arXiv:1908.01632v1 [math.AP] 5 Aug 2019

# INVISCID LIMIT TO THE SHOCK WAVES FOR THE FRACTAL BURGERS EQUATION

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Abstract. We show the vanishing viscosity limit to entropy shocks for the fractal Burgers equation in one space dimension. More precisely, we quantify the rate of convergence of the inviscid limit in $L^2$ for large initial perturbations around the entropy shock on any bounded time interval. This is the first result on the inviscid limit to entropy shock for the fractal Burgers equation with the quantified convergence, for large initial perturbations.

## 1. Introduction and main results

We consider the Burgers equation with the fractional Laplacian in one space dimension:

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = \Delta_x^{\alpha/2} u, \quad t > 0, \ x \in \mathbb{R}, \tag{1.1}$$

where $\alpha$ denotes the fractional power of the Laplacian in one dimension, and the fractional Laplacian can be written as a singular integral operator:

$$\Delta_x^{\alpha/2} u(x) = c_\alpha \text{P.V.} \int_{\mathbb{R}} \frac{u(y) - u(x)}{|y - x|^{1+\alpha}} dy. \tag{1.2}$$

The equation (1.1) is sometimes called the fractal Burgers equation. It has been extensively used as a toy model for the study of the fractal (anomalous) diffusion for a variety of physical phenomena where shock creation is an important ingredient. This includes the growth of molecular interfaces, traffic jams and the mass distribution for the large scale structure of the universe (see for example, Biler et al. [3] for a discussion of this model).

For $0 < \alpha \leq 1$, the well-posedness theory of (1.1) has been established in Alibaud [1] and in Kiselev-Nazarov-Shterenberg [19] for a different class of initial data and with further analysis about finite time blowup for $\alpha < 1$ and analyticity for $\alpha \geq 1$ (see Chan-Czubak [4] for $\alpha = 1$). In the case of $1 < \alpha < 2$, which is the focus of our work, prior to [19] was the work of Droniou-Gallouet-Vovelle [10], where the authors used a semi-group approach to obtain existence, uniqueness, smoothness and boundedness of solutions to (1.1) as well as their derivatives. Concerning time-asymptotic stability to rarefaction waves, we refer to Alibaud-Imbert-Karch [2] and Karch-Miao-Xu [18].

*Date*: August 6, 2019.

2010 *Mathematics Subject Classification.* 35L65, 35L67, 35B35, 35B40.

*Key words and phrases.* fractal Burgers equation, fractional Laplacian, scalar conservation laws, shock waves, inviscid limit, large perturbation, relative entropy.

**Acknowledgment.** S. Akopian and M.-J. Kang were partially supported by the NRF-2019R1C1C1009355. A. F. Vasseur was partially supported by the NSF Grant DMS 1209420.

In this article, we study the vanishing viscosity limit ($\varepsilon \to 0$) of the scaled equation

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \left( \frac{u_\varepsilon^2}{2} \right) = \varepsilon \Delta_x^{\alpha/2} u_\varepsilon, \quad t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(x,0) = u_0(x), \end{cases} \tag{1.3}$$

in the case of $1 < \alpha < 2$.
Note that for a solution $u$ to the equation (1.1), $u_\varepsilon(x,t) := u(x/\varepsilon^\beta, t/\varepsilon^\beta)$ solves the scaled equation (1.3), where

$$\beta = \frac{1}{\alpha - 1}, \quad \alpha \in (1,2).$$

We aim to quantify the vanishing viscosity limit ($\varepsilon \to 0$) of (1.3) with a general initial datum towards entropy shock waves of the (inviscid) Burgers equation:

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0. \tag{1.4}$$

We are particularly interested in the case where the initial datum carries too much entropy for the structure of the layer to be preserved in the inviscid limit.

It is well known that for any constants $u_-$ and $u_+$ with $u_- > u_+$, the equation (1.4) admits the entropy shock wave $S_0(x - \sigma t)$ connecting the two end states $u_\pm$ as follows (for example, see [21]) :

$$S_0(x - \sigma t) = \begin{cases} u_- & \text{if } x < \sigma t \\ u_+ & \text{if } x > \sigma t, \end{cases} \tag{1.5}$$

where the velocity $\sigma$ is determined by the Rankine-Hugoniot condition:

$$\sigma = \frac{A(u_-) - A(u_+)}{u_- - u_+}, \quad \text{where } A(u) := \frac{u^2}{2}.$$

Note that the condition $u_- > u_+$ ensures that the shock wave (1.5) is an entropy solution to (1.4).
On the other hand, we refer to Chmaj [5] for the existence of shock layer to the fractal Burgers equation (1.1) in the case of $1 < \alpha < 2$. That is, the following was proved: for any $u_- > u_+$, there exists a travelling wave $S_1(x - \sigma t)$ as a smooth solution to

$$\begin{cases} -\sigma S_1' + \left( \frac{S_1^2}{2} \right)' = \Delta_\xi^{\alpha/2} S_1, \\ \lim_{\xi \to \pm\infty} S_1(\xi) = u_\pm. \end{cases} \tag{1.6}$$

However, the rate of convergence of the shock layer to the two end states $u_\pm$ is not known.

We now present our main result.

**Theorem 1.1.** *Assume $1 < \alpha < 2$ in the equation (1.3). For any constants $u_-$ and $u_+$ with $u_- > u_+$, let $u_0$ be the initial datum such that*

$$u_0 \in L^\infty(\mathbb{R}), \qquad u_0 - S_0 \in L^2(\mathbb{R}) \qquad \text{and} \qquad \left( \frac{d}{dx} u_0 \right)_+ \in L^2(\mathbb{R}).$$

*For any $T > 0$, there exists a constant $C(T) > 0$ such that the following holds:*
*For any solution $u_\varepsilon$ to (1.3), there exists a Lipschitz continuous shift $t \in [0,T] \mapsto X(t)$ with $X(0) = 0$ such that for all $t \le T$,*

$$\|u_\varepsilon(\cdot + X(t), t) - S_0(\cdot - \sigma t)\|_{L^2(\mathbb{R})} \le \|u_0 - S_0\|_{L^2(\mathbb{R})} + C(T)\psi(\varepsilon). \tag{1.7}$$

*Here,*

$$\psi(\varepsilon) := \inf_{\delta \geq 4} \left( \sqrt{\delta\varepsilon^{\beta} + \sqrt{\delta\varepsilon^{\beta}} + \left( S_1\left(\sqrt{\delta}\right) - u_+ \right) + \left( u_- - S_1\left(-\sqrt{\delta}\right) \right) + \left(\frac{1}{\delta}\right)^{\alpha-1}} \right) + \varepsilon^{\beta/2},$$

*where $S_1$ denotes the viscous shock satisfying* (1.6).

**Remark 1.1.** *Since $\lim_{\xi\to\pm\infty} S_1(\xi) = u_\pm$ and $1 < \alpha < 2$, note that (for example, by choosing $\delta = \varepsilon^{-\beta/2}$)*

$$\psi(\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

*Therefore, Theorem 1.1 provides an explicit rate of convergence for the inviscid limit to the shock.*
*If the shock layer $S_1$ approaches the end states $u_\pm$ exponentially fast as in the case of the classical Laplacian, i.e., $\alpha = 2$ (for example, see* [16]*), then there exist constants $\varepsilon_0$ and $C > 0$ such that*

$$\psi(\varepsilon) \leq C\varepsilon^{\frac{1}{2(2\alpha-1)}}, \quad \forall \varepsilon < \varepsilon_0. \tag{1.8}$$

*Indeed, by choosing*

$$\delta = \varepsilon^{-\frac{1}{(\alpha-1)(2\alpha-1)}}$$

*there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$\begin{aligned} &\delta\varepsilon^{\beta} \ll \sqrt{\delta\varepsilon^{\beta}} = \varepsilon^{\frac{1}{2\alpha-1}}, \\ &\left| S_1\left(\sqrt{\pm\delta}\right) - u_\pm \right| \leq Ce^{-c_\pm\sqrt{\delta}} \ll \left(\frac{1}{\delta}\right)^{\alpha-1} = \varepsilon^{\frac{1}{2\alpha-1}}. \end{aligned} \tag{1.9}$$

**Remark 1.2.** *From a special layer study, one can see that the optimal rate of convergence is $\varepsilon^{\beta/2}$. Indeed, if we consider the shock layer $S_1(x-\sigma t)$ in* (1.6)*, then $S_1((x-\sigma t)/\varepsilon^{\beta})$ is a shock layer of* (1.3) *as a travelling wave solution of* (1.3) *with initial datum $S_1(x/\varepsilon^{\beta})$. In this case, the rate of convergence is of order $\varepsilon^{\beta/2}$, since*

$$\sqrt{\int_{\mathbb{R}} \left| S_1\left(\frac{x-\sigma t}{\varepsilon^{\beta}}\right) - S_0(x-\sigma t) \right|^2 \mathrm{d}x} = \sqrt{\varepsilon^{\beta} \int_{\mathbb{R}} \left| S_1(x) - S_0(x) \right|^2 \mathrm{d}x} = C\varepsilon^{\beta/2}. \tag{1.10}$$

*Therefore, if the shock layer $S_1$ approaches the end states exponentially fast, the rate of convergence $\varepsilon^{\frac{1}{2(2\alpha-1)}}$ in* (1.8) *is slightly worse than the optimal rate $\varepsilon^{\beta/2}$ above, because*

$$\varepsilon^{\beta/2} = \varepsilon^{\frac{1}{2(\alpha-1)}} < \varepsilon^{\frac{1}{2(2\alpha-1)}}.$$

*Note that such a layer study is the special case of small initial perturbations such as*

$$\|u_0 - S_0\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{\beta/2}).$$

*In the case where $u_0$ is the same initial data as the one of* (1.4)*, i.e, no initial perturbation, we refer to the result of Droniou* [9] *on the convergence of solution to* (1.3) *towards entropy solution to* (1.4).
*However, those studies collapse in the case of large initial perturbation as*

$$\|u_0 - S_0\|_{L^2(\mathbb{R})} \gg \varepsilon^{\beta/2}.$$

*In this situation, there is too much entropy for the asymptotic limit of the layer structure to be true. So, the physical layer may be destroyed. Therefore, Theorem 1.1 is the first result*

*on the inviscid limit to the entropy shock even for large initial perturbation, although the rate of convergence is not optimal.*

## 2. Proof of Theorem 1.1

We prove Theorem 1.1 for the case of a conservation law with a strictly flux: given a strictly convex flux $A$, consider

$$\begin{cases} \partial_t u_\varepsilon + \partial_x A(u_\varepsilon) = \varepsilon \Delta_x^{\alpha/2} u_\varepsilon, \quad t > 0,\ x \in \mathbb{R}. \\ u_\varepsilon(x,0) = u_0(x), \end{cases} \tag{2.1}$$

Although the existence issue of the shock layer of (2.1) is still open for general convex fluxes, we provide the proof of Theorem 1.1 in the general setting.
We also mention that Droniou-Gallouet-Vovelle [10] proved the global existence and uniqueness of smooth solutions to (2.1) with the $L^\infty$-bounded initial data in the case of $1 < \alpha < 2$, and

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})},$$

which will be used in our proof.

2.1. **Ideas and useful lemma.** Contrary to the proof of the result [6] for the case of the (local) Laplacian operator, i.e., $\alpha = 2$, the nonlocality of the fractional Laplacian leads us to first study on the convergence of the solution $u_\varepsilon$ towards the shock layer (of width $\varepsilon^\beta$) of (2.1). Once we prove it, the desired result (1.7) would be obtained by using the obvious convergence from the shock layer to the inviscid shock as in (1.10).

Without loss of generality, we only deal with the stationary shock wave $S_0$, i.e., $\sigma = 0$. We first see from (1.6) that the (stationary) shock layer $S_1$ of (1.1) is a solution to

$$\begin{cases} (A(S_1))' = \Delta_x^{\alpha/2} S_1, \\ \lim_{x\to\pm\infty} S_1(x) = u_\pm. \end{cases} \tag{2.2}$$

Then, $S_\varepsilon(x) := S_1(x/\varepsilon^\beta)$ is the associated shock layer of (2.1) as a solution to

$$\begin{cases} (A(S_\varepsilon))' = \varepsilon \Delta_x^{\alpha/2} S_\varepsilon, \\ \lim_{x\to\pm\infty} S_\varepsilon(x) = u_\pm. \end{cases} \tag{2.3}$$

In our analysis, we will use the monotonicity property of the shock layer, which is proved in the following lemma.

**Lemma 2.1.** *If $S_1(x)$ is a smooth shock layer of (2.2), then $S_1'(x) \le 0$ for all $x$.*

*Proof.* First, we take the derivative of both sides of (2.2) to get

$$A''(S_1)|S_1'|^2 + A'(S_1)S_1'' = \Delta^{\alpha/2} S.$$

Then, using the convexity of the function $y \mapsto y_+ \,(:= y\mathbf{1}_{y>0})$, we have

$$A''(S_1)((S_1')_+)^2 + A'(S_1)((S_1')_+)' \le \Delta^{\alpha/2}(S_1')_+.$$

Multiplying the above both sides by $(S_1')_+$, and integrating them over $\mathbb{R}$, we have

$$\int A''(S_1)((S_1')_+)^3 \,\mathrm{d}x + \int A'(S_1)\left(\frac{((S_1')_+)^2}{2}\right)^2 \mathrm{d}x \le \int (S_1')_+ \Delta^{\alpha/2}(S_1')_+ \,\mathrm{d}x,$$

which implies

$$\frac{1}{2}\int A''(S_1)((S_1')_+)^3\,\mathrm{d}x \le -\int |\Delta^{\alpha/4}(S_1')_+|^2\,\mathrm{d}x \le 0.$$

Therefore, using the strict convexity of the flux $A$, we have

$$(S_1')_+(x) = 0, \qquad \text{for a.e. } x \in \mathbb{R},$$

which completes the proof. □

The following lemma will be used in the proof of Proposition 2.1.

**Lemma 2.2.** *If $u_\varepsilon(x,t)$ is a solution of (2.1) with $\|(\partial_x u_0)_+\|_{L^2(\mathbb{R})} < \infty$, then*

$$\|(\partial_x u_\varepsilon(\cdot,t))_+\|_{L^2(\mathbb{R})} \le \|(\partial_x u_0)_+\|_{L^2(\mathbb{R})}.$$

*Proof.* Let $v := \partial_x u_\varepsilon$ and $v_+ := v\chi_{v\ge 0}$. Following the proof of [6, Lemma 3.2], we differentiate (2.1) with respect to $x$, multiply by $(\partial_x(u_\varepsilon))_+$ and integrate in $x$ to get

$$\begin{aligned}\int(\partial_t v)v_+\mathrm{d}x + \int \partial_x\left[A'(u_\varepsilon)v\right]v_+\mathrm{d}x &= \varepsilon\int(\Delta_x^{\alpha/2}v)v_+\mathrm{d}x \\ &\le \varepsilon\int(\Delta_x^{\alpha/2}v_+)v_+\mathrm{d}x = -\varepsilon\int|\Delta_x^{\alpha/4}v_+|^2\mathrm{d}x.\end{aligned} \tag{2.4}$$

Next, using the fact that $\chi_{f\ge 0} = \chi_{f\ge 0}^2$ and that

$$\partial\left(f(x,t)_+\right) = \left(\partial f(x,t)\chi_{f(x,t)\ge 0}\right) = \left(\partial f(x,t)\right)\chi_{f(x,t)\ge 0},$$

where the partial derivative above is taken with respect to either variable $x$ or $t$, the first and second terms in the left hand side of (2.4) are

$$\int(\partial_t v)v_+\mathrm{d}x = \frac{1}{2}\int\frac{\mathrm{d}}{\mathrm{d}t}(v_+)^2\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(v_+)^2\mathrm{d}x \tag{2.5}$$

$$\begin{aligned}\int\partial_x\left[A'(u_\varepsilon)v\right]v_+\mathrm{d}x &= \int A''(u_\varepsilon)vv_+\mathrm{d}x + \int A'(x)(\partial_x v)v_+\mathrm{d}x \\ &= \int A''(u_\varepsilon)(v_+)^2\mathrm{d}x + \frac{1}{2}\int A'(x)\partial_x(v_+)^2\mathrm{d}x \\ &= \frac{1}{2}\int A''(u_\varepsilon)(v_+)^2\mathrm{d}x.\end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x(u_\varepsilon(\cdot,t)\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int(v_+)^2\mathrm{d}x \le -2\varepsilon\int|\Delta_x^{\alpha/4}v_+|^2\mathrm{d}x - \int A''(u_\varepsilon)(v_+)^2\mathrm{d}x \le 0, \tag{2.7}$$

which completes the proof. □

2.2. **Evolution of the relative entropy.** Let $\varphi$ be a smooth nondecreasing function such that

$$\varphi(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x \ge 1. \end{cases} \tag{2.8}$$

To localize the layer, we consider a parametrized function $\varphi_\delta$, $\delta > 0$, defined by

$$\varphi_\delta(x) := \varphi\left(\frac{x}{\delta}\right).$$

The parameter $\delta$ will be determined as a function of $\varepsilon$ at the end of the proof.

For fixed $\varepsilon, \delta > 0$ and $X \in C^1([0,T])$, we will consider the evolution of

$$\mathcal{H}(t) := \int_{\mathbb{R}} \varphi_\delta^2 \left( \frac{|x|}{\varepsilon^\beta} \right) \frac{|u_\varepsilon(x + X(t), t) - S_\varepsilon(x)|^2}{2} \, dx. \tag{2.9}$$

Although the above functional is based on the $L^2$-norm, we take advantage of the relative entropy method to get the convergence of $\mathcal{H}'(t)$ as in [6].

The relative entropy method was introduced in the studies by Dafermos [7] and Diperna [8] of $L^2$-stability and uniqueness of Lipschitz solutions to hyperbolic conservation laws endowed with a convex entropy. Recently, this method was extensively used in studying the contraction and inviscid limit for large initial perturbations of viscous (or inviscid) shock waves (see [6, 11, 12, 13, 14, 15, 16, 17, 20, 22, 23, 24, 25, 26, 27]).

To use the relative entropy method, in particular we consider the quadratic entropy

$$\eta(u) := \frac{u^2}{2}, \tag{2.10}$$

where we note from the theory of conservation laws that any function is an entropy of the scalar conservation law (2.1).
In the general theory, for a strictly convex entropy $\eta$, we define the associated relative entropy function by

$$\eta(u|v) := \eta(u) - \eta(v) - \eta'(v)(u - v). \tag{2.11}$$

Likewise, we define the relative functional of the strictly convex flux $A$ by

$$A(u|v) := A(u) - A(v) - A'(v)(u - v). \tag{2.12}$$

Let $F(\cdot, \cdot)$ be the flux of the relative entropy defined by

$$F(u, v) := G(u) - G(v) - \eta'(v)(A(u) - A(v)), \tag{2.13}$$

where $G$ is the entropy flux of $\eta$, i.e., $G' = \eta' A'$.

Since, for the quadratic entropy (2.10), the associated relative entropy is

$$\eta(u|v) = \frac{|u - v|^2}{2}, \tag{2.14}$$

the function $\mathcal{H}(t)$ in (2.9) can be rewritten as

$$\mathcal{H}(t) := \int_{\mathbb{R}} \varphi_\delta^2 \left( \frac{|x|}{\varepsilon^\beta} \right) \eta(u_\varepsilon(x + X(t), t)|S_\varepsilon(x)) \, dx.$$

For simplification of our presentation, we use a change of variable as follows:

$$v_\varepsilon(x, t) := u_\varepsilon(x + X(t), t). \tag{2.15}$$

Then, it follows from (2.1) that $v_\varepsilon$ satisfies

$$\partial_t v_\varepsilon - \dot{X}(t) \partial_x v_\varepsilon + \partial_x A(v_\varepsilon) = \varepsilon \Delta_x^{\alpha/2} v_\varepsilon. \tag{2.16}$$

We now present the following lemma.

**Lemma 2.3.** *The function $\mathcal{H}(t)$ defined by (2.9) satisfies*

$$\begin{aligned}\mathcal{H}'(t) &= \int_{-\infty}^{\infty} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)\partial_x\Big(\eta(v_\varepsilon(x,t)|S_\varepsilon(x))\dot{X}(t) - F(v_\varepsilon(x,t),S_\varepsilon(x))\Big)\,\mathrm{d}x\\ &\quad + \int_{-\infty}^{\infty} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)(v_\varepsilon(x,t)-S_\varepsilon(x))S_\varepsilon'(x)\left(\dot{X}(t) - \frac{A(v_\varepsilon(x,t)|S_\varepsilon(x))}{v_\varepsilon(x,t)-S_\varepsilon(x)}\right)\mathrm{d}x\\ &\quad + \varepsilon\int_{-\infty}^{\infty} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)(v_\varepsilon(x,t)-S_\varepsilon(x))\Delta^{\alpha/2}(v_\varepsilon(x,t)-S_\varepsilon(x))\mathrm{d}x\\ &=: H_1 + H_2 + P.\end{aligned} \tag{2.17}$$

*Proof.* First of all, since

$$\mathcal{H}(t) = \int_{\mathbb{R}} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)\eta\left(v_\varepsilon(x,t)|S_\varepsilon(x)\right)\,dx,$$

we have

$$\mathcal{H}'(t) = \int_{\mathbb{R}} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)\partial_t\Big(\eta\left(v_\varepsilon(x,t)|S_\varepsilon(x)\right)\Big)\,dx.$$

Note from the definition (2.11) that

$$\partial_t\eta(v_\varepsilon(x,t)|S_\varepsilon(x)) = (\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\partial_t v_\varepsilon(x,t) - \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)\underbrace{\partial_t S_\varepsilon(x)}_{=0}.$$

To get a nice quadratic structure from the above right-hand side, we use (2.2) and (2.16) so that

$$\begin{aligned}\partial_t\eta(v_\varepsilon(x,t)|S_\varepsilon(x)) &= (\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\Big(\dot{X}(t)\partial_x v_\varepsilon - \partial_x A(v_\varepsilon) + \varepsilon\Delta_x^{\alpha/2}v_\varepsilon\Big)\\ &\quad - \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)\Big(-(A(S_\varepsilon))' + \Delta_x^{\alpha/2}S_\varepsilon\Big)\\ &= \dot{X}(t)(\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\partial_x v_\varepsilon\\ &\quad - (\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\partial_x A(v_\varepsilon) + \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)(A(S_\varepsilon))'\\ &\quad + \varepsilon\Big((\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\Delta_x^{\alpha/2}v_\varepsilon - \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)\Delta_x^{\alpha/2}S_\varepsilon\Big).\end{aligned}$$

Since a straightforward computation together with the definitions (2.12) and (2.13) yields the identity

$$-\partial_x F(v_\varepsilon,S_\varepsilon) - \eta''(S_\varepsilon)S_\varepsilon' A(v_\varepsilon|S_\varepsilon) = -(\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\partial_x A(v_\varepsilon) + \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)(A(S_\varepsilon))'$$

(which also appears in the proof of [16, Lemma 2.1]), we have

$$\begin{aligned}\partial_t\eta(v_\varepsilon(x,t)|S_\varepsilon(x)) &= \dot{X}(t)(\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\partial_x v_\varepsilon\\ &\quad - \partial_x F(v_\varepsilon,S_\varepsilon) - \eta''(S_\varepsilon)S_\varepsilon' A(v_\varepsilon|S_\varepsilon)\\ &\quad + \varepsilon\Big((\eta'(v_\varepsilon)-\eta'(S_\varepsilon))\Delta_x^{\alpha/2}v_\varepsilon - \eta''(S_\varepsilon)(v_\varepsilon-S_\varepsilon)\Delta_x^{\alpha/2}S_\varepsilon\Big).\end{aligned}$$

We now use the quadratic entropy (2.10) to obtain

$$\begin{aligned}\partial_t\eta(v_\varepsilon(x,t)|S_\varepsilon(x)) &= \dot{X}(t)(v_\varepsilon-S_\varepsilon)\partial_x v_\varepsilon - \partial_x F(v_\varepsilon,S_\varepsilon) - S_\varepsilon' A(v_\varepsilon|S_\varepsilon)\\ &\quad + \varepsilon(v_\varepsilon-S_\varepsilon)\Delta_x^{\alpha/2}\left(v_\varepsilon-S_\varepsilon\right).\end{aligned}$$

Therefore, using (2.14) and

$$(v_\varepsilon - S_\varepsilon)\partial_x v_\varepsilon = \partial_x \left( \frac{|v_\varepsilon - S_\varepsilon|^2}{2} \right) + (v_\varepsilon - S_\varepsilon)\partial_x S_\varepsilon,$$

we complete the proof. □

**Remark 2.1.** *Contrary to* [6, Lemma 2.1]*, we have a new hyperbolic part* $H_2$ *in* (2.17)*, because we are considering the viscous layer.*

2.3. **Estimate on the first hyperbolic part** $H_1$**.** Here, we estimate the first part $H_1$ of $\mathcal{H}'(t)$ in (2.17) by following the strategy in [6, Section 3]. For this, we consider the normalized relative entropy flux $f(\cdot,\cdot)$, given by

$$f(u,v) := \frac{F(u,v)}{\eta(u|v)}. \tag{2.18}$$

With such an $f$, we have the following properties.

**Lemma 2.4.** *For any* $L > 0$*, there exists a constant* $\Lambda > 0$ *such that for any* $u, v$ *with* $|u|, |v| \le L$*,*

$$0 \le \partial_u f(u,v) \le \Lambda,$$
$$\frac{1}{\Lambda} \le \partial_v f(u,v).$$

For the proof of the above lemma, we refer to [20].

We now define the shift $X$ as a solution to the ODE

$$\begin{cases} \dot{X}(t) = f(v_\varepsilon(0,t), S_\varepsilon(0)), \\ X(0) = 0. \end{cases} \tag{2.19}$$

Recall from (2.15) that $v_\varepsilon(0,t) = u_\varepsilon(X(t),t)$. As mentioned before, since, for any $\varepsilon > 0$, the equation (2.1) with $L^\infty$ initial datum admits a unique smooth solution, the Cauchy-Lipschitz theorem together with Lemma 2.4 implies the existence and uniqueness of the solution $X$ to the ODE (2.19).

We now present a bound on $H_1$ in (2.17). In what follows, $C$ denotes a positive constant which may change from line to line, but which is independent on $\varepsilon$.

**Proposition 2.1.** *Let* $\varphi$ *be a smooth nondecreasing function satisfying* (2.8)*. Under the same hypotheses as in Theorem 1.1, there exists a positive constant* $C = C(\|u_0\|_{L^\infty(\mathbb{R})}, u_\pm)$ *such that for any* $\varepsilon, \delta > 0$*,*

$$H_1 = \int_{-\infty}^{\infty} \varphi_\delta^2 \left( \frac{|x|}{\varepsilon^\beta} \right) \partial_x \left( \eta(v_\varepsilon(x,t)|S_\varepsilon(x))\dot{X}(t) - F(v_\varepsilon(x,t), S_\varepsilon(x)) \right) \mathrm{d}x \le C\sqrt{\delta \varepsilon^\beta}.$$

*Proof.* First of all, we separate $H_1$ into two parts:

$$\begin{aligned} H_1(t) &= \int_{-\infty}^{0} \left( \varphi_\delta \left( -\frac{x}{\varepsilon^\beta} \right) \right)^2 \partial_x \left( \eta(v_\varepsilon(x,t)|S_\varepsilon(x))\dot{X}(t) - F(v_\varepsilon(x,t)), S_\varepsilon(x) \right) \mathrm{d}x \\ &\quad + \int_{0}^{\infty} \left( \varphi_\delta \left( \frac{x}{\varepsilon^\beta} \right) \right)^2 \partial_x \left( \eta(v_\varepsilon(x,t)|S_\varepsilon(x))\dot{X}(t) - F(v_\varepsilon(x,t)), S_\varepsilon(x) \right) \mathrm{d}x \\ &=: H_1^L(t) + H_1^R(t). \end{aligned}$$

For $H_1^L$, by an integration by parts together with (2.18), (2.14) and (2.19), we have

$$\begin{aligned}
&H_1^L(t)\\
&= -\int_{-\infty}^{0} \partial_x \left[ \left( \varphi\left(-\frac{x}{\delta\varepsilon^\beta}\right)\right)^2 \right] \left( \eta(v_\varepsilon(x,t)|S_\varepsilon(x))\dot{X}(t) - \eta(v_\varepsilon(x,t)|S_\varepsilon(x)) f(v_\varepsilon(x,t), S_\varepsilon(x))\right) \mathrm{d}x\\
&= -\int_{-\infty}^{0} \left[ -\frac{2}{\delta\varepsilon^\beta}\varphi\left(-\frac{x}{\delta\varepsilon^\beta}\right)\varphi'\left(-\frac{x}{\delta\varepsilon^\beta}\right)\right] \frac{|v_\varepsilon(x,t) - S_\varepsilon(x)|^2}{2} \left( \dot{X}(t) - f(v_\varepsilon(x,t), S_\varepsilon(x))\right) \mathrm{d}x\\
&= \frac{1}{\delta\varepsilon^\beta}\int_{-\delta\varepsilon^\beta}^{0} \varphi\left(-\frac{x}{\delta\varepsilon^\beta}\right)\varphi'\left(-\frac{x}{\delta\varepsilon^\beta}\right) |v_\varepsilon(x,t) - S_\varepsilon(x)|^2 h(x,t)\mathrm{d}x,
\end{aligned}$$

where

$$h(x,t) := f(v_\varepsilon(0,t), S_\varepsilon(0)) - f(v_\varepsilon(x,t), S_\varepsilon(x)).$$

Notice that since $\|u_\varepsilon\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}$ by the maximum principle, and $|S_\varepsilon| \le \max\{|u_-|, |u_+|\}$, there is a constant $C = C(\|u_0\|_{L^\infty(\mathbb{R})}, u_\pm)$ such that

$$\|v_\varepsilon - S_\varepsilon\|_{L^\infty([0,T]\times\mathbb{R})} \le C. \tag{2.20}$$

To control $h(x,t)$, we first separate it into two parts:

$$h(x,t) = \underbrace{(f(v_\varepsilon(0,t), S_\varepsilon(0)) - f(v_\varepsilon(x,t), S_\varepsilon(0)))}_{=:h_1} + \underbrace{(f(v_\varepsilon(x,t), S_\varepsilon(0)) - f(v_\varepsilon(x,t), S_\varepsilon(x)))}_{=:h_2}.$$

To estimate $h_1$, using Lemma 2.2, we observe that for any $x < 0$,

$$v_\varepsilon(0,t) - v_\varepsilon(x,t) = \int_{x+X(t)}^{X(t)} \partial_y u_\varepsilon(y,t)dy \le \int_{x+X(t)}^{X(t)} (\partial_y u_\varepsilon)_+(y,t)dy \le \| (\partial_x u_0)_+ \|_{L^2(\mathbb{R})}\sqrt{|x|}.$$

Then, since $f$ is increasing with respect to the first variable by Lemma 2.4, we have

$$\begin{aligned}
h_1 &\le f\Big(v_\varepsilon(x,t) + \| (\partial_x u_0)_+ \|_{L^2(\mathbb{R})}\sqrt{|x|}, S_\varepsilon(0)\Big) - f(v_\varepsilon(x,t), S_\varepsilon(0))\\
&\le \Lambda\sqrt{|x|}\| (\partial_x u_0)_+ \|_{L^2(\mathbb{R})}.
\end{aligned}$$

Using Lemma 2.4 and Lemma 2.1, we have

$$h_2 \le \frac{1}{\Lambda}(S_\varepsilon(0) - S_\varepsilon(x)) \le 0 \quad \forall x \le 0.$$

Therefore, we have

$$\begin{aligned}
H_1^L(t) &\le \frac{C}{\delta\varepsilon^\beta}\int_{-\delta\varepsilon^\beta}^{0} \varphi\left(-\frac{x}{\delta\varepsilon^\beta}\right)\varphi'\left(-\frac{x}{\delta\varepsilon^\beta}\right)\sqrt{|x|}\,\mathrm{d}x \quad \le C\sqrt{\delta\varepsilon^\beta}\int_{-1}^{0}\varphi(-x)\varphi'(-x)\sqrt{|x|}dx\\
&\le C\sqrt{\delta\varepsilon^\beta},
\end{aligned}$$

where the last inequality is obtained by the definition of $\varphi$ as

$$\int_{-1}^{0}\varphi(-x)\varphi'(-x)\sqrt{|x|}dx \le \int_{-1}^{0}\varphi'(-x)dx = \varphi(1) - \varphi(0) = 1.$$

Likewise, using the same method as above, we have

$$H_1^R(t) \le C\sqrt{\delta\varepsilon^\beta}.$$

Hence, we complete the proof. □

2.4. **Estimate on the second hyperbolic part $H_2$.** Here we find a bound for convergence of the second part $H_2$ in (2.17). For this, we consider a specific choice of the monotone function $\varphi$ satisfying (2.8), defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 2x^2 & \text{if } 0 \le x \le 1/2, \\ 1 - 2(x-1)^2 & \text{if } 1/2 \le x \le 1, \\ 1 & \text{if } x \ge 1. \end{cases} \tag{2.21}$$

**Proposition 2.2.** *Let $\varphi$ be the function defined by* (2.21)*. Under the same hypotheses as in Theorem 1.1, there exists a positive constant $C$ such that for any $\delta \ge 4$,*

$$\begin{aligned} H_2 &= \int_{-\infty}^{\infty} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)(v_\varepsilon(x,t) - S_\varepsilon(x))S_\varepsilon'(x)\left(\dot X(t) - \frac{A(v_\varepsilon(x,t)|S_\varepsilon(x))}{v_\varepsilon(x,t) - S_\varepsilon(x)}\right)\mathrm{d}x \\ &\le C\left(\left(\frac{1}{\delta}\right)^{3/2} + \left(S_1\left(\sqrt{\delta}\right) - u_+\right) + \left(u_- - S_1\left(-\sqrt{\delta}\right)\right)\right), \end{aligned}$$

*Proof.* Since $v_\varepsilon$ and $S_\varepsilon$ are bounded as mentioned in (2.20), using the definition (2.12) of $A(\cdot|\cdot)$, and (2.19) with Lemma 2.4, we observe that there exists a positive constant $C = C(\|u_0\|_{L^\infty(\mathbb{R})}, u_\pm)$ such that for all $x \in \mathbb{R}$ and $t \le T$,

$$A(v_\varepsilon|S_\varepsilon) = (v_\varepsilon - S_\varepsilon)^2 \int_0^1 \int_0^1 A''(S_\varepsilon + st(v_\varepsilon - S_\varepsilon))t\,ds\,dt \le \|A''\|_{L^\infty(-C,C)}|v_\varepsilon - S_\varepsilon|^2,$$

and

$$|\dot X(t)| \le C.$$

Therefore, using (2.20), we have

$$H_2 \le C\int_{-\infty}^{\infty} \varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)|S_\varepsilon'(x)|\,\mathrm{d}x.$$

Note that, since $S_\varepsilon'(x) = \varepsilon^{-\beta}S_1'(x\varepsilon^{-\beta})$,

$$H_2 \le C\int_{-\infty}^{\infty} \varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x.$$

We now separate the right hand side into two parts:

$$\int_{-\infty}^{\infty} \varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x = \int_{|x|\le\sqrt{\delta}} \varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x + \int_{|x|\ge\sqrt{\delta}} \varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x.$$

For any $\delta \ge 4$, since

$$|x| \le \sqrt{\delta} \quad \Rightarrow \quad \frac{|x|}{\delta} \le \frac{1}{\sqrt{\delta}} \le \frac{1}{2},$$

we use (2.21) to get

$$\begin{aligned} \int_{|x|\le\sqrt{\delta}} \varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x &\le \|S_1'\|_{L^\infty(\mathbb{R})}\int_{|x|\le\sqrt{\delta}} \varphi^2\left(\frac{|x|}{\delta}\right)\mathrm{d}x \\ &= \|S_1'\|_{L^\infty(\mathbb{R})}\frac{4}{\delta^4}\int_{|x|\le\sqrt{\delta}}|x|^4\,\mathrm{d}x \le C\left(\frac{1}{\delta}\right)^{3/2}. \end{aligned}$$

Using Lemma 2.1, we have

$$\begin{aligned}\int_{|x|\geq\sqrt{\delta}}\varphi_\delta^2\left(|x|\right)|S_1'(x)|\,\mathrm{d}x &\leq \int_{|x|\geq\sqrt{\delta}}|S_1'(x)|\,\mathrm{d}x=-\int_{|x|\geq\sqrt{\delta}}S_1'(x)\,\mathrm{d}x\\ &=\left(S_1\left(\sqrt{\delta}\right)-u_+\right)+\left(u_--S_1\left(-\sqrt{\delta}\right)\right).\end{aligned}$$

Hence we complete the proof. □

## 2.5. **Estimate on the parabolic part $P$.**

**Proposition 2.3.** *Let $\varphi$ be a smooth nondecreasing function satisfying* (2.8)*. Under the same hypotheses as in Theorem 1.1, there exists a positive constant $C$ such that for any $\delta>0$,*

$$P=\varepsilon\int_{\mathbb{R}}\varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)(v_\varepsilon(x,t)-S_\varepsilon(x))\Delta^{\alpha/2}(v_\varepsilon(x,t)-S_\varepsilon(x))\mathrm{d}x\leq C\left(\frac{1}{\delta}\right)^{\alpha-1}.$$

*Proof.* For simplicity, here we set

$$w_\varepsilon(x,t):=v_\varepsilon(x,t)-S_\varepsilon(x).$$

First, using (1.2) and anti-symmetry, we have

$$\begin{aligned}P&=\varepsilon\int_{\mathbb{R}}\varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)\Delta^{\alpha/2}w_\varepsilon(x,t)\,\mathrm{d}x\\ &=\varepsilon\iint_{\mathbb{R}\times\mathbb{R}}\varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)\frac{w_\varepsilon(y,t)-w_\varepsilon(x,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\ &=-\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta^2\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)-\varphi_\delta^2\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]\frac{w_\varepsilon(x,t)-w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x,\end{aligned}$$

which can be rewritten into

$$\begin{aligned}P=&-\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)\frac{w_\varepsilon(x,t)-w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\ &-\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\frac{w_\varepsilon(x,t)-w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\ =:&\,P_1+P_2.\end{aligned}$$

Since

$$\begin{aligned}P_1=&-\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]^2\frac{1}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\ &+\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]\frac{\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x,\\ P_2=&-\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]\frac{\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\ &+\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)-\varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]^2\frac{w_\varepsilon(x,t)w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x,\end{aligned}$$

we have

$$P = -\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right)w_\varepsilon(x,t) - \varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)w_\varepsilon(y,t)\right]^2\frac{1}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x$$
$$+\frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right) - \varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]^2\frac{w_\varepsilon(x,t)w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x,$$

which gives

$$P \le \frac{\varepsilon}{2}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right) - \varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]^2\frac{w_\varepsilon(x,t)w_\varepsilon(y,t)}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x.$$

Using (2.20), we have

$$P \le C\varepsilon\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right) - \varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]^2\frac{1}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x.$$

Since

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi_\delta\left(\frac{|x|}{\varepsilon^\beta}\right) - \varphi_\delta\left(\frac{|y|}{\varepsilon^\beta}\right)\right]^2\frac{1}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\
&= \left(\frac{1}{\delta\varepsilon^\beta}\right)^{1+\alpha}\iint_{\mathbb{R}\times\mathbb{R}}\left[\varphi\left(\frac{|x|}{\delta\varepsilon^\beta}\right) - \varphi\left(\frac{|y|}{\delta\varepsilon^\beta}\right)\right]^2\frac{1}{|x\delta^{-1}\varepsilon^{-\beta} - y\delta^{-1}\varepsilon^{-\beta}|^{1+\alpha}}\mathrm{d}y\mathrm{d}x\\
&= \left(\frac{1}{\delta}\right)^{\alpha-1}\left(\frac{1}{\varepsilon}\right)^{\beta(\alpha-1)}\iint_{\mathbb{R}\times\mathbb{R}}\frac{\left[\varphi\left(|x|\right) - \varphi\left(|y|\right)\right]^2}{|x-y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x,
\end{aligned}$$

using $\beta = \frac{1}{\alpha-1}$, we have

$$P \le C\left(\frac{1}{\delta}\right)^{\alpha-1}\iint_{\mathbb{R}\times\mathbb{R}}\underbrace{\frac{\left[\varphi\left(|x|\right) - \varphi\left(|y|\right)\right]^2}{|x-y|^{1+\alpha}}}_{=:J}\mathrm{d}y\mathrm{d}x.$$

Now, it remains to show that $J$ is integrable on $\mathbb{R}\times\mathbb{R}$. To this end, we separate the integral into several parts:

$$\begin{aligned}
\iint J &= \iint_{|x|\le1,|y|\le2} J + \iint_{|x|\le1,|y|>2} J + \iint_{|x|>1,|y|>1} J + \iint_{1\le|x|\le2,|y|\le1} J + \iint_{|x|>2,|y|\le1} J\\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Using the smoothness of $\varphi$ and the boundedness of $\{|x|\le1, |y|\le2\}$, we have

$$I_1 \le C\iint_{|x|\le1,|y|\le2}\frac{1}{|x-y|^{\alpha-1}}\mathrm{d}y\mathrm{d}x,$$

which together with $1<\alpha<2$ implies $I_1<\infty$.
For $I_2$, observe that $|x-y| \ge |y|-|x| \ge |y|-1 \ge \frac{|y|}{2}$ for any $x, y$ with $|x|\le1, |y|>2$, and $|\varphi(y)-\varphi(x)| \le 2\|\varphi\|_\infty \le 2$. Thus, we have

$$I_2 \le C\int_{|x|\le1}\int_{|y|>2}\frac{1}{|y|^{1+\alpha}}\mathrm{d}y\mathrm{d}x < \infty.$$

Since $\varphi(|x|) = \varphi(|y|) = 1$ for all $|x|, |y| \ge 1$ by (2.8), we have $I_3 = 0$.
We use the same estimate as in $I_1$ to have $I_4 < \infty$.

Note that $J_5 = J_2$ by symmetry.
Hence we complete the proof. □

2.6. **Conclusion.** It follows from Lemma 2.3 and Propositions 2.1, 2.2, 2.3 that for any $\delta \geq 4$ and $\varepsilon > 0$,

$$\begin{aligned}\mathcal{H}'(t) &\leq C\left(\sqrt{\delta\varepsilon^\beta} + \left(\frac{1}{\delta}\right)^{3/2} + \left(S_1\left(\sqrt{\delta}\right) - u_+\right) + \left(u_- - S_1\left(-\sqrt{\delta}\right)\right) + \left(\frac{1}{\delta}\right)^{\alpha-1}\right)\\ &\leq \underbrace{C\left(\sqrt{\delta\varepsilon^\beta} + \left(S_1\left(\sqrt{\delta}\right) - u_+\right) + \left(u_- - S_1\left(-\sqrt{\delta}\right)\right) + \left(\frac{1}{\delta}\right)^{\alpha-1}\right)}_{=:E(\varepsilon,\delta)}.\end{aligned}$$

This, together with (2.9), implies

$$\begin{aligned}\int_{\{|x|\geq\delta\varepsilon^\beta\}} \frac{|u_\varepsilon(x+X(t),t) - S_\varepsilon(x)|^2}{2}\,dx &\leq \mathcal{H}(t) = \mathcal{H}(0) + \int_0^t \mathcal{H}'(s)ds\\ &\leq \mathcal{H}(0) + CTE(\varepsilon,\delta).\end{aligned}$$

Moreover, since (2.20) yields

$$\int_{\{|x|\leq\delta\varepsilon^\beta\}} \frac{|u_\varepsilon(x+X(t),t) - S_\varepsilon(x)|^2}{2}\,dx \leq \frac{1}{2}\|v_\varepsilon - S_\varepsilon\|^2_{L^\infty([0,T]\times\mathbb{R})}\int_{\{|x|\leq\delta\varepsilon^\beta\}} dx \leq C\delta\varepsilon^\beta,$$

we have

$$\int_{\mathbb{R}} \frac{|u_\varepsilon(x+X(t),t) - S_\varepsilon(x)|^2}{2}\,dx \leq \int_{\mathbb{R}} \frac{|u_0(x) - S_\varepsilon(x)|^2}{2}\,dx + CT\left(\delta\varepsilon^\beta + E(\varepsilon,\delta)\right).$$

Therefore,

$$\|u_\varepsilon(\cdot + X(t),t) - S_\varepsilon\|_{L^2(\mathbb{R})} \leq \|u_0 - S_\varepsilon\|_{L^2(\mathbb{R})} + C(T)\sqrt{\delta\varepsilon^\beta + E(\varepsilon,\delta)}.$$

Then, using

$$\|S_\varepsilon - S_0\|_{L^2(\mathbb{R})} = \varepsilon^{\beta/2}\|S_1 - S_0\|_{L^2(\mathbb{R})},$$

we have

$$\begin{aligned}\|u_\varepsilon(\cdot + X(t),t) - S_0\|_{L^2(\mathbb{R})} &\leq \|u_\varepsilon(\cdot + X(t),t) - S_\varepsilon\|_{L^2(\mathbb{R})} + \|S_\varepsilon - S_0\|_{L^2(\mathbb{R})}\\ &\leq \|u_0 - S_\varepsilon\|_{L^2(\mathbb{R})} + C(T)\sqrt{\delta\varepsilon^\beta + E(\varepsilon,\delta)} + C\varepsilon^{\beta/2}\\ &\leq \|u_0 - S_0\|_{L^2(\mathbb{R})} + \|S_\varepsilon - S_0\|_{L^2(\mathbb{R})} + C(T)\left(\sqrt{\delta\varepsilon^\beta + E(\varepsilon,\delta)} + \varepsilon^{\beta/2}\right)\\ &\leq \|u_0 - S_0\|_{L^2(\mathbb{R})} + C(T)\left(\sqrt{\delta\varepsilon^\beta + E(\varepsilon,\delta)} + \varepsilon^{\beta/2}\right).\end{aligned}$$

Therefore, for some constant $C(T)$,

$$\|u_\varepsilon(\cdot + X(t),t) - S_0\|_{L^2(\mathbb{R})} \leq \|u_0 - S_0\|_{L^2(\mathbb{R})} + C(T)\psi(\varepsilon),$$

where

$$\psi(\varepsilon) := \inf_{\delta\geq 4}\left(\sqrt{\delta\varepsilon^\beta + E(\varepsilon,\delta)} + \varepsilon^{\beta/2}\right).$$

This completes the proof.

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