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A REMARK ON THE SCATTERING THEORY FOR THE $2D$ RADIAL FOCUSING INLS

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Abstract. We revisit the scattering results of the radial solutions below the ground state to the focusing inhomogeneous nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |x|^{-b}|u|^p u = 0$$

when $0 < b < 1$ and $d = 2$ in the 2D case. We use a modified version of Arora-Dodson-Murphy's approach [1] to give a new proof that extends the scattering results of [10] and avoids concentration compactness.

Key Words: Schrödinger equation; Scattering theory.
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1. Introduction

Considering the initial value problem (IVP), also called the Cauchy problem for the inhomogenous nonlinear Schrödinger equation (INLS)

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^p u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^2 \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

with $0 < b < 1$ and $2 - b < p < \infty$. The equation (1.1) arises naturally in nonlinear optics for the propagation of laser beams, we refer to [5] [6].

Since the $\dot{H}^{s_p}(\mathbb{R}^2)$ norm is invariant under the standard scaling

$$u_\lambda(t,x) = \lambda^{\frac{2-b}{p}} u(\lambda^2 t, \lambda x),$$

where $s_p = \frac{d}{2} - \frac{2-b}{p}$, we call the equation is $\dot{H}^{s_p}(\mathbb{R}^2)$ critical. The solutions to equation (1.1) conserve the mass, defined by

$$M(u) := \int_{\mathbb{R}^2} |u|^2 dx = M(u_0),$$

and the energy, defined as the sum of the kinetic and potential energies:

$$E(u) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 - \frac{1}{p+2}|x|^{-b}|u|^{p+2} dx = E(u_0).$$

In particular, we have always have $s_p < 1$, which is called energy-subcritical. In the case $s_p = 0$ equation (1.1) is called mass-critical equation.

Now we recall the well-posedness theory of the equation (1.1). In [14], Genoud and Staurt proved that (1.1) is locally well-posed in $H^1(\mathbb{R}^d)$ for $0 < b < \min\{2, d\}$. More recently, Guzmán's [11] established the local well-posedness of (1.1) based on Strichartz estimates. In particular, he proved that, for $d \geq 4$ with $0 < b < 2$ or $d = 1, 2, 3$ with $0 < b < \frac{d}{3}$, the Cauchy problem (1.1) is locally well-posed in $H^1(\mathbb{R}^d)$. Dinh [4] extend Guzmán's result in dimension 3 for $0 < b < \frac{3}{2}$ and $\frac{4-2b}{3} < p < \frac{6-4b}{2b-1}$.

This equation admits a global nonscattering solution of the form $u(t) = e^{it}Q$, where Q is the ground state solution to the elliptic equation

$$-\Delta Q + Q - |x|^{-b}|Q|^p Q = 0.$$

The existence of the ground state was proved in Genoud [11],[12], Genoud and Staurt [14], while the uniqueness was handled in Yangida [19], Genoud [13]. More precisely, existence and uniqueness of the radial, positive, minimal mass solution Q hold for $d \geq 1$ and $0 \leq b < \min\{2, d\}$. Moreover, the global well-posedness and scattering theory of solutions to (1.1) under the ground states Q has been considered in [2],[9],[10], which extended the results of [3, 17] for nonlinear Schrodinger equation with $|u|^p u$ type nonlinearities.

In this paper, we consider the mass-supercritical range $2 - b < p < \infty$ and $0 < b < 1$, in two spatial dimension. We will give another proof of the following scattering results.

Theorem 1.1. *Let $0 < b < 1$ and $2 - b < p < \infty$. Suppose u_0 is radial and such that*

$$M(u_0)^{1-s_p} E(u_0)^{s_p} < M(Q)^{1-s_p} E(Q)^{s_p},$$

and

$$\|u_0\|_{L^2}^{1-s_p} \|\nabla u_0\|_{L^2}^{s_p} < \|Q\|_{L^2}\|^{1-s_p} \|\nabla Q\|_{L^2}^{s_p}.$$

Then the solution to equation (1.1) *with data u_0 is global and scattering.*

The proof in [10], with $0 < b < \frac{2}{3}$, was based on the concentration compactness. We extend the well-posedness results of (1.1). Then, inspired by the new approach of [1, 7], we present a simple proof that avoids the concentration compactness, which is based on the Virial/Morawetz estimate, the radial Sobolev embedding and a scattering criterion we establish.

The rest of this paper is organized as follows: In section 2, we set up some notations, recall some important properties for the inhomogeneous case. In section 3, we will establish a new scattering criterion for (1.1), Lemma 3.1. In section 4, by the Morawetz identity, we will establish the virial/Morawetz estimates to show the solution satisfy the scattering criterion of Lemma 3.1, thereby completing the proof of Theorem 1.1.

We conclude the introduction by giving some notations which will be used throughout this paper. We always use $X \lesssim Y$ to denote $X \leq CY$ for some constant $C > 0$. Similarly, $X \lesssim_u Y$ indicates there exists a constant $C := C(u)$ depending on u such that $X \leq C(u)Y$. We also use the big-oh notation $\mathcal{O}$. e.g. $A = \mathcal{O}(B)$ indicates $C_1 B \leq A \leq C_2 B$ for some constants $C_1, C_2 > 0$. The derivative operator ∇ refers to the spatial variable only. We use $L^r(\mathbb{R}^2)$ to denote the Banach space of functions $f : \mathbb{R}^2 \to \mathbb{C}$ whose norm

$$\|f\|_r := \|f\|_{L^r} = \Big(\int_{\mathbb{R}^2} |f(x)|^r dx \Big)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$. For any non-negative integer k, we denote by $H^{k,r}(\mathbb{R}^2)$ the Sobolev space defined as the closure of smooth compactly supported functions in the norm $\|f\|_{H^{k,r}} = \sum_{|\alpha| \leq k} \|\frac{\partial^\alpha f}{\partial x^\alpha}\|_r$, and we denote it by H^k when $r = 2$. For a time slab I, we use $L_t^q(I; L_x^r(\mathbb{R}^2))$ to denote the space-time norm

$$\|f\|_{L_t^q L_x^r (I \times \mathbb{R}^2)} = \Big(\int_I \|f(t,x)\|_{L_x^r}^q dt \Big)^{\frac{1}{q}}$$

with the usual modifications when q or r is infinite, sometimes we use $\|f\|_{L^q(I;L^r)}$ or $\|f\|_{L^q L^r (I \times \mathbb{R}^2)}$ for short.

2. Preliminaries

Let us start this section by introducing the notations used throughout the paper. We also recall some Strichartz estimates associated to the linear Schrödinger propagator.

We say the pair (q,r) is L^2-adimissible or simply admissible par if they satisfy the condition $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}$ and

$$\begin{cases} 2\leq r\leq \frac{2d}{d-2}, & d\geq 3;\\ 2\leq r<\infty, & d=2;\\ 2\leq r\leq\infty; & d=1. \end{cases} \tag{2.2}$$

We also call the pair (q,r) is $\dot{H}^s-$admissible if $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}-s$ and

$$\begin{cases} \frac{2d}{d-2s}\leq r\leq (\frac{2d}{d-2})^-, & d\geq 3;\\ \frac{2}{1-s}\leq r\leq ((\frac{2}{1-s})^+)', & d=2;\\ \frac{2}{1-2s}\leq r\leq\infty; & d=1. \end{cases} \tag{2.3}$$

Here, a^- is a fixed number slightly smaller than a ($a^-=a-\epsilon$, where ϵ is small enough) and, in the same way, we define a^+. Moreover $(a^+)'$ is the number such that

$$\frac{1}{a}=\frac{1}{(a^+)'}+\frac{1}{a^+},$$

that is, $(a^+)'=\frac{a\,a^+}{a^+-a}$. Finally we say that (q,r) is $\dot{H^{-s}}-$admissible if $\frac{2}{q}=\frac{d}{2}-\frac{d}{r}+s$ and

$$\begin{cases} (\frac{2d}{d-2s})^+\leq r\leq (\frac{2d}{d-2})^-, & d\geq 3.\\ (\frac{2}{1-s})^+\leq r\leq ((\frac{2}{1+s})^+)', & d=2;\\ (\frac{2}{1-2s})^+\leq r\leq\infty; & d=1. \end{cases} \tag{2.4}$$

Given $s\in\mathbb{R}$, let $\Lambda_s=\{(q,r):(q,r)\ is\ \dot{H}^s-admissible\}$. We define the following Strichartz norm

$$\|u\|_{S(\dot{H}^s)}=\sup_{(q,r)\in\Lambda_s}\|u\|_{L_t^qL_x^r}$$

and dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s})}=\sup_{(q,r)\in\Lambda_{-s}}\|u\|_{L_t^{q'}L_x^{r'}}.$$

Now, we recall the radial Sobolev embedding and the Strichartz estimates.

Lemma 2.1 (Radial Sobolev embedding)**.** *For radial $f\in H^1$, then*

$$\||x|^{\frac{d-1}{2}}f\|_{L^\infty}\lesssim ||f||_{H^1}.$$

Lemma 2.2 (Strichartz estimates[18])**.** *The following estimates hold.*

(i)(linear estimate).

$$\|e^{it\Delta}f\|_{S(\dot{H}^s)}\leq C\|f\|_{\dot{H}^s}.$$

(ii)(inhomogeneous estimates).

$$\left\|\int_0^t e^{i(t-s)\Delta}g(\cdot,s)ds\right\|_{S(\dot{H}^s)}\leq C\|g\|_{S'(\dot{H}^{-s})}.$$

Next, we recall some interpolation estimates for the nonlinear term.

Lemma 2.3. *Let $2-b<p<\infty$ and $0<b<2$. If $s_p<1$, then for $1\le j\le 3$, there exist $\theta_j\in(0,p)$ sufficiently small so that the following hold true:*

$$\||x|^{-b}|u|^p v\|_{S'(\dot H^{-s_p})}\le C\|u\|^{\theta_1}_{L_t^\infty H_x^1}\|u\|^{p-\theta_1}_{S(\dot H^{s_p})}\|v\|_{S(\dot H^{s_p})}, \tag{2.5}$$

$$\||x|^{-b}|u|^p v\|_{S'(L^2)}\le C\|u\|^{\theta_2}_{L_t^\infty H_x^1}\|u\|^{p-\theta_2}_{S(\dot H^{s_p})}\|v\|_{S(L^2)}, \tag{2.6}$$

where $C>0$ is some constant independent of u.

Proof. We begin defining the following numbers (depending only p and b)

$$\hat q=\frac{2p(p+2-\theta)}{p(p+b)-\theta(p-2+b)}\quad \hat r=\frac{2p(p+2-\theta)}{(p-\theta)(2-b)}$$

and

$$\tilde a=\frac{p(p+2-\theta)}{p(p+b-\theta)-(2-b)(1-\theta)}\quad \hat a=\frac{p(p+2-\theta)}{2-b}$$

where $\theta>0$ sufficiently small. It is easy to see that $(\hat q,\hat r)$ L^2-admissible, $(\hat a,\hat r)$ $\dot H^{s_p}-$admissible and $(\tilde a,\hat r)$ $\dot H^{-s_p}-$admissible. Moreover, we observe that

$$\frac{1}{\hat a}+\frac{1}{\tilde a}=\frac{2}{\hat q}$$

By Hölder and Sobolev inequalities (see [16] for details), we have

$$\||x|^{-b}|u|^p v\|_{L_x^{\hat r'}}\lesssim \|u\|^{\theta}_{H^1}\|u\|^{p-\theta}_{L_x^{\hat r}}\|v\|_{L_x^{\hat r}} \tag{2.7}$$

so that (2.5) and (2.6) follow. □

Lemma 2.4 ([4])**.** *Let $b\in(0,1)$ and $2-b<p<\infty$. Then there exist (p_1,q_1), $(p_1,q_2)\in\Lambda_0$ satisfying $2p+2>p_1,p_2$ and $q_1,q_2\in(2,\infty)$ such that*

$$\|\nabla(|x|^{-b}|u|^p u)\|_{S'(L^2)}\lesssim(\|u\|^p_{L_t^{m_1}L_x^{q_1}(I\times\mathbb{R}^2)}+\|u\|_{L_t^{m_2}L_x^{q_2}(I\times\mathbb{R}^2)})\|\langle\nabla\rangle u\|_{S(L^2)}$$

where $m_1=\frac{\alpha p_1}{p_1-2}$ and $m_2=\frac{\alpha p_2}{p_2-2}$.

Denote $C_{p,d,b}=\frac{\frac{dp}{2}+b}{p+2-(\frac{dp}{2}+b)}$, then we have the following Gagliardo-Nirenberg inequality.

Proposition 2.5 (Farah [8])**.** *Let $\frac{4-2b}{d}<p<\frac{4-2b}{d-2}$ and $0<b<\min\{2,d\}$, then the Gagliardo-Nirenberg inequality*

$$\int_{\mathbb{R}^d}|x|^{-b}|u|^{p+2}dx\le C_0\|\nabla u\|_{L^2}^{\frac{dp}{2}+b}\|u\|_{L^2}^{p+2-(\frac{dp}{2}+b)} \tag{2.8}$$

holds, and the sharp constant $C_0>0$ is explicitly given by

$$C_0=C_{p,d,b}^{\frac{4-pd-2b}{4}}\frac{p+2}{(\frac{dp}{2}+b)\|Q\|_{L^2}^p},$$

where Q is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$\Delta Q-Q+|x|^{-b}|Q|^pQ=0.$$

Moreover the solution Q satisfies the following relations

$$\|\nabla Q\|_{L^2}=C_{p,d,b}^{\frac12}\|Q\|_{L^2},$$

and

$$\int_{\mathbb{R}^d}|x|^{-b}|Q|^{p+2}dx=C_{p,d,b}\|Q\|_{L^2}^2.$$

Now, we recall the well posedness theory of equation in $H^1(\mathbb{R}^2)$ under the ground state Q, which is proved by Farah [8] and Dinh [4].

Theorem 2.6. *Let $d=2,\ 0<b<1,\ 2-b<p<\infty$, and $s_p=1-\frac{2-b}{p}$. Suppose that $u(t)$ is the solution of (1.1) with initial data $u_0\in H^1$ satisfying*

$$E(u_0)^{s_p}M(u_0)^{1-s_p}<E(Q)^{s_p}M(Q)^{1-s_p},$$

and

$$\|\nabla u_0\|_{L^2}^{s_p}\|u_0\|_{L^2}^{1-s_p}<\|Q\|_{L^2}^{s_p}\|Q\|_{L^2}^{1-s_p}.$$

Then $u(t)$ is globally well-posed in H^1. Moreover, the solution satisfy $u\in L^q_{loc}(\mathbb{R},L^r(\mathbb{R}^2))$ for any Schrödinger admissible pair (q,r) and

$$\|\nabla u(t)\|_{L^2}^{s_p}\|u(t)\|_{L^2}^{1-s_p}<\|\nabla Q\|_{L^2}^{s_p}\|Q\|_{L^2}^{1-s_p}.$$

3. Scattering Criterion

Now, we prove a new Scattering Criterion for the solutions of the equation (1.1).

Lemma 3.1 (Scattering Criterion)**.** *Let $0<b<1$ and $2-b<p<\infty$. Suppose $u:\mathbb{R}_t\times\mathbb{R}^2\to\mathbb{C}$ is a radial solution to (1.1) such that*

$$\|u\|_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^2)}\le E. \tag{3.9}$$

There exist $\epsilon=\epsilon(E)>0$ and $R=R(E)>0$ such that if

$$\liminf_{t\to\infty}\int_{|x|\le R}|u(t,x)|^2dx\le\epsilon^2, \tag{3.10}$$

and

$$\int_0^T\int_{\mathbb{R}^2}|x|^{-b}|u(t,x)|^{p+2}dx\le T^\alpha, \tag{3.11}$$

where $0<\alpha<1$, then u scatters forward in time.

Remark 3.1. *In fact, the inequality (3.10) follows from (3.11). Indeed, fix $R\gg1$, we may have*

$$R^{-b}\int_0^T\int_{|x|\le R}|u|^{p+2}dxdt\le T^\alpha$$

By the fundamental theorem of calculus on interval $[0,T]$, there exists time sequence $\{t_n\}\to\infty$ such that

$$\int_{|x|\le R}|u(t_n)|^{p+2}dx\to0,\quad n\to\infty$$

Using the inequality $\|u\|_{L^2_x(|x|\le R)}\le R^{1-\frac{2}{p+2}}\|u\|_{L_x^{p+2}(|x|\le R)}$, then (3.10) follows.

Proof. By a standard continuity argument, Sobolev embedding, Lemma 2.3 and Lemma 2.4, it suffices to show $\|u\|_{S(\dot H^{s_p})}<\infty$. In fact, let $s_1=1-\frac{2}{q_1}-\frac{2}{\alpha q_1}$ and $s_1=1-\frac{2}{q_2}-\frac{2}{\alpha q_2}$(see [4] for details), thus $0<s_1,s_2<1$. By interpolation, we obtain

$$\|u\|_{L_t^{m_1}L_x^{q_1}},\ \|u\|_{L_t^{m_2}L_x^{q_2}}<\infty.$$

Then a standard continuity argument and Lemma 2.4, we can obtain the global solution $u(t)$ scattering in energy spaces.

Thus, by the Duhamel formula and continuity argument, we need to show

$$\|e^{i(t-T_0)\Delta}u(T_0)\|_{S(\dot H^{s_p},[T_0,\infty))}\ll1.$$

We rewrite $e^{i(t-T_0)\Delta}u(T_0)$ as

$$e^{i(t-T_0)\Delta}u(T_0) = e^{it\Delta}u_0 - iF_1(t) - iF_2(t),$$

where

$$F_j(t) := \int_{I_j} e^{i(t-s)\Delta}(|x|^{-b}|u|^p u)(s)ds,\ j = 1, 2,$$

with $I_1 = [0, T_0 - \epsilon^{-\theta}]$ and $I_2 = [T_0 - \epsilon^{-\theta}, T_0]$.

Let T_0 be large enough, then we have

$$\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_p},[T_0,\infty))} \ll 1.$$

Hence, it remains to show

$$\|F_j(t)\|_{S(\dot{H}^{s_p},[T_0,\infty))} \ll 1, \quad \text{for } j = 1, 2.$$

Estimation of $F_1(t)$: We may use the dispersive estimate, Hölder's inequality, and the assumption (3.11) to obtain

$$\begin{aligned}
\left\| \int_0^{T_0-\epsilon^{-\theta}} e^{i(t-s)\Delta}|x|^{-b}|u|^p u ds \right\|_{L_x^\infty} &\lesssim \int_0^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \||x|^{-b}|u|^p u\|_{L_x^1} ds \\
&\lesssim \int_0^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \left(\||x|^{\frac{-b}{p+2}}u\|_{L_x^{p+2}}^{\frac{p+2}{2}} \||x|^{\frac{-b}{2}}\|_{L_x^{\frac{4}{b+}}(B_1)} \||u|^{\frac{p}{2}}\|_{\frac{4}{2-p-}} \right) ds \\
&\quad + \int_0^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \left(\||x|^{\frac{-b}{p+2}}u\|_{L_x^{p+2}}^{\frac{p+2}{2}} \| \||x|^{\frac{-b}{2}}\|_{L_x^{\frac{4}{b-}}(B_1^c)} \||u|^{\frac{p}{2}}\|_{\frac{4}{2-p+}} \right) ds \\
&\lesssim \int_0^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \||x|^{\frac{-b}{p+2}}u\|_{L_x^{p+2}}^{\frac{p+2}{2}} ds \\
&\lesssim T_0^{\frac{\alpha}{2}}\epsilon^{\frac{\theta}{2}},
\end{aligned}$$

which yields

$$\left\| \int_0^{T_0-\epsilon^{-\theta}} e^{i(t-s)\Delta}|x|^{-b}|u|^p u ds \right\|_{L_{t,x}^\infty(T_0,\infty)\times\mathbb{R}^2} \lesssim T_0^{\frac{\alpha}{2}}\epsilon^{\frac{\theta}{2}}.$$

On the other hand, we may rewrite F_1 as

$$F_1(t) = e^{i(t-T_0+\epsilon^{-\theta})\Delta}u(T_0 - \epsilon^{-\theta}) - e^{it\Delta}u_0. \tag{3.12}$$

For any $(q, r) \in \Lambda_{s_p}$, we define the pair (q_0, r_0) by

$$q_0 = (1 - s_p)q, \quad r_0 = (1 - s_p)r$$

then $(q_0, r_0) \in \Lambda_0$. By Strichartz and the equality (3.12), we have

$$\|F_1(t)\|_{L_t^{q_0}L_x^{r_0}} \lesssim 1.$$

Thus, by interpolation, we get

$$\|F_1(t)\|_{S(\dot{H}^{s_p},[T_0,\infty))} \lesssim (T_0^\alpha \epsilon^\theta)^{\frac{1-s_p}{2}}.$$

Estimation of $F_2(t)$: By the inequality (2.5), Sobolev and radial Sobolev embedding, we get

$$\begin{aligned}\|F_2(t)\|_{S(\dot{H}^{s_p},[T_0,\infty))} \lesssim & \||x|^{-b}|u|^p u\|_{S'(\dot{H}^{-s_p})}\\ \lesssim & \|u\|^{\eta}_{L_t^\infty H_x^1}\|u\|^{p+1-\eta}_{S(\dot{H}^{s_p})}\\ \lesssim & \sup_{(q,r)\in\Lambda_{s_p}}\|u\|^{p+1-\eta}_{L_t^q L_x^r(I_2)}\\ \lesssim & \sup_{(q,r)\in\Lambda_{s_p}}\|u\|^{p+1-\eta}_{L_t^\infty L_x^r(I_2)}\epsilon^{\frac{-(p+1-\eta)\theta}{q}}.\end{aligned}$$

Let T_0 be large enough. By the assumption (3.10) and identity $\partial_t|u|^2 = -2\nabla\cdot Im(\bar{u}\nabla u)$, together with integration by parts and Cauchy-Schwartz, we deduce

$$\left|\partial_t\int_{I_2}\chi_R|u|^2ds\right| \lesssim \frac{1}{R}.$$

Thus, choosing $R \gg \epsilon^{-(2+\theta)}$, we find

$$\|\chi_R u\|_{L_t^\infty L_x^2(I_2\times\mathbb{R}^2)} \lesssim \epsilon.$$

Using radial Sobolev inequality and choosing large enough, we have

$$\|u\|_{L_t^\infty L_x^r(I_2)} \lesssim \epsilon^{\frac{1}{r}}.$$

By the definition of Λ_{s_p}, there exists $\delta, c > 0$ such that $\frac{2}{1-s_p}+\delta \le r \le c$. Thus

$$\begin{aligned}\|F_2(t)\|_{S(\dot{H}^{s_p},[T_0,\infty))} \lesssim & \sup_r\|u\|^{p+1-\eta}_{L_t^\infty L_x^r(I_2)}\epsilon^{\frac{-(p+1-\eta)\theta}{2}}\\ \lesssim & \epsilon^{(p+1-\eta)(\frac{1}{c}-\frac{\theta}{2})}.\end{aligned}$$

Choosing $\theta = \frac{1}{c},\ \gamma$ such that $\epsilon = T^{-c(1+\gamma)}$ and $\gamma+\alpha<1$, we get

$$\|e^{i(t-T_0)\Delta}u(T_0)\|_{S(\dot{H}^{s_p},[T_0,\infty))} \ll 1.$$

The proof is completed. □

4. Proof of Theorem 1.1

Throughout this section, we suppose $u(t)$ is a solution to equation (1.1) satisfying the hypotheses of Theorem 1.1. In particular, we have that u is global and uniformly bounded in H^1. Furthermore, we will see that there exists $\delta > 0$ so that

$$\sup_{t\in\mathbb{R}}\|u(t)\|_{L^2}^{1-s_p}\|\nabla u(t)\|_{L^2}^{s_p} \ < \ (1-2\delta)\|Q\|_{L^2}^{1-s_p}\|\nabla Q\|_{L^2}^{s_p}. \tag{4.13}$$

We will prove that the potential energy of energy-critical escapes to spatial infinity as $t\to\infty$.

Proposition 4.1 (Morawetz estimates)**.** *Let $d=2$, $0<b<2$, $2-b<p<\infty$ and u be a solution to the focusing equation* (1.1) *on the space-time slab $[0,T]\times\mathbb{R}^d$. Then there exists constant $0<\beta<1$ such that*

$$\int_0^T\int_{\mathbb{R}^d}|x|^{-b}|u(t,x)|^{p+2}dxdt < T^\beta. \tag{4.14}$$

Using Proposition 4.1, Remark 3.1, and the scattering criterion in Section 3, we can quickly prove Theorem 1.1.

Next, we prove Proposition 4.1 by a virial/Morawetz estimate. First, we recall the necessary coercivity property, that is, the inequality (4.13) holds on large balls. Set $\chi_R(x) := \chi(\frac{x}{R})$ for $R > 0$, where $\chi(x)$ is a radial smooth function such that

$$\chi(x) = \begin{cases} 1, & |x| \le \frac{1}{2}, \\ 0, & |x| > 1. \end{cases}$$

Lemma 4.2 (Coercivity on balls)**.** *There exists $R = R(\delta, M(u), Q) > 0$ sufficiently large so that*

$$\sup_{t\in\mathbb{R}} \|\chi_R u\|_{L_x^2}^{1-s_p} \|\chi_R u\|_{\dot{H}^1}^{s_p} < (1-\delta)\|Q\|_{L_x^2}^{1-s_p}\|Q\|_{\dot{H}^1}^{s_p}.$$

In particular, there exists δ' so that

$$\int |\nabla(\chi_R u)|^2 dx - \frac{b+p}{p+2}\int |x|^{-b}|\chi_R u|^{p+2}dx \ge \delta' \int |x|^{-b}|u|^{p+2}.$$

Proof. The proof follows from the conservation of mass and energy and the Gagliardo-Nirenberg inequality (2.8). We refer to [1] for an analogous proof. □

By a direct computation, we have the following Morawetz identity.

Lemma 4.3 (Morawetz identity)**.** *Let u be the solution of (1.1) and $a(x)$ be a smooth function. We denote the Morawetz action $M_a(t)$ by*

$$M_a(t) = 2\int_{\mathbb{R}^2} \nabla a(x) Im(\bar{u}\nabla u)(x)dx.$$

Then we have

$$\frac{d}{dt}M_a(t) = -\int \Delta^2 a(x)|u|^2 dx + 4\int \partial_{jk}a(x) Re(\partial_k u \partial_j \bar{u})dx \tag{4.15}$$

$$-\frac{2p}{p+2}\int \Delta a(x)|x|^{-b}|u|^{p+2}dx + \frac{4}{p+2}\int \nabla a(x)\cdot\nabla(|x|^{-b})|u|^{p+2}dx, \tag{4.16}$$

where the repeated indices are summed.

Using this identity, we may prove proposition 4.1, that's following:

Proof. Let $R \gg 1$ to be chosen later. We take $a(x)$ to be a radial function satisfying

$$a(x) = \begin{cases} |x|^2, & |x| \le R; \\ 3R|x|, & |x| > 2R, \end{cases} \tag{4.17}$$

and when $R < |x| \le 2R$, there holds

$$\partial_r a \ge 0, \ \ \partial_{rr} a \ge 0 \quad \text{and} \quad |\partial^\alpha a| \lesssim R|x|^{-|\alpha|+1}.$$

Here ∂_r denotes the radial derivative. Under these conditions, the matrix (a_{jk}) is non-negative. It is easy to verify that

$$\begin{cases} a_{jk} = 2\delta_{jk}, \quad \Delta a = 4, \quad \Delta\Delta a = 0, & |x| \le R, \\ a_{jk} = \frac{3R}{|x|}[\delta_{jk} - \frac{x_j x_k}{|x|^2}], \quad \Delta a = \frac{3R}{|x|}, \quad \Delta\Delta a = \frac{3R}{|x|^3}, & |x| > 2R. \end{cases}$$

Thus, we can divide $\frac{dM(t)}{dt}$ as follows:

$$\frac{dM(t)}{dt} =8\int_{|x|\leq R} |\nabla u|^2 - \frac{p+b}{p+2}|x|^{-b}|u|^{p+2}dx \tag{4.18}$$

$$+\int_{|x|>2R} \frac{-6pR}{|x|^{b+1}}|u|^{p+2}dx + \int \frac{-12Rb}{(p+2)|x|^{b+1}}|u|^{p+2}dx \tag{4.19}$$

$$+\int_{|x|>2R} \frac{-3R}{|x|^3}|u|^2dx + \int_{|x|>2R} \frac{12R}{|x|}|\not\nabla u|^2dx \tag{4.20}$$

$$+\int_{R<|x|\leq 2R} 4Re\bar{u}_i a_{ij}u_j + \mathcal{O}(\frac{R}{|x|^{b+1}}|u|^{p+2} + \frac{R}{|x|^3}|u|^2)dx, \tag{4.21}$$

where $\not\nabla$ denotes the angular derivative and subscripts denote partial derivatives.

Thus, we have

$$\int_{|x|<R} |\nabla u|^2dx - \frac{p+b}{p+2}\int_{|x|\leq R} |x|^{-b}|u|^{p+2}dx \lesssim \frac{dM(t)}{dt} + \frac{1}{R^\alpha}. \tag{4.22}$$

where $\alpha = \min\{2, b+\frac{p}{2}\}$. Using the following identity

$$\int \chi_R^2|\nabla u|^2 = \int |\nabla(\chi_R u)|^2) + \chi_R\Delta(\chi_R)|u|^2dx \tag{4.23}$$

we have

$$\|\chi_R u\|_{\dot{H}^1}^2 \lesssim \|u\|_{\dot{H}^1}^2 + \frac{1}{R^2}.$$

Thus, by radial Sobolev inequality, Sobolev embedding and Lemma 4.2, we get

$$\int_{|x|\leq \frac{R}{2}} |x|^{-b}|u|^{p+2}dx \lesssim \frac{dM(t)}{dt} + \frac{1}{R^\alpha}, \tag{4.24}$$

From the estimate (4.24) and radial Sobolev inequality, we have

$$\int_{\mathbb{R}^2} |x|^{-b}|u|^{p+2}dx \lesssim \frac{dM(t)}{dt} + \frac{1}{R^\alpha}.$$

Note that the uniform H^1-bounds for u, and the choice of the weight function a, we have

$$\sup_{t\in\mathbb{R}} |M(t)| \lesssim R.$$

Then, we can use the fundamental theorem of calculus on an interval $[0,T]$ to obtain

$$\int_0^T \int_{\mathbb{R}^2} |x|^{-b}|u|^{p+2}dxdt \lesssim R + \frac{T}{R^\alpha}.$$

Let $R = T^{\frac{1}{1+\alpha}}$, we have the estimate (4.14), which finishes the proof. □

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