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SOME HOMOLOGICAL PROPERTIES OF CATEGORY $\mathcal{O}$, V

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Abstract. We compute projective dimension of translated simple modules in the regular block of the BGG category $\mathcal{O}$ in terms of Kazhdan-Lusztig combinatorics. This allows us to determine which projectives can appear at the last step of a minimal projective resolution for a translated simple module, confirming a conjecture by Johan Kåhrström. We also derive some inequalities, in terms of Lusztig's $\mathbf{a}$-function, for possible degrees in which the top (or socle) of a translated simple module can live. Finally, we relate Kostant's problem with decomposability and isomorphism of translated simple modules, addressing yet another conjecture by Johan Kåhrström.

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1. Introduction, motivation and description of the results

1.1. **Setup.** Let $\mathfrak{g}$ be a semi-simple complex Lie algebra with a fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

and $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathcal{O}$ denote the associated Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, see [BGG, Hu]. Let, further, $\mathcal{O}_0$ denote the *principal block* $\mathcal{O}$, that is, the indecomposable direct summand containing the trivial $\mathfrak{g}$-module.

Let W be the Weyl group of $\mathfrak{g}$. It acts naturally on $\mathfrak{h}^*$ via $(w, \lambda) \mapsto w(\lambda)$. We consider the *dot-action* $(w, \lambda) \mapsto w \cdot \lambda$ of W which is obtained by shifting the usual action by the half of the sum of all positive roots.

The category $\mathcal{O}_0$ is equivalent to A-mod, for some finite-dimensional associative basic algebra A, unique up to isomorphism. Simple objects in $\mathcal{O}_0$ are exactly the (pairwise non-isomorphic) simple highest weight modules $L_w := L(w \cdot 0)$ of highest weight $w \cdot 0$, for $w \in W$. The category $\mathcal{O}_0$ is equipped with the action of the monoidal category $\mathscr{P}$ of *projective functors*, as defined in [BG]. Up to isomorphism, indecomposable projective functors are also in bijection with the elements in W, where the indecomposable projective functor θ_w, for $w \in W$, is normalized such that it sends the projective cover P_e of L_e to the projective cover P_w of L_w.

The algebra A is Koszul by [So1], in particular, it admits a positive $\mathbb{Z}$-grading. Denote by $\mathcal{O}_0^{\mathbb{Z}}$ the category of finite dimensional $\mathbb{Z}$-graded A-modules with morphisms being homogeneous homomorphisms of degree zero, see e.g., [St].

1.2. **Motivation.** The first major motivation for the present paper is the following:

Conjecture 1.1 ([KM]). *Assume that $\mathfrak{g}$ is of type A. Then, for $x, y \in W$, the module $\theta_x L_y$ is either indecomposable or zero.*

Various approaches to Conjecture 1.1 were considered in [KM] and [CMZ]. In the latter paper, the conjecture was confirmed in the cases $\mathfrak{g} = \mathfrak{sl}_n$, where $n = 2, 3, 4, 5, 6$. For other values of n, a number of special results are obtained in [KM] and [CMZ].

The second major motivation for the present paper is the so-called *Kostant's problem*, as popularized in [Jo], the content of which is to determine all $w \in W$ for which the universal enveloping algebra surjects onto the space of linear endomorphism of L_w that are locally finite with respect to the adjoint action (we will denote this property by $\mathbf{K}(w)$, see § 2.3 for details). This problem was studied in [Ma1, Ma3, MS2, Kh, KhM], see also the references therein. In March 2019, the second author received an email from Johan Kåhrström with the following conjecture (based on extensive computer computation).

Conjecture 1.2 (J. Kåhrström). *For a Duflo element $d \in W$, the following assertions are equivalent:*

(*a*) $\mathbf{K}(d)$.

(*b*) $\theta_x L_d \not\cong \theta_y L_d$, *for all* $x \neq y \in W$ *such that* $\theta_x L_d \neq 0$ *and* $\theta_y L_d \neq 0$.

(*c*) *For all* $x \neq y \in W$ *such that* $\theta_x L_d \neq 0$ *and* $\theta_y L_d \neq 0$, *there exists* $z \in W$ *and* $i \in \mathbb{Z}$ *such that* $[\theta_x L_d : L_z\langle i\rangle] \neq [\theta_y L_d : L_z\langle i\rangle]$ *in* $\mathcal{O}_0^{\mathbb{Z}}$.

(*d*) *For all* $x \neq y \in W$ *such that* $\theta_x L_d \neq 0$ *and* $\theta_y L_d \neq 0$, *there is* $z \in W$ *such that* $[\theta_x L_d : L_z] \neq [\theta_y L_d : L_z]$ *in* $\mathcal{O}_0$.

The final piece of motivation for the present paper is the problem to determine the projective dimension for all modules of the form $\theta_x L_y \in \mathcal{O}_0$, as formulated in [Ma3, Problem 24]. In connection to this problem, the email from Johan Kåhrström mentioned above contained the following conjecture (also based on extensive computer computation):

Conjecture 1.3 (J. Kåhrström). *Let $x, y \in W$ and k be the projective dimension of $\theta_x L_y$. Assume that $z \in W$ is such that $\mathrm{Ext}^k(\theta_x L_y, L_z) \neq 0$. Then z and x belong to the same Kazhdan-Lusztig left cell.*

1.3. **Description of the results.** The first main result of the present paper, see Proposition 6.2, Formula (21), Theorem 6.1 and Corollary 6.3, is:

Theorem A. *Conjecture 1.3 is true. Moreover, the projective dimension of $\theta_x L_y$, for $x, y \in W$ such that $\theta_x L_y \neq 0$, is given by*

$$\mathbf{a}(w_0 x) + \mathbf{b}(y^{-1} w_0, w_0 x^{-1}),$$

where w_0 is the longest element in W, $\mathbf{a}$ denotes Lusztig's $\mathbf{a}$-function, and $\mathbf{b}(u, v)$ is defined as the maximal degree shift of the composition factors in $\theta_u L_v$ (see Subsection 6.3).

Note that both $\mathbf{a}(w_0 x)$ and $\mathbf{b}(y^{-1} w_0, w_0 x^{-1})$ are "combinatorial" in the sense that they are defined explicitly in terms of Kazhdan-Lusztig polynomials.

Our second main result is the following statment, see Theorem 8.21, which combines ingredients of Conjecture 1.1 with parts of Conjecture 1.2:

Theorem B. *For $y \in W$, the assertion $\mathbf{K}(y)$ is true if and only if both of the following conditions hold.*

(i) for all $x \neq z$, we have $\theta_x L_y \not\cong \theta_z L_y$ whenever nonzero (we refer to this property as $\mathbf{Kh}(y)$);

(ii) for each $x \in W$, the module $\theta_x L_y$ is either indecomposable or zero (we refer to this property as $\mathbf{KM}(*, y)$).

In particular, if Conjecture 1.1 is true, then (a)⇔(b) in Conjecture 1.2 holds in type A.

In Theorem 2.2 we obtain some bounds (given in terms of the $\mathbf{a}$-function) on the degrees of simple constituents in the top (or socle) of $\theta_x L_y$. When the bound prescribed by Theorem 2.2 is achieved, an interesting question is the multiplicity of the corresponding simple constituent in the top. We formulate a condition on this multiplicity that we call $\mathbf{KMM}$, see (3), which would imply Conjecture 1.1 in type A.

We also relate Kostant's problem for different simple highest weight modules from the same Kazhdan-Lusztig left cell, see Corollary 8.19 and Proposition 8.26 for the condition (e):

Theorem C. *Let $d \in W$ be a Duflo element and let $\mathcal{L}$ be the Kazhdan-Lusztig left cell containing d.*

(i) If $\mathbf{K}(d)$ is not true, then $\mathbf{K}(y)$ is not true, for all $y \in \mathcal{L}$;

(ii) If $\mathbf{K}(d)$ is true, then, for each $y \in \mathcal{L}$, the following conditions are equivalent:

- *(a) $\mathbf{K}(y)$;*
- *(b) $\theta_y L_{y^{-1}} \cong \theta_d L_d$;*
- *(c) $t_y t_{y^{-1}} = t_d$ in the asymptotic ring for W (see §3.4);*
- *(d) $\mathbf{KM}(y, y^{-1})$;*
- *(e) if, furthermore, $\mathfrak{g}$ is of classical type: the $\mathtt{H}$-cell of y and the $\mathtt{H}$-cell of d contain the same number of elements (here, an $\mathtt{H}$-cell is the intersection of a Kazhdan-Lusztig right cell and a Kazhdan-Lusztig left cells, see Subsection 3.3).*

Theorem B and Theorem C give a conjectural answer to Kostant's problem in terms of Kazhdan-Lusztig combinatorics as follows (see Corollary 8.23).

Corollary 1.4. *Suppose (a) ⇔ (c) in Conjecture 1.2 is true for W. Then, for each $y \in W$, the condition $\mathbf{K}(y)$ is equivalent to the conjunction of the following conditions:*

(i) $t_y t_{y^{-1}} = t_d$ in the asymptotic ring for W;

(ii) For the Duflo element $d \sim_{\mathtt{L}} y$ and $x, x' \leq_{\mathtt{R}} d$, if $h_{x,z,d} = h_{x',z,d}$, for all $z \in W$, then $x = x'$.

Here $h_{x,z,d} \in \mathbb{Z}[v, v^{-1}]$ is the structure coefficient for the Kazhdan-Lusztig basis (see (6)).

In the last section, we use Theorem B, Theorem C, Kazhdan-Lusztig combinatorics, and some other results in the paper to determine $\mathbf{K}(y)$, as well as $\mathbf{KM}(y)$ and $\mathbf{Kh}(y)$, for all $y \in W$ in a number of small rank cases. The result confirms Conjecture 1.2 in type A_n for $n \leq 5$ and in type BCD in rank ≤ 4. In particular, we completely solve Kostant's problem in type A_5, see Corollary 10.1. This question was considered before in [KhM, Kh], where it was solved completely for A_n, where $n \leq 4$, and a partial answer for A_5 was given.

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2. A zoo of questions about $\mathcal{O}_0$

In this sections we both recall some classical open problems and questions about $\mathcal{O}_0$ and propose some new ones. In the rest of the paper we look deeper into connection between these problems and questions.

For any function $\mathbf{F} : W \to \{\mathrm{false}, \mathrm{true}\}$, we write $\mathbf{F}(*)$ for the conjunction of all $\mathbf{F}(w)$, where $w \in W$, and similarly for functions of several variables. We also write $\mathbf{F}(*_d)$ for the conjunction of all $\mathbf{F}(w)$, where $w \in W$ is a Duflo element.

2.1. **Indecomposability of translation of simple modules.** For $x, y \in W$, we denote by $\mathbf{KM}(x, y)$ the statement "the module $\theta_x L_y$ is either indecomposable or zero". The following problem is still open:

Problem 2.1. *Determine all $x, y \in W$, for which $\mathbf{KM}(x, y)$ is true.*

Conjecture 1.1 asserts that $\mathbf{KM}(x, y)$ is always true in type A. We note that $\theta_x L_y$ is non-zero if and only if $x^{-1} \leq_{\mathtt{L}} y$, where $\leq_{\mathtt{L}}$ denotes the Kazhdan-Lusztig left order on W from [KL], see, for example, [KM, Formula (1)] and the references preceding this formula. If $\mathfrak{g}$ is of type B_2 and $1, 2$ are the two simple reflections in W, then $\mathbf{KM}(12, 21)$ is known to be false, see [KM, Subsection 5.1].

2.2. **Graded simple tops of translated simple modules.** Denote by $\langle 1 \rangle$ the grading shift on $\mathcal{O}_0^{\mathbb{Z}}$ normalized such that it maps degree one to degree zero. Fix standard graded lifts L_w, where $w \in W$, of simple modules concentrated in degree zero. Fix standard graded lifts P_w, where $w \in W$, of indecomposable projective modules such that their tops are concentrated in degree zero. According to [St], each θ_w, where $w \in W$, also has a graded lift (unique up to isomorphism and shift of grading), which we normalize such that $\theta_w P_e \cong P_w$ holds as graded modules. Morphisms in $\mathcal{O}^{\mathbb{Z}}$ is denoted by hom.

We will use Lusztig's $\mathbf{a}$-function $\mathbf{a} : W \to \mathbb{Z}_{\geq 0}$ from [Lu1, Lu2] (see §3.3 for details). One of our principal observations in this paper is the following:

Theorem 2.2. *For $x, y, z \in W$ and $i \in \mathbb{Z}$, the condition*

$$\mathrm{hom}(\theta_x L_y, L_z\langle i \rangle) \neq 0 \quad \textit{implies} \quad i \geq \mathbf{a}(x). \tag{1}$$

Theorem 2.2 is proved in Section 4.

Recall that each Kazhdan-Lusztig left (and right) cell contains a unique distinguished involution, called the *Duflo element*. In type A, all involutions are Duflo elements. We propose the following:

Conjecture 2.3. *Let $d, y \in W$ be such that d is a Duflo element. Let, further, M be an indecomposable summand of $\theta_d L_y$. Then*

$$\dim \mathrm{hom}(M, L_y\langle \mathbf{a}(d) \rangle) = 1. \tag{2}$$

Let $d \in W$ be a Duflo element. Then there is a (unique up to a non-zero scalar and homogeneous of degree zero) non-zero natural transformation $\zeta : \theta_d \to \theta_e\langle \mathbf{a}(d) \rangle$, see [MM3, Section 7]. If $y \in W$ is such that $\theta_d L_y \neq 0$, then ζ_{L_y} is non-zero because the cokernel of ζ is killed by θ_d, see [MM1, Proposition 17], and hence this cokernel must annihilate L_y. Therefore, $\dim \mathrm{hom}(\theta_d L_y, L_y\langle \mathbf{a}(d) \rangle) \geq 1$.

For fixed $x, y \in W$, we denote by $\mathbf{KMM}(x, y)$ the property

$$\dim \mathrm{hom}(\theta_x L_y, L_y\langle \mathbf{a}(x) \rangle) \leq 1. \tag{3}$$

2.3. **Kostant's problem.** For any $\mathfrak{g}$-modules M and N, the vector space $\mathrm{Hom}_{\mathbb{C}}(M,N)$ has the natural structure of a $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule. The subspace $\mathscr{L}(M,N)$ consisting of all vectors of $\mathrm{Hom}_{\mathbb{C}}(M,N)$, the adjoint action of $\mathfrak{g}$ on which is locally finite, is a $U(\mathfrak{g})$-$U(\mathfrak{g})$-subbimodule. If $M=N$, then the image of $U(\mathfrak{g})$ in $\mathrm{Hom}_{\mathbb{C}}(M,M)$ belongs to $\mathscr{L}(M,M)$ and this $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodule map is also an algebra map. The following is known as *Kostant's problem*, see [Jo]:

Problem 2.4. *For which $w\in W$, the image of $U(\mathfrak{g})$ in $\mathrm{Hom}_{\mathbb{C}}(L_w,L_w)$ coincides with $\mathscr{L}(L_w,L_w)$?*

This problem was studied, for example, in [GJ, MS1, MS2, Ma1, Ma3, KhM, Kh], where several partial results were obtained. However, the general case is very much open. As already mentioned, we write $\mathbf{K}(w)$ for the statement "the image of $U(\mathfrak{g})$ in $\mathrm{Hom}_{\mathbb{C}}(L_w,L_w)$ coincides with $\mathscr{L}(L_w,L_w)$".

2.4. **Kåhrström's conditions.** For $y\in W$, we write $\mathbf{Kh}(y)$ for the statement "for all $x\neq z\in W$ such that $\theta_x L_y\neq 0$ and $\theta_z L_y\neq 0$, we have $\theta_x L_y\ncong\theta_z L_y$" and write $[\mathbf{Kh}](y)$ for the statement "for all $x\neq z\in W$ such that $\theta_x L_y\neq 0$ and $\theta_z L_y\neq 0$, we have $[\theta_x L_y]\neq[\theta_z L_y]$ in the Grothendieck group $\mathrm{Gr}(\mathcal{O}_0^{\mathbb{Z}})$".

In particular, the equivalence of Conjecture 1.2(a) and Conjecture 1.2(b) can be expresses as $\mathbf{K}(d)\Leftrightarrow\mathbf{Kh}(d)$, and the equivalence of Conjecture 1.2(b) and Conjecture 1.2(c) can be expresses as $\mathbf{Kh}(d)\Leftrightarrow[\mathbf{Kh}](d)$, for a Duflo element $d\in W$.

3. Kazhdan-Lusztig combinatorics

The Kazhdan-Lusztig conjecture from [KL], proved in [BB, BK] (see also [EW]), tells us that a large amount of information on $\mathcal{O}_0^{\mathbb{Z}}$ is encoded in the Hecke algebra of W and thus can be computed combinatorially. We recall in this section some well-known constructions and facts around the Hecke algebra and their relation to $\mathcal{O}_0^{\mathbb{Z}}$.

3.1. **Hecke algebra and Kazhdan-Lusztig basis.** Let $S\subset W$ be the set of simple reflections. Then (W,S) is a Coxeter group. The Hecke algebra $H(W,S)$ associated to (W,S) is a $\mathbb{Z}[v,v^{-1}]$-algebra generated by H_s, for $s\in S$, which satisfy the (Coxeter) braid relations and the quadratic relation

$$(H_s+v)(H_s-v^{-1})=0,$$

for all $s\in S$. Given a reduced expression $w=st\cdots u$ of $w\in W$, we let $H_w=H_sH_t\cdots H_u$. The element H_w is, in fact, independent of the choice of the reduced expression, and $\{H_w\}_{w\in W}$ is a ($\mathbb{Z}[v,v^{-1}]$-)basis of $H(W,S)$ called the *standard basis*. Now consider the ($\mathbb{Z}$-algebra-)involution

$$\overline{}: H(W,S)\to H(W,S)$$

determined by $\overline{v}=v^{-1}$ and $\overline{H_s}=H_s^{-1}$. Then there is a unique element $\underline{H}_w$ in $H(W,S)$ such that $\overline{\underline{H}_w}=\underline{H}_w$ and

$$\underline{H}_w=H_w+\sum_y p_{y,w}H_y,$$

for some $p_{y,w}\in v\mathbb{Z}[v]$. The elements $\underline{H}_w$, where $w\in W$, form a basis of $H(W,S)$ called the *Kazhdan-Lusztig (KL) basis*. We refer to [KL], or [Lu3, §3-5] for details (if referring to [Lu3], note that we are in the special case $L(s)=1$ for all $s\in S$ and that our v is denoted by v^{-1}, H_w is denoted by T_w, and our $\underline{H}_w$ is denoted by c_w in [Lu3]).

We denote by $\boldsymbol{\mu}$ the Kazhdan-Lusztig μ-function.

3.2. **Kazhdan-Lusztig theory.** Let $\mathscr{P}$ be the monoidal category of (graded) projective functors on $\mathcal{O}_0^{\mathbb{Z}}$ where indecomposables θ_w, for $w \in W$, are normalized such that $\theta_w P_e \cong P_w$ holds in $\mathcal{O}_0^{\mathbb{Z}}$. Then, for the split Grothendieck ring $\mathrm{Gr}_{\oplus}(\mathscr{P})$, we have

$$\begin{aligned} \mathrm{Gr}_{\oplus}(\mathscr{P})^{\mathrm{op}} &\xrightarrow{\sim} H(W,S) \\ [\theta_w] &\mapsto \underline{H}_w \end{aligned} \tag{4}$$

as $\mathbb{Z}[v,v^{-1}]$-algebras so that $\langle 1\rangle$ on the left corresponds to v on the right. For example, $[\theta_w\langle m\rangle] \mapsto v^m\underline{H}_w$. Furthermore,

$$\begin{aligned} \mathrm{Gr}(\mathcal{O}_0^{\mathbb{Z}}) &\xrightarrow{\sim} H(W,S) \\ [P_w] &\mapsto \underline{H}_w \end{aligned} \tag{5}$$

as (right) modules over $\mathrm{Gr}_{\oplus}(\mathscr{P})^{\mathrm{op}} \cong H(W,S)$ so that $\langle 1\rangle$ on the left hand side corresponds to v on the right hand side. We note that, for $w \in W$, we have $[\Delta_w] \mapsto H_w$, where Δ_w denotes the (graded) Verma module with simple top L_w.

Using (5), various multiplicities in $\mathcal{O}_0^{\mathbb{Z}}$ can be described in terms of the Kazhdan-Lusztig polynomials from [KL]. In particular, the graded composition multiplicities in P_e are given by the corresponding coefficients of the Kazhdan-Lusztig polynomials.

3.3. **Kazhdan-Lusztig cells and the a-function.** For $x, y \in W$, write

$$\underline{H}_x\underline{H}_y = \sum_z h_{x,y,z}\underline{H}_z, \tag{6}$$

where $h_{x,y,z} \in \mathbb{Z}[v^{\pm 1}]$. In fact, we have $h_{x,y,z} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \cap \mathbb{Z}[v+v^{-1}]$ by [KL].

Given $x, y \in W$, we say $x \leq_{\mathtt{L}} y$ if there exists $z \in W$ such that $\underline{H}_y$ appears with a nonzero coefficient in $\underline{H}_z\underline{H}_x$, that is, $h_{z,x,y} \neq 0$. This defines the equivalence relation $\sim_{\mathtt{L}}$ on W. A (Kazhdan-Lusztig) left cell is an equivalence class for the relation $\sim_{\mathtt{L}}$. Similarly, define the right preorder $\leq_{\mathtt{R}}$ and (Kazhdan-Lusztig) right cells using multiplication on the right. Finally, we define the two-sided preorder $\leq_{\mathtt{J}}$ and (Kazhdan-Lusztig) two-sided cells using multiplication on both sides. The equivalence relation $\sim_{\mathtt{J}}$ is the minimum equivalence relation containing both $\sim_{\mathtt{L}}$ and $\sim_{\mathtt{R}}$. We define the equivalence relation $\sim_{\mathtt{H}}$ as the intersection of $\sim_{\mathtt{L}}$ and $\sim_{\mathtt{R}}$. Equivalence classes for $\sim_{\mathtt{H}}$ are called $\mathtt{H}$-cells. A two-sided cell is called *strongly regular* if the intersection of each left and each right cell inside this two-sided cell is a singleton. A left (right) cell is *strongly regular* if it belongs to a strongly regular two-sided cell.

We note that the left and right orders are the opposite of that in [Lu3]. Our conventions are consistent with the previous papers [Ma2, Ma3, CM1, CM2] of the series.

Lusztig's a-function $\mathbf{a} : W \to \mathbb{Z}_{\geq 0}$ is defined as follows:

$$\mathbf{a}(z) := \max_{x,y\in W}\{\deg h_{x,y,z}\}. \tag{7}$$

The following facts can be found in [Lu3] (note that the conjectures P1-15 in [Lu3, §13] are proved in [Lu3, §14-15] in our setting). Let w_0 denote the longest element in W.

Proposition 3.1. *Let $x, y \in W$.*

(i) $x \leq_{\mathtt{L}} y \iff x^{-1} \leq_{\mathtt{R}} y^{-1}$.

(ii) $x \leq_{\mathtt{L}} y \iff w_0x \geq_{\mathtt{L}} w_0y$. *Furthermore,* $x \leq_{\mathtt{R}} y \iff xw_0 \geq_{\mathtt{R}} yw_0$.

(iii) Let $\mathtt{X} \in \{\mathtt{L}, \mathtt{R}, \mathtt{J}\}$. *If* $x \leq_{\mathtt{X}} y$, *then* $\mathbf{a}(x) \leq \mathbf{a}(y)$. *If* $x <_{\mathtt{X}} y$, *then* $\mathbf{a}(x) < \mathbf{a}(y)$. *In particular, the* $\mathbf{a}$*-function is* $\mathtt{J}$*-cell invariant.*

(iv) If $\mathbf{a}(x) = \mathbf{a}(y)$, *then* $x \leq_{\mathtt{L}} y \Rightarrow x \sim_{\mathtt{L}} y$. *Furthermore, if* $\mathbf{a}(x) = \mathbf{a}(y)$, *then* $x \leq_{\mathtt{R}} y \Rightarrow x \sim_{\mathtt{R}} y$.

Recall from [Lu1, Lu2], that the $\mathbf{a}$-function can be also defined as the maximal (or minimal, depending on the normalization) possible degree of a Kazhdan-Lusztig polynomial between the identity and an element of a given left (or right) cell. As a consequence of this definition, for any $w \in W$, we have

$$[P_e : L_w\langle i\rangle] \neq 0 \quad \text{implies} \quad -\mathbf{a}(w) \geq i \geq -\ell(w), \tag{8}$$

where $\ell(w)$ denotes the length of w, moreover, the inequality $-\mathbf{a}(w) \geq i$ is strict unless w is a Duflo element and in the latter case

$$[P_e : L_w\langle -\mathbf{a}(w)\rangle] = 1. \tag{9}$$

For each right cell $\mathcal{R}$ in W, we have the Serre subcategory $\mathcal{O}_0^{\hat{\mathcal{R}}}$ of $\mathcal{O}_0$ whose simples are L_w, for all $w \in W$ such that $w \leq_{\mathtt{R}} \mathcal{R}$, see [MS1].

3.4. **Asymptotic rings.** We introduce the asymptotic ring $A(W) = A(W, S)$ defined by Lusztig (and called "the ring J" in [Lu3]). It has a ($\mathbb{Z}$-)basis $\{t_w\}_{w\in W}$ whose multiplication is defined as

$$t_x t_y = \sum_{z\in W} \gamma_{x,y,z^{-1}} t_z, \tag{10}$$

where $\gamma_{y,x,z^{-1}} \in \mathbb{Z}_{\geq 0}$ is the coefficient of $\theta_z\langle \mathbf{a}(z)\rangle$ in the decomposition of $\theta_x\theta_y$, i.e., the top degree coefficient in $h_{x,y,z}$ (see (6)). One can check that $\gamma_{x,y,z^{-1}} = 0$ unless y and x^{-1} belong to the same right cell. The basis elements t_d corresponding to Duflo elements $d \in W$ are local identities in $A(W)$ in the following sense:

Lemma 3.2. *Let $d \in W$ be a Duflo element in $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$. Here, $\mathcal{L}$ is the left cell containing d and $\mathcal{L}^{-1}$ is the right cell containing $d^{-1} = d$. We have*

$$t_d t_x = t_x \quad \text{and} \quad t_y t_d = t_y. \tag{11}$$

for each $y \in \mathcal{L}$ and $x \in \mathcal{L}^{-1}$ and

$$t_{x^{-1}} t_x = t_d + \sum_{d\neq z\in \mathcal{L}\cap\mathcal{L}^{-1}} \gamma_{x^{-1},x,z^{-1}} t_z. \tag{12}$$

Proof. This follows from P5, P7, P8, P13 in [Lu3, §14.1] and positivity of $h_{x,y,z}$. □

3.5. **Graded composition multiplicities in $\mathcal{O}_0^{\mathbb{Z}}$.** By §3.2 (and [KL, BB, BK]), composition multiplicities of many important objects in $\mathcal{O}_0^{\mathbb{Z}}$ can be computed purely inside the Hecke algebra $H(W, S)$.

Proposition 3.3. *For $x, y, w \in W$, we have*

$$[\theta_x L_y : L_z] = h_{x^{-1},z,y}, \tag{13}$$

where $[\theta_x L_y : L_z]$ denotes the graded composition multiplicity viewed as an element in $\mathbb{Z}[v, v^{-1}]$. In particular, for $i \in \mathbb{Z}$, we have

$$[\theta_x L_y : L_z\langle i\rangle] = [\theta_z L_{y^{-1}} : L_x\langle i\rangle]. \tag{14}$$

Proof. Formula (13) follows, by adjunction, from the observation that the multiplicity of $L_z\langle i\rangle$ in $\theta_x L_y$ equals

$$\dim \hom(\theta_z P_e\langle i\rangle, \theta_x L_y) = \dim \hom(\theta_{x^{-1}}\theta_z P_e\langle i\rangle, L_y).$$

By a similar adjunction, (14) is equivalent to

$$\hom(\theta_{x^{-1}}\theta_z L_{y^{-1}}, I_e\langle i\rangle) \cong \hom(\theta_{z^{-1}}\theta_x L_y, I_e\langle i\rangle),$$

where I_e is the indecomposable injective envelope of L_e with socle concentrated in degree zero. Also, by adjunction, for $a, b \in W$ and $j \in \mathbb{Z}$, we have

$$\dim \hom(\theta_{a^{-1}} L_b, I_e\langle j\rangle) = \delta_{a,b}\delta_{j,0}.$$

As $\theta_{x^{-1}}\theta_z$ is adjoint to $\theta_{z^{-1}}\theta_x$ and θ_y is adjoint to $\theta_{y^{-1}}$, the multiplicity of θ_y in $\theta_{x^{-1}}\theta_z$ coincides with the multiplicity of $\theta_{y^{-1}}$ in $\theta_{z^{-1}}\theta_x$. The claim follows. □

4. Proof of Theorem 2.2

For $w \in W$, we denote by T_w the indecomposable tilting module in $\mathcal{O}_0$ with highest weight $w \cdot 0$.

We record a lemma which follows directly from, for example, [MM1, Lemma 13(a)].

Lemma 4.1. *Let θ be a projective functor on $\mathcal{O}_0$ and $M \in \mathcal{O}_0$. If $y \in W$ is such that $[\theta M : L_y] \neq 0$, then there exists $x \in W$ such that $[M : L_x] \neq 0$ and $y \leq_{\mathtt{R}} x$.*

We also need the following statement.

Lemma 4.2. *Assume $M \in \mathcal{O}_0$. Let $i \in \mathbb{Z}$ and $w \in W$ be such that $\mathrm{Ext}^i_{\mathcal{O}}(M, T_w) \neq 0$. Then we have*

$$i \geq \min\{\mathbf{a}(x^{-1}w_0) \,:\, x \in W \text{ such that } [M : L_x] \neq 0\}. \tag{15}$$

Proof. As $T_w = \theta_{w_0 w} T_{w_0}$, by adjunction, we have

$$\mathrm{Ext}^i_{\mathcal{O}}(M, T_w) \cong \mathrm{Ext}^i_{\mathcal{O}}(\theta_{w^{-1}w_0} M, T_{w_0}).$$

By Lemma 4.1, any $y \in W$ such that $[\theta_{w^{-1}w_0} M : L_y] \neq 0$ satisfies $y \leq_{\mathtt{R}} x$, for some $x \in W$ such that $[M : L_x] \neq 0$. For such x and y, we have $w_0 y \geq_{\mathtt{R}} w_0 x$ and thus $\mathbf{a}(w_0 y) \geq \mathbf{a}(w_0 x)$ (see Proposition 3.1). In particular,

$$\min\{\mathbf{a}(x^{-1}w_0) \,:\, x \in W \text{ such that } [M : L_x] \neq 0\} =$$
$$\min\{\mathbf{a}(x^{-1}w_0) \,:\, x \in W \text{ such that } [\theta_{w^{-1}w_0} M : L_x] \neq 0\}.$$

Therefore, it is enough to prove the claim for $w = w_0$, in which case $T_w = T_{w_0} = L_{w_0}$.

Let $\mathcal{I}^\bullet_{w_0}$ be the minimal injective resolution of L_{w_0}. The assumption implies that there is a nonzero morphism from M to $\mathcal{I}^i_{w_0}$, i.e., there is $x \in W$ such that $[M : L_x] \neq 0$ and $I_x \subseteq \mathcal{I}^i_{w_0}$. Recall from [So1] that $\mathcal{O}_0$ is Koszul self-dual and this self-duality maps L_w to $P_{w^{-1}w_0}$ and I_w to $L_{w^{-1}w_0}$. Consequently, the minimal injective resolution $\mathcal{I}^\bullet_{w_0}$ of L_{w_0} corresponds to the dominant projective module P_e in the Koszul dual picture. Thus, for $x \in W$ as above, I_x appearing as a summand of $\mathcal{I}^i_{w_0}$ implies that $L_{x^{-1}w_0}$ appears as a composition subquotient of P_e in degree i. By (8), we have $\mathbf{a}(x^{-1}w_0) \leq i \leq \ell(x^{-1}w_0)$, for such x. This implies the inequality in (15) and completes the proof. □

Proof of Theorem 2.2. Let $x, y, z \in W$. Koszul-Ringel self-duality of $\{\theta_a L_b \,:\, a, b \in W\}$ from [Ma3, Theorem 16] maps a non-zero homomorphism from $\theta_x L_y$ to $L_z\langle i\rangle$ to a non-zero element in

$$\mathrm{Ext}^i_{\mathcal{O}}(\theta_{y^{-1}w_0} L_{w_0 x^{-1}}, T_{w_0 z^{-1} w_0}).$$

By Lemma 4.1, $\theta_{y^{-1}w_0} L_{w_0 x^{-1}}$ consists of composition factors isomorphic to L_z with $z \leq_{\mathtt{R}} w_0 x^{-1}$. Equivalently, we have $z^{-1} \leq_{\mathtt{L}} x w_0$ yielding $x \leq_{\mathtt{L}} z^{-1} w_0$ by Proposition 3.1(ii). For all such z, we have $\mathbf{a}(z^{-1}w_0) \geq \mathbf{a}(x)$ by Proposition 3.1(iii). Now the claim of Theorem 2.2 follows from Lemma 4.2. □

5. $\mathbf{KMM}$ vs $\mathbf{KM}$

In this section we establish a connection between $\mathbf{KMM}$ and $\mathbf{KM}$.

5.1. **The graded endomorphism algebra of $\theta_x L_y$.** For $x, y \in W$, we consider the module $\theta_x L_y \in \mathcal{O}_0^{\mathbb{Z}}$ and its endomorphism algebra $\mathrm{End}(\theta_x L_y)$ which is naturally $\mathbb{Z}$-graded.

Lemma 5.1. *The natural $\mathbb{Z}$-grading on $\mathrm{End}(\theta_x L_y)$ is non-negative in the sense that all components with negative degrees are zero.*

Proof. By [Ma3, Theorem 16], the module $\theta_x L_y$ is Koszul-Ringel dual to $\theta_{y^{-1}w_0} L_{w_0 x^{-1}}$, where w_0 is the longest element of W. Under this duality, the endomorphism algebra of $\theta_x L_y$ is mapped to the algebra of diagonal self-extensions for $\theta_{y^{-1}w_0} L_{w_0 x^{-1}}$. The grading of the latter algebra is manifestly non-negative, which implies the claim. □

Corollary 5.2. *If $\dim \mathrm{End}(\theta_x L_y)_0 = 1$, then $\mathbf{KM}(x, y)$ is true.*

Proof. Every idempotent of the non-negatively graded algebra $\mathrm{End}(\theta_x L_y)$ must be homogeneous of degree zero. Therefore the assumption $\dim \mathrm{End}(\theta_x L_y)_0 = 1$ implies that $\mathrm{End}(\theta_x L_y)_0 \cong \mathbb{C}$, that is, the only idempotents of $\mathrm{End}(\theta_x L_y)$ are 0 and 1. This means that $\mathrm{End}(\theta_x L_y)$ is local and hence $\theta_x L_y$ is indecomposable. □

5.2. $\mathbf{KMM}(*_d, y)$ **implies** $\mathbf{KM}(*, y)$ **in type** A**.** Assume now that $\mathfrak{g}$ is of type A. Then the cell structure of W is especially nice. In particular, if $\mathcal{L}$ and $\mathcal{R}$ are a left and a right cell inside the same two-sided cell, then $|\mathcal{L} \cap \mathcal{R}| = 1$. Moreover, if $\mathcal{R} = \{w^{-1} : w \in \mathcal{L}\}$, then $\mathcal{L} \cap \mathcal{R} = \{d\}$, where d is a Duflo element.

Lemma 5.3. *For $x \in W$ in type A and d a Duflo element such that $x \sim_R d$, we have*

$$\theta_{x^{-1}} \theta_x \cong \theta_d \langle -\mathbf{a}(d) \rangle \oplus \bigoplus_{w \in W} \bigoplus_{i > -\mathbf{a}(w)} \theta_w \langle i \rangle^{\oplus m_{w,i}}.$$

Proof. Since $\mathcal{R} \cap \mathcal{R}^{-1} = \{d\}$, where $x \in \mathcal{R}$, the claim follows from (12) in Lemma 3.2. □

Proposition 5.4. *Assume that $\mathfrak{g}$ is of type A and $x, y \in W$. Then $\mathbf{KMM}(*_d, y)$ implies $\mathbf{KM}(x, y)$.*

Proof. By adjunction, we have

$$\hom(\theta_x L_y, \theta_x L_y) \cong \hom(\theta_{x^{-1}} \theta_x L_y, L_y).$$

Given Theorem 2.2, from Lemma 5.3 it follows that the only term in the decomposition of $\theta_{x^{-1}} \theta_x$ which can contribute to a non-zero element of $\hom(\theta_{x^{-1}} \theta_x L_y, L_y)$ is $\theta_{d'} \langle -\mathbf{a}(x) \rangle$, where $d' \sim_{\mathtt{R}} x$ is a Duflo element. Therefore, we just need to show that

$$\dim \hom(\theta_{d'} L_y, L_y \langle \mathbf{a}(x) \rangle) = 1.$$

If $\theta_{d'} L_y \neq 0$, this is guaranteed by $\mathbf{KMM}(d', y)$. □

6. Extensions between translated simple modules

6.1. **Monotonicity of projective dimension.** One of our main observations in this section is the following statement in the spirit of [Ma3] and [CM2].

Theorem 6.1. *Let $x, x', y \in W$ be such that $x >_{\mathtt{R}} x'$ and $\theta_{x'} L_y \neq 0$. Then*

$$\mathrm{proj.dim}(\theta_x L_y) < \mathrm{proj.dim}(\theta_{x'} L_y).$$

Note that, in the setup of Theorem 6.1, the inequality $\mathrm{proj.dim}(\theta_x L_y) \leq \mathrm{proj.dim}(\theta_{x'} L_y)$ follows easily applying projective functors to a minimal projective resolution of $\theta_{x'} L_y$, where $\theta_x L_y$ can be eventually found as a direct summand of the homology in the homological position zero, since $x >_{\mathtt{R}} x'$.

We note that the main result of [Ma3] provides an explicit formula for $\mathrm{proj.dim}(\theta_y L_{w_0})$ in terms of the $\mathbf{a}$-function and thus implies Theorem 6.1 in the case $y = w_0$ by since the $\mathbf{a}$-function is strictly monotone along any of the Kazhdan-Lusztig orders (see Proposition 3.1 (ii)). More generally, in the case $y = w_0^{\mathfrak{p}} w_0$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, Theorem 6.1 follows from [CM2, Table 2 and Theorem 4.1(i)].

Let M be a $\mathbb{Z}$-graded module. Then the *graded length* of M is defined as the difference between the maximal and the minimal degrees of non-zero components of M. If $M = 0$, then the graded length is, by convention, $-\infty$. For modules concentrated in a single degree, the graded length is 0. For example, the projective dimension of M is the same as the graded length of a minimal projective resolution of M consider as a module in an appropriate graded category with a $\mathbb{Z}$-grading given by the homological degree.

For $x, y \in W$, denote by $g_{x,y}$ the graded length of $\theta_x L_y$, viewed as an object of the category of linear complexes of tilting modules in $\mathcal{O}$. From [Ma3, Theorem 16], it follows that $g_{x,y}$ is always even. Applying projective functors, it is easy to see that

$$g_{x,y} \leq g_{x',y} \quad \text{if} \quad x \geq_{\mathtt{R}} x'. \tag{16}$$

Proposition 6.2. *Consider $x, y \in W$ such that $\theta_x L_y \neq 0$ and let $k := \mathrm{proj.dim}(\theta_x L_y)$. Then we have:*

(i) $k = \mathbf{a}(w_0 x) + \frac{1}{2} g_{x,y} =: k'$.

(ii) For any $z \in W$ such that $z \not\sim_{\mathtt{L}} x$, we have $\mathrm{Ext}^k(\theta_x L_y, L_z) = 0$.

Proof. Let $z \in W$ be such that $\mathrm{Ext}^k(\theta_x L_y, L_z) \neq 0$. Let d be the Duflo element in the left cell of x. Then $\theta_d \theta_x = \theta_x \oplus \theta$, for some projective functor θ, in particular, $\mathrm{Ext}^k(\theta_d \theta_x L_y, L_z) \neq 0$. By adjunction, we obtain $\mathrm{Ext}^k(\theta_x L_y, \theta_d L_z) \neq 0$. We use the Koszul-Ringel duality of [Ma3, Theorem 16] and represent both $\theta_x L_y$ and $\theta_d L_z$ as complexes of tilting modules. We call these complexes $\mathcal{X}^\bullet$ and $\mathcal{Y}^\bullet$, respectively. The minimal non-zero position of $\mathcal{X}^\bullet$ is $-\frac{1}{2} g_{x,y}$. The maximal non-zero position of $\mathcal{Y}^\bullet$ is $\frac{1}{2} g_{d,z}$. As $d \sim_{\mathtt{J}} x$, from [Ma3, Proposition 1] we get the inequality $\frac{1}{2} g_{d,z} \leq \mathbf{a}(w_0 d) = \mathbf{a}(w_0 x^{-1}) = \mathbf{a}(w_0 x)$, moreover, $\frac{1}{2} g_{d,z} = \mathbf{a}(w_0 x)$ if and only if $w_0 z^{-1} \sim_{\mathtt{L}} w_0 d$. Note that $w_0 z \sim_{\mathtt{L}} w_0 d$ is equivalent to $z \sim_{\mathtt{L}} d$ in which case we have $z \sim_{\mathtt{L}} x$. This implies that $k \leq k'$ and that $k = k'$ holds exactly when $z \sim_{\mathtt{L}} x$.

Therefore, we just need to find some $z \in W$ such that $z \sim_{\mathtt{L}} x$ and

$$\mathrm{Ext}^{k'}(\theta_x L_y, L_z) \neq 0.$$

Let L_u be some simple module which appears in the graded degree $-\frac{1}{2} g_{x,y}$, which is the extreme degree, of the module $\theta_{y^{-1} w_0} L_{w_0 x^{-1}}$. By Lemma 4.1 u belongs to the right cell $\mathcal{R}$ of $w_0 x^{-1}$. Since all indecomposable projective-injective modules in $\mathcal{O}^{\hat{\mathcal{R}}}$ are of the form $\theta_{z^{-1} w_0} L_{w_0 d}$ for $z \sim_{\mathtt{L}} x$ (see [Ma3]), there exists $z \sim_{\mathtt{L}} x$ such that $\theta_{z^{-1} w_0} L_{w_0 d}$ is the injective envelope of L_u in $\mathcal{O}^{\hat{\mathcal{R}}}$. For this z, the factor L_u appears in the graded degree $\frac{1}{2} g_{d,z}$ of the module $\theta_{z^{-1} w_0} L_{w_0 d}$. Since any isomorphism between the corresponding summands of $\mathcal{X}^{-\frac{1}{2} g_{x,y}}$ and $\mathcal{Y}^{\frac{1}{2} g_{d,z}}$ gives rise a non-zero extension of degree k' between $\theta_x L_y$ and $\theta_d L_z$ (e.g., by the argument from [MO, Theorem 1], a), the claim follows. □

Proof of Theorem 6.1. By Proposition 6.2(i) we need to prove that $\mathbf{a}(w_0 x) + \frac{1}{2} g_{x,y}$ is strictly monotone along the right Kazhdan-Lusztig order with respect to x. The term $\mathbf{a}(w_0 x)$ is strictly monotone by Proposition 3.1 (ii),(iii). The claim follows by (16). □

Directly from Proposition 6.2(ii), we obtain:

Corollary 6.3. *Conjecture 1.3 is true.*

We record a special case of Proposition 6.2(i).

Corollary 6.4. *Let $x, y \in W$ be such that $\theta_x L_y \neq 0$ and $x \sim_{\mathtt{J}} y$. Then*

$$\mathrm{proj.dim}(\theta_x L_y) = 2\mathbf{a}(w_0 x) = \mathbf{a}(w_0 x) + \mathbf{a}(y^{-1} w_0).$$

Proof. This follows from Proposition 6.2 (i) and [Ma3, Proposition 1]. □

6.2. **Theorem 6.1 does not naively extend to the Koszul-Ringel dual.** An alternative description of $g_{x,y}$ is given by:

Proposition 6.5. *For $x, y \in W$ such that $\theta_x L_y \neq 0$, we have*

$$g_{x,y} = \max\{i \,:\, \mathrm{Ext}^i(\theta_x L_y, \theta_x L_y) \neq 0\}.$$

Proof. By representing $\theta_x L_y$ as a complex $\mathcal{X}^\bullet$ of tilting modules, we can compute $\mathrm{Ext}^i(\theta_x L_y, \theta_x L_y)$ via homomorphisms from $\mathcal{X}^\bullet$ to $\mathcal{X}^\bullet[i]$ in the homotopy category of titling modules. If $i > g_{x,y}$, then the homomorphism space is obviously zero as non-zero components of $\mathcal{X}^\bullet$ and $\mathcal{X}^\bullet[i]$ never match.

Because of the self-duality of $\theta_x L_y$ combined with the Koszul-Ringel self-duality of $\{\theta_u L_v \,:\, u, v \in W\}$ from [Ma3, Theorem 16], the maximal and the minimal non-zero components of $\mathcal{X}^\bullet$ are isomorphic as objects in $\mathcal{O}$. Therefore, when $i = g_{x,y}$, the argument from the proof of [MO, Theorem 1] shows that an isomorphism from the minimal component in $\mathcal{X}^\bullet$ to the maximal component in $\mathcal{X}^\bullet[i]$ induces a non-zero homomorphism between complexes in the homotopy category of complexes of tilting modules. The claim follows. □

In contrast to Theorem 6.1, the function $x \mapsto g_{x,y}$, for y fixed, does not have to be strictly monotone with respect to the right order on W. Indeed, if $u \in W$ is the longest element of some parabolic subgroup, then the graded length of each non-zero $\theta_u L_v$, where $v \in W$, equals $2\ell(u)$. By Koszul-Ringel duality, this gives, in the case $y = w_0 u^{-1}$, that $g_{x,y} = 2\ell(u)$, for any x for which $g_{x,y} \neq -\infty$. In other words, in this case, the function $x \mapsto g_{x,y}$ is, in fact, constant. This example and the discussion above imply the following corollary:

Corollary 6.6. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ and T any non-zero tilting module in the parabolic category $\mathcal{O}_0^{\mathfrak{p}}$, then*

$$\max\{i \,:\, \mathrm{Ext}^i_{\mathcal{O}}(T, T) \neq 0\} = 2\ell(w_0^{\mathfrak{p}}).$$

Proof. This follows from the above discussion noting that all indecomposable tilting modules in $\mathcal{O}_0^{\mathfrak{p}}$ are of the form $\theta_x L_{w_0^{\mathfrak{p}'} w_0}$, for some x, where $\mathfrak{p}'$ is the parabolic subalgebra obtained from $\mathfrak{p}$ using w_0. □

We emphasize that the extensions in the formulation of Corollary 6.6 are taken in $\mathcal{O}$ and not in $\mathcal{O}^{\mathfrak{p}}$. Indeed, T is assumed to be a tilting module in $\mathcal{O}^{\mathfrak{p}}$ and, as such, is ext-self-orthogonal in $\mathcal{O}^{\mathfrak{p}}$.

6.3. **The $\mathbf{b}$-function.** We look further into the number $g_{x,y}$ appear in Proposition 6.5. We introduce a function which refines the $\mathbf{a}$-function. For $x, y \in W$, consider the non-negative integer

$$\mathbf{b}(x, y) := \max\{\deg h_{x,z,y} \,:\, z \in W\} \tag{17}$$

which we take as the definition of the function $\mathbf{b} : W \times W \to \mathbb{N} \sqcup \{-\infty\}$ (by our convention the degree of the zero polynomial is $-\infty$).

By (13), $\mathbf{b}(x,y)$ gives the height (the highest degree of the composition factors) of the module $\theta_x L_y$. Since the latter is self-dual, we have

$$2\mathbf{b}(x,y) = \text{the graded length of } \theta_x L_y. \tag{18}$$

Using the Koszul-Ringel duality from [Ma3], we also have

$$2\mathbf{b}(x,y) = \mathrm{tilt.dim}\, \theta_{y^{-1}w_0} L_{w_0 x^{-1}} \tag{19}$$

where $\mathrm{tilt.dim}\, M$ denotes the length of a minimal complex of tilting modules representing M. Consequently, we have

$$2\mathbf{b}(x,y) = g_{y^{-1}w_0, w_0 x^{-1}}. \tag{20}$$

Thus we can rewrite Proposition 6.2(i) as

$$\mathrm{proj.dim}(\theta_x L_y) = \mathbf{a}(w_0 x) + \mathbf{b}(y^{-1}w_0, w_0 x^{-1}). \tag{21}$$

On the one hand, this provides a formula for the projective dimension of $\theta_x L_y$ purely in terms of Kazhdan-Lusztig combinatorics. On the other hand, this gives another characterization of the $\mathbf{b}$-function.

Proof. This follows directly from (21) and Proposition 6.8(i). □

Proposition 6.7. *The value $\mathbf{b}(x,y)$ is $-\infty$ if and only if $x^{-1} \not\leq_{\mathtt{L}} y$. Moreover, if $y \sim_{\mathtt{L}} y'$, then $\mathbf{b}(x,y) = \mathbf{b}(x,y')$.*

Proof. The first statement follows from (18). If $y \sim_{\mathtt{L}} y'$, then $y^{-1}w_0 \sim_{\mathtt{R}} (y')^{-1}w_0$. The second statement follows from (19). Alternatively, given a tilting resolution $T_\bullet$ of $\theta_{y^{-1}w_0}L$, a tilting resolution of $\theta_{(y')^{-1}w_0}L$ is obtained as a direct summand of $\theta T_\bullet$ for some projective functor θ. This implies $\mathbf{b}(x,y) \geq \mathbf{b}(x,y')$ and the desired equality holds by symmetry. □

Proposition 6.8. *Let $x, y \in W$ be such that $x^{-1} \leq_{\mathtt{L}} y$.*

(i) If $x \sim_{\mathtt{J}} y$ (i.e., $x^{-1} \sim_{\mathtt{L}} y$), then we have $\mathbf{a}(x) = \mathbf{b}(x,y) = \mathbf{a}(y)$;

(ii) If $x \not\sim_{\mathtt{J}} y$ (i.e., $x^{-1} <_{\mathtt{L}} y$), then $\mathbf{a}(x) \leq \mathbf{b}(x,y) < \mathbf{a}(y)$;

(iii) $\mathbf{b}(x,y) \leq \ell(x)$;

(iv) If $x = w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$, then $\mathbf{a}(x) = \mathbf{b}(x,y) = \ell(x)$;

(v) If $y = w_0 w_0^{\mathfrak{p}}$, for some parabolic subalgebra $\mathfrak{p}$, then $\mathbf{b}(x,y) = \mathbf{s}_{\lambda_{\mathfrak{p}^{w_0}}}(w_0 x^{-1}) - \mathbf{a}(y)$, where $\mathbf{s}_{\lambda_{\mathfrak{p}^{w_0}}}(w)$ denotes the projective dimension of $L(w.\lambda_{\mathfrak{p}^{w_0}})$ for a singular weight $\lambda_{\mathfrak{p}^{w_0}}$ with stabilizer given by the w_0-conjugate of the parabolic subgroup of W corresponding to $\mathfrak{p}$, cf. [CM2].

Proof. By (18), we have $\mathbf{a}(x) \leq \mathbf{b}(x,y) \leq \ell(x)$ and $\mathbf{b}(x,y) \leq \mathbf{a}(y)$ is clear from the definition. The latter is equality if and only if $x \sim_{\mathtt{J}} y$ by Proposition 6.4. This proves claims (i), (ii) and (iii). Claim (iv) follows from claims (i), (ii) and (iii) since $\mathbf{a}(w_0^{\mathfrak{p}}) = \ell(w_0^{\mathfrak{p}})$.

Claim (v) follows from Proposition 6.2 and (20), after two observations. The first observation is that $L(w_0 x^{-1}.\lambda)$ has a linear projective resolution in its singular block $\mathcal{O}_\lambda$ and this resolution is mapped to a linear projective resolution of $\theta_{w_0^{\mathfrak{p}}} L_{w_0 x^{-1}}$ by translating out of the wall from $\mathcal{O}_\lambda$ to $\mathcal{O}_0$. The second observation is that $\mathbf{a}(y) = \mathbf{a}(w_0 y^{-1} w_0)$. □

Remark 6.9. In the case $y = w_0^{\mathfrak{p}} w_0$, we alternatively have $\theta_x L_y = \theta_x L_{w_0^{\mathfrak{p}} w_0}$ is the indecomposable tilting module with highest weight $w_0^{\mathfrak{p}} w_0 x$ in the principal block $\mathcal{O}_0^{\mathfrak{p}}$ of the parabolic category $\mathcal{O}^{\mathfrak{p}}$. According to [CM2, Table 2], the graded length of this parabolic tilting module is given by $2(\mathbf{s}_\lambda((w_0^{\mathfrak{p}} w_0 x)^{-1} w_0^{\mathfrak{p}}) - \mathbf{a}(w_0 w_0^{\mathfrak{p}}))$. Therefore we have $\mathbf{b}(x,y) = \mathbf{s}_\lambda(w_0(w_0 x^{-1})w_0) - \mathbf{a}(w_0 y w_0)$ which agrees with Proposition 6.8(v).

Note that Proposition 6.8(v) provides an explicit description of the function $\mathbf{s}_\lambda$ in terms of Kazhdan-Lusztig combinatorics.

7. $\mathbf{K}$ AND $\mathbf{KMM}$

7.1. **Category $\mathcal{O}$ via Harish-Chandra bimodules.** Denote by $\mathcal{H}$ the category of *Harish-Chandra* $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules, that is, finitely generated $U(\mathfrak{g})$-$U(\mathfrak{g})$-bimodules on which the adjoint action of $\mathfrak{g}$ is locally finite and has finite multiplicities, see [Ja, Kapitel 6]. Note that $\mathcal{H}$ is equipped with a tensor product given by tensoring over $U = U(\mathfrak{g})$ (however, $\mathcal{H}$ is not monoidal as the regular bimodule $U(\mathfrak{g})$ is not a Harish-Chandra bimodule).

Let $\mathbf{m}$ denote the maximal ideal of $Z(\mathfrak{g})$ which annihilates the trivial $\mathfrak{g}$-module. Let I be the kernel of the surjection from $Z(\mathfrak{g})$ onto $\mathrm{End}(P_{w_0})$, see Endomorphismensatz in [So1]. Let ${}^\infty_0\mathcal{H}^\infty_0$ denote the full subcategory of $\mathcal{H}$ consisting of all bimodules X such that $X\mathbf{m}^i = 0$ and $\mathbf{m}^i X = 0$, for $i \gg 0$. Similarly let ${}^\infty_0\mathcal{H}^1_0$ be the full subcategory of $\mathcal{H}$ (or of ${}^\infty_0\mathcal{H}^\infty_0$) consisting of bimodules X such that $X\mathbf{m} = 0$ and $\mathbf{m}^i X = 0$ for $i \gg 0$. Let ${}^\infty_0\mathcal{H}^I_0$ be the full subcategory of ${}^\infty_0\mathcal{H}^\infty_0$ consisting of bimodules X such that $XI = 0$ and $\mathbf{m}^i X = 0$ for $i \gg 0$. By [BG], we have mutually inverse equivalences of categories as follows:

$$\mathcal{O}_0 \underset{-\otimes_{U(\mathfrak{g})} P_e}{\overset{\mathscr{L}(P_e,-)}{\rightleftarrows}} {}^\infty_0\mathcal{H}^1_0 \tag{22}$$

Similarly, we have a monoidal equivalence

$$(\overline{\mathscr{P}}, \circ) \cong ({}^\infty_0\mathcal{H}^I_0, \otimes_U), \tag{23}$$

where $\overline{\mathscr{P}}$ denotes the projective abelianization of $\mathscr{P}$, see [MM1, § 3.5].

7.2. **Internal hom of projective functors.** The monoidal category $({}^\infty_0\mathcal{H}^I_0, \otimes_U)$ acts (in the sense of [EGNO, Section 7]) on the left on $\mathcal{O}_0$ in the obvious way, and we can apply the theory of internal hom as in [Os] (see [EGNO, §7.8-7.10]). This action restricts to the projective objects, that is, $\mathrm{proj}({}^\infty_0\mathcal{H}^I_0) \cong \mathscr{P}$ acts on $\mathrm{proj}(\mathcal{O}_0)$ as well as on $\mathcal{O}_0$. The action of $\mathscr{P}$ on $\mathcal{O}_0$ is given by exact functors, and the action of ${}^\infty_0\mathcal{H}^I_0$ on $\mathcal{O}_0$ is given by right exact functors. For $M \in \mathcal{O}_0$, the internal hom functor $[M, -] : \mathcal{O}_0 \to {}^\infty_0\mathcal{H}^I_0$ is defined as the right adjoint of the right exact functor $-\otimes_U M : {}^\infty_0\mathcal{H}^I_0 \to \mathcal{O}_0$. The object $[M, N]$, for $M, N \in \mathcal{O}_0$, agrees with the subspace $\mathscr{L}(M, N)$ of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ where the action of $\mathfrak{g}$ is locally finite, as defined in §2.3. The following adjunction confirms this fact.

Proposition 7.1. *For all $M, N \in \mathcal{O}_0$ and $X \in {}^\infty_0\mathcal{H}^I_0$, we have:*

$$\mathrm{Hom}_{\mathcal{O}}(X \otimes_U M, N) \cong \mathrm{Hom}_{{}^\infty_0\mathcal{H}^I_0}(X, \mathscr{L}(M, N)).$$

Proof. The Hom-tensor adjunction gives

$$\mathrm{Hom}_{\mathcal{O}}(X \otimes_U M, N) = \mathrm{Hom}_{U-}(X \otimes_U M, N) \cong \mathrm{Hom}_{U-U}(X, \mathrm{Hom}_{\mathbb{C}}(M, N)).$$

Now, since $X \in {}^\infty_0\mathcal{H}^I_0$, any $U - U$-bimodule map $X \to \mathrm{Hom}_{\mathbb{C}}(M, N)$ factors through $\mathscr{L}(M, N)$, that is, we have

$$\mathrm{Hom}_{\mathcal{O}}(X \otimes_U M, N) \cong \mathrm{Hom}_{{}^\infty_0\mathcal{H}^I_0}(X, \mathscr{L}(M, N)). \qquad \square$$

Note that $\mathscr{L}(\Delta_e, \Delta_e)$, as well as $\mathscr{L}(M, M)$, for any $M \in \mathcal{O}_0$, is an algebra in ${}^\infty_0\mathcal{H}^I_0$. The multiplication coming from the internal hom construction and the multiplication restricted from $\mathrm{Hom}_{\mathbb{C}}(M, M)$ coincide. We denote by $\mathrm{mod}_{{}^\infty_0\mathcal{H}^I_0}(\mathscr{L}(M, M))$ the category of (right) $\mathscr{L}(M, M)$-modules in ${}^\infty_0\mathcal{H}^I_0$. We denote by $\mathrm{proj}_{{}^\infty_0\mathcal{H}^I_0}(\mathscr{L}(M, M))$ the category of projective objects in $\mathrm{mod}_{{}^\infty_0\mathcal{H}^I_0}(\mathscr{L}(M, M))$. The morphism spaces in both categories are denoted

$\mathrm{Hom}_{\mathscr{L}(M,M)}(-,-)$. There is a natural action of ${}^{\infty}_{0}\mathcal{H}^{I}_{0}$ on $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M))$ on the left given by the monoidal structure of ${}^{\infty}_{0}\mathcal{H}^{I}_{0}$. We refer to [EGNO] for further details on module categories in monoidal categories.

Proposition 7.2. *For $M \in \mathcal{O}_0$, we have*

$$\mathrm{add}_{\mathcal{O}_0}\{\theta_x M \mid x \in W\} \cong \mathrm{proj}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M))$$

as module categories over $\mathscr{P}$. Moreover, we have

$$\mathrm{add}_{\mathcal{O}_0}\{FM \mid F \in {}^{\infty}_{0}\mathcal{H}^{I}_{0}\} \cong \mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M))$$

as module categories over ${}^{\infty}_{0}\mathcal{H}^{I}_{0}$.

Proof. Since $\mathscr{P} \cong \mathrm{proj}\, {}^{\infty}_{0}\mathcal{H}^{I}_{0}$ has monoidal duals (i.e., rigid in the sense of [EGNO]; it is also fiat in the sense of [MM1]), the internal hom theory as in [EGNO, Section 7] and [MMMT] applies. Consider the functor

$$\mathscr{L}(M,-) : \mathcal{O}_0 \to {}^{\infty}_{0}\mathcal{H}^{I}_{0}.$$

Objects of the form $\mathscr{L}(M,N)$ can be viewed as objects in $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M))$ since composition defines an action $\mathscr{L}(M,N) \otimes \mathscr{L}(M,M) \to \mathscr{L}(M,N)$. Now, consider the restriction $\overline{\Phi}$ of $\mathscr{L}(M,-)$ as follows:

$$\overline{\Phi} : \mathrm{add}_{\mathcal{O}_0}\{FM \mid F \in {}^{\infty}_{0}\mathcal{H}^{I}_{0}\} \to \mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M)).$$

Then we have that

(i) $\overline{\Phi}$ is a $\mathscr{P}$-module functor;

(ii) $\mathscr{L}(M,M)$ is projective in $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{\mathrm{I}}_{0}}(\mathscr{L}(M,M))$.

This yields a $\mathscr{P}$-module functor

$$\Phi : \mathrm{add}_{\mathcal{O}_0}\{\theta M \mid \theta \in \mathscr{P}\} \to \mathrm{proj}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(M,M))$$

(see the proofs in [MMMT, §4.2] for the dual version). Since $\mathrm{add}_{\mathcal{O}_0}\{\theta M \mid \theta \in \mathscr{P}\}$ is, by definition, a transitive representation of $\mathscr{P}$, an algebra analog of [MMMT, Theorem 4.7] gives the two equivalences in the statement. □

The following is a slight variation of (22).

Proposition 7.3. *We have* $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(\Delta_e,\Delta_e)) \cong \mathcal{O}_0$.

Proof. By Proposition 7.2, we have equivalences

$$\mathrm{add}_{\mathcal{O}_0}\{\theta_x \Delta_e \mid x \in W\} \cong \mathrm{proj}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(\Delta_e,\Delta_e))$$

and

$$\mathrm{add}_{\mathcal{O}_0}\{F \otimes_U \Delta_e \mid F \in {}^{\infty}_{0}\mathcal{H}^{I}_{0}\} \cong \mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^{I}_{0}}(\mathscr{L}(\Delta_e,\Delta_e)). \tag{24}$$

But $\theta_x \Delta_e \cong P_x$, and therefore $\mathrm{add}_{\mathcal{O}_0}\{\theta_x \Delta_e \mid x \in W\} \cong \mathrm{proj}\, \mathcal{O}_0$. The claim follows. □

7.3. **The image of $\mathscr{L}(L_w, L_w)$ in $\mathcal{O}_0$.** Graded characters of the projective modules P_w, $w \in W$, can be computed using Kazhdan-Lusztig combinatorics. Recall from Subsection 3.3 the left $\leq_{\mathtt{L}}$, right $\leq_{\mathtt{R}}$ and the two-sided order $\leq_{\mathtt{J}}$ on W.

Note that $\mathrm{Ann}(L_x) = \mathrm{Ann}(L_y)$ if and only if $x \sim_{\mathtt{L}} y$, and moreover, the inclusion order of primitive ideals for simple modules in $\mathcal{O}_0$ is the opposite of $\leq_{\mathtt{L}}$, see [BV1, BV2]. In particular, the image of $U(\mathfrak{g})$ in $\mathscr{L}(L_w, L_w)$ depends (up to isomorphism) only on the left cell of w.

Let $d \in W$ be a Duflo element. The module $\theta_d L_d$ is indecomposable with simple top $L_d\langle \mathbf{a}(d)\rangle$ and simple socle $L_d\langle -\mathbf{a}(d)\rangle$. It has a unique simple subquotient isomorphic to the trivial $\mathfrak{g}$-module, and this subquotient is in degree zero. Therefore there is a unique, up to scalar, non-zero homomorphism from P_e to $\theta_d L_d$. We denote by D_d the image of this homomorphism. The module D_d is indecomposable with simple top L_e and simple socle $L_d\langle -\mathbf{a}(d)\rangle$. All other simple subquotients of D_d have the form $L_w\langle i\rangle$, where $-\mathbf{a}(d) < i < 0$ and $x <_{\mathtt{R}} d$ (and hence also $x <_{\mathtt{L}} d$ since d is an involution). We refer to [Ma3, Section 3] for details.

Proposition 7.4. *We have*

$$(U(\mathfrak{g})/\mathrm{Ann}(L_d)) \otimes_{U(\mathfrak{g})} P_e \cong D_d.$$

Proof. The equivalence (22) sends, by construction, $U(\mathfrak{g})/(U(\mathfrak{g})\mathbf{m})$ to P_e inducing a bijection between subobjects. Note that subobjects of the former are exactly the two-sided ideals and that D_d is, by definition, a quotient of P_e. Therefore, it is enough to argue that $\mathrm{Ann}(D_d) = \mathrm{Ann}(L_d)$. This follows from [KhM, Lemma 6]. □

Denote by $\overline{D}_d$ the intersection of the kernels of all possible homomorphisms

$$\varphi : \theta_d L_d \to \theta_w L_d\langle i\rangle, \quad w \sim_{\mathtt{R}} d, \quad i \in \mathbb{Z},$$

satisfying $\varphi(D_d) = 0$. By construction, $D_d \subset \overline{D}_d$. In particular, L_d is the simple socle of $\overline{D}_d$ and all other composition factors of $\overline{D}_d$ are of the form $L_w\langle i\rangle$, where $-\mathbf{a}(d) < i$ and $w <_{\mathtt{R}} d$ (and also $w <_{\mathtt{L}} d$).

Proposition 7.5. *We have*

$$\mathscr{L}(L_d, L_d) \otimes_{U(\mathfrak{g})} P_e \cong \overline{D}_d.$$

Proof. By (22), it is enough to prove that

$$\mathscr{L}(P_e, \overline{D}_d) \cong \mathscr{L}(L_d, L_d). \tag{25}$$

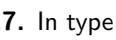
The natural projection $P_e \twoheadrightarrow D_d$ induces an embedding

$$\mathscr{L}(D_d, \overline{D}_d) \subset \mathscr{L}(P_e, \overline{D}_d). \tag{26}$$

Let K be the kernel of $P_e \twoheadrightarrow D_d$. From [Ma3, Corollary 3] it follows that θK has no non-zero homomorphisms to $\overline{D}_d$, for any projective functor θ. Therefore the inclusion (26) is, in fact, an isomorphism. Consequently, we have $\mathscr{L}(P_e, \overline{D}_d) \cong \mathscr{L}(D_d, \overline{D}_d)$. Now (25) follows from [KhM, Lemma 11] and [KhM, Lemma 12]. □

As an immediate corollary from Propositions 7.4 and 7.5, we have the following statement (which is a reformulation of the main result of [KhM]):

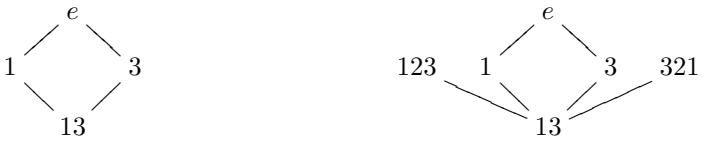
Corollary 7.6. *For $d \in W$ Duflo element, $\mathbf{K}(d)$ is equivalent to $D_d = \overline{D}_d$.*

Example 7.7. In type A_3 with Dynkin diagram 

, where $1, 2, 3$ are simple reflections, it is known from [MS2] that $\mathbf{K}(13)$ is false. In this case one can compute that the graded composition multiplicities of D_{13} and $\overline{D}_{13}$

are, respectively, as follows (here e is in degree zero, and we also abbreviate L_w by w, for simplicity):

(27)



7.4. $\mathbf{K}(d)$ implies $\mathbf{KMM}(*,d)$.

Proposition 7.8. *Let $d \in W$ be a Duflo element. Then $\mathbf{K}(d)$ implies $\mathbf{KMM}(*,d)$.*

Proof. By Proposition 7.5, the claim $\mathbf{K}(d)$ is equivalent to $D_d = \overline{D}_d$. Note that D_d is a quotient of P_e, and hence (8) and (9) implies that $[D_d : L_x\langle -\mathbf{a}(x)\rangle] \leq 1$. This implies $\mathbf{KMM}(*,d)$. □

From Example 7.7 we see that $\mathbf{K}$ is strictly stronger than $\mathbf{KMM}$. Indeed, the two additional elements on the right picture in (27) (compared with the left picture) both have the $\mathbf{a}$-value 1 and have multiplicity one (they are not Duflo elements either). Therefore, in this case, we have that $\mathbf{K}(13)$ is false while $\mathbf{KMM}(*,13)$ is true.

8. $\mathbf{K}$ vs. $\mathbf{KM}$ and $\mathbf{Kh}$

8.1. Some computations in $\mathcal{O}^{\hat{\mathcal{R}}}$.

Lemma 8.1. *Let $\mathcal{L}$ be a left cell in W and $d \in \mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$ the corresponding Duflo element. For each $y \in \mathcal{L}$ and $y' \in \mathcal{L}^{-1}$, we have*

$$\theta_d\theta_y = \theta_y\langle \mathbf{a}(d)\rangle \oplus \theta, \qquad \theta_{y'}\theta_d = \theta_{y'}\langle \mathbf{a}(d)\rangle \oplus \theta', \tag{28}$$

where $\theta, \theta' \in \mathscr{P}$ are (possibly empty) direct sums of (some) $\theta_z\langle a\rangle$ with $z \not\leq_{\mathrm{J}} d$ or $a < \mathbf{a}(d)$.

Proof. This follows from Lemma 3.2. □

The following lemma slightly extends [Ma3, Theorem 6].

Lemma 8.2. *Let $\mathcal{R} \subseteq W$ be a right cell, $y \in \mathcal{R}$ and $x \in W$ be such that $x \sim_{\mathrm{J}} y$. Then $\theta_x L_y$ is a projective-injective object in $\mathcal{O}^{\hat{\mathcal{R}}}$.*

Proof. If $y = d$ is Duflo, the claim follows directly from [Ma3, Theorem 6]. Moreover, we also have that $\theta_x L_d$, where $x^{-1} \in \mathcal{R}$, is a complete and irredundant list of indecomposable projective-injective modules in $\mathcal{O}^{\hat{\mathcal{R}}}$. Set $a := \mathbf{a}(d)$. Each of $\theta_x L_d$ has simple top $L_x\langle a\rangle$ and simple socle $L_x\langle -a\rangle$ and no other subquotients in these extreme non-zero components.

For an arbitrary y, by [Ma3, Proposition 1], the module $\theta_x L_y$ is either zero (in which case the claim is obvious) or the top of $\theta_x L_y$ is concentrated in degree $-a$ and the socle of $\theta_x L_y$ is concentrated in degree a, moreover, these top and socle have the same length due to self-duality of $\theta_x L_y$. Therefore, from the previous paragraph it follows that the minimal projective cover of $\theta_x L_y$ is an isomorphism. The claim of the lemma follows. We note that, alternatively, the claim of the lemma follows from [KMMZ, Theorem 2]. □

Proposition 8.3. *Let $\mathcal{L}$ be a left cell and $y, d \in \mathcal{L}$ be such that d is Duflo. Let d' be the Duflo element in the right cell $\mathcal{R}$ of y. Then $\theta_d L_y \cong \theta_y L_{d'} \cong P_y^{\hat{\mathcal{R}}}\langle \mathbf{a}(d)\rangle$, where $P_y^{\hat{\mathcal{R}}}$ denotes the projective cover of L_y in $\mathcal{O}_0^{\hat{\mathcal{R}}}$.*

Proof. By Lemma 8.1, we have

$$\theta_d P_y \cong \theta_d \theta_y P_e \cong P_y\langle \mathbf{a}(d)\rangle \oplus P,$$

where P is a direct sum of (some) $P_z\langle a\rangle$ with either $z \not\leq_{\mathtt{J}} d$ or $a < \mathbf{a}(d)$. Applying θ_d to the canonical map $P_y \twoheadrightarrow L_y$, we get

$$P_y\langle \mathbf{a}(d)\rangle \oplus P \cong \theta_d P_y \twoheadrightarrow \theta_d L_y. \tag{29}$$

By [Ma3, Proposition 1], the top of $\theta_d L_y$ is concentrated in degree $\mathbf{a}(d) = \mathbf{b}(d,y) = \mathbf{a}(y)$. Therefore, this top is a direct sum of $L_w\langle \mathbf{a}(d)\rangle$, for $w \sim_{\mathtt{R}} y$. It follows that the image of any possible map from P to $\theta_d L_y$ does not contain any top component of $\theta_d L_y$, showing that (29) restricts to $P_y\langle \mathbf{a}(d)\rangle \twoheadrightarrow \theta_d L_y$. Now the claim of the proposition follows combining the facts that $\theta_d L_y$ is projective in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, see Lemma 8.2, and that $\theta_y L_{d'}$ is the projective cover of $L_y\langle \mathbf{a}(d)\rangle$ in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, see [Ma3, Theorem 6]. □

Proposition 8.4. *Let $\mathcal{R} \subseteq W$ be a right cell and $d, y \in \mathcal{R}$ with d Duflo. Let $\mathcal{L}$ be the left cell of y and $x \in W$ be such that $x \sim_{\mathtt{J}} y$. Then $\theta_x L_y = 0$ if $x^{-1} \notin \mathcal{L}$ and, in case $x^{-1} \in \mathcal{L}$, we have*

$$\theta_x L_y \cong \bigoplus_{z \in \mathcal{L}\cap\mathcal{R}} \theta_z L_d^{\oplus \gamma_{y,x,z^{-1}}}. \tag{30}$$

Proof. That $\theta_x L_y = 0$, for all $x \in W$ such that $x \sim_{\mathtt{J}} y$ and $x^{-1} \notin \mathcal{L}$ follows from § 2.1. From [Ma3, Proposition 1 and Theorem 6] it follows that $\theta_x L_y$ is a direct sum of $\theta_z L_d$, where $z \in \mathcal{L} \cap \mathcal{R}$, with some non-negative integer multiplicities, denote them by c_z. Applying θ_x to $\theta_y L_d\langle -\mathbf{a}(d)\rangle \twoheadrightarrow L_y$ and using [Ma3, Proposition 1] and § 3.4, we obtain $c_z = \gamma_{y,x,z^{-1}}$. □

8.2. **Computation with simple bimodules.** Denote by β_x the simple Harish-Chandra bimodule corresponding to $x \in W$, that is, β_x is the simple top of θ_x in ${}^\infty_0\mathcal{H}^I_0$. Denote by $\star : {}^\infty_0\mathcal{H}^I_0 \to {}^\infty_0\mathcal{H}^I_0$ the simple preserving duality satisfying $(\beta\langle a\rangle)^\star = \beta^\star\langle -a\rangle$, see [MM, § 4.1]. This duality restricts to $\mathcal{O}_0$ and is compatible with the action in the sense that, for $M \in \mathcal{O}_0$ and $\theta \in {}^\infty_0\mathcal{H}^I_0$, we have a natural isomorphism $\theta^\star M^\star \cong (\theta M)^\star$. The following statement provides a different interpretation of the internal hom.

Proposition 8.5. *For any $y \in W$, we have $\mathscr{L}(L_y, (_)^\star I_e)^\star \cong - \otimes_U \beta_{y^{-1}}$.*

Proof. Consider the functor $- \otimes_U \beta_{y^{-1}} : {}^\infty_0\mathcal{H}^I_0 \to {}^\infty_0\mathcal{H}^I_0$. It naturally commutes (in the sense of [Kho]) with the left action of ${}^\infty_0\mathcal{H}^I_0$ on itself. Therefore, by the main result of [Kho], it is determined, up to isomorphism, by its value at θ_e. Clearly, this value is $\theta_e \otimes_U \beta_{y^{-1}} = \beta_{y^{-1}}$.

By construction, $\mathscr{L}(L_y, (_)^\star I_e)^\star$ is also right exact and naturally commutes with the left action of ${}^\infty_0\mathcal{H}^I_0$ by [Ja, § 6.8]. The value

$$\mathscr{L}(L_y, (\theta_e)^\star I_e)^\star \cong \mathscr{L}(L_y, (\theta_e P_e)^\star)^\star \cong \mathscr{L}(L_y, I_e)^\star$$

at θ_e can be identified studying homomorphisms from projective objects:

$$\mathrm{Hom}_{{}^\infty_0\mathcal{H}^I_0}(\theta_w, \mathscr{L}(L_y, I_e)) \cong \mathrm{Hom}_{\mathcal{O}}(\theta_w L_y, I_e) \cong \mathrm{Hom}_{\mathcal{O}}(L_y, \theta_{w^{-1}} I_e) \cong \mathrm{Hom}_{\mathcal{O}}(L_y, I_{w^{-1}}) = \delta_{y,w^{-1}}\mathbb{C}.$$

It follows that $\mathscr{L}(L_y, I_e) \cong \mathscr{L}(L_y, I_e)^\star \cong \beta_{y^{-1}}$. The claim follows. □

Let $y, z \in W$. Evaluating both sides of Proposition 8.5 at β_z and noticing that all simples are self-dual, we obtain

$$\mathcal{L}(L_y, L_z) \cong (\beta_z \beta_{y^{-1}})^\star. \tag{31}$$

Evaluating both sides of Proposition 8.5 at θ_z, we obtain

$$\mathcal{L}(L_y, I_z) \cong (\theta_z \beta_{y^{-1}})^\star. \tag{32}$$

Corollary 8.6. *We have the canonical inclusion $\mathscr{L}(L_y, L_z) \otimes_U P_e \hookrightarrow \theta_z L_{y^{-1}}$, for $y, z \in W$.*

Proof. Applying the left exact functor $\mathscr{L}(L_y, _)$ to the canonical inclusion $L_z \hookrightarrow I_z$, we obtain

$$\mathscr{L}(L_y, L_z) \hookrightarrow \mathscr{L}(L_y, I_z)$$

Applying now the equivalence $_ \otimes_U P_e$, the right term becomes $(\theta_z L_{y^{-1}})^\star \cong \theta_z L_{y^{-1}}$. □

Note that $\mathscr{L}(L_y, L_z) = 0$ unless $y \sim_{\mathtt{R}} z$. Proposition 8.5 provides the following description of $\mathscr{L}(L_y, L_z)$ in the case when it is nonzero:

Proposition 8.7. *Let $y, z \in W$ be such that $y \sim_{\mathtt{R}} z$. Then the object $\mathscr{L}(L_y, L_z) \otimes_U P_e$ admits a copresentation*

$$0 \to \mathscr{L}(L_y, L_z) \otimes_U P_e \to \theta_z L_{y^{-1}} \to \bigoplus_{w \in W} \boldsymbol{\mu}(z, w) \theta_w L_{y^{-1}} \langle 1 \rangle. \tag{33}$$

Proof. The simple object $L_z \in \mathcal{O}$ admits an injective copresentation

$$L_z \hookrightarrow I_z \to \bigoplus_{w \in W} \boldsymbol{\mu}(z, w) I_w \langle 1 \rangle. \tag{34}$$

Applying the left exact functor $\mathscr{L}(L_y, -) \otimes_U P_e$ to (34), we obtain (33) using (32) by the same arguments as in Corollary 8.6. □

Next lemma give us additional information on the copresentation in (33).

Lemma 8.8. *For $y, z \in W$ such that $y \sim_{\mathtt{R}} z$, we have* $\operatorname{soc} \mathscr{L}(L_y, L_z) = \operatorname{soc} \mathscr{L}(L_y, I_z)$.

Proof. We prove the equivalent statement in $\mathcal{O}$, namely, $\operatorname{soc} \mathscr{L}(L_y, L_z) \otimes_U P_e = \operatorname{soc} \mathscr{L}(L_y, I_z) \otimes_U P_e$. Since $y \sim_{\mathtt{R}} z$, the socle of $\mathscr{L}(L_y, I_z) \otimes_U P_e \cong \theta_z L_{y^{-1}}$ is concentrated in degree $\mathbf{a}(z)$. By Proposition 8.7, it is thus enough to note that each $\theta_w L_{y^{-1}} \langle 1 \rangle$ has extreme degree $\mathbf{b}(w, y^{-1}) - 1 \leq \mathbf{a}(y^{-1}) - 1 < \mathbf{a}(z)$, see Proposition 6.8. □

From Proposition 8.7 and Theorem 2.2, we have that, for $y, z \in W$ such that $y \sim_{\mathtt{R}} z$, the assumption

$$[\mathscr{L}(L_y, L_z) \otimes_U P_e : L_w \langle -a \rangle] \neq 0,$$

for some $w \in W$ and $a \in \mathbb{Z}$, implies $\mathbf{a}(w) \leq a \leq \mathbf{a}(y)$.

The following statement generalizes [KhM, Lemma 8(i)].

Corollary 8.9. *Let $y, z \in W$ be such that $y \sim_{\mathtt{R}} z$, and $\mathcal{R}$ be the right cell of y^{-1}. Then $\mathscr{L}(L_y, L_z) \otimes_U P_e$ is isomorphic to the largest submodule M of $\theta_z L_{y^{-1}} \in \mathcal{O}_0^{\hat{\mathcal{R}}}$ such that any simple subquotient of $M / \operatorname{soc} M$ is, up to shift of grading, of the form L_w, for some $w \notin \mathcal{R}$.*

Proof. We start by showing that, for $w \in \mathcal{R}$, the module L_w can only appear in the socle of $\mathscr{L}(L_y, L_z) \otimes_U P_e$, that is in degree $\mathbf{a}(w)$. This follows directly from [Ma3, Proposition 1] and the adjunction

$$\operatorname{Hom}_{\mathcal{O}}(\theta_w L_y, L_z) \cong \operatorname{Hom}_{{}^\infty_0 \mathcal{H}^I_0}(\theta_w, \mathscr{L}(L_y, L_z)),$$

given by Proposition 7.1. Therefore $\mathscr{L}(L_y, L_z) \otimes_U P_e \subset M$ by Corollary 8.6. Now, the necessary isomorphism $\mathscr{L}(L_y, L_z) \otimes_U P_e = M$ follows from Proposition 8.7 since the socle of each summand in the second term of the injective (in $\mathcal{O}_0^{\hat{\mathcal{R}}}$) copresentation of $\mathscr{L}(L_y, L_z) \otimes_U P_e$ is, up to shift of grading, of the form L_w, for some $w \in \mathcal{R}$. □

For a Duflo element $d \in W$, let $\mathcal{R}$ be the right cell of d. Denote by $\Psi : \mathcal{O}_0^{\hat{\mathcal{R}}} \to \mathcal{O}_0^{\hat{\mathcal{R}}}$ the functor of partial approximation with respect to projective-injective modules in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, see [KM, § 2.4]. The easiest way to define this functor is as follows: we let $\mathcal{X}$ denote the Serre subcategory of $\mathcal{O}_0^{\hat{\mathcal{R}}}$ generated by all simple which do not appear in the socle of projective-injective objects. Then Ψ is the composition of the (exact) natural projection $\mathcal{O}_0^{\hat{\mathcal{R}}} \twoheadrightarrow \mathcal{O}_0^{\hat{\mathcal{R}}}/\mathcal{X}$ followed by the right adjoint of this projection. In particular, Ψ is left exact and is equipped with a natural transformation η from the identity to it which is non-zero exactly on those simple modules which appear in the socle of a projective-injective module. By [Ma3, Theorem 6], Corollary 8.9 says precisely that $\mathscr{L}(L_y, L_z) \otimes_U P_e$ is isomorphic to $\Psi(\mathrm{soc}(\theta_z L_{y^{-1}}))$.

We can now relate our discussion closer to the results of [KhM].

Proposition 8.10. *Let $y \in W$ and $\mathcal{R}$ be the right cell containing y^{-1}. Then the left exact functor*

$$\mathscr{L}(L_y, _) \otimes_U P_e : \mathcal{O}_0 \to \mathcal{O}_0$$

has image in $\mathcal{O}^{\hat{\mathcal{R}}}$ and maps $I_z \in \mathcal{O}_0$, for $w \in W$, to zero unless $z \leq_{\mathtt{R}} y$.

Proof. Note that, for any $M \in \mathcal{O}_0$, for L_x to appear in the composition series of $\mathscr{L}(L_y, M) \otimes_U P_e$, the space

$$\mathrm{Hom}_{\mathcal{O}}(P_x, \mathscr{L}(L_y, M) \otimes_U P_e) \cong \mathrm{Hom}_{{}^{\infty}_{0}\mathcal{H}^I_0}(\theta_x, \mathscr{L}(L_y, M)) \cong \mathrm{Hom}_{\mathcal{O}}(\theta_x L_y, M) \tag{35}$$

should be nonzero, which requires $\theta_x L_y \neq 0$ and thus $x \leq_{\mathtt{R}} y^{-1}$. This proves the first claim.

Note that all simple subquotients of $\theta_x L_y$ have the form L_z, where $z \leq_{\mathtt{R}} y$. This implies the second claim. □

For $y \in W$, denote by $\mathcal{R}$ the right cell of y and by $\mathcal{R}'$ the right cell of y^{-1}. Proposition 8.10 says that the functor $\mathscr{L}(L_y, -) \otimes_U P_e$ restricts to the functor

$$\Psi_y : \mathcal{O}_0^{\hat{\mathcal{R}}} \to \mathcal{O}_0^{\hat{\mathcal{R}}'}. \tag{36}$$

If y is a Duflo element, then $\mathcal{R} = \mathcal{R}'$ and the functor Ψ_y agrees with the functor of partial coapproximation with respect to the projective-injective modules in $\mathcal{O}_0^{\hat{\mathcal{R}}}$, cf. [KhM, Corollary 7.21].

8.3. **Kostant's problem via internal hom.** By [Ja, § 6.9], we have $U/\operatorname{Ann}(P_e) \cong \mathscr{L}(P_e, P_e)$ as algebras in ${}^{\infty}_{0}\mathcal{H}^I_0$. Recall that $\operatorname{Ann}(P_e) = UI$. More generally, for every $M \in \mathcal{O}_0$, there is an injective map of algebras $\phi_M : U/\operatorname{Ann}(M) \to \mathscr{L}(M, M)$. In the case $M = L(w)$, since $U/\operatorname{Ann}(P_e) \twoheadrightarrow U/\operatorname{Ann}(L_w)$, we have an algebra homomorphism

$$\mathscr{L}(P_e, P_e) \to \mathscr{L}(L_w, L_w) \tag{37}$$

which is surjective if and only if $\mathbf{K}(w)$ holds.

Fix $y \in W$, let $\mathcal{L}$ be the left cell of y and $\mathcal{R} = \mathcal{L}^{-1}$, the right cell of y^{-1}. For simplicity, we write $U_y := U/\operatorname{Ann}(L_y)$ and $A_y := \mathscr{L}(L_y, L_y)$, and let $\phi_y := \phi_{L_y} : U_y \hookrightarrow A_y$. This induces the restriction functor

$$\mathrm{Res}^{A_y}_{U_y} : \mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^I_0}(A_y) \to \mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^I_0}(U_y).$$

Proposition 8.11. *The category $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^I_0}(U_y)$ can be identified with ${}^{\infty}_{0}\mathcal{H}_0^{\mathrm{Ann}(L_y)}$, i.e., the category of Harish-Chandra bimodules that are annihilated by $\mathrm{Ann}(L_y)$ on the right.*

Proof. Note that $_ \otimes_U U_y$ is the identity functor on the full subcategory of ${}^{\infty}_{0}\mathcal{H}^I_0$ consisting of all M such that $M \cdot \mathrm{Ann}(L_y) = 0$. In particular, we have $\mu : U_y \otimes_U U_y \cong U_y$ and this, together with the unit map $\epsilon : U/UI \twoheadrightarrow U_y$, makes U_y an algebra object in ${}^{\infty}_{0}\mathcal{H}^I_0$. For this algebra object, the corresponding module category $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^I_0}(U_y)$ is a full subcategory of ${}^{\infty}_{0}\mathcal{H}^I_0$. The objects in $\mathrm{mod}_{{}^{\infty}_{0}\mathcal{H}^I_0}(U_y)$ are exactly those $M \in {}^{\infty}_{0}\mathcal{H}^I_0$ for which $M \otimes_U U_y \cong M$, i.e., $M \cdot \mathrm{Ann}(L_y) = 0$. In fact, if $M \otimes_U U_y \cong M$, then $1_M \otimes_U \mu$ defines a U_y-module structure on M, and, if we

have a U_y-modules structure $m : M \otimes_U U_y \to M$, then the unit law gives $1_M = m \circ (1_M \otimes_U \epsilon)$, which implies that m is an isomorphism since $1_M \otimes_U \epsilon$ is surjective. The claim follows. □

Recall that, for an additive category $\mathcal{A}$, its projective abelianization is denoted $\overline{\mathcal{A}}$, see [MM1, § 3.5].

Proposition 8.12. *We have* $\mathrm{mod}_{\infty_0\mathcal{H}_0^I}(U_y) \cong \overline{\mathrm{add}_{\infty_0\mathcal{H}_0^I}\{\theta_x \otimes_U U_y \,:\, x \in W\}} \cong \mathcal{O}^{\hat{\mathcal{R}}}$.

Proof. Since β_x, for $x \in W$ such that $x \leq_{\mathtt{R}} y^{-1}$, are exactly the simples that are killed by $\mathrm{Ann}(L_y)$ on the right, the equivalence $_ \otimes_U P_e$ given by (22) restricts to the desired equivalence via Proposition 8.11. □

Corollary 8.13. *The indecomposable projectives in* $\mathrm{mod}_{\infty_0\mathcal{H}_0^I}(U_y)$ *are exactly* $\theta_x \otimes_U U_y$*, for* $x \in \hat{\mathcal{R}}$.

Proof. This follows from the fact that the nonzero $\theta_x \otimes_U U_y$ corresponds in $\mathcal{O}_0$ to a quotient of P_x. Therefore such modules are indecomposable and mutually non-isomorphic. □

Proposition 8.14. *For* $y \in W$*, we have that* $\mathbf{K}(y)$ *implies both* $\mathbf{Kh}(y)$ *and* $\mathbf{KM}(*, y)$.

Proof. Suppose $\mathbf{K}(y)$ is true. Then we have

$$\mathrm{proj}_{\infty_0\mathcal{H}_0^I}(A_y) \cong \mathrm{proj}_{\infty_0\mathcal{H}_0^I}(U_y), \tag{38}$$

where $\theta_x \otimes_U A_y$ corresponds to $\theta_x \otimes_U U_y$. By Proposition 7.2, we also have

$$\mathrm{add}_{\mathcal{O}_0}\{\theta_x L_y \,:\, x \in W\} \cong \mathrm{proj}_{\infty_0\mathcal{H}_0^I}(A_y), \tag{39}$$

where $\theta_x L_y$ corresponds to $\theta_x A_y$. Combining (38) and (39), we get an equivalence that identifies $\theta_x L_y$ with $\theta_x \otimes_U U_y \in {}^{\infty}_0\mathcal{H}_0^I$. Thus, by Corollary 8.13, $\theta_x L_y$ are indecomposable and mutually non-isomorphic. □

Proposition 8.15. *For a Duflo element* $d \in W$*, if both* $\mathbf{Kh}(d)$ *and* $\mathbf{KM}(*, d)$ *are true, then* $\mathrm{Res}_{U_d}^{A_d}$ *is an equivalence.*

Proof. Let $\mathcal{R}$ be the right cell of d. The equivalence in Proposition 8.12 restricts to the equivalence

$$\mathrm{proj}(\mathcal{O}^{\hat{\mathcal{R}}}) \cong \mathrm{proj}_{\infty_0\mathcal{H}_0^I}(U_d), \tag{40}$$

which sends $P_w^{\hat{\mathcal{R}}}$, for $w \in W$, to $\theta_w \otimes_U U_d$. By [Ma3, Corollary 3], we have $L_d \hookrightarrow P_e^{\hat{\mathcal{R}}}$ and any simple subquotient of the cokernel of this inclusion is of the form L_x, for $x <_{\mathtt{R}} d$, up to shift of grading. For $w \in W$, applying θ_w gives the inclusion $\theta_w L_d \hookrightarrow P_w^{\hat{\mathcal{R}}}$ such that any simple subquotient of the cokernel of this inclusion is of the form L_x, for $x <_{\mathtt{R}} d$, up to shift of grading. In particular, there are no homomorphisms from any such cokernel to any object in $\mathrm{proj}(\mathcal{O}^{\hat{\mathcal{R}}})$. This means that, for $w, u \in W$, any non-zero homomorphism from $P_w^{\hat{\mathcal{R}}}$ to $P_u^{\hat{\mathcal{R}}}$ restricts to a non-zero homomorphism from $\theta_w L_d$ to $\theta_u L_d$. Using (39) and (40), this gives a faithful functor

$$\Upsilon : \mathrm{proj}_{\infty_0\mathcal{H}_0^I}(U_d) \to \mathrm{proj}_{\infty_0\mathcal{H}_0^I}(A_d).$$

Because of $\mathbf{Kh}(d)$ and $\mathbf{KM}(*, d)$, this functor sends (pairwise non-isomorphic) indecomposable objects to (pairwise non-isomorphic) indecomposable objects.

Using (39), Proposition 8.12 and Proposition 7.2, the functor $\mathrm{Res}_{U_d}^{A_d}$ gives rise to the faithful functor

$$\Theta : \mathrm{add}_{\mathcal{O}}\{\theta_x L_d \mid x \in \hat{\mathcal{R}}\} \to \mathrm{add}_{\infty_0\mathcal{H}_0^I}\{\theta_x \otimes_U U_d \mid x \in \hat{\mathcal{R}}\}$$

which sends $\theta_x L_d$ to $\theta_x \otimes_U U_d$. The combination of $\mathbf{Kh}(d)$ and $\mathbf{KM}(*, d)$ implies that Θ sends (pairwise non-isomorphic) indecomposable objects to (pairwise non-isomorphic) indecomposable objects. Putting the faithful functors Υ and Θ together, we thus conclude that they are equivalences. This means that $\mathrm{Res}_{U_d}^{A_d}$ is an equivalence when restricted to projective objects and hence is an equivalence. This completes the proof. □

Theorem 8.16. *Let $d \in W$ be a Duflo element. Then $\mathbf{K}(d)$ is true if and only if both $\mathbf{Kh}(d)$ and $\mathbf{KM}(*, d)$ are true.*

Proof. The "only if" direction is Proposition 8.14. For the "if" direction, assume that both $\mathbf{Kh}(d)$ and $\mathbf{KM}(*, d)$ are true. Then, by Proposition 8.15, $\mathrm{Res}_{U_d}^{A_d}$ is an equivalence sending A_d to U_d. Therefore A_d and U_d are isomorphic as objects in ${}_0^{\infty}\mathcal{H}_0^I$, which is exactly the claim $\mathbf{K}(d)$. □

8.4. **More on K, Kh, and KM.** For $y \in W$, we set $E_y := \mathscr{L}(L_y, L_y) \otimes_U P_e \cong \overline{D_y}$ and $D_y := U_y \otimes_U P_e \in \mathcal{O}$. Note that D_y depends only on the left cell of y. From $U_y \to \mathscr{L}(L_y, L_y)$, we have $D_y \hookrightarrow E_y$.

Proposition 8.17. *Let $y \in W$ and d be the Duflo element in the left cell of y. Then E_d is a summand of E_y.*

Proof. Let $\mathcal{R}$ be the right cell of d. By adjunction, we have

$$\mathrm{Hom}(L_d, \theta_y L_{y^{-1}}) = \mathrm{Hom}(\theta_{y^{-1}} L_d, L_{y^{-1}}).$$

The right hand side has dimension one by [Ma3, Theorem 6]. Therefore $\theta_d L_d$, which is the indecomposable injective envelope of L_d in $\mathcal{O}_0^{\mathcal{R}}$, is a summand of the injective module $\theta_y L_{y^{-1}}$ in $\mathcal{O}_0^{\mathcal{R}}$ with multiplicity one.

Now we can use Proposition 8.7 and Corollary 8.9. The socle of the module E_d coincides with the socle of $\theta_d L_d$ and $E_d \cong \Psi(\mathrm{Soc}(\theta_d L_d))$. The socle of E_y coincides with the socle of $\theta_y L_{y^{-1}}$ and we have $E_y \cong \Psi(\mathrm{Soc}(\theta_y L_{y^{-1}}))$. Now the claim of the proposition follows from the additivity of Ψ be applying the latter to the unique up to a scalar split injection of $\mathrm{Soc}(\theta_d L_d)$ in $\mathrm{Soc}(\theta_y L_{y^{-1}})$ given by the previous paragraph. □

We summarize the above discussions in the following statement.

Theorem 8.18. *Let $y \in W$ and d be the Duflo element in the left cell of y, and denote by $\mathcal{H}$ the* H*-cell of y. Then we have a commutative diagram*

$$\begin{array}{ccccc} D_y & \hookrightarrow & E_y & \hookrightarrow & \theta_y L_{y^{-1}} \cong \bigoplus_{h \in \mathcal{H}} \theta_h L_{y^{-1}} \ {}^{\oplus \gamma_{y,y^{-1},h^{-1}}} \\ \| & & \uparrow & & \uparrow \\ D_d & \hookrightarrow & E_d & \hookrightarrow & \theta_d L_d, \end{array} \tag{41}$$

whose vertical maps are split. Moreover, the diagram (41) *restricts to*

$$\begin{array}{ccccc} L_d = \mathrm{soc}\, D_y & \hookrightarrow & \mathrm{soc}\, E_y & = & \mathrm{soc}\, \theta_y L_{y^{-1}} \\ \| & & \uparrow & & \uparrow \\ L_d = \mathrm{soc}\, D_d & = & \mathrm{soc}\, E_d & = & \mathrm{soc}\, \theta_d L_d \end{array} \tag{42}$$

and the object E_y is determined by its socle.

Proof. The first claim is given by Proposition 8.4 , Corollary 8.6, and Proposition 8.17. The second claim is given by Lemma 8.8 and Corollary 8.9. □

Corollary 8.19. *Let $d \in W$ be a Duflo element and $\mathcal{L}$ its left cell.*

(i) If $\mathbf{K}(d)$ is not true, then $\mathbf{K}(y)$ is not true, for all $y \in \mathcal{L}$.

(ii) If $\mathbf{K}(d)$ is true, then $\mathbf{K}(y)$ is equivalent to $\mathbf{KM}(y, y^{-1})$, for $y \in \mathcal{L}$.

Proof. This is a direct consequence of Proposition 8.17 and Theorem 8.16. □

Proposition 8.20. *Let d be the Duflo element in a left cell $\mathcal{L}$ in W and $y, z \in \mathcal{L}$. If $t_y t_{y^{-1}} = t_z t_{z^{-1}}$, then, for each $x \in W$, we have $\mathbf{KM}(x,y) \Leftrightarrow \mathbf{KM}(x,z)$ and $\mathbf{Kh}(y) \Leftrightarrow \mathbf{Kh}(z)$. In particular, both $\mathbf{KM}(*,_)$ and $\mathbf{Kh}(_)$ are constant on strongly regular left cells.*

Proof. By Proposition 8.17, the assumption $t_y t_{y^{-1}} = t_z t_{z^{-1}}$ implies $E_y = E_z$. Thus, by Proposition 7.5 and Proposition 7.2, we have equivalences

$$\mathrm{add}\{\theta_x L_y \colon x \in W\} \cong \mathrm{proj}_{\infty_0 \mathcal{H}_0^I}(A_y) = \mathrm{proj}_{\infty_0 \mathcal{H}_0^I}(A_z) \cong \mathrm{add}\{\theta_x L_z \colon x \in W\},$$

under which $\theta_x L_y \mapsto \theta_x L_z$, for each x. The claim follows. □

At this point we can strengthen Theorem 8.16.

Theorem 8.21. *Let $y \in W$. Then $\mathbf{K}(y)$ is true if and only if both $\mathbf{Kh}(y)$ and $\mathbf{KM}(*,y)$ are true.*

Proof. The "only if" direction is Proposition 8.14. For the "if" direction, assume that both $\mathbf{Kh}(y)$ and $\mathbf{KM}(*,y)$ are true. Let d' be the Duflo element in the right cell of y and d be the Duflo element in the left cell of y. We will first prove that $\mathbf{K}(d) \Leftrightarrow \mathbf{K}(y)$.

Because of $\mathbf{KM}(*,y)$, the module $\theta_{y^{-1}} L_y$ is indecomposable and hence is isomorphic to $\theta_{d'} L_{d'}$. In particular, $t_{y^{-1}} t_y = t_{d'}$ in the asymptotic ring $A(W)$, see (30). We claim that this implies $t_y t_{y^{-1}} = t_d = t_d t_d$. Indeed, since $t_{d'}$ is a local identity, $t_y t_{y^{-1}}$ is an idempotent of $A(W)$. Using (12), we can write $t_y t_{y^{-1}}$ as $t_d + x$, where x is a linear combination of various t_u with non-negative integer coefficients. Then

$$t_d + x = (t_d + x)(t_d + x) = t_d + t_d x + x t_d + x^2 = t_d + 2x + x^2$$

by Lemma 3.2, and thus $0 = x + x^2$. Since the structure coefficients with respect to the t_u's are non-negative, we conclude $x = 0$ and $t_y t_{y^{-1}} = t_d$. By Proposition 8.17, this implies $E_y = E_d$, and hence we have $\mathbf{K}(d) \Leftrightarrow \mathbf{K}(y)$.

Now, by Proposition 8.20, $\mathbf{Kh}(y)$ implies $\mathbf{Kh}(d)$, and $\mathbf{KM}(*,y)$ implies $\mathbf{KM}(*,d)$. Therefore $\mathbf{K}(d)$ holds by Theorem 8.16 and we are done. □

In most small rank examples, we have $\mathbf{K}(y)$ if and only if $\mathbf{Kh}(y)$. However, the situation is slightly more complicated in type G_2.

Example 8.22. Let $\mathfrak{g}$ be of type G_2, and let (W,S) be its Weyl group. Write $S = \{1,2\}$. The two-sided cells in W are given by

$\boxed{e}$

1 121 12121	21 2121
12 1212	2 212 21212

$\boxed{212121}$

where the rows are left, and the columns are right cells. For the right cell $\mathcal{R}$ containing 1, the indecomposables in $\mathrm{add}\{\theta_x L_y \,:\, x^{-1}, y \in R\}$ are exactly

$$\theta_1 L_1, \quad \theta_1 L_{12}, \quad \theta_1 L_{121}, \quad \theta_1 L_{1212}, \quad \theta_1 L_{12121}.$$

We describe the situation for $y \in \mathcal{R}$ in Table 1, the rest is either trivial or symmetric to what is given. Note that, for dihedral types, $[\theta_x L_y]$ determines the isomorphism class of $\theta_x L_y$.

y	$\mathbf{K}(y)$	$\mathbf{KM}(*,y)$	$\mathbf{Kh}(y)$	$\theta_{y^{-1}}L_y$	E_y
1	True	True	True	$\theta_1 L_1$	D_1
12	False	False	True	$\theta_1 L_1 \oplus \theta_1 L_{121}$	$D_2 \oplus L_{212}\langle -1\rangle$
121	False	False	False	$\theta_1 L_1 \oplus \theta_1 L_{121} \oplus \theta_1 L_{12121}$	$D_1 \oplus L_{121}\langle -1\rangle \oplus L_{12121}\langle -1\rangle$
1212	False	False	True	$\theta_1 L_1 \oplus \theta_1 L_{121}$	$D_2 \oplus L_{212}\langle -1\rangle$
12121	True	True	True	$\theta_1 L_1$	D_1

TABLE 1. $\mathbf{K}$, $\mathbf{KM}$ and $\mathbf{Kh}$ for G_2

8.5. **A combinatorial statement.** Although we reduce $\mathbf{K}(y)$ to $\mathbf{Kh}(y)$ and $\mathbf{KM}(*,y)$, it is not easy to determine $\mathbf{Kh}(y)$ and $\mathbf{KM}(*,y)$ in general. But the combinatorial version $[\mathbf{Kh}](y)$ of $\mathbf{Kh}(y)$ can be determined by computing $h_{x,y,z}$ (see (13)). Recall that Conjecture 1.2 claims $[\mathbf{Kh}](d) = \mathbf{K}(d)$ for Duflo elements $d \in W$. When this is true, the problem $\mathbf{K}(y)$ has a purely combinatorial solution as follows.

Corollary 8.23. *Let d be a Duflo element and let $y \sim_{\mathtt{L}} d$. Suppose $[\mathbf{Kh}(d)] = \mathbf{K}(d)$. Then the condition $\mathbf{K}(y)$ is equivalent to the conjunction of the following conditions:*

(i) $t_y t_{y^{-1}} = t_d$ in $A(W)$;

(ii) For $x, x' \leq_{\mathtt{R}} d$, if $h_{x,z,d} = h_{x',z,d}$, for all $z \in W$, then $x = x'$.

Proof. Suppose $\mathbf{K}(y)$ is true. Then Theorem 8.18 says that $E_y = E_d$, Condition (i) is true, and $\mathbf{K}(d)$ is true. By assumption, we also have $[\mathbf{Kh}(d)]$ which is equivalent to Condition (ii) by (13).

For the other direction, suppose Conditions (i), (ii) are true. Theorem 8.18 and Condition (i) implies $\mathbf{K}(d) = \mathbf{K}(y)$. Since Condition (ii) implies $\mathbf{K}(d)$ by assumption, it follows that $\mathbf{K}(y)$ is true. □

8.6. **Extra results in classical types.** In this subsection, unless explicitly stated otherwise, we assume $\mathfrak{g}$ to be of (classical) type A, B, C, or D.

The asymptotic algebra endows each diagonal $\mathtt{H}$-cell $\mathcal{H}$ with the structure of an abelian group. This group depends only on the two-sided cell containing $\mathcal{H}$ and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$, for some $k \in \mathbb{N}$, see [Lu3]. The off-diagonal $\mathtt{H}$-cells in the same two-sided cell have order 2^l, for $l \leq k$ (see [MMMTZ, Section 7] for more information). If $\mathcal{H}$ is an $\mathtt{H}$-cell, $\mathcal{H}_l$ is the diagonal $\mathtt{H}$-cell in the left cell of $\mathcal{H}$, and $\mathcal{H}_r$ is the diagonal $\mathtt{H}$-cell in the right cell of $\mathcal{H}$, then the asymptotic algebra endows $\mathcal{H}$ with the structure of a transitive $\mathcal{H}_l$-set via right multiplication, and with the structure of a transitive $\mathcal{H}_r$-set via left multiplication. In particular, we can speak about stabilizers of elements from $\mathcal{H}$ in both, $\mathcal{H}_l$ and $\mathcal{H}_r$. This combinatorics allows us to formulate a necessary condition for $\mathbf{Kh}(y)$, and thus also for $\mathbf{K}(y)$.

Proposition 8.24. *Let $\mathfrak{g}$ be of classical type and $y \in W$. Assume that the $\mathtt{H}$-cell of y does not have the maximal cardinality among the $\mathtt{H}$-cells inside the two-sided cell of y. Then both $\mathbf{Kh}(y)$ and $\mathbf{K}(y)$ are false.*

Proof. Let $y \in \mathcal{H}$ and $\mathcal{H}_l$ and $\mathcal{H}_r$ be as above, in particular, $|\mathcal{H}| < |\mathcal{H}_l| = |\mathcal{H}_r|$. Let $d \in \mathcal{H}_l$ be a Duflo element. As $\mathbf{K}(y)$ implies $\mathbf{Kh}(y)$ by Proposition 8.14, we only need to show that $\mathbf{Kh}(y)$ is false. Taking (30) into account, it is enough to find $z \in \mathcal{H}$ such that $z \neq d$ and $t_y t_d = t_y t_z$. As $|\mathcal{H}| < |\mathcal{H}_l|$, the stabilizer of y in $\mathcal{H}_l$ is non-trivial and hence we can take as z any element from this stabilizer different from d. The claim follows. □

Lemma 8.25. *Suppose W is of classical type, $\mathcal{L}$ is a left cell in W and $y \in \mathcal{L}$. Set $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$. Then*

$$\theta_y \theta_{y^{-1}} \cong \bigoplus_{z \in \mathrm{Stab}_{\mathcal{H}}(y)} \theta_z \langle \mathbf{a}(y) \rangle \oplus \theta,$$

where each summand of θ is of the form $\theta_w\langle a\rangle$, where $a < \mathbf{a}(w)$.

Proof. The element $t_y t_{y^{-1}}$ of the asymptotic ring is a linear combination of t_u, for $u \in \mathcal{H}$, and has the same left stabilizer as y. Therefore $t_y t_{y^{-1}}$ must be a scalar multiple of $\sum_{z \in \mathrm{Stab}_{\mathcal{H}}(y)} t_z$. But the multiplicity of θ_d, where $d \in \mathcal{H}$ is the Duflo element, in $\theta_y \theta_{y^{-1}}$ is one. Therefore the scalar in question is one which implies our claim. □

Lemma 8.26. *Suppose W is of classical type, $\mathcal{L}$ is a left cell in W and $y \in \mathcal{L}$. Set $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$ and let $d \in \mathcal{H}$ be the Duflo element. Then*

$$\theta_y L_{y^{-1}} \cong \bigoplus_{z \in \mathrm{Stab}_{\mathcal{H}}(y)} \theta_z L_d.$$

In particular, $\theta_y L_{y^{-1}}$ is indecomposable if and only if $\mathcal{H}$ is of maximal cardinality in its two-sided cell.

Proof. This follows from Proposition 8.4 and Lemma 8.25. □

Lemma 8.27. *Let W be of any type and $x, y \in W$. Then we have*

$$\hom(\theta_x L_y, \theta_x L_y) = \bigoplus_{w \in \mathcal{R} \cap \mathcal{R}^{-1}} \hom(\theta_w L_y, L_y\langle \mathbf{a}(x)\rangle)^{\oplus \gamma_{x^{-1},x,w^{-1}}}.$$

Proof. This follows from Theorem 2.2 and the definition of the asymptotic ring by adjunction. □

Lemma 8.28. *Let W be of any type, $x, y \in W$ and d the Duflo element such that $d \sim_{\mathtt{R}} x$. Assume that $\mathbf{KMM}(d, y)$ is true. Then $\mathbf{KM}(x^{-1}, x)$ implies $\mathbf{KM}(x, y)$.*

Proof. If $\theta_{x^{-1}} L_x$ is indecomposable, then, by Proposition 8.4, $t_{x^{-1}} t_x = t_d$. Hence, by Lemma 8.27, $\mathrm{end}(\theta_x L_y)$ is isomorphic to $\hom(\theta_d L_y, L_y\langle \mathbf{a}(x)\rangle)$ and thus has dimension one by $\mathbf{KMM}(d, y)$. This implies the claim. □

Lemma 8.29. *Let W be of any type, $y, z \in W$ and d be the Duflo element such that $d \sim_{\mathtt{R}} w_0 y^{-1}$. Suppose $\mathbf{KMM}(d, z)$ and $\mathbf{KM}(y^{-1}, y)$ are true. Then $\mathbf{KM}(w_0 z^{-1}, y)$ is true.*

Proof. The indecomposability of $\theta_{y^{-1}} L_y$ is, by Proposition 8.4, equivalent to $t_{y^{-1}} t_y = t_{d'}$, where d' is an appropriate Duflo element. As we saw in the proof of Theorem 8.21, this, in turn, is equivalent to $t_y t_{y^{-1}} = t_{d''}$, where d'' is an appropriate Duflo element. The latter is equivalent to the indecomposability of $\theta_y L_{y^{-1}}$. By Koszul-Ringel duality, we get that $\theta_{y w_0} L_{w_0 y^{-1}}$ is indecomposable. Then $\mathbf{KMM}(d, z)$ and Lemma 8.28 implies that $\theta_{w_0 y^{-1}} L_z$ is indecomposable. Then, by Koszul-Ringel duality again, $\theta_{z^{-1} w_0} L_{w_0 y w_0}$ is indecomposable. Since conjugation by w_0 corresponds to an automorphism of the Dynkin diagram, we obtain that $\theta_{w_0 z^{-1}} L_y$ is indecomposable. □

We can now establish a general sufficient condition for equivalence of $\mathbf{K}(y)$ is equivalent to $\mathbf{Kh}(y)$.

Proposition 8.30. *Let $\mathfrak{g}$ be of classical type, $y \in W$ and d be the Duflo element such that $d \sim_{\mathtt{R}} w_0 y^{-1}$. Assume that $\mathbf{KMM}(d, z)$ is true, for all $z \in W$. Then $\mathbf{K}(y)$ is equivalent to $\mathbf{Kh}(y)$.*

Proof. By Theorem 8.21, it is enough to show that $\mathbf{Kh}(y)$ implies $\mathbf{KM}(*, y)$. The latter assertion would follow from Lemma 8.29 provided that we can show that $\theta_{y^{-1}} L_y$ is indecomposable. Given $\mathbf{Kh}(y)$, from Proposition 8.24 it follows that the H-cell of y is of maximal cardinality inside the J-cell of y. Hence indecomposability of $\theta_{y^{-1}} L_y$ follows from Lemma 8.26. □

In this subsection, we do need the assumption on $\mathfrak{g}$ to be of classical type (compare with Example 8.22).

degree: composition factors in this degree:

$-\mathbf{3}$: $123121,\ 23123121,\ \mathbf{231232}^{\oplus \mathbf{2}}$

-2: $12321,\ 23123^{\oplus 2},\ 121,\ 2312312^{\oplus 2},\ 2312321^{\oplus 4},\ 23121,\ 12312^{\oplus 2}$

-1: $23123121^{\oplus 4},\ 1231^{\oplus 3},\ 2312^{\oplus 2},\ 2321,\ 12,\ 231231^{\oplus 3}, 123121^{\oplus 4},\ 231232^{\oplus 4},\ 21$

0: $12321^{\oplus 3},\ 23123^{\oplus 4},\ 121^{\oplus 2},\ 1,\ 123,\ 2312312^{\oplus 4},\ 2312321^{\oplus 8},\ 231^{\oplus 2},\ 23121^{\oplus 2},\ 12312^{\oplus 4}$

1: $23123121^{\oplus 4},\ 1231^{\oplus 3},\ 2312^{\oplus 2},\ 2321,\ 12,\ 231231^{\oplus 3}, 123121^{\oplus 4},\ 231232^{\oplus 4},\ 21$

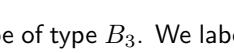
2: $12321,\ 23123^{\oplus 2},\ 121,\ 2312312^{\oplus 2},\ 2312321^{\oplus 4},\ 23121,\ 12312^{\oplus 2}$

3: $123121,\ 23123121,\ 231232^{\oplus 2}$

FIGURE 1. The character of $\theta_x L_y$ from Subsection 9.1

9. Further discussion and speculation on $\mathbf{KMM}$

9.1. **A counterexample.** In this subsection we give an example in which $\mathbf{KMM}$ is false.

Let (W,S) be of type B_3. We label the simple reflections in the following way: 

. Consider $y = 231232$ and a Duflo element $x = 2312312$. We have $\mathbf{a}(x) = 3$. One can check (using a computer) that the graded character of $\theta_x L_y$ is as given in Figure 1 and directly see that $\dim \hom(\theta_x L_y, L_y\langle \mathbf{a}(x)\rangle) = 2$. We note also that we have $\theta_x L_y \cong \theta_{12312} L_{23123121} \oplus \theta_{12312} L_{123121}$ is decomposable, supporting Conjecture 2.3.

9.2. **A homological approach to $\mathbf{KMM}$.** Here is a general criterion for $\mathbf{KMM}$ given in homological terms, in addition to KL combinatorics.

Proposition 9.1. *Let $x, y \in W$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$, where $\mathcal{L}$ is the left cell of $y^{-1}w_0$. Then $\mathbf{KMM}(x,y)$ is equivalent to*

$$\sum_{z \in \mathcal{H}} \gamma_{w_0 y, y^{-1} w_0, z^{-1}} \cdot \dim \operatorname{ext}^{\mathbf{a}(x)}(\theta_z L_{w_0 x^{-1}}, L_{w_0}\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle) \leq 1. \tag{43}$$

We note that, in type A, the formula (43) simplifies to

$$\dim \operatorname{ext}^{\mathbf{a}(x)}(\theta_d L_{w_0 x^{-1}}, L_{w_0}\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle) \leq 1, \tag{44}$$

where $d \in \mathcal{H}$ is the Duflo element (in fact, $\mathcal{H} = \{d\}$ in this case). In classical types, the formula (43) reads

$$\sum_{z \in \mathrm{Stab}_{\mathcal{H}}(y^{-1}w_0)} \dim \operatorname{ext}^{\mathbf{a}(x)}(\theta_d L_{w_0 x^{-1}}, L_{w_0}\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle) \leq 1.$$

Proof of Proposition 9.1. We realize $\theta_x L_y$ as a linear complex of tilting modules. Using Koszul-Ringel duality and adjunction, we have

$$\begin{aligned} \hom(\theta_x L_y, L_y\langle \mathbf{a}(x)\rangle) &\cong \operatorname{ext}^{\mathbf{a}(x)}(\theta_{y^{-1}w_0} L_{w_0 x^{-1}}, \theta_{y^{-1}w_0} L_{w_0}\langle -\mathbf{a}(x)\rangle) \\ &\cong \operatorname{ext}^{\mathbf{a}(x)}(\theta_{w_0 y}\theta_{y^{-1}w_0} L_{w_0 x^{-1}}, L_{w_0}\langle -\mathbf{a}(x)\rangle). \end{aligned} \tag{45}$$

For $z \in \mathcal{H}$, the composition $\theta_{w_0 y}\theta_{y^{-1}w_0}$ contains $\theta_z\langle -\mathbf{a}(y^{-1}w_0)\rangle$ with multiplicity $\gamma_{w_0 y, y^{-1}w_0, z^{-1}}$. All other summands of this composition are of the form $\theta_w\langle -a\rangle$, where $w \geq_{\mathtt{J}} d$ and $a < \mathbf{a}(w)$. By (43), the contribution of all $\theta_z\langle -\mathbf{a}(y^{-1}w_0)\rangle$ to the right hand side of (45) is at most one-dimensional. So, to complete the proof we only need to show that the contribution of all other summands of $\theta_{w_0 y}\theta_{y^{-1}w_0}$ to the right hand side of (45) is zero.

To this end, fix $\theta_w\langle -a\rangle$ as above and consider the linear complex $\mathcal{T}_\bullet$ of tilting modules representing $\theta_w L_{w_0x^{-1}}$. By Koszul-Ringel duality, each indecomposable summand of $\mathcal{T}_{-\mathbf{a}(x)}$ is of the form $T_u\langle -\mathbf{a}(x)\rangle$, for some $u \in W$ such that $u \leq_{\mathtt{J}} w_0w \leq_{\mathtt{J}} w_0d$.

For such u, we claim that the condition $\hom(T_u, L_{w_0}\langle b\rangle) \neq 0$ necessarily implies $b \geq \mathbf{a}(w_0u) = \mathbf{a}(d)$. Indeed, we have $\hom(T_u, L_{w_0}\langle b\rangle) = \hom(T_u, T_{w_0}\langle b\rangle)$ which, in turn, equals $\hom(P_{w_0u}, P_e\langle b\rangle)$ by Soergel's character formula for tilting modules, see [So2, Theorem 6.7]. Now our $b \geq \mathbf{a}(w_0u)$ follows from (8).

For such b, we have $b - a > \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)$ since $-a > -\mathbf{a}(w)$. This implies that the contribution of $\theta_w\langle -a\rangle$ to the right hand side of (45) is zero and completes the proof. □

9.3. **$\mathbf{KMM}$ via twisting functors.** Consider the full twisting functor $\mathbf{T}_{w_0}$, see [AS]. The following statement provides a reformulation of $\mathbf{KMM}$ in terms of the cohomology of the derived twisting functor $\mathscr{L}\mathbf{T}_{w_0}$ evaluated at a simple module.

Proposition 9.2. *Let $x, y \in W$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^{-1}$, where $\mathcal{L}$ is the left cell of $y^{-1}w_0$. Then $\mathbf{KMM}(x, y)$ is equivalent to*

$$\sum_{z\in\mathcal{H}} \gamma_{w_0y,y^{-1}w_0,z^{-1}} \cdot [\mathscr{L}_{\mathbf{a}(x)}\mathbf{T}_{w_0}(L_{w_0x^{-1}}) : L_{z^{-1}}\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle] \leq 1. \tag{46}$$

In type A, the formula (46) simplifies to

$$[\mathscr{L}_{\mathbf{a}(x)}\mathbf{T}_{w_0}(L_{w_0x^{-1}}) : L_d\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle] \leq 1,$$

where d is the Duflo element in $\mathcal{H}$. In classical types, the formula (46) reads

$$\sum_{z\in \mathrm{Stab}_{\mathcal{H}}(y^{-1}w_0)} [\mathscr{L}_{\mathbf{a}(x)}\mathbf{T}_{w_0}(L_{w_0x^{-1}}) : L_{z^{-1}}\langle \mathbf{a}(y^{-1}w_0) - \mathbf{a}(x)\rangle] \leq 1.$$

Proof of Proposition 9.2. In this proof we use standard properties of twisting functors, see [AS] for details. To start with, we note that $\mathscr{L}\mathbf{T}_{w_0}$ is an auto-equivalence of the bounded derived category of $\mathcal{O}_0$. Let us apply this auto-equivalence to the left hand side of (43). In the second argument of the extension space, we get $\mathscr{L}\mathbf{T}_{w_0}(T_{w_0}) = I_e$. In the first argument of the extension space, we note that $\mathscr{L}\mathbf{T}_{w_0}$ commutes with projective functors, so we can move θ_z out. By adjunction, we move θ_z over to the second argument obtaining $\theta_{z^{-1}}I_e \cong I_{z^{-1}}$. The leftover in the first argument is $\mathscr{L}\mathbf{T}_{w_0}(L_{w_0x^{-1}})$. Evaluating at the correct degree of the extension and noting that homomorphisms to $I_{z^{-1}}$ give exactly the composition multiplicity of $L_{z^{-1}}$ for the homology, we obtain (46). □

10. Small rank results

The results in this paper, together with (computer-assisted computations for) Kazhdan-Lusztig combinatorics, enable us to determine $\mathbf{K}(y)$, $\mathbf{KM}(*, y)$, and $\mathbf{Kh}(y)$ in many cases. We present some of the results in this section.

In computer-assisted calculations, SageMath v.9.0 has been used.

10.1. **Type A.** It is verified in [CMZ] that $\mathbf{KM}(x, y)$ is true for all x, y in type A_n for $n \leq 5$. Therefore, in this case we only need to determine either $\mathbf{Kh}(y)$ or $\mathbf{K}(y)$, which are equivalent by Theorem 8.21.

In [KhM, Kh], Kostant's problem is solved for simple highest weight modules in $\mathcal{O}_0$, for $\mathfrak{sl}_n$, $n \leq 5$. For $\mathfrak{sl}_6$, i.e., type A_5, out of 76 Duflo elements, 47 have positive answer to Kostant's problem, 20 have negative, and the

following 9 Duflo elements were left as an open problem (the notation for simple reflections is analogous to the one in Example 7.7):

$$\begin{array}{ll} 23432, & 4523412, \\ 234312, & 12342321, \\ 452342, & 23454232, \\ 234512342312, & 3451234231, \\ & 2345412312. \end{array} \tag{47}$$

We can now solve these remaining cases.

For all y in the first column in (47), one can check (by computer) that $[\mathbf{Kh}](y)$ holds, and therefore $\mathbf{Kh}(y)$ holds. From [CMZ] and Theorem 8.21 it follows that $\mathbf{K}(y)$ is true.

For the second column in (47), it is enough to consider only $x := 4523412$, $y := 12342321$ and $z := 3451234231$, because of the symmetry of the Dynkin diagram. By [CMZ, Proposition 46.b)], we have

$$\theta_{23}L_x \cong \theta_3 L_{x3} \cong \theta_{43}L_x \qquad \text{and} \qquad \theta_{32}L_z \cong \theta_2 L_{z2} \cong \theta_{12}L_z.$$

Therefore both $\mathbf{Kh}(x)$ and $\mathbf{Kh}(z)$ fail. So again, using [CMZ] and Theorem 8.21, we conclude that neither $\mathbf{K}(x)$ nor $\mathbf{K}(z)$ holds.

A calculation shows that $\theta_{45342} \cong \theta_{342}\theta_{45}$. One can check that $\theta_{45}L_y$ has height 1. Moreover, its top and socle are simple, consisting of $L_{y'}$ with $y' := 123452321$, and L_y in the middle. Since $342 \not\leq_{\mathtt{R}} (y')^{-1}$, it follows that

$$\theta_{45342}L_y \cong \theta_{342}L_y.$$

Therefore $\mathbf{Kh}(y)$ does not hold, and so $\mathbf{K}(y)$ also does not hold. Since the property $\mathbf{K}$ in type A is invariant for the left cells, we have:

Corollary 10.1. *Kostant's problem has a positive solution, for a simple highest weight module L_w for $\mathfrak{sl}_6$, if and only if w does not belong to the left cells containing one of the following 25 Duflo elements:*

$$\begin{array}{lllll} 31, & 531, & 45341, & 512321, & 23454232, \\ 42, & 3431, & 52312, & 3453431, & 34541231, \\ 53, & 4121, & 234232, & 4523412, & 2345412312, \\ 232, & 4542, & 345431, & 5123121, & 3451234231, \\ 343, & 5232, & 454121, & 12342321, & 12345343121. \end{array}$$

10.2. **Types BCD.** We completely determine $\mathbf{K}(w)$, $\mathbf{KM}(*,w)$ and $\mathbf{Kh}(w)$, for each $w \in W$ in types B_3 and D_4. We also determine $\mathbf{K}(w)$ and $\mathbf{Kh}(w)$ completely in type B_4. In all above examples, we have $\mathbf{Kh}(w) \Leftrightarrow \mathbf{KM}(*,w)$, supporting Conjecture 1.2. We provide below the results without details for type B_3 and D_4.

A good way to present these results is to mark the H-cells in W in the following way (note that, since W is of classical type, by Proposition 8.20 and Lemma 8.26, the function $\mathbf{K}(w)$ (resp., $\mathbf{KM}(*,w)$ and $\mathbf{Kh}(w)$) has the same value for w in the same H-cell) :

The elements w in the H-cells colored in

- this color satisfy $\mathbf{K}(w)$, $\mathbf{KM}(*,w)$ and $\mathbf{Kh}(w)$,
- this color satisfy $\mathbf{KM}(*,w)$ and do not satisfy $\mathbf{Kh}(w)$, thus do not satisfy $\mathbf{K}(w)$,
- white do not satisfy any of the above properties.

Figure 2 presents type B_3 results and Figure 3 gives type D_4 results.

The following example is potentially related to Conjecture 1.2 (b)$\Leftrightarrow$(c):

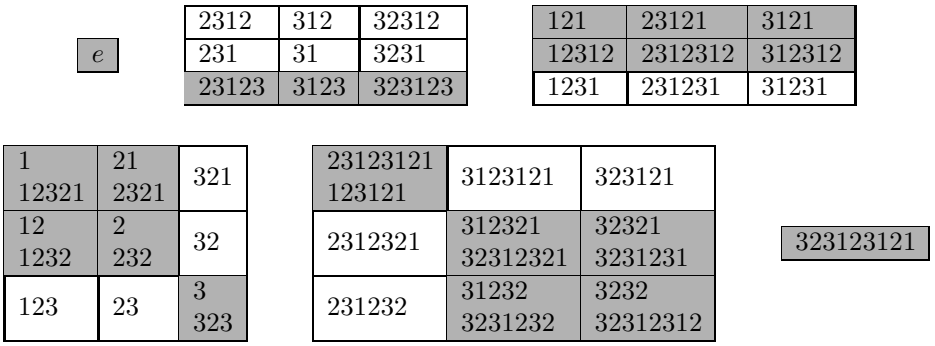


FIGURE 2. Cells in type B_3 with the labeling $\overset{1}{\circ}\!\!-\!\!\!-\overset{2}{\circ}\!\!=\!\!\overset{3}{\circ}$. Rows are left, and columns are right cells. Duflo elements are the top elements in the diagonal blocks.

Example 10.2. Let W be of type B_3 as in Figure 2. In $\mathrm{Gr}(\mathcal{O}_0^{\mathbb{Z}})$ we have the following equalities:

$$\begin{aligned}[\theta_{123}L_{121}] &= [\theta_3 L_{231}] + [\theta_{3231}L_{1231}]\\ &= [\theta_3 L_{23123}] + [\theta_{323123}L_{1231}].\end{aligned}$$

However, as shown in Figure 2, the object $\theta_{123}L_{121}$ is indecomposable.

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1	21	321	421
12	2	32	42
123	23	3	423
124	24	324	4

2312	312	42312
231	31	4231
23124	3124	423124

2412	32412	412
24123	324123	4123
241	3241	41

124321	24321	4321
12432	2432	432
1243	243	43

121 1243121	3121 31243121	4121 12423121	23121 3243121	24121 2423121	423121 324121	243121	43121
1231 12412321	12321 312412321	41231 1242321	2321 32412321	241231 242321	42321 3241231	2412321	412321
1241 23124321	31241 3124321	12421 423124321	324321 231241	2421 42312421	32421 4231241	2312421	312421
12312 1241232	1232 31241232	124232 412312	232 3241232	24232 2412312	4232 32412312	241232	41232
12412 2312432	312432 312412	1242 42312432	32432 2312412	242 4231242	3242 42312412	231242	31242
124123 231243	31243 3124123	12423 4231243	3243 23124123	2423 42312423	32423 423124123	2312423	312423
124312	3124312	1242312	324312	242312	3242312	24312 231242312	4312 31242312
12431	312431	124231	32431	24231	324231	2431 23124231	431 3124231

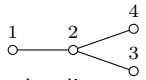
123121	24123121	4123121
23124312	4231242312	423124312
2312431	423124231	42312431

23124123121	324123121	3124123121	124123121
231241232	42312412312	4231241232	423124232
2312412321	4231241231	42312412321	4231242321
231243121	423124121	4231243121	42312423121

2312423121	312423121	32423121
231242321	31242321	3242321
23124232	3124232	324232

124121	23124121	3124121
12412312	2312412312	312412312
1241231	231241231	31241231

423124123121

FIGURE 3. Cells in type D_4 with labeling 

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