there is a functor $\mathtt{V}_{\mathbf{G}} \colon \mathbf{G}^{\mathrm{op}} \to \mathrm{Fin}_* \simeq \mathbf{F}_*$ from just above Proposition 2.2.23, which is one part of the morphism $\Theta(f)$. That is, we have a morphism $q \colon \langle n \rangle \to \langle m \rangle$, so that $q(a) = b$ means that $v_a \in \mathtt{V}(G)$ is a vertex of the structured subgraph $f(C_{w_b}) \in \mathtt{Sb}(G)$. On the other hand, for $b = 1, \dots, m$, we

write $G_b$ for the structured subgraph $f(C_{w_b}) \in \mathtt{Sb}(G)$ together with the following data:

- the induced bijection

$$\begin{array}{ccc} q^{-1}(b) & \xrightarrow{\cong} & \mathtt{V}(f(C_{w_b})) \\ \downarrow\subseteq & & \downarrow\subseteq \\ \langle n \rangle & \xrightarrow{\cong} & \mathtt{V}(G), \end{array}$$

- for each $v \in \mathtt{V}(f(C_{w_b}))$, an ordering on $\mathtt{in}(v)$ and $\mathtt{out}(v)$ from the corresponding orderings in $G$, and
- an ordering on $\mathtt{in}(f(C_{w_b}))$ and $\mathtt{out}(f(C_{w_b}))$ from the corresponding orderings on $\mathtt{in}(w_b)$ and $\mathtt{out}(w_b)$.

In this way, $G_b$, for $b = 1, \dots, m$ is considered as an element of

$$\mathbf{Prpd}(\{\vec{k}_a\}_{a \in q^{-1}(b)}; \vec{\jmath}_b).$$

Declare the value of $\Theta(f)$ to be

$$(G_1, \dots, G_m) \in \prod_{b=1}^{m} \mathbf{Prpd}(\{\vec{k}_a\}_{q^{-1}(b)}; \vec{\jmath}_b) \subseteq_q \mathcal{P}rpd(\vec{k}_1, \dots, \vec{k}_n; \vec{\jmath}_1, \dots, \vec{\jmath}_m).$$

**Lemma 7.1.13.** *$\Theta$ is a functor.*

*Proof.* For composition, suppose we have

$$\begin{array}{lll} K \xrightarrow{g} H \xrightarrow{f} G & & \text{in } \mathbf{G} \\ \langle \ell \rangle \xleftarrow{p} \langle m \rangle \xleftarrow{q} \langle n \rangle & & \text{in } \mathbf{F}_* \end{array}$$

with

$$(H_1, \dots, H_\ell) \in \prod_{c=1}^{\ell} \mathbf{Prpd}(\{\vec{\jmath}_b\}_{p^{-1}(c)}; \vec{\imath}_c) \subseteq_p \mathcal{P}rpd(\vec{\jmath}_1, \dots, \vec{\jmath}_m; \vec{\imath}_1, \dots, \vec{\imath}_\ell)$$

equaling $\Theta(g)$,

$$(J_1, \dots, J_\ell) \in \prod_{c=1}^{\ell} \mathbf{Prpd}(\{\vec{k}_a\}_{(pq)^{-1}(c)}; \vec{\imath}_c) \subseteq_{pq} \mathcal{P}rpd(\vec{k}_1, \dots, \vec{k}_n; \vec{\imath}_1, \dots, \vec{\imath}_\ell)$$

equaling $\Theta(fg)$, and $\Theta(f) = (G_1, \dots, G_m)$ as given in Definition 7.1.12. Note that $J_c$ is the graph $f(g(C_{u_c})) \in \mathtt{Sb}(G)$ together with the bijection $(pq)^{-1}(c) \to \mathtt{V}(f(g(C_{u_c})))$ and the above indicated orderings on inputs and outputs.

On the other hand, the composition $\Theta(g)\Theta(f)$ is also in the $pq$ component of $\mathcal{P}rpd(\vec{k}_1, \dots, \vec{k}_n; \vec{\imath}_1, \dots, \vec{\imath}_\ell)$. Its $c$th projection is given by applying the operadic composition

$$\begin{array}{c} \mathbf{Prpd}(\{\vec{\jmath}_b\}_{p^{-1}(c)}; \vec{\imath}_c) \times \displaystyle\prod_{b \in p^{-1}(c)} \mathbf{Prpd}(\{\vec{k}_a\}_{q^{-1}(b)}; \vec{\jmath}_b) \\ \Big\downarrow \\ \mathbf{Prpd}(\{\vec{k}_a\}_{(pq)^{-1}(c)}; \vec{\imath}_c) \end{array}$$

to $H_c, \{G_b\}_{b\in p^{-1}(c)}$. This graph substitution $H_c\{G_b\}_{b\in p^{-1}(c)}$ is isomorphic to $J_c$ since

$$\begin{array}{ccc} \coprod_{p^{-1}(c)} q^{-1}(b) & \xrightarrow{\cong} & (pq)^{-1}(c) \\ \downarrow\cong & & \downarrow\cong \\ \coprod_{\mathtt{V}(g(C_{u_c}))} \mathtt{V}(f(C_{w_b})) & \xrightarrow{\cong} & \mathtt{V}(f(g(C_{u_c}))) \end{array}$$

commutes, and the input / output orderings are induced from the same places. Thus $\Theta(fg) = \Theta(g)\Theta(f)$. □

**Definition 7.1.14.** Given a set $S$, we let $\mathbf{G}_S$ be the category where an object consists of a graph $G$ in $\mathbf{G}$ together with a function $\mathtt{E}(G) \to S$. Morphisms should respect the coloring function. In other words, if $\mathtt{E}\colon \mathbf{G} \to \mathrm{Set}$ is the functor which takes $G$ to its set of edges $\mathtt{E}(G)$, then $\mathbf{G}_S$ is the comma category $\mathtt{E} \downarrow S$. Analogously we define the categories $\mathbf{G}_{\mathrm{out},S}$ and $\mathbf{G}_{\mathrm{sc},S}$.

The category $\mathbf{G}_S$ is given by applying the construction of Notation 3.2.31 to the special case where $\mathbf{\Xi} = \mathbf{G}$ and $S$ is a discrete space. We will not need such generality here, and our $\mathbf{G}_S^{\mathrm{op}}$ (for $S$ a set) will be an ordinary category, rather than an $\infty$-category. Moreover, notice that $\mathbf{G}_S \cong \mathbf{G}$ if $S = *$.

**Notation 7.1.15.** For a set $S$, the functor $\Theta\colon \mathbf{G}^{\mathrm{op}} \to \mathcal{P}rpd$ from Definition 7.1.12 naturally extends to a functor $\mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$ which we denote by $\Theta_S$.

We want to prove that the functor $\Theta_S$ is an *approximation* in the sense of [Lur, Definition 2.3.3.6]:

**Definition 7.1.16.** Given an $\infty$-operad $p\colon \mathcal{O} \to \mathbf{F}_*$ and an $\infty$-category $\mathcal{C}$. We say a functor $f\colon \mathcal{C} \to \mathcal{O}$ is an *approximation to* $\mathcal{O}$, if it satisfies the following conditions:

(1) Suppose $p' = p \circ f$, $c \in \mathcal{C}$ is an object and $p'(c) = \langle n\rangle$. For every $1 \le i \le n$, the inert map $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ has a locally $p'$-cocartesian lift $\tilde{\rho}^i\colon c \to c_i$ in $\mathcal{C}$ such that $f(\tilde{\rho}^i)$ in $\mathcal{O}$ is inert.
(2) Every active morphism $\alpha\colon x \to f(c)$ in $\mathcal{O}$ has an $f$-cartesian lift $\tilde{\alpha}\colon \tilde{x} \to c$ in $\mathcal{C}$.

**Proposition 7.1.17.** *The functor $\Theta_S\colon \mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$ is an approximation to $\mathcal{P}rpd_S$.*

*Proof.* For simplicity, we restrict the proof to the case where $S$ is a point. The general case is similar.

(1) Let $p\colon \mathcal{P}rpd \to \mathbf{F}_*$ be the structure map and let $p' = p \circ \Theta$. Then it follows from the definition of $p$ and $\Theta$ that for each $G$ with $p'(G) = \langle n\rangle$, every inert map $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ has a unique inert lift $g\colon G \to C_i$ in $\mathbf{G}^{\mathrm{op}}$. Notice that $\Theta(g)$, which is represented by

$$\mathrm{id}_{\vec{k}_i} \in \mathbf{Prpd}(\vec{k}_i; \vec{k}_i) \subseteq_{\rho^i} \mathcal{P}rpd(\vec{k}_1, \dots, \vec{k}_n; \vec{k}_i)$$

is $p$-cocartesian, hence it is inert in $\mathcal{P}rpd$. It remains to show that $g$ is locally $p'$-cocartesian. That is, given the pullback

$$\begin{array}{ccc} \mathbf{G}_i^{\mathrm{op}} & \longrightarrow & \mathbf{G}^{\mathrm{op}} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p'} \\ \Delta^1 & \xrightarrow{\rho^i} & \mathbf{F}_* \end{array}$$

we must show that the morphism $(g, 0 \to 1)$ in $\mathbf{G}_i^{\mathrm{op}}$ is $q$-cocartesian. Hence, for every corolla $C' \in \mathbf{G}^{\mathrm{op}}$, we have to show that the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{G}_i^{\mathrm{op}}}((C,1),(C',1)) & \longrightarrow & \mathrm{Map}_{\mathbf{G}_i^{\mathrm{op}}}((G,0),(C',1)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\Delta^1}(1,1) & \longrightarrow & \mathrm{Map}_{\Delta^1}(0,1) \end{array}$$

is a pullback square. This is automatic as the horizontal maps are equivalences.

(2) Let $G$ be an object of $\mathbf{G}$ and let

$$(\vec{j}_1, \dots, \vec{j}_m) \to \Theta(G) = (\vec{k}_1, \dots, \vec{k}_n) \tag{♡}$$

be an active morphism of $\mathcal{P}rpd$ lying over $\alpha \colon \langle m \rangle \to \langle n \rangle$, which is exhibited by

$$(H_a)_a \in \prod_{a=1}^{n} \mathbf{Prpd}(\{\vec{j}_b\}_{b \in \alpha^{-1}(a)}; \vec{k}_a).$$

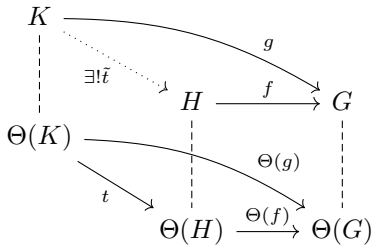
Suppose that $G'$ is $G$ with some ordering of $\mathtt{in}(G)$ and $\mathtt{out}(G)$ and that $\vec{k}$ is $(|\mathtt{in}(G)|, |\mathtt{out}(G)|)$. Let $H'$ be the image of $(G', (H_a)_a)$ under the operadic composition

$$\mathbf{Prpd}(\vec{k}_1, \dots, \vec{k}_n; \vec{k}) \times \prod_{a=1}^{n} \mathbf{Prpd}(\{\vec{j}_b\}_{b \in \alpha^{-1}(a)}; \vec{k}_a) \to \mathbf{Prpd}(\vec{j}_1, \dots, \vec{j}_m; \vec{k})$$

and let $H \in \mathbf{G}$ be the graph obtained by forgetting the orderings on inputs and outputs of $H'$. We then have an active map $f \colon H \to G$ of $\mathbf{G}^{\mathrm{op}}$ which sends $H_a \in \mathtt{Sb}(H)$ to $C_a \in \mathtt{Sb}(G)$ (using [HRY15, Theorem 6.50] since $G\{H_a\} = H$), and the image of this map under $\Theta$ is (♡). The map $f$ is our proposed $\Theta$-cartesian lift of (♡).

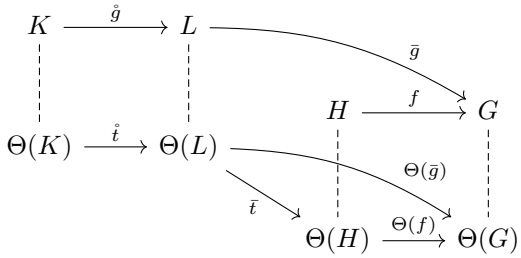
Suppose that we are in the situation of having $g \colon K \to G$ in $\mathbf{G}^{\mathrm{op}}$ and $t \colon \Theta(K) \to \Theta(H)$ in $\mathcal{P}rpd$ satisfying $\Theta(f)t = \Theta(g)$.

(♢)

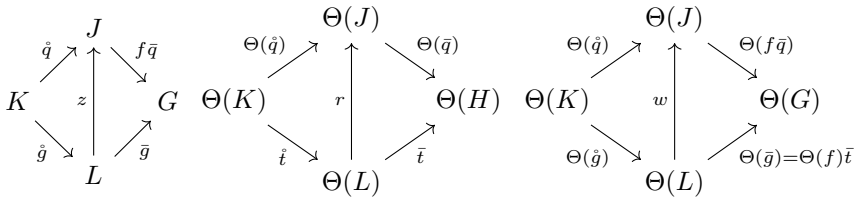


Our goal is to show there exists a unique $\tilde{t}\colon K \to H$ so that $\Theta(\tilde{t}) = t$ and $f\tilde{t} = g$.

We first reduce to the active case. Observe that if $g = \bar{g}\mathring{g}\colon K \to L \to G$ is a decomposition of $g$ with $\mathring{g}$ inert and $\bar{g}$ active, then $\Theta(\mathring{g}) =: \mathring{t}$ is part of a similar decomposition $t = \bar{t}\mathring{t}$ since $\Theta(f)$ is active.




If we knew that there is a unique $s\colon L \to H$ so that $fs = \bar{g}$ and $\Theta(s) = \bar{t}$, then $s\mathring{g}$ gives existence of a solution to ($\diamondsuit$). Further, $s$ must be active since $f$ and $\bar{g}$ are active. Suppose that $q$ is some other solution satisfying $\Theta(q) = t$ and $fq = g$, and write $q = \bar{q}\mathring{q}$ for an inert-active factorization $K \to J \to H$. There are unique isomorphisms $z\colon L \to J$, $r\colon \Theta(L) \to \Theta(J)$, and $w\colon \Theta(L) \to \Theta(J)$ making the following diagrams commute:




As replacing $w$ in the third diagram by either $\Theta(z)$ or $r$ makes it commute, we have that $r = \Theta(z)$. Since $f\bar{q}z = \bar{g} = fs$ and $\Theta(\bar{q}z) = \Theta(\bar{q})r = \bar{t}$, we see that $\bar{q}z = s$. It follows that $s\mathring{g} = \bar{q}z\mathring{g} = \bar{q}\mathring{q} = q$. Hence we have showed that $s\mathring{g}$ is the *unique* solution to ($\diamondsuit$).

It remains to show that ($\diamondsuit$) has a unique solution in the case when $g$ is active. Let the morphism $t\colon \Theta(K) = (\vec{\imath}_1, \dots, \vec{\imath}_\ell) \to \Theta(H)$ lying over $\beta\colon \langle \ell \rangle \to \langle m \rangle$ be exhibited by

$$(K_b) \in \prod_{b=1}^{m} \mathbf{Prpd}(\{\vec{\imath}_c\}_{c \in \beta^{-1}(b)}; \vec{\jmath}_b).$$

Commutativity of the bottom triangle of ($\diamondsuit$) is simply the assertion if $L_a$ is the image of $H_a, \{K_b\}_{b \in \alpha^{-1}(a)}$ under operadic composition

$$\mathbf{Prpd}(\{\vec{\jmath}_b\}_{b \in \alpha^{-1}(a)}; \vec{k}_a) \times \prod_{b \in \alpha^{-1}(a)} \mathbf{Prpd}(\{\vec{\imath}_c\}_{c \in \beta^{-1}(b)}; \vec{\jmath}_b) \to \mathbf{Prpd}(\{\vec{\imath}_c\}_{c \in \beta^{-1}\alpha^{-1}(a)}; \vec{k}_a),$$

then

$$(L_a) \in \prod_{a=1}^{n} \mathbf{Prpd}(\{\vec{\imath}_c\}_{c \in \beta^{-1}\alpha^{-1}(a)}; \vec{k}_a).$$

exhibits $\Theta(g)\colon \Theta(K) \to \Theta(G)$. But as graphs,

$$g(C_a) \cong L_a \cong H_a\{K_b\}_{b \in \alpha^{-1}(a)}, \tag{17}$$

so since $g(C_a)$ is a structured subgraph of $K$, so is $K_b$. Define $\tilde{t}\colon K \to H$ in $\mathbf{G}^{\mathrm{op}}$ by setting $\tilde{t}(C_b) = K_b$. This is a graphical map by [HRY15, Theorem 6.50] since its image under $H\{K_b\} \cong G\{H_a\}\{K_b\} \cong G\{H_a\{K_b\}_{b\in\alpha^{-1}(a)}\} \cong G\{L_a\} \cong K$ is a structured subgraph of $K$. By construction, $\Theta(\tilde{t}) = t$, and $f\tilde{t} = g$ by (17). Finally, $\Theta$ is a faithful functor, so $\tilde{t}$ is unique. □

**Corollary 7.1.18.** *Let $\mathcal{P}rpd_S^{\mathtt{out}}$ be the $\infty$-operad associated to the operad $\mathbf{Prpd}_S^{\mathtt{out}}$ defined in Example 7.1.6. Then $\Theta_S\colon \mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$ restricts to an approximation $\mathbf{G}_{\mathtt{out},S}^{\mathrm{op}} \to \mathcal{P}rpd_S^{\mathtt{out}}$.*

*Proof.* According to [Lur, Remark 2.3.3.9] the pullback of the approximation $\Theta_S\colon \mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$ along the morphisms $\mathcal{P}rpd_S^{\mathtt{out}} \to \mathcal{P}rpd_S$ induced by the canonical inclusions $\mathbf{Prpd}_S^{\mathtt{out}} \to \mathbf{Prpd}_S$ is again an approximation and the construction of $\Theta_S$ implies that the pullback $\mathbf{G}_S^{\mathrm{op}} \times_{\mathcal{P}rpd_S} \mathcal{P}rpd_S^{\mathtt{out}}$ coincides with $\mathbf{G}_{\mathtt{out},S}^{\mathrm{op}}$ introduced in Definition 7.1.14. □

Let $\mathcal{DO}pd_S$ be the $\infty$-operad associated to the operad $\mathbf{DOpd}_S$ defined in Example 7.1.6. Contrary to the previous corollary $\mathbf{G}_{\mathrm{sc},S}^{\mathrm{op}}$ is not given by the pullback of $\Theta_S$ along the canonical inclusion $\mathcal{DO}pd_S \to \mathcal{P}rpd_S$. Nevertheless, a small adaptation of the proof of Proposition 7.1.17 yields the following result.

**Lemma 7.1.19.** *The functor $\Theta_S\colon \mathbf{G}_S^{\mathrm{op}} \to \mathcal{P}rpd_S$ restricts to an approximation $\mathbf{G}_{\mathrm{sc},S}^{\mathrm{op}} \to \mathcal{DO}pd_S$.*

*Proof.* We assume that $S = *$ as the general case can be proven analogously. After replacing **Prpd** with **DOpd** in the proof of Proposition 7.1.17 we see that the first part of the proof is still valid since the graph corresponding to $\mathrm{id}_{\vec{k}_i}$ is a corolla and in particular simply-connected. The second part of the proof is also not affected by the change because $\mathbf{G}_{\mathrm{sc}}^{\mathrm{op}}$ is closed under graph substitutions ($\mathbf{G}_{\mathrm{sc}}^{\mathrm{op}}$ is a full subcategory of $\mathbf{G}^{\mathrm{op}}$). □

The following corollary is an easy application of Proposition 7.1.17.

**Corollary 7.1.20.** *For every symmetric monoidal $\infty$-category $\mathcal{V}$, the precomposition with $\Theta_S$ induces an equivalence*

$$\Theta_S^*\colon \mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbf{G}_S^{\mathrm{op}}}(\mathcal{V}).$$

*Proof.* Since $\Theta_S$ obviously restricts to an equivalence $\mathbf{G}_S^{\mathrm{op}} \times_{\mathbb{F}_*} \{\langle 1\rangle\} \xrightarrow{\sim} \mathcal{P}rpd_{S,\langle 1\rangle}$ of fibres over $\langle 1\rangle$, by [Lur, Theorem 2.3.3.23] the functor $\Theta_S^*$ is an equivalence. □

A similar statement holds in the $\mathbf{G}_{\mathrm{out},S}^{\mathrm{op}}$ / $\mathcal{P}rpd_S^{\mathtt{out}}$ and $\mathbf{G}_{\mathrm{sc},S}^{\mathrm{op}}$ / $\mathcal{DO}pd_S$ contexts as well, by using Corollary 7.1.18 and Lemma 7.1.19 instead of Proposition 7.1.17. Indeed, the four remaining items from this subsection have analogues in both of these contexts.

**Definition 7.1.21.** We write $\mathrm{Alg}_{\mathcal{P}rpd/\mathrm{Set}}(\mathcal{V}) \to \mathrm{Set}$ for the cartesian fibration corresponding to the functor $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ taking $S$ to $\mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{V})$, and we let $\mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathrm{Set}}(\mathcal{V}) \to \mathrm{Set}$ denote the pullback of the cartesian fibration $\mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathcal{S}}(\mathcal{V}) \to \mathcal{S}$ along the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$. Since the functors $\Theta_S$ are natural in $S$, they induce a functor for which we write

$$\Theta^*\colon \mathrm{Alg}_{\mathcal{P}rpd/\mathrm{Set}}(\mathcal{V}) \to \mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathrm{Set}}(\mathcal{V})$$

of cartesian fibrations over Set.

**Corollary 7.1.22.** *The functor*

$$\Theta^*\colon \mathrm{Alg}_{\mathcal{P}rpd/\mathrm{Set}}(\mathcal{V}) \to \mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathrm{Set}}(\mathcal{V})$$

*is an equivalence.*

*Proof.* The functor $\Theta^*$ is an equivalence because it is one at each fibre by Corollary 7.1.20. □

**Proposition 7.1.23.** *For every presentably symmetric monoidal $\infty$-category $\mathcal{V}$, the inclusion $\mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathrm{Set}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathcal{S}}(\mathcal{V})$ induces an equivalence*

$$\mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Alg}_{\mathbf{G}^{\mathrm{op}}/\mathcal{S}}(\mathcal{V})[\mathrm{FFES}^{-1}]$$

*after localizing with respect to the class* FFES *of fully faithful and essential surjective functors.*

*Proof.* It can be proven in as in [GH15, Theorem 5.3.17]. □

**Corollary 7.1.24.** *For every presentably symmetric monoidal $\infty$-category $\mathcal{V}$, there is equivalence of $\infty$-categories*

$$\mathrm{Alg}_{\mathcal{P}rpd/\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}] \simeq \mathrm{Seg}^{\mathrm{rep}}(\mathbf{G}^{\mathcal{V}})[\mathrm{FFES}^{-1}] \simeq \mathrm{Prpd}^{\mathcal{V}}_{\infty}.$$

*Proof.* The first equivalence is induced by that of Corollary 7.1.22, Proposition 7.1.23 and Theorem 3.2.33, while the second is given by Theorem 6.2.9 and Notation 6.2.10. □

7.2. **Rectification.** In this subsection we compare our $\infty$-categorical definition of $\mathcal{V}$-enriched $\infty$-properads with the strict notion of properad enriched in a symmetric monoidal model category $\mathbf{V}$. One of our main findings, Theorem 7.2.5, is that it is not always possible to rectify an enriched $\infty$-properad to a strict one. We show in Theorem 7.2.10 that it is possible to perform such rectification when working over a field of characteristic zero. The situation is very different for dioperads and for output properads, where we prove a rectification result over an arbitrary base in Theorem 7.2.9.

**Definition 7.2.1.** Let $\mathbf{V}$ be a simplicial symmetric monoidal model category. We call an operad $\mathbf{O}$ *admissible in* $\mathbf{V}$ (alternatively, $\mathbf{V}$ is admissible for $\mathbf{O}$) if there is a model structure on $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ such that the weak equivalences and fibrations are those maps whose underlying maps in $\mathbf{V}$ are weak equivalences and fibrations, respectively.

**Examples 7.2.2.** According to [PS18b, §7] following model categories are admissible for *all* operads:

(i) the category of simplicial sets, equipped with the Kan–Quillen model structure,
(ii) the category of compactly generated weak Hausdorff spaces, equipped with the usual model structure,
(iii) the category of chain complexes of $k$-vector spaces, where $k$ is a field of characteristic 0 (or more generally a commutative ring containing $\mathbb{Q}$), equipped with the projective model structure, and
(iv) the category of symmetric spectra, equipped with the positive stable model structure.

Moreover, by [NS17] we know that for any presentably symmetric monoidal $\infty$-category $\mathcal{V}$ there exists a symmetric monoidal simplicial combinatorial model category modeling $\mathcal{V}$ for which all (simplicial) operads are admissible.

We are of course especially interested in the operad $\mathbf{Prpd}$. The relevant model structure for properads in chain complexes over a field of characteristic 0 was first constructed in the appendix of [MV09].

**Definition 7.2.3.** If $\mathbf{V}$ is a simplicial symmetric monoidal model category and an operad $\mathbf{O}$ which is admissible for $\mathbf{V}$, and let $\mathcal{V}$, $\mathcal{O}$ denote the associated $\infty$-category and $\infty$-operad, respectively. We refer to the map

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

as the *canonical map*. Here, we regard $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ as the $\infty$-category associated to the model category $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$, i.e. we implicitly identify it with the nerve of the localization of the full subcategory of $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V})$ spanned by cofibrant objects with respect to weak equivalences.

**Theorem 7.2.4.** *If* $\mathbf{O}$ *is either of* $\mathbf{Prpd}_S^{\mathrm{out}}$ *or* $\mathbf{DOpd}_S$ *and is admissible in* $\mathbf{V}$*, then the canonical map*

$$\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

*is an equivalence.*

*Proof.* The proof of the theorem is based on the verification of the assumptions of [PS18a, Theorem 7.11] which gives a necessary and sufficient condition for under which the canonical map is an equivalence. By Remark 7.1.7, the operads $\mathbf{Prpd}_S^{\mathrm{out}}$ and $\mathbf{DOpd}_S$ are $\Sigma$-free which implies that the symmetric flatness condition of [PS18a, Theorem 7.11] is satisfied. This gives the equivalence $\mathrm{Alg}_{\mathbf{O}}(\mathbf{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$. □

Theorem 7.2.4 is a key component in the proof of our Rectification Theorem 7.2.9 below. Unfortunately it is a statement about dioperads and output (or input) properads, rather than about general properads. This is simply because it does not hold in generality:

**Theorem 7.2.5.** *Let* sSet *denote the model category of simplicial sets equipped with the Kan–Quillen model structure, and suppose that* $S$ *is a nonempty set. Then the canonical map of Definition 7.2.3*

$$\mathrm{Alg}_{\mathbf{Prpd}_S}(\mathrm{sSet}) \to \mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{S})$$

*is* not *an equivalence.*

*Proof.* Let $\mathbf{P} = \mathbf{Prpd}_S$ and let $\phi\colon \mathbf{O} \to \mathbf{P}$ be a $\Sigma$-cofibrant replacement of $\mathbf{P}$ in the category of $T$-colored simplicial operads (by taking, for instance, a suitable product of $\mathbf{P}$ with the Barratt–Eccles operad), where $T$ is the color set of $\mathbf{P}$ (see Definition 7.1.4). We let $x \in \mathbf{P}(a, a, b, b; c)$ be the left graph from Example 2.2.13, where all edges are colored by some fixed $s \in S$ (so $a = (\ ; s, s)$, $b = (s, s;\ )$, and $c = (\ ;\ )$ are elements of $T$). Let $V = \{e, (12), (34), (12)(34)\} \subseteq \Sigma_4$ be the group of permutations fixing the profile $a, a, b, b$ and let $\Sigma_2 = \{e, (12)(34)\} \subseteq V$. As mentioned in Remark 7.1.7, the stabilizer group of $x$ is $\Sigma_2$. Let $X \subseteq \mathbf{O}(a, a, b, b; c)$ be the fiber over $x$, which is a summand of $\mathbf{O}(a, a, b, b; c)$ (since $\mathbf{P}$ is a discrete colored operad). We have that $X = \phi^{-1}(x) \to \{x\}$ is a weak equivalence, and using that $\mathbf{O}$ is $\Sigma$-cofibrant, one can show that the $\Sigma_2$-action on $X$ is free.

Let $F_{\mathbf{P}}\colon \mathrm{sSet}^T \to \mathrm{Alg}_{\mathbf{P}}(\mathrm{sSet})$ be the free $\mathbf{P}$-algebra functor, and likewise for $F_{\mathbf{O}}$. Letting $*_T$ be the terminal $T$-colored object, we have a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \{x\} \\ \downarrow & & \downarrow \\ \mathbf{O}(a,a,b,b;c) & \longrightarrow & \mathbf{P}(a,a,b,b;c) \\ \downarrow & & \downarrow \\ F_{\mathbf{O}}(*_T)_c & \longrightarrow & F_{\mathbf{P}}(*_T)_c \end{array}$$

where the vertical composites are surjective onto summands. More specifically, $\mathbf{O}(a,a,b,b;c)/V$ is naturally a summand of $F_{\mathbf{O}}(*_T)_c$, and $\phi^{-1}(x\cdot V) \subseteq \mathbf{O}(a,a,b,b;c)$ is a $V$-summand, so

$$X/\Sigma_2 \cong \phi^{-1}(x\cdot V)/V$$

is a summand of $\mathbf{O}(a,a,b,b;c)/V$.

As the left vertical map factors through $X/\Sigma_2 \simeq B\Sigma_2$, the map $F_{\mathbf{O}}(*_T)_c \to F_{\mathbf{P}}(*_T)_c$ has a summand of the form $B\Sigma_2 \to \{x\}/\Sigma_2 = *$. It follows that $F_{\mathbf{O}}(*_T) \to \phi^* F_{\mathbf{P}}(*_T)$ is not a weak equivalence of $\mathbf{O}$-algebras.

We are now in a position to see that

$$\phi_!\colon \mathrm{Alg}_{\mathbf{O}}(\mathrm{sSet}) \rightleftarrows \mathrm{Alg}_{\mathbf{P}}(\mathrm{sSet})\colon \phi^*$$

is not a Quillen equivalence. Both model structures are right-induced from the forgetful functors to $\mathrm{sSet}^T$, hence $\phi^*$ creates weak equivalences and fibrations. In particular, $\phi^*$ is right Quillen and reflects weak equivalences between fibrant objects, so by [Hov99, Corollary 1.3.16], $\phi_! \dashv \phi^*$ is a Quillen equivalence if and only if $A \to \phi^*(\phi_! A)^f$ is a weak equivalence for all cofibrant $A$ in $\mathrm{Alg}_{\mathbf{O}}(\mathrm{sSet})$. But the fibrant replacement $\phi_! A \to (\phi_! A)^f$ is a weak equivalence and $\phi^*$ preserves weak equivalences, so by 2-of-3, $\phi_! \dashv \phi^*$ is a Quillen equivalence if and only if $A \to \phi^*\phi_! A$ is a weak equivalence for all cofibrant $A$ in $\mathrm{Alg}_{\mathbf{O}}(\mathrm{sSet})$.

The algebra $A = F_{\mathbf{O}}(*_T)$ is cofibrant. Since the diagram

$$\begin{array}{ccc} \mathrm{Alg}_{\mathbf{O}}(\mathrm{sSet}) & \xleftarrow{\phi^*} & \mathrm{Alg}_{\mathbf{P}}(\mathrm{sSet}) \\ & \searrow \quad \swarrow & \\ & \mathrm{sSet}^T & \end{array}$$

commutes, we have $\phi_! A = \phi_! F_{\mathbf{O}}(*_T) \cong F_{\mathbf{P}}(*_T)$. But as we saw, $A = F_{\mathbf{O}}(*_T) \to \phi^*\phi_! F_{\mathbf{O}}(*_T) = \phi^* F_{\mathbf{P}}(*_T)$ is not a weak equivalence, so $\phi_! \dashv \phi^*$ is not a Quillen equivalence.

Since this is not a Quillen equivalence, [PS18a, Theorem 7.5] shows that $\phi$ is not symmetric flat in sSet. Using their terminology, $\mathbf{O} = Q\mathbf{P}$, so the fact that $\phi\colon \mathbf{O} \to \mathbf{Prpd}_S$ is not symmetric flat implies, by [PS18a, Theorem 7.11], that $\mathrm{Alg}_{\mathbf{Prpd}_S}(\mathrm{sSet}) \to \mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{S})$ is not an equivalence. □

**Definition 7.2.6.** Given a symmetric monoidal model category $\mathbf{V}$ for which the operads $\mathbf{Prpd}_S$ are admissible for all sets $S$. Let $\mathcal{V}$ denote the associated $\infty$-category. We write $\mathrm{Prpd}(\mathbf{V}) \to \mathrm{Set}$ for the Grothendieck fibration (and opfibration) which corresponds to the functor $\mathrm{Set} \to \mathrm{Cat}$ taking $S$ to $\mathrm{Alg}_{\mathbf{Prpd}_S}(\mathbf{V})$. It follows from [Hau15, Proposition 4.25] or [HP15, Theorem 3.0.12] that the model structure on $\mathrm{Alg}_{\mathbf{Prpd}_S}(\mathbf{V})$ induces one on $\mathrm{Prpd}(\mathbf{V})$ where weak equivalences are those morphisms

that are bijective on objects and weak equivalences on all multimorphism objects. We write $\mathrm{Prpd}(\mathcal{V})$ for the $\infty$-category associated to the model category $\mathrm{Prpd}(\mathbf{V})$. Similarly, we define $\infty$-categories $\mathrm{Prpd}^{\mathrm{out}}(\mathcal{V})$ and $\mathrm{DOpd}(\mathcal{V})$ when $\mathbf{V}$ is a symmetric monoidal model category so that the operads $\mathbf{Prpd}_S^{\mathrm{out}}$ (respectively, $\mathbf{DOpd}_S$) are admissible for all sets $S$.

We should emphasize here that the model structures $\mathrm{Prpd}(\mathbf{V})$ and so on are only *intermediate* model structures, and not of independent interest for us. The reader should also be careful to distinguish between this model structure and others that may exist on the same underlying category. For example (with some restriction on $\mathbf{V}$) there is a model structure on $\mathrm{Prpd}(\mathbf{V})$ whose weak equivalences are the Dwyer–Kan equivalences from Definition 7.2.8 below (see [HRY17] for the case $\mathbf{V} = \mathrm{sSet}$ and [Yau] for certain other $\mathbf{V}$).

**Corollary 7.2.7.** *Let $\mathbf{V}$ be a symmetric monoidal model category. If $\mathbf{Prpd}_S^{\mathrm{out}}$ is admissible in $\mathbf{V}$ for all sets $S$, then there is an equivalence*

$$\mathrm{Prpd}^{\mathrm{out}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{P}rpd^{\mathrm{out}}/\mathrm{Set}}(\mathcal{V})$$

*over* Set*. Likewise, if $\mathbf{DOpd}_S$ is admissible in $\mathbf{V}$ for all sets $S$, then there is an equivalence*

$$\mathrm{DOpd}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{DO}pd/\mathrm{Set}}(\mathcal{V})$$

*over* Set*.*

*Proof.* We prove the first statement, as the second is entirely analogous. According to [Hau15, Corollary 4.22] or [Hin16, Proposition 2.1.4], $\mathrm{Prpd}^{\mathrm{out}}(\mathbf{V}) \to \mathrm{Set}$ is the cartesian (and cocartesian) fibration corresponding to the functor taking $S$ to the $\infty$-category $\mathrm{Alg}_{\mathcal{P}rpd_S^{\mathrm{out}}}(\mathcal{V})$ associated to $\mathrm{Alg}_{\mathbf{Prpd}_S^{\mathrm{out}}}(\mathbf{V})$. This shows that the functor

$$\mathrm{Prpd}^{\mathrm{out}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{P}rpd^{\mathrm{out}}/\mathrm{Set}}(\mathcal{V})$$

over Set is a functor between cartesian fibrations that preserves cartesian morphisms which is then an equivalence as it is one on each fibre by Theorem 7.2.4. □

Suppose that $\mathbf{V}$ is a symmetric monoidal model category. The functor $\mathbf{V} \to h\mathbf{V}$ to the homotopy category is symmetric monoidal, so to a $\mathbf{V}$-enriched operad $\mathbf{P}$ we can associate an $h\mathbf{V}$-enriched operad $h\mathbf{P}$.

**Definition 7.2.8.** If $\mathbf{V}$ is a symmetric monoidal model category, we say a morphism $F\colon \mathbf{P} \to \mathbf{P}'$ of $\mathbf{V}$-enriched properads is a *Dwyer–Kan equivalence* if:

(1) The map

$$\mathbf{P}(x_1, \dots, x_m; y_1, \dots, y_n) \to \mathbf{P}'(F(x_1), \dots, F(x_m); F(y_m), \dots, F(y_n))$$

is a weak equivalence in $\mathbf{V}$ for all $x_1, \dots, x, y_1, \dots, y_n$ in $\mathbf{P}$.

(2) The induced functor of $h\mathbf{V}$-enriched operads $hF\colon h\mathbf{P} \to h\mathbf{P}'$ is essentially surjective (i.e. its underlying functor of enriched categories is essentially surjective).

Dwyer–Kan equivalences for $\mathbf{V}$-enriched dioperads are defined similarly.

We are now ready to prove our main rectification result.

**Theorem 7.2.9.** *Suppose $\mathbf{V}$ is a symmetric monoidal model category for which the operads $\mathbf{Prpd}_S^{\mathrm{out}}$ are admissible for all sets $S$. Then the $\infty$-category $\mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}}$*

*(Notation 6.2.10) can be identified with the localization of* $\mathrm{Prpd}^{\mathrm{out}}(\mathcal{V})$ *at the class of Dwyer–Kan equivalences:*

$$\mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}} \simeq \mathrm{Prpd}^{\mathrm{out}}(\mathcal{V})[\mathrm{DK}^{-1}].$$

*Likewise, if* $\mathbf{DOpd}_S$ *is admissible in* $\mathbf{V}$ *for all sets* $S$*, then the* $\infty$*-category* $\mathrm{DOpd}_\infty^{\mathcal{V}}$ *of* $\mathcal{V}$*-enriched* $\infty$*-dioperads can be identified with the localization of* $\mathrm{DOpd}(\mathcal{V})$ *at the Dwyer–Kan equivalences.*

*Proof.* We prove the first statement, as the second is similar. Under the equivalence of Corollary 7.2.7 the class of Dwyer–Kan equivalences corresponds to the class of fully faithful and essentially surjective morphisms in $\mathrm{Alg}_{\mathcal{P}rpd^{\mathrm{out}}/\mathrm{Set}}(\mathcal{V})$. Hence, the localization of $\mathrm{Prpd}^{\mathrm{out}}(\mathcal{V})$ with respect to Dwyer–Kan equivalences is equivalent to $\mathrm{Alg}_{\mathcal{P}rpd^{\mathrm{out}}/\mathrm{Set}}(\mathcal{V})[\mathrm{FFES}^{-1}]$ which can be identified with $\mathrm{Prpd}_\infty^{\mathrm{out},\mathcal{V}}$ by the appropriate variation of Corollary 7.1.24. □

In light of Theorem 7.2.5, the method of proof used in this theorem breaks down if one tries to prove that $\mathrm{Prpd}_\infty^{\mathcal{V}}$ is equivalent to $\mathrm{Prpd}(\mathcal{V})[\mathrm{DK}^{-1}]$ in generality. However, we are able to adapt this proof when working over very special bases.

**Theorem 7.2.10.** *Let* $k$ *be a commutative ring containing* $\mathbb{Q}$*, let* $\mathbf{Ch}_k$ *be the category of unbounded chain complexes equipped with the projective model structure, and let* $\mathcal{C}h_k$ *denote the* $\infty$*-category associated to* $\mathbf{Ch}_k$*. Then there is an equivalence (see Notation 6.2.10)*

$$\mathrm{Prpd}(\mathcal{C}h_k)[\mathrm{DK}^{-1}] \simeq \mathrm{Prpd}_\infty^{\mathcal{C}h_k}$$

*after localizing at the class of Dwyer–Kan equivalences.*

*Proof.* It was observed in [PS18b, §7.4] that $\mathbf{Ch}_k$ is symmetric flat, so the proof of Theorem 7.2.4 can be adapted to show that the canonical map

$$\mathrm{Alg}_{\mathbf{Prpd}_S}(\mathbf{Ch}_k) \to \mathrm{Alg}_{\mathcal{P}rpd_S}(\mathcal{C}h_k)$$

is an equivalence for all $S$. A variation on the proof of Corollary 7.2.7 gives that

$$\mathrm{Prpd}(\mathbf{Ch}_k) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{P}rpd/\mathrm{Set}}(\mathcal{C}h_k)$$

is an equivalence. The remainder of the proof follows that of Theorem 7.2.9. □

**Remark 7.2.11.** The key to the proof of the previous theorem is that the model structure on $\mathbf{Ch}_k$ is symmetric flat. In particular, it is also true that

$$\mathrm{Prpd}(\mathcal{V})[\mathrm{DK}^{-1}] \simeq \mathrm{Prpd}_\infty^{\mathcal{V}}$$

when $\mathbf{V}$ is the positive stable model structure on symmetric spectra. Indeed, [PS19, Proposition 3.5.1] shows that this $\mathbf{V}$ is symmetric flat (actually this holds for spectra over more general 'nice' bases as in [PS19, Definition 2.3.1]), so the proof of Theorem 7.2.10 is readily adapted.

**Question 7.2.12.** In Theorem 7.2.9, we showed that the $\infty$-category of $\infty$-dioperads, $\mathrm{DOpd}_\infty^{\mathcal{S}}$, is equivalent to $\mathrm{DOpd}(\mathcal{S})[\mathrm{DK}^{-1}]$. On the other hand, [HRY17, Theorem 5.4] shows that the category $\mathrm{DOpd}(\mathrm{sSet})$ (see Definition 7.2.6) admits a model structure whose weak equivalences are the Dwyer–Kan equivalences from Definition 7.2.8. We do not know if the $\infty$-category presented by this model category is equivalent to $\mathrm{DOpd}(\mathcal{S})[\mathrm{DK}^{-1}]$ or not, but it would be interesting to explore. One difficulty in addressing this in the present context is simply that the model structure from [HRY17, Theorem 5.4] is not a Bousfield localization of the model structure in

Definition 7.2.6. In fact, more is true: the identity functor on this category is not a Quillen functor between the two model structures.

Similar questions can be raised for other ground symmetric monoidal categories $\mathbf{V}$ and for properads instead of dioperads.

## Appendix A. Equivalence of $\mathbf{G}$ with the properadic graphical category

This section is devoted to a proof of the following theorem.

**Theorem A.1.** *The category* $\mathbf{G}$ *is equivalent to the category* $\mathbf{\Gamma}$ *from* [HRY15].

One difference between the setup of [HRY15] and that of the present paper is that the graphs in that book always come equipped with orderings of $\mathtt{in}(v)$, $\mathtt{out}(v)$, $\mathtt{in}(G)$, and $\mathtt{out}(G)$. As such, we first replace $\mathbf{G}$ with an equivalent category $\mathbf{G}'$ whose objects are graphs together with orderings on the sets $\mathtt{in}(v)$, $\mathtt{out}(v)$, $\mathtt{in}(G)$, and $\mathtt{out}(G)$. Morphisms ignore this extra structure entirely.

We will show that $\mathbf{G}'$ is isomorphic to the category $\mathbf{\Gamma}$ from [HRY15]. Both categories have the same set of objects.

Given a morphism $f : G \to K$ in $\mathbf{G}'$, we define a corresponding properadic graphical map $f^\gamma : G \to K$ in $\mathbf{\Gamma}$. This map has $f_0^\gamma = f_0$, while $f_1^\gamma(v) \coloneqq f_1(C_v)$ (see [HRY15, Lemma 5.19]). At the moment, this is just a map of the corresponding colored properads.

**Lemma A.2.** *If* $H \in \mathtt{Sb}(G)$*, then* $f^\gamma(H)$ *is equal to* $f_1(H)$*.*

*Proof.* We induct on $\deg(H) = |\mathtt{V}(H)|$. The result is either automatic for $\deg(H) \in \{0, 1\}$. Suppose that $\deg(H) \geq 2$. By [HRY15, Corollary 2.76], $H$ has an almost isolated vertex $v$. We have $H = C_v \tilde{\cup} H'$, where $H' \in \mathtt{Sb}(H) \subseteq \mathtt{Sb}(G)$[6] has $\mathtt{V}(H) \setminus \{v\}$ as its set of vertices (see [HRY15, Definition 2.60 & §6.1.2]). Then by the induction hypothesis, we have

$$f_1(H) = f_1(C_v) \tilde{\cup} f_1(H') = f^\gamma(C_v) \tilde{\cup} f^\gamma(H'). \tag{18}$$

If $f^\gamma(C_v) = \downarrow_e$, then $f^\gamma(H') = f^\gamma(H)$ and we are done. Otherwise, the subgraph from (18) is an open subgraph with the same set of vertices as $f^\gamma(H)$ (see [HRY15, Definition 6.40]), hence must be equal to $f^\gamma(H)$. □

*Proof of Theorem A.1.* As $\mathbf{G}'$ and $\mathbf{\Gamma}$ have the same set of objects it suffices to show that there is a bijection of morphisms which respects compositions and identities. The previous lemma shows $f^\gamma(G) = f_1(G)$ is a structured subgraph of $K$, so that the assignment $f \mapsto f^\gamma$ takes a morphism in $\mathbf{G}'$ to a properadic graphical map in $\mathbf{\Gamma}$ introduced in [HRY15, Definition 6.46].

In the reverse direction, suppose that $f : G \to K$ is a properadic graphical map in $\mathbf{\Gamma}$. Define $f^v$ by $f_0^v = f_0$ and $f_1^v(H) = f(H)$, which we know is a structured subgraph by [HRY15, Theorem 6.50]. The fact that Definition 2.2.11(1) holds follows from the fact that $f$ is a map between the corresponding colored properads, as in the second part of [HRY15, Lemma 5.19]. We have that

$$\mathtt{V}(f(H)) \cong \coprod_{v \in \mathtt{V}(H)} \mathtt{V}(f_1(v))$$

[6] Using Proposition 2.2.5.

for any $H \in \mathtt{Sb}(G)$ by [HRY15, Definition 6.40 & Remark 2.42(1)], thus

$$\mathtt{V}(f_1^\upsilon(H_1 \tilde{\cup} H_2)) = \mathtt{V}(f_1^\upsilon(H_1)) \cup \mathtt{V}(f_1^\upsilon(H_2)) = \mathtt{V}(f_1^\upsilon(H_1) \uplus f_1^\upsilon(H_2)).$$

Further, we have $f_1^\upsilon(H_1) \uplus f_1^\upsilon(H_2)$ is an open subgraph of $f_1^\upsilon(H_1 \tilde{\cup} H_2)$ since $f_1^\upsilon$ preserves the partial order $\sqsubset$. Here we have two open subgraphs which have the same set of vertices. If that vertex set is non-empty, then the containment becomes an equality.

We now show that the operations $f \mapsto f^\gamma$ and $f \mapsto f^\upsilon$ are inverse to each other. First suppose that $f$ is a morphism in $\mathbf{G}'$, we wish to show that $(f^\gamma)^\upsilon = f$. Since this is true by definition on edge sets, we must show $(f^\gamma)_1^\upsilon = f_1$. Let $H \in \mathtt{Sb}(G)$. Then we have

$$(f^\gamma)_1^\upsilon(H) = f^\gamma(H) = f_1(H)$$

where the first equality is the definition and the second equality is given by Lemma A.2.

Likewise, if $f$ is a properadic graphical map in $\mathbf{\Gamma}$, we wish to show that $(f^\upsilon)^\gamma = f$. But

$$(f^\upsilon)_0^\gamma = f_0^\upsilon = f_0$$

hence $(f^\upsilon)^\gamma = f$ by [HRY15, Corollary 6.62]. Thus we have established a bijective correspondence between morphisms of $\mathbf{G}'$ and properadic graphical maps of $\mathbf{\Gamma}$.

It remains to show that this bijection constitutes a functor between the two categories in question. But if $f, g$ are two composable morphisms in $\mathbf{G}'$, then $f^\gamma g^\gamma = (fg)^\gamma$ using [HRY15, Corollary 6.62], since

$$(f^\gamma g^\gamma)_0 = f_0^\gamma g_0^\gamma = f_0 g_0 = (fg)_0 = (fg)_0^\gamma.$$

Likewise, $(\mathrm{id}_G)_0^\gamma = (\mathrm{id}_G)_0$ is an identity, hence $(\mathrm{id}_G)^\gamma$ must be an identity. □

**Remark A.3.** By Theorem A.1, the category $\mathbf{G}$ is equivalent to the category $\mathbf{\Gamma}$ which can be identified with a wide subcategory of Kock's category $\mathbf{K}$ (from Remark 2.0.4) containing fewer inert morphisms and the same active morphisms (see [Koc16, 2.4.14]).

## Appendix B. Proof of Proposition 2.3.2

In this section, *graph* will always mean *acyclic* graph. The following is a variation of the construction in Definition 2.0.4, which was only about *connected* graphs.

**Definition B.1.** If $G$ is a (possibly disconnected) graph, then there is an $\mathtt{E}(G)$-colored properad $\mathfrak{P}(G)$ generated by the vertices of $G$. Each connected, open subgraph $H$ of $G$ determines (up to a choice of orderings on inputs and outputs) an element in $\mathfrak{P}(G)$.

In general, there are many other elements of $\mathfrak{P}(G)$ that do not come from connected, open subgraphs. The reader concerned about disconnected graphs can either define $\mathfrak{P}(\coprod_i G_i) \coloneqq \coprod_i \mathfrak{P}(G_i)$ where each $G_i$ is connected, or note that Definition 5.3 and 5.7 of [HRY15] make no real use of connectedness of $G$. The properads acquired in these two ways will be isomorphic.

Many examples of this construction, as well as for properad maps $\mathfrak{P}(G) \to \mathfrak{P}(H)$, are given for connected graphs in [HRY15, Chapter 5]. Thus we will give a single example for a disconnected graph here.

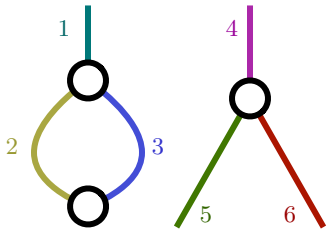



FIGURE 9. The graph $G$ from Example B.2

**Example B.2.** Consider the disconnected graph $G$ from Figure 9 with edge set $\{1,2,3,4,5,6\}$. Up to orderings, $\mathfrak{P}(G)$ has elements precisely in the following profiles (where $n \geq 1$):

$$\mathfrak{P}(G)(1;2,3) \qquad \mathfrak{P}(G)(2,3;\ )$$
$$\mathfrak{P}(G)(1,\overset{n}{\cdots},1;\ ) \qquad \mathfrak{P}(G)(1,\overset{n}{\cdots},1,2;2)$$
$$\mathfrak{P}(G)(1,\overset{n}{\cdots},1,2,3;\ ) \qquad \mathfrak{P}(G)(1,\overset{n}{\cdots},1,3;3)$$
$$\mathfrak{P}(G)(1,\overset{n}{\cdots},1;2,3) \qquad \mathfrak{P}(G)(4;5,6).$$

See Figure 10 for pictures of some of these elements.

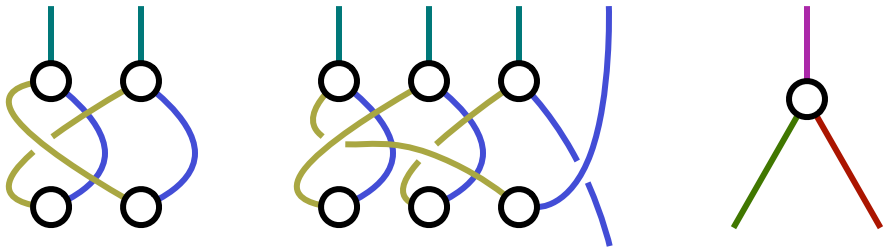


FIGURE 10. Elements of $\mathfrak{P}(G)(1,1;\ )$, $\mathfrak{P}(G)(1,1,1,3;3)$, and $\mathfrak{P}(G)(4;5,6)$ in the graph $G$ from Figure 9.

If $(\alpha,\eta): G \to H$ is in $\mathbf{L}$, then there is a well-defined map of properads $\mathfrak{P}(G) \to \mathfrak{P}(H)$ sending $v \in G_{i,i+1}$ to $\eta_{i,i+1}(v) \in H_{\alpha(i),\alpha(i+1)} \hookrightarrow \mathfrak{P}(H)$. The goal of this section is to show, when $G \to H$ is in $\mathbf{L}_{\mathrm{c}}$, that the map $f: \mathfrak{P}(G) \to \mathfrak{P}(H)$ is actually a 'graphical map.' That is, the *functor* from $\mathbf{L}_{\mathrm{c}} \to \mathrm{Prpd}(\mathrm{Set})$ factors through the (non-full) subcategory $\mathbf{\Gamma}$. Thus we are using Theorem A.1 in an essential way in this section.

Each $\mathfrak{C}$-colored properad $P$ has an underlying bimodule (see §3.6.1 of [HRY15]), that is, for each pair of lists of colors $c_1,\dots,c_n$ and $c'_1,\dots,c'_m$, a set

$$P(c_1,\dots,c_n;c'_1,\dots,c'_m),$$

along with isomorphisms

$$P(c_1,\dots,c_n;c'_1,\dots,c'_m) \xrightarrow[(\sigma';\sigma)]{\cong} P(c_{\sigma'(1)},\dots,c_{\sigma'(n)};c'_{\sigma^{-1}(1)},\dots,c'_{\sigma^{-1}(m)})$$

for $\sigma' \in \Sigma_n$ and $\sigma \in \Sigma_m$. These give a right $\Sigma_n$ action and a left $\Sigma_m$ action on

$$P(n;m) \coloneqq \coprod_{\mathfrak{C}^n \times \mathfrak{C}^m} P(c_1, \dots, c_n; c'_1, \dots, c'_m).$$

**Definition B.3.** If $P$ is a $\mathfrak{C}$ colored properad (in Set), let $\overline{P}$ denote the bigraded set

$$\overline{P}(n;m) \coloneqq P(n;m)/(\Sigma_n^{\mathrm{op}} \times \Sigma_m)$$

along with the induced functions

$$\overline{P}(n;m) \to (\mathfrak{C}^n/\Sigma_n) \times (\mathfrak{C}^m/\Sigma_m).$$

Another way of phrasing this is that if $M\mathfrak{C}$ is the free commutative monoid on $\mathfrak{C}$, then $\overline{P}$ is a set equipped with a function $\overline{P} \to M\mathfrak{C} \times M\mathfrak{C}$. As we've taken the underlying object of $P$ and removed all symmetry, the object $\overline{P}$ will not typically admit a properad structure.

Why do we consider this structure? Each vertex $v \in \mathtt{V}(G)$ with $n$ inputs and $m$ outputs will generate $n!m!$ different morphisms in the properad $\mathfrak{P}(G)$ generated by a graph $G$, one for each choice of ordering on $\mathtt{in}(v)$ and $\mathtt{out}(v)$. For our current purposes, this extra data is a distraction, hence we have use for $\overline{\mathfrak{P}(G)}$, which admits a well-defined function

$$\mathtt{V}(G) \hookrightarrow \overline{\mathfrak{P}(G)}$$

sending $v$ to the set of morphisms it generates.

**Definition B.4.** Let us say that a map of properads $f : \mathfrak{P}(H) \to \mathfrak{P}(G)$ *has a vertex lift* if the dotted arrow $\widetilde{f}$ exists in the diagram

$$\begin{array}{ccc} \mathtt{V}(H) & \overset{\widetilde{f}}{\dashrightarrow} & \mathtt{V}(G) \\ \downarrow & & \downarrow \\ \overline{\mathfrak{P}(H)} & \xrightarrow{\overline{f}} & \overline{\mathfrak{P}(G)}. \end{array} \tag{19}$$

**Lemma B.5.** *Let $H$ be a connected graph, and let $f : \mathfrak{P}(H) \to \mathfrak{P}(G)$ be a map of properads. If $f$ has a vertex lift, then $f$ comes from an étale map $H \to G$. If, in addition, $f$ is injective on colors (that is, if $\mathtt{E}(H) \to \mathtt{E}(G)$ is injective), then $f$ comes from an étale monomorphism $H \to G$.*

*Proof.* Let

$$x \in \mathfrak{P}(H)(e_1, \dots, e_n; e'_1, \dots, e'_m)$$

be a representative for a vertex $v$. This implies that $\mathtt{in}(v) = \{e_1, \dots, e_n\}$ is an $n$-element subset of $\mathtt{E}(H)$ and $\mathtt{out}(v) = \{e'_1, \dots, e'_m\}$ is an $m$-element subset of $\mathtt{E}(H)$. We know

$$fx \in \mathfrak{P}(G)(fe_1, \dots, fe_n; fe'_1, \dots, fe'_m)$$

and that this element represents the vertex $\widetilde{f}(v) \in \mathtt{V}(G)$. As is the case whenever we look at an element representing a vertex in $\mathfrak{P}(G)$, we have that $\mathtt{in}(\widetilde{f}(v)) = \{fe_1, \dots, fe_n\}$ is an $n$-element set and $\mathtt{out}(\widetilde{f}(v)) = \{fe'_1, \dots, fe'_m\}$ is an $m$-element set. Define a naïve morphism $f' : H \to G$ which is $f|_{\mathtt{E}(H)}$ on edges and $\widetilde{f}$ on vertices. We have seen that $f'$ preserves numbers of inputs and outputs, hence $f'$ is an étale map (Definition 2.0.3).

With the first statement proved, let us now suppose that $\mathtt{E}(H) \to \mathtt{E}(G)$ is injective. For the second statement, we must show that $\widetilde{f} : \mathtt{V}(H) \to \mathtt{V}(G)$ is injective.

Suppose $\widetilde{f}(v_1) = \widetilde{f}(v_2)$, from which it follows that $\mathtt{in}(\widetilde{f}(v_1)) = \mathtt{in}(\widetilde{f}(v_2))$ and $\mathtt{out}(\widetilde{f}(v_1)) = \mathtt{out}(\widetilde{f}(v_2))$. If $\mathtt{in}(\widetilde{f}(v_k)) = \varnothing = \mathtt{out}(\widetilde{f}(v_k))$, then we also have that $\mathtt{in}(v_k) = \varnothing = \mathtt{out}(v_k)$. Since $H$ is connected, this implies that $\mathtt{V}(H)$ has a unique element, so $v_1 = v_2$.

We may thus assume that one of the two sets $\mathtt{in}(\widetilde{f}(v_k))$ or $\mathtt{out}(\widetilde{f}(v_k))$ is nonempty; without loss of generality, assume $\mathtt{in}(\widetilde{f}(v_1)) = f(\mathtt{in}(v_1)) = f(\mathtt{in}(v_2))$ is nonempty. Since $f|_{\mathtt{E}(H)}$ is injective, this set is inhabited by an element $f(e)$ with $e \in \mathtt{in}(v_1) \cap \mathtt{in}(v_2)$, so $v_1 = v_2$. Thus $\widetilde{f}$ is injective, so the étale map $f'$ given in the first paragraph is a monomorphism. □

We now turn to the case when $f$ is not injective on colors.

**Lemma B.6.** *Let $H$ be a connected graph, and let $f : \mathfrak{P}(H) \to \mathfrak{P}(G)$ be a map of properads. Suppose that $f$ has a vertex lift $\widetilde{f} : \mathtt{V}(H) \to \mathtt{V}(G)$ which is injective. If $e_1 \neq e_2$ are edges of $H$ so that $f(e_1) = f(e_2)$, then $(e_1, e_2)$ or $(e_2, e_1)$ is an element of $\mathtt{in}(H) \times \mathtt{out}(H)$.*

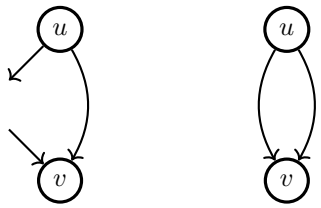
*Proof.* Suppose $e_1 \neq e_2$ and $f(e_1) = f(e_2)$. It suffices to show that $(e_1, e_2) \notin \mathtt{in}(H) \times \mathtt{out}(H)$ implies $(e_2, e_1) \in \mathtt{in}(H) \times \mathtt{out}(H)$. For this we first prove that $(e_1, e_2) \notin \mathtt{in}(H) \times \mathtt{out}(H)$ implies $e_1 \notin \mathtt{in}(H)$ and $e_2 \notin \mathtt{out}(H)$.

The existence of $e_1 \neq e_2$ shows that $H \ncong \downarrow$, hence, if $e_1 \notin \mathtt{in}(H)$ then $e_1 \in \mathtt{out}(v)$ for some vertex $v$. If $e_2 \in \mathtt{out}(H)$ then there exists a vertex $u$ with $e_2 \in \mathtt{out}(u)$. On the one hand, the injectivity of $f|_{\mathtt{out}(v)}$ implies that $u \neq v$. On the other, $f(e_1) = f(e_2)$ can be an output for at most one vertex. Thus, $\widetilde{f}(u) = \widetilde{f}(v)$ and the injectivity of $\widetilde{f}$ gives $u = v$. This contradiction proves that $e_1 \notin \mathtt{in}(H)$ implies $e_2 \notin \mathtt{out}(H)$ and a symmetric argument shows that the reverse implication also holds. Therefore, if $(e_1, e_2) \notin \mathtt{in}(H) \times \mathtt{out}(H)$ then $e_1 \notin \mathtt{in}(H)$ and $e_2 \notin \mathtt{out}(H)$.

Now let us assume that $e_1 \in \mathtt{out}(v)$ and $e_2 \in \mathtt{in}(w)$ for some vertices $v, w$. To show that $(e_2, e_1) \in \mathtt{in}(H) \times \mathtt{out}(H)$ it is necessary to exclude that possibility that $e_1 \in \mathtt{in}(u)$ for some $u$. If such a vertex $u$ exists, then $f(e_1) \in \mathtt{in}(\widetilde{f}(u))$ equals $f(e_2) \in \mathtt{in}(\widetilde{f}(w))$, so we must have $\widetilde{f}(u) = \widetilde{f}(w)$ which implies that $u = w$. But $f|_{\mathtt{in}(w)}$ is a monomorphism, so this would imply $e_1 = e_2$, contrary to our hypothesis. Thus no such $u$ exists and $e_1 \in \mathtt{out}(H)$. The symmetric argument establishes that $e_2 \in \mathtt{in}(H)$, so $(e_2, e_1) \in \mathtt{in}(H) \times \mathtt{out}(H)$. □

The following example, which appeared as Example 5.25 of [HRY15], shows that the behavior of Lemma B.6 actually occurs.

**Example B.7.** There is an evident étale map from the graph $H$ on the left to the graph $G$ on the right which is not injective on edge sets. The induced map of properads $f : \mathfrak{P}(H) \to \mathfrak{P}(G)$ has an injective vertex lift, but does not satisfy the hypothesis of the next corollary.



**Corollary B.8.** *Let $H$ be a connected graph, and let $f : \mathfrak{P}(H) \to \mathfrak{P}(G)$ be a map of properads. Suppose that $f$ has a vertex lift $\widetilde{f} : \mathtt{V}(H) \to \mathtt{V}(G)$ which is injective. If $f|_{\mathtt{in}(H)\cup\mathtt{out}(H)}$ is injective, then $f$ comes from an étale monomorphism.*

*Proof.* Lemma B.6 shows that $f$ is injective on edges, so Lemma B.5 applies. □

**Lemma B.9.** *Suppose $H_1$ and $H_2$ are connected open subgraphs of a connected graph $G$. If $\mathtt{in}(H_1) = \mathtt{in}(H_2)$ and $\mathtt{out}(H_1) = \mathtt{out}(H_2)$, then $H_1 = H_2$.*

*Proof.* Notice that if $H$ is an open *connected* subgraph of $G$, then $\mathtt{in}(H) \cap \mathtt{out}(H) \neq \varnothing$ if and only if $H$ is an edge. Moreover, if this holds then $H$ is uniquely determined by $\mathtt{in}(H)$, a one element set. Thus it suffices to consider the case when $\mathtt{in}(H_k) \cap \mathtt{out}(H_k) = \varnothing$ $(k = 1, 2)$. Forgetting the direction, the two inclusions become embeddings of undirected graphs in the sense of [HRY20]. The conclusion follows of the lemma follows by applying [HRY20, Proposition 1.25] to these two inclusions (in their notation we have $\eth(f_k) = \mathtt{in}(H_k) \amalg \mathtt{out}(H_k)$, $k = 1, 2$). □

**Theorem B.10.** *Let $\varphi = (\alpha, \eta) : G \to H$ be a morphism of $\mathbf{L}_{\mathrm{c}}$. Suppose that $x \in G_{i,j}$ is an $(i, j)$-level subgraph with associated level graph $K$. Let $\mathtt{V}(K) \cong \{v_1, \dots, v_n\} \subseteq \coprod_{p=i}^{j-1} G_{p,p+1}$ be the set of vertices that map to $x$, and let $H_\ell$ be the graph associated to $\eta(v_\ell) \in H_{\alpha(p),\alpha(p+1)}$. Then there is an étale monomorphism $f' : K\{H_1, \dots, H_n\} \to H$ whose image is $\eta(x)$.*

*Proof.* We know that

$$\mathtt{V}(K\{H_1, \dots, H_n\}) = \coprod_{\ell} \mathtt{V}(H_\ell)$$

and we can use this to define a properad map

$$f : \mathfrak{P}(K\{H_1, \dots, H_n\}) \to \mathfrak{P}(H)$$

whose restriction to the generators in $\mathtt{V}(H_\ell)$ is just the inclusion $\mathtt{V}(H_\ell) \subseteq \mathtt{V}(H)$. Since $\mathtt{V}_{\mathbf{L}}(\varphi)(w) = v_\ell$ when $w$ is in $H_\ell$, we see that the graphs $H_\ell$ have disjoint sets of vertices. Thus the properad map $f$ induces a monomorphism

$$\widetilde{f} : \mathtt{V}(K\{H_1, \dots, H_n\}) \to \mathtt{V}(H).$$

Moreover, we have

$$\mathtt{in}(K\{H_1, \dots, H_n\}) = \mathtt{in}(K)$$
$$\mathtt{out}(K\{H_1, \dots, H_n\}) = \mathtt{out}(K)$$

and the map $f|_{\mathtt{in}(K\{H_1,\dots,H_n\})}$ is just the restriction of $\eta_{i,i}$ and likewise for outputs. Since $\eta_{i,i}$ and $\eta_{j,j}$ are monomorphisms, this gives that $f|_{\mathtt{in}\cup\mathtt{out}}$ is a monomorphism except in the case when $\alpha(i) = \alpha(j)$. If $\alpha(i) = \alpha(j)$ then $\eta(v_\ell) = \eta(x)$ is always an edge, as is $K\{H_1, \dots, H_n\}$. Thus the requirements of Corollary B.8 are satisfied, and we have that $f' : K\{H_1, \dots, H_n\} \to H$ is an étale monomorphism.

We will now show that the image of this monomorphism is $\eta(x)$. The set $\mathtt{in}(K\{H_1, \dots, H_n\}) \cong \mathtt{in}(K)$ maps to the inputs (in $H_{\alpha(i),\alpha(i)}$) of $\eta(x) \in H_{\alpha(i),\alpha(j)}$. A similar statement holds for outputs. By Lemma B.9, the result follows. □

*Proof of Proposition 2.3.2.* By Theorem A.1, it is sufficient to show that this becomes a morphism in $\mathbf{\Gamma}$. Let $f : \mathfrak{P}(G) \to \mathfrak{P}(H)$ be the properad morphism associated to a morphism $(\alpha, \eta) : G \to H$ in $\mathbf{L}_{\mathrm{c}}$, where $G$ has height $k$. Apply Theorem B.10 to the unique element $x \in G_{0,k}$, representing $G$ itself. As $\eta(x)$ represents a structured subgraph of $H$ by Lemma 2.3.1, it follows that $f(G) \cong K\{H_1, \dots, H_n\}$ can be

considered as a structured subgraph of $H$. Thus we have verified the condition from [HRY15, Definition 6.46], so $f$ is a morphism of $\mathbf{\Gamma}$. □

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