The two-sided exit problem for a random walk on $\mathbb{Z}$ with infinite variance I

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Abstract

Let $S = (S_n)$ be an oscillatory random walk on the integer lattice $\mathbb{Z}$ with i.i.d. increments. Let $V_{\rm d}(x)$ be the renewal function of the strictly descending ladder height process for S. We obtain several sufficient conditions—given in terms of the distribution function of the increment $S_1 - S_0$—so that as $R \to \infty$

$$(*) \quad P[S \text{ leaves } [0,R] \text{ on its upper side} \,|\, S_0 = x] \sim V_{\rm d}(x)/V_{\rm d}(R)$$

uniformly for $0 \le x \le R$. When S is attracted to a stable process of index $0 < \alpha \le 2$ and there exists $\rho = \lim P[S_n > 0]$, the sufficient condition obtained are also necessary for $(*)$ and fulfilled if and only if $(\alpha \vee 1)\rho = 1$, and some asymptotic estimates of the probability on the left side of $(*)$ are given in case $(\alpha \vee 1)\rho \ne 1$.

KEYWORDS: exits from interval; relatively stable; infinite variance; renewal functions for ladder heights.

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1 Introduction

Let $S = (S_n, n = 0, 1, 2, \ldots)$ be a random walk (r.w.) on the integer lattice $\mathbb{Z}$ with i.i.d. increments and an initial point S_0 which is an unspecified integer. Let X be a generic random variable having the same law as the increment $S_1 - S_0$. For $x \in \mathbb{Z}$ let P_x denote the law of the r.w. S started at x and E_x the expectation by P_x; the subscript x is dropped from P_x and E_x if $x = 0$. We suppose throughout the paper that S is irreducible (as a Markov chain on $\mathbb{Z}$) and *oscillating* (i.e., the sequence (S_n) changes signs infinitely often with probability 1). For a subset $B \subset (-\infty, \infty)$ such that $B \cap \mathbb{Z} \ne \emptyset$, denote by σ_B (sometimes by σB when it appears as a subscript) the first time when the r.w. S visits B after time zero, namely $\sigma_B = \inf\{n \ge 1 : S_n \in B\}$. We shall mainly work with the negative half line $(\infty, -1]$ which we denote by Ω:

$$\Omega = (-\infty, -1].$$

Put

$$T = \sigma_\Omega.$$

This paper concerns the asymptotic form as $R \to \infty$ of the probability of

$$\Lambda_R = \{\sigma_{(R,\infty)} < T\},$$

the event the r.w. S exits from an interval $[0, R]$ through the right boundary. (R will always denote a positive integer). The classical result given in [14, Theorem 22.1] says that if the

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variance $\sigma^2 := EX^2$ is finite, then $P_x(\Lambda_R) - x/R \to 0$ uniformly for $0 \le x < R$. This can be refined to

$$P_x(\Lambda_R) \sim \frac{V_{\rm d}(x)}{V_{\rm d}(R)} \quad \text{uniformly for } 0 \le x \le R \text{ as } R \to \infty \tag{1.1}$$

as is given in [15, Proposition 2.2] (the proof is easy: see Section 3.1 of the present article). Here $V_{\rm d}(x)$ denotes the renewal function of the weakly descending ladder height process for S and $\sim$ means that the ratio of its two sides tends to unity. Our primary purpose is to find a sufficient condition for (1.1) to hold that applies to a wide class of r.w.'s on $\mathbb{Z}$ with $\sigma^2 = \infty$.

Rogozin [13] studied the corresponding problem for strictly stable processes and derived an explicit analytic expression of the probability of the process exiting the unit interval on the right side (together with the law of the overshoot distribution) and, as an application of the result, obtained the scaled limit of $P_x(\Lambda_R)$ for asymptotically stable r.w.'s (see Remark 1.3). There are a few other results concerning the asymptotic behaviour of $P_x(\Lambda_R)$. Griffin and MacConnel [7] gave a criterion for $P_x(\Lambda_{2x})$ to approach unity (see Remark 1.1). When X is in the domain of attraction of a stable law, Doney [4] derived the asymptotic form of $P(\Lambda_R)$. Kesten and Maller [10] gave an analytic condition for $\liminf P_{R/2}(\Lambda_R) \wedge [1 - P_{R/2}(\Lambda_R)] > 0$, i.e., for S started at the centre of a long interval to exit it either side of it with positive probability.

Let Z (resp. $\hat{Z}$) be the first ladder height of the strictly ascending (resp. descending ladder) process: $Z = S_{\sigma[1,\infty)}$, $\hat{Z} = S_T$, if $S_0 = 0$. Both Z and $\hat{Z}$ are proper random variables since the r.w. S is assumed to oscillate. The product $EZE|\hat{Z}|$ is finite if $\sigma^2 := EX^2 < \infty$, and infinite otherwise (cf [14, Section 17]). Z is said to be *relatively stable (r.s.)* if there is a norming constants B_n such that $(Z_1 + \cdots + Z_n)/B_n \to 1$ in probability, where Z_k are independent copies of Z. Let X be a generic random variable having the same law as the increment $S_1 - S_0$ and F be the distribution function of X. If $E|X| < \infty$, put

$$\eta_+(x) = \int_0^x (1 - F(t))dt, \quad m_+(x) = \int_0^x \eta_+(t)dt,$$

for $x \ge 0$; define η_- and m_- similarly with $F(-t)$ in place of $1 - F(t)$ and let $\eta = \eta_- + \eta_+$ and $m = m_- + m_+$. In the sequel we always assume

$$\sigma^2 = EX^2 = \infty.$$

Because of the oscillation of S, $EX = 0$ if $E|X| < \infty$. (Below, we shall write $EX = 0$ simply with the understanding that $E|X| < \infty$ is implicitly assumed.) A positive Borel function f on a neighbourhood of ∞ is said to be *slowly varying (s.v.)* (at infinity) if $\lim f(\lambda x)f(x) = 1$ for any $\lambda > 1$. Our main result is stated in Theorem 1 below and some results complementary to it are provided in Proposition 1.

Theorem 1. *Let $\sigma^2 = \infty$. In order for (1.1) to be true each of the following are sufficient*

(C1) *$EX = 0$ and $m_+(x)/m(x) \to 0$ as $x \to \infty$.*

(C2) *$EX = 0$ and m is s.v. as $x \to \infty$.*

(C3) *Z is r.s. and. $V_{\rm d}$ is s.v.*

(C4) *$F(-x)/(1 - F(x) \to 0$ as $x \to \infty$ and $1 - F(x)$ is regularly varying at infinity with index $-\alpha$, $0 < \alpha < 1$.*

The sufficiency of (C1) is proved by the present author in [17, Proposition 47]. The primary objective of this note is to verify the sufficiency of (C2) to (C4); proof will be made independently of the result under (C1).

It is warned that the second condition in (C4) does not mean that the left tail $F(-x)$ may get small arbitrarily fast as $x \to \infty$, the random walk S being supposed to oscillate so that according to [5] the growth condition $\int_{-\infty}^{0} \big(|t| / \int_0^{|t|}(1-F(s))ds\big)dF(t) = \infty$ must be satisfied (valid for general r.w.s with $E[X; X > 0] = \infty$). In case (C3) (1.1) is of interest only when $x/R \to 0$, for if $V_{\rm d}$ is s.v., (1.1) entails $P_x(\Lambda_R) \to 1$ whenever x/R is bounded away from zero.

Condition (C2) holds if and only if F belongs to the domain of attraction of a centred normal law [17, Appendix A]. If F is attracted to a stable law of exponent α and there exists $\rho := \lim P[S_n > 0]$, then (C3) is equivalent to $\alpha = \rho = 1$ (see Remark 1.6 and [17, Remark 11], or [18].)

In what follows we shall omit '$x \to \infty$' or '$R \to \infty$' when it is obvious.

Remark 1.1. Put

$$A(x) = \int_0^x [1 - F(t) - F(-t)]dt \quad \text{and} \quad H(x) = P_x[|X| > x].$$

Then condition (C3) holds if

$$\frac{A(x)}{xH(x)} \longrightarrow \infty. \tag{1.2}$$

Indeed, (1.2) is equivalent to the positive relative stability of X according to [9, p.1478] and the latter implies that Z is r.s. [12, Theorem10]. Moreover the condition

$$A(x)/xF(-x) \longrightarrow \infty, \tag{1.3}$$

apparently weaker than (1.2), is equivalent to $\lim P_x(\Lambda_{2x}) = 1$ [9, p.1477] and, therefore, entails that $V_{\rm d}$ is s.v. (see (2.3 below). In view of Theorem 1, (C3) implies $P_x(\Lambda_{2x}) \to 1$, hence (1.3), whereas it is not clear if (1.2) holds under (C3).

Remark 1.2. If $U_{\rm d}$ denotes the renewal function of the strictly descending ladder height process, then $U_{\rm d}(x) = V_{\rm d}(x)/V_{\rm d}(0)$. [The renewal function of the strictly ascending ladder height process will be denoted by $U_{\rm a}(x)$.] We write v_0 for $V_{\rm d}(0)$, which is given by

$$1/v_0 = P[S_{\sigma(-\infty,0]} < 0] = P[S_{\sigma[0,\infty)} > 0] = \exp\Big\{ -\sum k^{-1}p^k(0)\Big\}$$

(cf., e.g., [6, Sections XII.9, XVIII.3]). Put

$$\ell^*(x) = \int_0^x P[Z > t]dt \quad \text{and} \quad \hat{\ell}^*(x) = \frac{1}{v_0}\int_0^x P[-\hat{Z} > t]dt.$$

Then

$$V_{\rm d}(x) \sim \begin{cases} a(x)\ell^*(x) & \text{in case} \quad \text{(C1)}, \\ x/\hat{\ell}^*(x) & \text{in case} \quad \text{(C2)}, \\ \left(\alpha \displaystyle\int_x^\infty \frac{U_{\rm a}(t)F(-t)}{t}dt\right)^{-1} & \text{in cases} \quad \text{(C3), (C4)}, \end{cases} \tag{1.4}$$

and in case (C4) (as for (C3)) $V_{\rm d}$ is s.v. so that $P_{R/2}(\Lambda_R) \to 1$. Here $\alpha = 1$ in case (C3) and

$$a(x) = \sum_{n=0}^{\infty} \big(P[S_n = 0] - P[S_n = -x]\big),$$

the potential function of S (if S is recurrent) (see [17, Section 7.3], [12] and Lemma 3.1 for the first, second and third formula, respectively). In each case ℓ^* or $\hat{\ell}^*$ or $F(-\cdot)U_{\rm a}$ is s.v. ($U_{\rm a}$ denotes the renewal function of the strictly ascending ladder height process, as mentioned before.) Under (C1) $a(x) \asymp x/m(x)$ (cf. [17]), where $\asymp$ means that the ratio of its two sides is bounded away from zero and infinity. If (1.2) holds, the first formula of (1.4) has the following extension (see Remark 3.4): if $p := \lim[1-F(x)]/H(x), q = 1-p$, F is recurrent and $p \neq 1/2$ (in addition to (1.2)), then

$$V_{\rm d}(x) \sim (1 - q^{-1}p)a(x)\ell^*(x).$$

In all the cases (C1) to (C4) either Z is r.s. or $P[S_n > 0] \to 1$. (the latter is equivalent to $P_x(\Lambda_{2x}) \to 1$ under the present setting (cf. [9]; see [17, Theorem 8(i)]) for the case (C1)). Below we suppose the asymptotic stability condition

$$\text{(AS)} \begin{cases} \text{a. } X \text{ is attracted to a stable law of exponent } 0 < \alpha \le 2. \\ \text{b. } EX = 0 \text{ if } E|X| < \infty. \\ \text{c.} \quad \text{there exists} \quad \rho := \lim P[S_n > 0], \end{cases}$$

and derive some upper and lower bounds of $P_x(\Lambda_R)$. Note that the assumption of S being oscillating is always in force.

Suppose condition (ASab)—the conjunction of (ASa) and (ASb)—to hold with $\alpha < 2$. It then follows that

$$1 - F(x) \sim pL(x)x^{-\alpha} \quad \text{and} \quad F(-x) \sim qL(x)x^{-\alpha} \tag{1.5}$$

for some s.v. function L and constant $p = 1-q \in [0,1]$. Let the positive numbers c_n be chosen so that $\lim H(c_n)n = (2-\alpha)/\alpha$. For $\alpha \neq 1$, (ASc) is valid with $[1-\alpha^{-1}] \vee 0 \le \rho \le \alpha^{-1} \wedge 1$ and S_n/c_n converges in law to a no-degenerate stable variable. For $\alpha = 1$, (ASc) holds with $\rho = 1$ at least if either $p < 1/2, EX = 0$ or $p > 1/2, E|X| = \infty$; if $p = 1/2$, in order that S_n/c_n converges in law to a non-degenerate variable it is necessary and sufficient that (ASc) holds with $0 < \rho < 1$ and this is equivalent to the existence of $\lim nE[\sin X/c_n] \in \mathbb{R}$. (Cf. [6, Section XVII.5].)

If (AS) holds, then it follows that $\alpha\rho \le 1$ if $\alpha = 2$; Z is r.s. if and only if $\alpha\rho = 1$; for $\alpha = 2$, $\rho = 1/2$; for $1 < \alpha < 2$, (C1) $\Leftrightarrow$ $p = 0$ $\Leftrightarrow$ $\alpha = \rho = 1$, and that

$$\text{(C2)} \Leftrightarrow \alpha = 2; \ \text{(C3)} \Leftrightarrow \alpha = \rho = 1; \ \text{(C4)} \Leftrightarrow \alpha < 1 = p \Rightarrow \rho = 1; \tag{1.6}$$

in particular, either one of (C1) to (C4) holds if and only if $(\alpha \vee 1)\rho = 1$. We consider the case $(\alpha \vee 1)\rho < 1$ in the next proposition which combined with Theorem 1 shows that (1.1) holds if and only if $(\alpha \vee 1)\rho = 1$.

Proposition 1. *Suppose (AS) to hold with $0 < \alpha < 2$. Let δ be a constant arbitrarily chosen so that $\frac{1}{2} < \delta < 1$.*

(i) *If $0 < (\alpha \vee 1)\rho < 1$ (equivalently, $p > 0$ for $\alpha > 1$ and $0 < \rho < 1$ for $\alpha \le 1$), then there exist constants $\theta_* > 0$ and $\theta^* < 1$ such that for all sufficiently large R,*

$$\theta_* \le \frac{P_x(\Lambda_R)V_{\rm d}(R)}{V_{\rm d}(x)} \le \theta^* \qquad \text{for} \quad 0 \le x \le \delta R. \tag{1.7}$$

(ii) *Let $\alpha \le 1$. In (1.7), one can choose θ_* and θ^* so that $\theta^* \to 0$ as $\rho \to 0$ and $\theta_* \to 1$ as $\rho \to 1$. [The statement "$\theta^* \to 0$ as $\rho \to 0$" means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\theta^* < \varepsilon$ for any 'admissible' F with $0 < \rho < \delta$, and similarly for "$\theta^* \to 1$ as $\rho \to 1$".] If $\rho = 0$, then $P_x(\Lambda_R)V_{\rm d}(R)/V_{\rm d}(x) \to 0$ uniformly for $0 \le x < \delta R$, and $V_{\rm d}(x) \sim x^\alpha/\hat{\ell}(x)$ for some s.v. function $\hat{\ell}$. [For $\alpha = 1$, one can take $\hat{\ell}^*$ for $\hat{\ell}$ (see (4.4).]*

REMARK 1.3. Suppose that (AS) holds and that $0 < \rho < 1$ if $\alpha \neq 1$ (i.e., limiting stable law is strictly stable). Let $Y = (Y_t)_{t\geq 0}$ be the limiting stable process (which we suppose to be defined on the same probability space as S), denote by Λ_r^Y, $r > 0$ the corresponding event for Y:

$$\Lambda_r^Y = \big\{\text{the first exit of } Y \text{ from the interval } [0, r] \text{ is on the right side}\big\},$$

and put $Q(\xi) = P(\Lambda_1^Y \,|\, Y_0 = \xi)$ $(0 < \xi < 1)$. Rogozin [13] established the overshoot law given below in (1.9) and from it derived an analytic expression of $Q(\xi)$ which, if $0 < \alpha\rho < 1$, must read

$$Q(\xi) = \frac{1}{B(\alpha\rho, \alpha\hat{\rho})} \int_0^\xi t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}dt \quad (0 < \xi < 1), \tag{1.8}$$

where $\hat{\rho} = 1 - \rho$ and $B(\cdot, \cdot)$ is the beta function. (For $\alpha\rho = 1$ when (1.1) holds, $Q(\xi) = \xi^{\alpha-1}$; (1.8) is valid for $\alpha\rho = 1$ if $\rho > 1$ but of course not if $\alpha = 1$.) By the functional limit theorem one deduces that $P_{\xi c_n}(\Lambda_{c_n}) \to P[\Lambda_1^Y \,|\, Y_0 = \xi]$ for each $0 < \xi < 1$. This convergence, though uniform since $P_x(\Lambda_R)$ is monotone in x and $Q(\xi)$ is continuous, does not imply (1.7).

Let $Z^Y(r)$ denote the overshoot of Y as Y crosses the level r from the left to the right of it (the analogue of $Z(R)$ given at the beginning of the next section). Then the overshoot law of Rogozin [13] mentioned above may read: if $0 < \alpha\rho < 1$, then for $0 < \xi < r$, $\eta > 0$,

$$P\big[Z^Y(1) < \eta, \Lambda_1^Y \big| Y_0 = \xi\big] = \frac{\sin \pi\alpha\rho}{\pi}(1-\xi)^{\alpha\rho}\xi^{\alpha\hat{\rho}} \int_0^\eta \frac{t^{-\alpha\rho}(t+1)^{-\alpha\hat{\rho}}}{t+1-\xi}dt. \tag{1.9}$$

The verification of (1.8), given in [13], does not seem to be finished and the formula obtained there differs from (1.8)—ρ and $\hat{\rho}$ are interchanged. Here we provide proof of (1.8), based on (1.9). Suggested by proof of Lemma 1 of Kesten [8] we prove instead of (1.8)

$$P(\Lambda_{1+c}^Y \,|\, Y_0 = 1) = \frac{1}{B(\alpha\rho, \alpha\hat{\rho})} \int_0^{1/(1+c)} t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}dt \qquad (c > 0), \tag{1.10}$$

which is equivalent to (1.8). By (1.9) the conditional probability on the LHS is expressed as

$$\frac{\sin \pi\alpha\rho}{\pi} c^{\alpha\rho} \int_0^\infty \frac{t^{-\alpha\rho}(t+1+c)^{-\alpha\hat{\rho}}}{t+c}dt.$$

On changing the variable $t = c/s$ this yields

$$P(\Lambda_{1+c}^Y \,|\, Y_0 = 1) = \frac{\sin \pi\alpha\rho}{\pi} \int_0^\infty \frac{s^{\alpha-1}(s+(1+s)c)^{-\alpha\hat{\rho}}}{1+s}ds.$$

Using the identity $\int_0^\infty s^{\gamma-1}(1+s)^{-\nu}ds = B(\gamma, \nu-\gamma)$ $(\nu > \gamma > 0)$ one sees from the above expression that the two sides of (1.10) equal 1 for $c = 0$ and their derivatives with respect to c coincide.

REMARK 1.4. With the help of our Theorem 1 we obtain from (1.9) (cf. also [14]) that there exists $Q(\xi, \eta) = \lim_{R\to\infty} P_{\xi R}[Z(R) > \eta R, \Lambda_R]$ $(0 \leq \xi \leq 1, \eta > 0)$,

$$Q(\xi) = \xi^{\alpha-1} \text{ or } 0 \quad \text{according as } \alpha\rho = 1 \text{ or } 0 \quad (\xi < 1), \text{ and}$$

$$Q(\xi, \eta) = \begin{cases} 0 & (\xi < 1, \eta > 0) \quad \text{if} \quad \alpha\rho \in \{0, 1\}, \\ C(\xi)\eta^{-\alpha}\{1 + o(1)\} & \text{as } \eta \to \infty \quad \text{if} \quad 0 < \alpha\rho < 1. \end{cases}$$

where $C(\xi)$ is a continuous function positive for $0 < \xi < 1$ and $C(0) = C(1) = 0$.

As mentioned above, the formula (1.8) yields the explicit asymptotic form of $P_x(\Lambda_R)$ if x/R is bounded away from zero provided $0 < \rho < 1$ if $\alpha = 1$, but when $x/R \to 0$ it says only $P_x(\Lambda_R) \to 0$. One may especially be interested in the case $x = 0$. Doney [4, Corollary 3] obtained under (AS) with $0 < \rho < 1$ a functional limit theorem of the scaled r.w. conditioned on various events (that depend on the behaviour of the r.w. only up to T), and derived, as an application of it, that

$$P\big[\max_{n<T} S_n \geq b_n\big] = P(\Lambda_{b_n}) \sim c/n \tag{1.11}$$

where c is a positive constant and $b_1, b_2, \ldots$ are norming constants determined by the relation $nP[-\hat{Z} > b_n] \to 1$ if $\alpha\hat{\rho} < 1$ and $\int_0^{b_n} P[-\hat{Z} > t]dt/b_n \sim b_n/n$ if $\alpha\hat{\rho} = 1$. If (1.1) holds, taking $x = 0$ therein gives $P(\Lambda_R) \sim v_0/V_{\rm d}(R)$. [It holds that $V_{\rm d}(b_n)/n$ tends to a positive constant (see (2.4) of the next section) and one can easily ascertain that the equivalence in (1.11) may be written as $P(\Lambda_R) \sim c/V_{\rm d}(R)$.]

The corollary below gives the corresponding consequence of Proposition 1 which improves on the result of [4] mentioned right above by extending it to the case $\rho \in \{0, 1\}$ and providing information how the limit depend on ρ.

Suppose (AS) to hold. We distinguish the following three cases:

Case *I*: $(\alpha \vee 1)\rho = 1$.

Case *II*: $0 < (\alpha \vee 1)\rho < 1$.

Case *III*: $\rho = 0$

If $\alpha = 2$, *I* is always the case. For $1 < \alpha < 2$, *I* or *II* is the case according as $p = 0$ or $p > 0$. If $\alpha = 1$, we have Case *I* only if $p = 1/2$, otherwise all the cases *I* to *III* are possible. For $\alpha < 1$, *I* or *III* is the case according as $p = 1$ or $p = 0$, otherwise we have Case *II*.

The following result is obtained as a corollary of Theorem 1 and Proposition 1 except for the case $\alpha > 1$ of its last assertion.

Corollary 1. *Suppose (AS) to hold. Then*

(i) *there exists* $c = \lim P(\Lambda_R)V_{\rm d}(R)$*;*

(ii) $c = 1$ *in Case I,* $0 < c < 1$ *in Case II and* $c = 0$ *in Case III;*

(iii) $c \to 0$ *as* $\rho \to 0$ *(necessarily* $\alpha \leq 1$*), and* $c \to 1$ *as* $\rho \to 1/(\alpha \vee 1)$.

Proof. As to the proof of (i) and (ii) Cases *I* and *III* are covered by Theorem 1 and by the second half of Proposition 1(ii), respectively, and Case *II* by the above mentioned result of [4] combined with Proposition 1. The assertion (iii) follows from Proposition 1(ii) if $\alpha \leq 1$. In case $1 < \alpha < 2$, we need to use a result from [19]. There we obtain the asymptotic estimates of $P_x^{\Omega}[\sigma_R < \infty]$ for $x < R$ in Proposition 1.4(i) of [19] (deduced from the estimates of $u_{\rm a}$ and $v_{\rm d}$ gotten independently of the preset work), which entails that as $R \to \infty$

$$\frac{P(\Lambda_R)V_{\rm d}(R)}{v_0} \geq \frac{P[\sigma_{\{R\}} < T]V_{\rm d}(R)}{v_0} \longrightarrow \frac{\alpha - 1}{\alpha(1-\rho)}. \qquad \square$$

Remark 1.5. (a) In [4] $EX = 0$ is assumed if $\alpha = 1$. However, by examining the arguments therein, one sees that this assumption is superfluous (not only for Corollary 3 of [4] but for the main theorem of it), so that (1.11) is valid under (AS) with $0 < \rho < 1$.

(b) In Corollary 3 of [4] some related results are stated, which are paraphrased as follows: $P(\Lambda_R) \sim c/R$ with $c > 0$ if and only if $E|\hat{Z}| < \infty$; and $\int_0^x P(\Lambda_t)dt$ is s.v. if $\alpha\hat{\rho} = 1$ and $0 < \alpha < 2$. These follow immediately from our Corollary 1 (recall that $V_{\rm d}(x) \sim x/\hat{\ell}^*(x)$, $\lim \hat{\ell}^*(x) = E|\hat{Z}| \leq \infty$ and ℓ^* is s.v. if $\alpha\rho = 1$).

The present paper has been written simultaneously with [17]. The two are closely related, and we make use of some results from one in the other: we shall apply Proposition 40 of [17] in the proof of Theorem 1 (in case (C2)) of the present one, while Lemmas 3.7 and 4.1 of the present one are used in Section 7.4 of [17]. These however does not give rise to any risk of circular arguments.

The rest of this paper is organised as follows. In Section 2, we state several known facts and provide a lemma; these will be fundamental in the later discussions. Proof of Theorem 1 is given in Sections 3, where we prove lemmas as to miscellaneous matters. In Section 4 we prove Proposition 1 by further showing several lemmas.

2 Preliminaries

Let v_{d} denote the renewal sequence for the weakly descending ladder height process: $v_{\mathrm{d}}(x) = V_{\mathrm{d}}(x) - V_{\mathrm{d}}(x-1)$ and $v_{\mathrm{d}}(0) = v_0$. Here and in the sequel $V_{\mathrm{d}}(x)$ is set to be zero for $x \leq -1$. Similarly let $U_{\mathrm{a}}(x)$ and $u_{\mathrm{a}}(x)$ be the renewal function and sequence for the strictly ascending ladder height process. We shall be also concerned with the overshoot which we define by

$$Z(R) = S_{\sigma[R+1,\infty)} - R.$$

The function V_{d} is harmonic for the r.w. S killed as it enters Ω in the sense that

$$\sum_{y=0}^{\infty} V_{\mathrm{d}}(y)p(y-x) = V_{\mathrm{d}}(x) \qquad (x \geq 0), \tag{2.1}$$

where $p(x) = P[X = x]$ [14, Proposition 19.5], so that the process $M_n = V_{\mathrm{d}}(S_{n\wedge T})$ is a martingale under P_x for $x \geq 0$. By the optional stopping theorem one deduces

$$E_x[V_{\mathrm{d}}(S_{\sigma(R,\infty)}); \Lambda_R] = V_{\mathrm{d}}(x). \tag{2.2}$$

Indeed, on passing to the limit in $E_x[M_{n\wedge\sigma(R,\infty)}] = V_{\mathrm{d}}(x)$ Fatou's lemma shows that the expectation on the LHS is less than or equal to $V_{\mathrm{d}}(x)$, which fact in turn shows that the martingale M_n is uniformly integrable since $M_{n\wedge\sigma_{(R,\infty)}}$ is bounded by the summable random variable $V_{\mathrm{d}}(R) \vee M_{\sigma_{(R,\infty)}}$. Obviously the second expectation in (2.2) is not less than $V_{\mathrm{d}}(R)P_x(\Lambda_R)$ so that

$$P_x(\Lambda_R) \leq V_{\mathrm{d}}(x)/V_{\mathrm{d}}(R). \tag{2.3}$$

If either V_{d} or $\int_0^x P[-\hat{Z} > t]dt$ is regularly varying, it follows [2, Eq(8.6.6)] that

$$\frac{V_{\mathrm{d}}(x)}{xv_0}\int_0^x P[-\hat{Z} > t]dt \longrightarrow \frac{1}{\Gamma(1+\alpha\hat{\rho})\Gamma(2-\alpha\hat{\rho})}. \tag{2.4}$$

For a non-empty $B \subset \mathbb{Z}$ we define the Green function $g_B(x,y)$ of the r.w. killed as it hits B by

$$g_B(x,y) = \sum_{n=0}^{\infty} P_x[S_n = y, n < \sigma_B]. \tag{2.5}$$

(Thus if $x \in B$, $g_B(x,y)$ is equal to $\delta_{x,y}$ for $y \in B$ and to $\sum p(z-x)g_B(z,y)$ for $y \notin B$.) We shall repeatedly apply the formula

$$g_\Omega(x,y) = \sum_{k=0}^{x\wedge y} v_{\mathrm{d}}(x-k)u_{\mathrm{a}}(y-k) \quad \text{for } x,y \geq 0 \tag{2.6}$$

[14, Propositions 18.7, 19.3]. It follows that for $x \geq 1$,

$$P[\hat{Z} = -x] = \sum_{y=0}^{\infty} g_\Omega(0, y)p(-x - y) = v_0 \sum_{y=0}^{\infty} u_{\rm a}(y)p(-x - y) \tag{2.7}$$

and, by the duality relation $g_{[1,\infty)}(-x, -y) = g_\Omega(y, x)$,

$$P[Z = x] = \sum_{y=0}^{\infty} g_{[1,\infty)}(0, -y)p(x + y) = \sum_{y=0}^{\infty} v_{\rm d}(y)p(x + y). \tag{2.8}$$

Let ℓ^* be as in Remark 1.2, namely $\ell^*(x) = \int_0^x P[Z > t]dt$. It is known that Z is r.s. if and only if ℓ^* is s.v. [12] and if this is the case

$$u_{\rm a}(x) \sim 1/\ell^*(x) \tag{2.9}$$

[17, Appendix(B)], [16].

Lemma 2.1. *For $0 \leq x \leq R$,*

$$\sum_{y=0}^{R} g_\Omega(x, y) \leq V_{\rm d}(x)U_{\rm a}(R).$$

Tale a positive constant $\delta < 1$ arbitrarily. If $U_{\rm a}$ is regularly varying with positive index, then there exists a constant $c > 0$ such that for $0 \leq x \leq \delta R$

$$\sum_{y=0}^{R} g_\Omega(x, y) \geq c\, V_{\rm d}(x)U_{\rm a}(R),$$

and if $U_{\rm a}$ is s.v., then uniformly for $0 \leq x < \delta R$

$$\sum_{y=0}^{R} g_\Omega(x, y) \sim V_{\rm d}(x)U_{\rm a}(R).$$

Proof. Split the sum on the LHS of the inequality of the lemma at x. By (2.6) and (2.9) the sum over $y > x$ and that over $y \leq x$ are equal to, respectively,

$$\sum_{y=x+1}^{R} \sum_{k=0}^{x} v_{\rm d}(k)u_{\rm a}(y - x + k) = \sum_{k=0}^{x} v_{\rm d}(k)\big[U_{\rm a}(R - x + k) - U_{\rm a}(k)\big]$$

and

$$\sum_{k=0}^{x} \sum_{y=k}^{x} v_{\rm d}(x - y + k)u_{\rm a}(k) = \sum_{k=0}^{x} \big[V_{\rm d}(x) - V_{\rm d}(k - 1)\big]u_{\rm a}(k).$$

Summing by parts yields $\sum_{k=0}^{x} \big[V_{\rm d}(x) - V_{\rm d}(k - 1)\big]u_{\rm a}(k) = \sum_{k=0}^{x} v_{\rm d}(k)U_{\rm a}(k)$, so that

$$\sum_{y=0}^{R} g_\Omega(x, y) = \sum_{k=0}^{x} v_{\rm d}(k)U_{\rm a}(R - x + k)$$

by which the assertions of the lemma follow immediately. □

3 Proof of Theorem 1

Proof of Theorem 1 is given in two subsections, the first for the case (C2) and the second for cases (C3) and (C4).

3.1. Case (C2).

Suppose $EX = 0$ and recall $\eta_{\pm}(x) = \int_x^{\infty} P[\pm X > t]dt$ and $\eta = \eta_- + \eta_+$. It follows that if $\lim x\eta_+(x)/m(x) = 0$, then Z is r.s. [17, Proposition 40], so that $u_{\rm a}(x) \sim 1/\ell^*(x)$ is s.v. Thus (C2), which is equivalent to $\lim x\eta(x)/m(x) = 0$, entails both $v_{\rm d}$ and $u_{\rm a}$ are s.v., so that $\sum_{k=0}^{y} v_{\rm d}(k)u_{\rm a}(k) \sim yv_{\rm d}(y)u_{\rm a}(y) \sim V_{\rm d}(y)u_{\rm a}(y)$ $(y \to \infty)$ and one can easily deduce that as $R \to \infty$

$$g_{\Omega}(x, R) \sim \sum_{k=0}^{x} \frac{v_{\rm d}(k)}{\ell^*(R - x + k)} \sim \frac{V_{\rm d}(x)}{\ell^*(R)} \quad \text{uniformly for } \ 0 \le x < R.$$

This leads to the lower bound

$$P_x(\Lambda_R) \ge P_x[\sigma_{\{R\}} < T] = \frac{g_{\Omega}(x, R)}{g_{\Omega}(R, R)} = \frac{V_{\rm d}(x)}{V_{\rm d}(R)}\{1 + o(1)\}, \tag{3.1}$$

which combined with (2.3) shows (1.1) as desired.

3.2. Cases (C3) and (C4).

Each of conditions (C3) and (C4) implies that $U_{\rm a}(x)/x^{\alpha}$ (necessarily $0 < \alpha \le 1$) is s.v. and that $limP[S_n > 0] = 1$. Let ℓ be a s.v. function such that

$$U_{\rm a}(x) \sim x^{\alpha}/\ell(x) \quad \text{if } \ \alpha < 1 \quad \text{and} \quad \text{if } \ \alpha = 1, \quad \ell = \ell^*. \tag{3.2}$$

Here and in the sequel $\alpha = 1$ in cases (C3). We then bring in the function

$$\ell_{\sharp}(t) = \alpha \int_t^{\infty} \frac{s^{\alpha-1}F(-s)}{\ell(s)}ds \quad (t > 0), \tag{3.3}$$

where the summability of the integral is assured by $U_{\rm a}(x) = \sum_{-\infty}^{0} U_{\rm a}(y)p(y + x)$, the dual relation of (2.1). In cases (C1) and (C2) we have $P_x[\sigma_{\{R\}} < T] \sim V_{\rm d}(x)/V_{\rm d}(R)$ which combined with (2.3) entails (1.1), but in cases (C3) and (C4) this equivalence does not generally hold (see Proposition 1.4 of [19]) and for proof of (1.1) we shall make use of (2.2) not only for the upper bound but for the lower bound.

Lemma 3.1. *Suppose that either (C3) or (C4) holds. Then $\ell_{\sharp}(x)$ is s.v., and*

$$(a)\ \ P[-\hat{Z} \ge x] \sim v_0\ell_{\sharp}(x) \quad \textit{and} \quad (b)\ \ V_{\rm d}(x) \sim 1/\ell_{\sharp}(x).$$

Proof. We have only to show (a), in addition to $\ell_{\sharp}$ being s.v., (a) entailing (b) in view of (2.4) applied with $\alpha\hat{\rho} = 0$. If (C3) or (C4) holds, then $V_{\rm d}$ is s.v. (cf. [16] in case (C4)), hence $P[-\hat{Z} \ge x]$ is also s.v. [12]. Because of this fact together with the expression $P[-\hat{Z} \ge x] = v_0 \sum_{y=0}^{\infty} u_{\rm a}(y)F(-x-y)$, the assertion follows from a general result involving regularly varying functions that we state and verify in Lemma 5.1 of Appendix (A) in order not to break the continuation of the discourse. [Under (C3), Lemma 3.1 is a special case of Lemma 50 of [17] where a different proof is found.] □

Lemma 3.2. (i) *Let (C3) hold. Then for any* $\delta > 0$,

$$P[Z > x] \geq V_{\rm d}(x)[1 - F(x + \delta x)]\{1 + o(1)\}. \tag{3.4}$$

If $\limsup[1 - F(\lambda x)]/[1 - F(x)] < 1$ *for some* $\lambda > 1$ *in addition, then*

$$P[Z > x] \leq V_{\rm d}(x)[1 - F(x)]\{1 + o(1)\}. \tag{3.5}$$

(ii) *If (C4) holds, then* $P[Z > x] \sim V_{\rm d}(x)[1 - F(x)]$.

Proof. First we prove (ii). Suppose (C4) to hold and let $1 - F(x) \sim L_+(x)/x^\alpha$ with a s.v. function L_+. By (2.8)

$$P[Z > x] = \sum_{y=0}^{\infty} v_{\rm d}(y)\big[1 - F(x + y)\big]. \tag{3.6}$$

For $M > 1$, the sum over $y \leq Mx$ is asymptotically equivalent to

$$\sum_{y=0}^{Mx} \frac{v_{\rm d}(y)L_+(x + y)}{(x + y)^\alpha} \sim L_+(x) \sum_{y=0}^{Mx} \frac{v_{\rm d}(y)}{(x + y)^\alpha} = \frac{V_{\rm d}(x)L_+(x)}{x^\alpha}\big\{1 + o(1)\big\},$$

since $V_{\rm d}$ is s.v. owing to Lemma 3.1 and hence $\sum_{y=\varepsilon x}^{Mx} v_{\rm d}(y) = o(V_{\rm d}(x))$ for any $\varepsilon > 0$. Choosing L_+ so that $L'_+(x) = o(L_+(x)/x)$ we also see that the sum over $y > Mx$ is at most $V_{\rm d}(x)[1 - F(Mx)]$. Since M can be arbitrarily large, this concludes the asserted equivalence.

The inequality (3.4) is obtained by restricting the range of summation in (3.4) to $y \leq \delta x$. (3.5) follows from

$$P[Z > x] \leq V_{\rm d}(x)[1 - F(x)] + \sum_{y=x}^{\infty} v_{\rm d}(y)\big[1 - F(y)\big]. \tag{3.7}$$

Indeed, under the assumption for (3.5), the sum of the infinite series is $o\big(V_{\rm d}(x)[1 - F(x)]\big)$ according to a general result, Lemma 5.2 of Appendix (A), concerning s.v. functions. □

REMARK 3.1. Lemma 3.2 says that if the positive tail of F satisfies a mild regularity condition, then $P[Z > x] \sim [1 - F(x)]\ell_\sharp(x)$ under (C3). For the integrals $\int_0^x [1 - F(t)]/\ell_\sharp(t)dt$ and $\int_0^x P[Z > t]\ell_\sharp(t)dt$, however, we have the corresponding relations with no additional condition assumed (see Lemmas 3.5 and 3.6).

The slow variation of $\ell^*(x) = \int_0^x P[Z > t]dt$ implies $P[Z > x] = o(\ell^*(x)/x)$ so that

$$Z \text{ is r.s.} \implies \lim U_{\rm a}(x)P[Z > x] = 0. \tag{3.8}$$

Lemma 3.3. (i) *Under (C3)*

$$V_{\rm d}(x)U_{\rm a}(x)H(x) \to 0. \tag{3.9}$$

(ii) *Under (C4),* $V_{\rm d}(x)U_{\rm a}(x)[1 - F(x)]\big] \to (\sin \pi\alpha)/\pi\alpha$.

One may compare the results above with the known one under (AS) with $0 < \rho < 1$ stated in (4.11) below; (i) will be refined in Lemma 4.1(iii) under (AS).

Proof. Under (C4), Lemma 3.2(ii) says $P[Z > x] \sim V_{\rm d}(x)[1 - F(x)]$ so that by Lemma 3.1 $P[Z > x]$ is regularly varying with index $-\alpha$, which implies that $P[Z > x]U_{\rm a}(x)$ approaches $(\sin \pi\alpha)/\pi\alpha$ (cf. [2, Eq(8.6.4)]. Hence we have (ii).

As for (i), on recalling the definition of $\ell_\sharp$, the slow variation of $\ell_\sharp$ entails $xF(-x)/\ell^*(x) = o(\ell_\sharp(x))$, so that on the one hand

$$V_{\rm d}(x)U_{\rm a}(x)F(-x) \sim V_{\rm d}(x)xF(-x)/\ell^*(x) \sim xF(-x)/[\ell^*(x)\ell_\sharp(x)] \to 0. \tag{3.10}$$

On the other hand, applying (3.7)—with $\delta = 1$ and with $\frac{1}{2}x$ in place of x—we deduce

$$\begin{aligned} V_{\rm d}(x)U_{\rm a}(x)[1-F(x)] &\sim U_{\rm a}(x)V_{\rm d}(\tfrac{1}{2}x)[1-F(x)] \\ &\le U_{\rm a}(x)P[Z > \tfrac{1}{2}x]\{1+o(1)\} \to 0. \end{aligned}$$

where the convergence to zero is due to (3.8). Thus we have (i). □

REMARK 3.2. (3.9) holds also under (C2). Indeed, we know $V_{\rm d}(x)U_{\rm a}(x) \sim x^2/\int_0^x tH(t)dt$ (see (4.11)), and hence (3.9) follows since $x^2H(x) = o\big(\int_0^x tH(t)dt\big)$. Under (C1), (3.9) may fail to hold, but it holds (see [17, Proposition 50(i)]) that

$$V_{\rm d}(x)U_{\rm a}(x)[1-F(x)] \to 0.$$

Lemma 3.4. (i) *Under (C3), for each $\varepsilon > 0$ as $R \to \infty$*

$$E_x\big[V_{\rm d}(S_{\sigma(R,\infty)}); Z(R) \ge \varepsilon R,\, \Lambda_R\big]/V_{\rm d}(x) \longrightarrow 0 \quad \textit{uniformly for} \quad 0 \le x \le R. \tag{3.11}$$

(ii) *Under (C4), as $M \wedge R \to \infty$*

$$E_x\big[V_{\rm d}(S_{\tau(R)}); Z(R) \ge MR,\, \Lambda_R\big]/V_{\rm d}(x) \longrightarrow 0 \quad \textit{uniformly for} \quad 0 \le x \le R. \tag{3.12}$$

Proof. We first prove (ii). The expectation on the LHS of (3.12) is less than

$$\sum_{w \ge R+MR} \sum_{z=0}^{R-1} g_\Omega(x,z)p(w-z)V_{\rm d}(w) \tag{3.13}$$

because of the trivial inequality $g_{\mathbb{Z}\setminus[0,R)}(x,z) < g_\Omega(x,z)$. By Lemma 3.1 $V_{\rm d}(x) \sim 1/\ell_\sharp(x)$ and the derivative $(1/\ell_\sharp)'(x) = o(1/x\ell_\sharp(x))$. Hence for each $M > 0$ there exists a constant R_0 such that for all $R > R_0$ and $z < R$,

$$\begin{aligned} \sum_{w \ge R+MR} p(w-z)V_{\rm d}(w) &\le \frac{1-F(MR)}{\ell_\sharp(R)\{1+o(1)\}} + \sum_{w \ge MR} \frac{1-F(w)}{w\ell_\sharp(w)} \times o(1) \\ &\le \frac{1-F(R)}{M^\alpha \ell_\sharp(R)}\{1+o(1)\} \le 2\frac{V_{\rm d}(R)[1-F(R)]}{M^\alpha}, \end{aligned} \tag{3.14}$$

where we employ the summation by parts for the first inequality and (C4) for the equality. Owing to Lemma 3.3 the last member is at most $3/M^\alpha U_{\rm a}(R)$ and hence on applying Lemma 2.1 the double sum in (3.13) is dominated by $3V_{\rm d}(x)/M^\alpha$, showing (3.12).

In case (C3), by the inequality (3.4) the sum of the infinite series in the second member of (3.14) is at most a constant multiple of $\sum_{w \ge MR} P[Z \ge \frac{1}{2}w]/w$, which is $o(\ell^*(R)/R)$ as one can easily check by summing by parts. This together with the equivalence $U_{\rm a}(x) \sim x/\ell^*(x)$ and Lemma 3.3(i) shows that the sum on the left-most member of (3.14) is $o(1/U_{\rm a}(R))$ and on returning to (3.13) we conclude (3.11) by Lemma 2.1 again. □

Proof of Theorem 1 in cases (C3), (C4). We have the upper bound (2.3) of $P_x(\Lambda_R)$. To obtain the lower bound we observe that for $x \le R/2$,

$$\frac{V_{\rm d}(R)}{1+o(1)} \ge E_x\big[V_{\rm d}(S_{\sigma(R,\infty)}); Z(R) < MR \,\big|\, \Lambda_R\big] \ge \frac{(1-\varepsilon)V_{\rm d}(x)}{P_x(\Lambda_R)\{1+o(1)\}} :$$

the left-had inequality follows immediately from the slow variation of $V_{\rm d}$ (see Lemma 3.1 in case (C4)), and the right-hand one from (2.2) and Lemma 3.4. The inequalities above yield the lower bound for $x \le R/2$ since ε can be made arbitrarily small. In view of the slow variation of $V_{\rm d}$ the result for $\lfloor R/2 \rfloor \le x < R$ follows from $P_{\lfloor R/2 \rfloor}(\Lambda_R) \to 1$. The proof is complete. □

REMARK 3.3. Theorem 1 together with Lemma 3.4 shows that uniformly for $0 \le x \le R$

$$\begin{aligned} &\text{if (C3) holds,} \quad P_x[Z(R) > \varepsilon R \,|\, \Lambda_R] \to 0 \qquad \text{as} \quad R \to \infty \quad \text{for each} \quad \varepsilon > 0; \\ &\text{if (C4) holds,} \quad P_x[Z(R) > MR \,|\, \Lambda_R] \to 0 \qquad \text{as} \quad R \wedge M \to \infty. \end{aligned} \tag{3.15}$$

3.3. MISCELLANEOUS LEMMAS.

In the rest of this section we provide several lemmas for later citations. In the following three lemmas we put for $x \ge 0$

$$H_+(x) = 1 - F(x). \tag{3.16}$$

Lemma 3.5. *Under (C3),* $\ell^*(x) \sim \int_0^x V_{\rm d}(t)[1 - F(t)]dt$.

Proof. Summation by parts yields

$$P[Z > t] = \sum_{y=0}^{\infty} v_{\rm d}(y)[1 - F(y+t)] = \sum_{y=0}^{\infty} V_{\rm d}(y) p(y + 1 + \lfloor t \rfloor).$$

Since $\int_0^x p(y + 1 + \lfloor t \rfloor)dt = F(y + 1 + x) - F(y) + O(p(\lfloor x \rfloor + y + 1))$, putting

$$V(x) = 1/\ell_\sharp(x) \quad \text{and} \quad \tilde{\ell}(x) = \int_0^{\infty} V(t)[F(t+x) - F(t)]dt,$$

we have $\ell^*(x) = \tilde{\ell}(x) + o(1)$. By splitting the integral at x and employing a change of variables a simple manipulation yields

$$\tilde{\ell}(x) = \int_0^x V(t) H_+(t)dt + \int_0^{\infty} \big(V(y+x) - V(y)\big) H_+(t+x)dt. \tag{3.17}$$

In a similar way, taking the difference for the integral defining $\tilde{\ell}(x)$, we deduce

$$\begin{aligned} \tilde{\ell}(x) - \tilde{\ell}(\tfrac{1}{2}x) &= \int_0^{\infty} V(t)\big[H_+(t + \tfrac{1}{2}x) - H_+(t+x)\big]dt \\ &= \int_0^{x/2} V(t) H_+(t + \tfrac{1}{2}x)dt + \int_0^{\infty} \big(V(t + \tfrac{1}{2}x) - V(t)\big) H_+(t+x)dt. \end{aligned} \tag{3.18}$$

Since the integrals on the last member of (3.18) are positive and $\tilde{\ell}$ is s.v., both are of smaller order than $\tilde{\ell}$, in particular the second integral restricted to $y \ge x$ is $o(\tilde{\ell}(x))$, which fact leads to

$$\int_0^{\infty} \big(V(y+x) - V(y)\big) H_+(t+x)dt = o(\tilde{\ell}(x)),$$

for by the slow variation of V as well as of $1/\ell^*$ and the monotonicity of $F(-t)$, it holds that if $y \ge \frac{1}{2}x$,

$$V(t+x) - V(t) \sim \frac{V^2(t)}{\ell^*(t)} \int_0^x F(-t-x)dt \le 2\big[V(t + \tfrac{1}{2}x) - V(t)\big]\{1 + o(1)\}.$$

Returning to (3.17) we can now conclude $\tilde{\ell}(x) \sim \int_0^x V(t)[1 - F(t)]dt$, hence by Lemma 3.1 $\ell^*(x) \sim \int_0^x V_{\rm d}(t)[1 - F(t)]dt$ as desired. □

Lemma 3.6. *Suppose (C3) to hold. Then*

$$\int_0^x \ell_\sharp(t)P[Z>t]dt \sim \int_0^x [1-F(t)]dt; \tag{3.19}$$

and if either $E[X;X>0]$ or $E[|X|;X<0]$ is finite in addition, then

$$E|X|<\infty, \quad \int_0^\infty \ell_\sharp(t)P[Z>t]dt = E[X;X>0]$$

and

$$\int_x^\infty \ell_\sharp(t)P[Z>t]dt \sim \int_x^\infty [1-F(t)]dt. \tag{3.20}$$

Proof. Denote by $J(x)$ the integral on the RHS of (3.19). We first derive the lower bound for it. By (3.4) and $\lim \ell_\sharp(x)V_{\rm d}(x) \to 1$, for any $\lambda>1$,

$$\frac{J(x)}{1+o(1)} \geq \int_0^x \ell_\sharp(t)V_{\rm d}(t)H_+(\lambda t)dt = \frac{1}{\lambda}\int_0^{\lambda t} H_+(t)dt\{1+o(1)\}.$$

Since λ may be arbitrarily close to unity, we have $J(x) \geq \int_0^x H_+(t)dt\{1+o(1)\}$.

According to Lemma 5.3 given in Appendix (A), by the slow variation of $V_{\rm d}$ and the monotonicity of H_+ it follows that

$$\sum_{y=x}^\infty v_{\rm d}(y)H_+(y) = o\biggl(V_{\rm d}(x)H_+(x) + \int_x^\infty V_{\rm d}(t)H_+(t)\frac{dt}{t}\biggr).$$

In view of (3.7), it therefore suffices for the upper bound to show that

$$\int_0^x \ell_\sharp(t)dt \int_t^\infty V_{\rm d}(t)H_+(t)\frac{dt}{t} \leq C\int_0^x H_+(t)dt. \tag{3.21}$$

Integrating by parts we may write the LHS as

$$\biggl[x\ell_\sharp(x)\int_t^\infty V_{\rm d}(t)H_+(t)\frac{dt}{t} + \int_0^x H_+(t)dt\biggr]\{1+o(1)\}.$$

Integrating by parts again and using Lemma 3.5 one can easily deduce that the first integral in the big square brackets is less than $\int_x^\infty \ell^*(t)t^{-2}\{1+o(1)\}dt \sim \ell^*(x)/x \leq V_{\rm d}(x)\int_0^x H_+(t)dt \times O(1/x)$. Thus we obtain (3.21), hence (3.19).

Let $E[X;X>0] \wedge E[|X|;X<0] < \infty$. One can show (3.20) in an analogous way to the above, but, under $E|X|<\infty$— so that $EX=0$, (3.20) follows immediately from (3.19) since

$$\int_0^\infty \ell_\sharp(t)P[Z>t]dt = \int_0^\infty \frac{F(-s)}{\ell^*(s)}ds\int_0^s P[Z>t]dt = \int_0^\infty F(-s)ds = \int_0^\infty H_+(t)dt.$$

The verification of $E|X|<\infty$ also follows from (3.19) by using an estimate of $A(x)$ and is postponed to the proof of the next lemma. □

Let $\ell_\sharp$ be the function defined by (3.3) in the preceding section.

Lemma 3.7. *Suppose that either (C3) or (C4) holds. Then*

$$\ell^*(t)\ell_\sharp(t) = -\int_0^t F(-s)ds + \int_0^t P[Z>s]\ell_\sharp(s)ds = A(t) + o\biggl(\int_0^t [1-F(s)]ds\biggr) \tag{3.22}$$

and in case $EX=0$, both η_- and η are s.v. and

$$\ell^*(t)\ell_\sharp(t) = \int_t^\infty \bigl[F(-s) - P[Z>s]\ell_\sharp(s)\bigr]ds = A(t) + o(\eta_+(t)). \tag{3.23}$$

Proof. Recall $\ell^*(t) = \int_0^t P[Z > s]ds$ and note that as $t \downarrow 0$, $\ell^*(t) = O(t)$ and $\ell_\sharp(t) \sim O(\log 1/t)$. Then integration by parts leads to $\int_0^t P[Z > s]\ell_\sharp(s)ds = \ell^*(t)\ell_\sharp(t) + \int_0^t F(-s)ds$, hence the first equality of (3.22). The second one follows from (3.19). [Note that in case (C4) (3.19) is immediate from Lemma 3.2(ii).

Here we complete the proof of the second half of Lemma 3.6. Indeed, if $E[|X|; X < 0] < \infty$, then (3.22) verifies $E[X; X > 0] < \infty$ as is clear from

$$\ell^*(t)\ell_\sharp(t) < \int_t^\infty F(-s)ds \, (\leq \infty), \tag{3.24}$$

while if $E[X; X > 0] < \infty$, then $E[|X|; X < 0] < \infty$, since the right-most member of (3.22) is positive.

Let $EX = 0$. Then, because of (3.24), the first equality of (3.23) is deduced from the identity

$$[\ell^*(t)\ell_\sharp(t)]' = P[Z > t]\ell_\sharp(t) - F(-t) \tag{3.25}$$

(valid almost every $t > 1$). For the second one we have only to apply (3.20). Lemma 3.3(i) together with (b) of Lemma 3.1 entails that under (C3)

$$xP[|X| \geq x] \ll \ell^*(x)\ell_\sharp(x) \leq \eta_-(x).$$

Thus both η_- and η are *s.v.* □

REMARK 3.4. Suppose that (1.2) and $[1-F(x)]/H(x) \to p$, and let $q = 1-p$. Then combining Lemmas 3.1 and 3.7 shows that for $p \neq 1/2$,

$$A(x) \sim \ell_\sharp(x)\ell^*(x) \sim x/[U_{\rm a}(x)V_{\rm d}(x)].$$

If F is recurrent (necessarily $p \leq 1/2$), we have $a(x) \sim \int_{x_0}^x F(-t)dt/A^2(t)$ and $a(-x) = \int_{x_0}^x [1 - F(t)]dt/A^2(t) + o(a(x))$ as $x \to \infty$ ([17]). This leads to

$$\frac{1}{A(x)} = -(p-q)\int_{x_0}^x \frac{H(t)}{A^2(t)}dt + \frac{1}{A(x_0)} \sim (1 - q^{-1}p)a(x)$$

for $p < 1/2$, so that $U_{\rm a}(x)V_{\rm d}(x)/x \sim (1 - q^{-1}p)a(x)$.

Lemma 3.8. *Suppose that (1.5) holds with $\alpha = 1$. Let $EX = 0$ and $p \leq 1/2$. Then*

$$\int_1^x \frac{F(-t)dt}{[\ell^*(t)\ell_\sharp(t)]^2} \sim g_\Omega(x, x).$$

Proof. We suppose $p > 0$ for simplicity. [If $EX = 0$ and $p < 1/2$, the result follows from (3.23) because of Theorem 1 of [19] that is shown independently of the present issue .] Recall $[\ell^*(t)]' = P[Z > t] \sim [1 - F(t)]/\ell_\sharp(t)$ and then observe that

$$\begin{aligned} g_\Omega(x, x) &= \sum_{k=0}^x v_{\rm d}(k)u_{\rm a}(k) = \sum_{k=0}^x \frac{v_{\rm d}(k)}{\ell^*(k)}\{1 + o_k(1)\} \\ &= \frac{V_{\rm d}(x)}{\ell^*(x+1)}\{1 + o(1)\} + \sum_{k=0}^x V_{\rm d}(k)\left(\frac{1}{\ell^*(k)} - \frac{1}{\ell^*(k+1)}\right)\{1 + o_k(1)\} \\ &= \frac{1}{\tilde{A}(x)}\{1 + o(1)\} + \sum_{k=1}^x \frac{1 - F(k)}{[\tilde{A}(k)]^2}\{1 + o_k(1)\}, \end{aligned}$$

where $\tilde{A}(t) = \ell^*(t)\ell_\sharp(t)$ and $o_t(1) \to 0$ as $t \to \infty$. By (3.30) $P[Z > t]\ell_\sharp(t) \sim 1 - F(t)$ if $p > 0$, and by (3.25) it follows that

$$\frac{1}{\tilde{A}(x)} = \int_1^x \frac{F(-t) - 1 + F(t)}{[\tilde{A}(t)]^2} dt + \int_1^x \frac{[1 - F(t)] \times o_t(1)}{[\tilde{A}(t)]^2} dt + \frac{1}{\tilde{A}(1)} \tag{3.26}$$

and hence, on absorbing $1/\tilde{A}(1)$ into the second integral on the RHS,

$$g_\Omega(x, x) = \int_1^x \frac{F(-t) - 1 + F(t)}{[\tilde{A}(t)]^2} dt\{1 + o(1)\} + \int_1^x \frac{H_+(t)}{[\tilde{A}(t)]^2}\{1 + o_t(1)\} dt, \tag{3.27}$$

which leads to the equivalences of the lemma. □

For F recurrent, under (1.2) we shall see that $g_\Omega(x, x) \sim a(x)$ $(x \to \infty)$ in [19], so that the equivalence of Lemma 3.8, which we do not use Lemma 3.8 in this paper, gives

$$\int_1^x \frac{F(-t)dt}{[\ell^*(t)\ell_\sharp(t)]^2} \sim a(x) \sim \int_1^x \frac{F(-t)dt}{A^2(t)}$$

that is of interest in the critical case $\lim a(-x)/a(x) = 1$ when we have little information about the behaviour of $A(x)/\ell^*(x)\ell_\sharp(x)$.

Lemma 3.9. *If either (C3) or (C4) holds, then for any $\delta < 1$, as $R \to \infty$*

$$1 - P_{R-x}(\Lambda_R) = o\left(\frac{U_{\rm a}(x)}{U_{\rm a}(R)}\right) \quad \textit{uniformly for } 0 \le x < \delta R.$$

Proof. We prove the dual assertion. Suppose $U_{\rm a}$ and $v_{\rm d}$ are s.v. We show

$$P_x(\Lambda_R)V_{\rm d}(R)/V_{\rm d}(x) \to 0. \tag{3.28}$$

Since for $\frac{1}{3}R \le x < \delta R$, $V_{\rm d}(R)/V_{\rm d}(x)$ is bounded and $P_x(\Lambda_R) \to 0$ as $R \to \infty$, we have only to consider the case $x < \frac{1}{3}R$. Putting $R' = \lfloor R/3 \rfloor$ and $R'' = 2R'$, we have

$$P_x(\Lambda_R) = J_1(R) + J_2(R),$$

where

$$J_1(R) = P_x\big[S_{\sigma(R',\infty)} > R'',\, \Lambda_R\big] \quad \text{and} \quad J_2(R) = P_x\big[S_{\sigma(R',\infty)} \le R'',\, \Lambda_R\big].$$

Obviously

$$J_1(R) = \sum_{z=0}^{R'} g_{\mathbb{Z}\setminus[0,R']}(x, z)P[X > R'' - z] \le \big[1 - F(R')\big] \sum_{z=0}^{R'} g_\Omega(x, z).$$

The last sum is at most $V_{\rm d}(x)U_{\rm a}(R)$ by Lemma 2.1. Using Lemma 3.3(i) we accordingly infer that

$$J_1(R)\,\frac{V_{\rm d}(R)}{V_{\rm d}(x)} \le V_{\rm d}(R)U_{\rm a}(R)\big[1 - F(R')\big] \;\longrightarrow\; 0.$$

As for J_2, we decompose $J_2(R) = P_x(\Lambda_{R'}) \sum_{z=R'}^{R''} P_x\big[S_{[R',\infty)} = z \,\big|\, \Lambda_{R'}\big] P_z(\Lambda_R)$. On applying the upper bound (2.3) to $P_x(\Lambda_{R'})$ it then follows that as $R \to \infty$

$$J_2(R)\,\frac{V_{\rm d}(R)}{V_{\rm d}(x)} \le \frac{V_{\rm d}(R)}{V_{\rm d}(R')} \sum_{z=R'}^{R''} P_x\big[S_{[R',\infty)} = z \,\big|\, \Lambda_{R'}\big] P_z(\Lambda_R) \;\longrightarrow\; 0, \tag{3.29}$$

for $P_z(\Lambda_R) \to 0$ uniformly for $R' \le z \le R''$. Thus (3.28) is verified.

In case (C4), $V_{\rm d}(x) \sim x^\alpha/\hat{\ell}(x)$ and $U_{\rm a}(x) \sim 1/\hat{\ell}_\sharp(x)$ (see (4.3) and (4.4)) and we can proceed as above. □

The next lemma presents some of the results obtained above in a neat form under condition (AS).

Lemma 3.10. *Suppose (AS) to hold with $\alpha = 1$.*

(i) *The following are equivalent*

$$\textit{(a)}\ P[-\hat{Z} \ge x]\ \textit{is s.v.} \qquad \textit{(b)}\ \ Z\ \ \textit{is r.s.} \qquad \textit{(c)}\quad \lim P[S_n > 0] = 1,$$

and each of (a) to (c) above implies $P[-\hat{Z} \ge x] \sim \ell_\sharp(x)$ and $V_{\rm d}(x) \sim 1/\ell_\sharp(x)$; in particular these two asymptotic equivalences hold if either $EX = 0, p < 1/2$ or $E|X| = \infty$, $p > 1/2$.

(ii) *If either of (a) to (c) of* (i) *holds, then*

$$\begin{aligned} &\sup_{x>(1+\varepsilon)R} P_x[\sigma_{\{R\}} < T] \to 0 \qquad \textit{for each } \varepsilon > 0\textit{; and} \\ &\begin{cases} P[Z > x] \sim V_{\rm d}(x)[1 - F(x)] & \textit{if}\ \ p > 0, \\ P[Z > x] = o\big(V_{\rm d}(x)F(-x)\big) & \textit{if}\ \ p = 0. \end{cases} \end{aligned} \tag{3.30}$$

Proof. The equivalence of (a), (b) and (c) is observed in [17]. [The same result can also deduced from what is mentioned in Remark 1.2 combined with our Theorem 1, where one uses the latter to ensure that the slow variation of $V_{\rm d}$-equivalent to (a)—implies (c).] The other assertion of (i) is contained in Lemma 3.1.

From the slow variation of $V_{\rm d}$ it follows that $\lim_{y\to\infty} P_y[S_{\sigma(-\infty,0]} < -My] = 1$ for each $M > 1$ [12, Theorem 3], which entails the first relation of (ii). The second one follows immediately from Lemma 3.2(i) if $p > 0$, since one can take $\delta = 0$ in (3.4) under (AS). In case $p = 0$, examine the proof of (3.5) by noting that $\sum_y v_{\rm d}(y)[1 - F(y)] \ll \sum_y v_{\rm d}(y)F(-y)$. □

4 Proof of Proposition 1

Throughout this section we suppose (AS) to hold. Let p, q and L be as in (1.5). Recall that $\alpha\rho$ ranges exactly over the interval $[(\alpha - 1) \vee 0, \alpha \wedge 1]$.

It is known that $P[Z > x]$ varies regularly at infinity with index $-\alpha\rho$ if $\alpha\rho < 1$ (which if $0 < \alpha < 2$ is equivalent to $p > 0$) and Z is r.s. if $\alpha\rho = 1$ (cf. [12, Theorem 9] in case $\rho > 0$; also Lemma 3.1 for the first statement in case $\rho = 0$); and in either case

$$\frac{U_{\rm a}(x)\int_0^x P[Z > t]dt}{x} \longrightarrow \frac{1}{\Gamma(1 + \alpha\rho)\Gamma(2 - \alpha\rho)} \tag{4.1}$$

(the dual of (2.4)). We choose a s.v. function $\ell(x)$ so that

$$\begin{aligned} &\ell(x) = \ell^*(x) && \text{if}\ \ \alpha\rho = 1 \\ &P[Z > x] \sim \frac{\sin \pi\alpha\rho}{\pi\alpha\rho} x^{-\alpha\rho}\ell(x) && \text{if}\ \ \alpha\rho < 1. \end{aligned} \tag{4.2}$$

Here $t^{-1}\sin t$ is understood to equal unity for $t = 0$. [Note that in case $\alpha = 1$ with $\rho < 1$, ℓ is not asymptotically equivalent to ℓ^*; the constant factor in the second formula above is chosen

so as to have the choice conform to that in (3.2)—see (4.4) below.] As the dual relation we have a s.v. function $\hat{\ell}$ such that

$$\hat{\ell} = \hat{\ell}^*(x) \quad (\alpha\hat{\rho} = 1); \quad P[-\hat{Z} > x]/v_0 \sim \frac{\sin \pi\alpha\hat{\rho}}{\pi\alpha\hat{\rho}} x^{-\alpha\hat{\rho}} \hat{\ell}(x) \quad (\alpha\hat{\rho} < 1).$$

Note that according to Lemma 3.1,

$$\hat{\ell}(x) \sim \ell_\sharp(x) \quad \text{if } \rho = 1;$$

similarly $\ell(x) \sim \hat{\ell}_\sharp(x)$ if $\rho = 0$ (necessarily $\alpha \le 1$), where

$$\hat{\ell}_\sharp(x) = \alpha \int_x^\infty \frac{1 - F(t)}{t^{1-\alpha}\hat{\ell}(t)} dt \qquad (x > 0). \tag{4.3}$$

It then follows that for all $0 \le \rho \le 1$,

$$U_{\rm a}(x) \sim x^{\alpha\rho}/\ell(x) \quad \textit{and} \quad V_{\rm d}(x) \sim x^{\alpha\hat{\rho}}/\hat{\ell}(x). \tag{4.4}$$

The two functions ℓ and $\hat{\ell}$ are linked as we are going to see in Lemma 4.1 below. We bring in the constant

$$\kappa = \frac{\Gamma(\alpha)\pi^{-1}\sin \pi\alpha\rho}{p\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)} = \frac{\Gamma(\alpha)\pi^{-1}\sin \pi\alpha\hat{\rho}}{q\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)} = \frac{\Gamma(\alpha)[\sin \pi\alpha\rho + \sin \pi\alpha\hat{\rho}]}{\pi\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)},$$

where only the case $p > 0$ or $q > 0$ of the two expressions above is adopted if $pq = 0$; if $pq \ne 0$ the two coincide, namely $p/q = \sin \pi\alpha\rho / \sin \pi\alpha\hat{\rho}$ ($pq \ne 0$) as is implicit in the proof of the next result or directly derived (cf. Appendix (A) of [17]). Note that $\kappa = 0$ if and only if either $\alpha = 1$ and $\rho\hat{\rho} = 0$ or $\alpha = 2$. Recall $H(t)$ stands for $P[|X| > t]$.

Lemma 4.1. (i) *For all* $0 < \alpha \le 2$, $\ell(x)\hat{\ell}(x)/[x^\alpha H(x)] \to \kappa$, *namely*

$$U_{\rm a}(x)V_{\rm d}(x) = \frac{\kappa + o(1)}{H(x)}.$$

(ii) *If* $\alpha = 2$, *then* $\ell(x)\hat{\ell}(x) \sim \int_0^x tH(t)dt$, *or what amounts to the same,*

$$U_{\rm a}(x)V_{\rm d}(x) \sim \frac{x^2}{\int_0^x tH(t)dt}.$$

(iii) *If* $\alpha = 1$ *and* $p \ne 1/2$ *(entailing* $\rho \in \{0, 1\}$ *so that* $\kappa = 0$*), then*

$$U_{\rm a}(x)V_{\rm d}(x) \sim \frac{x}{|A(x)|} \sim \begin{cases} \dfrac{x}{|p-q|\int_x^\infty H(t)dt} & (EX = 0), \\ \dfrac{x}{|p-q|\int_0^x H(t)dt} & (E|X| = \infty). \end{cases}$$

Lemma 4.1 covers all the cases of (AS) other than the case $\alpha = 2p = 1$ and $\rho \in \{0, 1\}$.

Proof. The last assertion (iii) follows immediately from Lemmas 3.1 and 3.7 (see also Remark 3.4).

Proof of (i). We have only to consider the case $0 < \rho < 1$ (of $0 < \alpha < 2$), since for $\rho = 1$ the result follows from Lemma 3.3 and for $\rho = 0$ by duality, and since for $\alpha = 2$

$x^2H(x) = o(\int_0^x tH(t)dt)$ so that the result follows from (ii). By symmetry we may and do suppose $q > 0$ (entailing $\alpha\hat{\rho} < 1$). Recalling (2.7) we have

$$\frac{P[-\hat{Z} \geq x]}{v_0} = \sum_{z=0}^{\infty} u_{\rm a}(z)F(-x-z). \tag{4.5}$$

We may suppose that ℓ and $L_- := qL$ are differentiable and satisfy $\ell'(t) = o(\ell(t)/t)$ and $L'_-(t) = o(L_-(t)/t)$. Summing by parts deduces that the sum on the RHS is equal to

$$\sum_{z=0}^{\infty} U_{\rm a}(z)\big[F(-x-z) - F(-x-z-1)\big]. \tag{4.6}$$

On substituting from (4.4) and then summing by parts back, the above sum is equivalent to

$$\alpha\rho\sum_{z=0}^{\infty} \frac{z^{\alpha\rho-1}L_-(x+z)}{\ell(z)(x+z)^{\alpha}} \sim \frac{qL(x)}{\ell(x)}\int_0^{\infty} \frac{\alpha\rho z^{\alpha\rho-1}}{(x+z)^{\alpha}}dz.$$

The integral on the RHS equals $e^{-\alpha\hat{\rho}}\alpha\rho B(\alpha\rho, \alpha\hat{\rho})$, where $B(t,s) = \Gamma(t)\Gamma(s)/\Gamma(t+s)$ (the Bessel function). We accordingly obtain

$$P[-\hat{Z} \geq x]/v_0 \sim q\alpha\rho B(\alpha\rho, \alpha\hat{\rho})x^{-\alpha\hat{\rho}}\, L(x)/\ell(x). \tag{4.7}$$

Thus by (2.4) (the dual of (4.1)) and the identity $\Gamma(1-t)\Gamma(1+t) = \pi t/\sin t$ ($|t| < 1$)

$$V_{\rm d}(x) \sim \frac{\sin\pi\alpha\hat{\rho}}{(\pi\alpha^2\rho\hat{\rho})B(\alpha\rho,\alpha\hat{\rho})}\cdot\frac{x^{\alpha\hat{\rho}}\ell(x)}{qL(x)} = \frac{\Gamma(\alpha)\pi^{-1}\sin\pi\alpha\hat{\rho}}{\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)}\cdot\frac{x^{\alpha\hat{\rho}}\ell(x)}{qL(x)}, \tag{4.8}$$

which combined with $U_{\rm a}(x) \sim x^{\alpha\rho}/\ell(x)$ shows the asserted convergence.

Proof of (ii). Let $\alpha = 2$. Then $\ell = \ell^*$, $\hat{\ell} = \hat{\ell}^*$. Suppose that $EZ = \infty$ so that $\ell^*(x) \to \infty$, otherwise the proof below being merely simplified. By (4.5) and its dual

$$\begin{aligned}(\ell\hat{\ell})'(x) &= \ell^*(x)P[-\hat{Z} > x]/v_0 + \hat{\ell}^*(x)P[Z > x] \qquad (4.9)\\ &= \ell^*(x)\int_0^{\infty}\frac{F(-x-y)}{\ell^*(y)\vee 1}dy\{1+o(1)\} + \hat{\ell}^*(x)\int_1^{\infty}\frac{1-F(x+y)}{\hat{\ell}^*(y)}dy\{1+o(1)\}.\end{aligned}$$

We claim that for any constant $\varepsilon \in (0,1)$,

$$\int_0^x \ell^*(t)dt\int_0^{\infty}\frac{F(-t-y)}{\ell^*(y)\vee 1}dy \begin{cases} \leq \big[\varepsilon\int_0^x tF(-t)dt + \int_0^x \eta_-(t)dt\{1+o(1)\},\\ \geq \int_0^x tF(-t)dt\{1-\varepsilon+o(1)\}.\end{cases} \tag{4.10}$$

By splitting the inner integral of the LHS at εt and using the monotonicity of ℓ^* as well as of $F-t$) it follows that

$$\ell^*(t)\int_0^{\infty}\frac{F(-t-y)}{\ell^*(y)\vee 1}dy \leq \big[\varepsilon F(-t) + \eta_-(t)\big]\{1+o(1)\},$$

showing the first inequality of (4.10). On the other hand the LHS of (4.10) is greater than

$$\begin{aligned}\int_0^x \ell^*(t)dt\int_0^{x-y}\frac{F(-t-y)}{\ell^*(y)\vee 1}dy &= \int_0^x F(-w)dw\int_1^w \frac{\ell^*(w-y)}{\ell^*(y)\vee 1}dy\\ &\geq \int_0^x F(-w)dw\int_0^{(1-\varepsilon)w}\{1+o(1)\}dy,\end{aligned}$$

which yields the second inequality of (4.10).

Put $L^*(x) = \int_0^x tH(t)dt$. Then (AS) (with $\alpha = 2$) is equivalent to the slow variation of L^* and in this case $m(x) = \int_0^x \eta(t)dt \sim L^*(x)$ (cf. [17, Appendix (A)]), which entails $x\eta(x) = o(L^*(x))$. By (4.10) and its dual that follows because of symmetric roles of two tails of F, integrating the second expression of $(\ell\hat{\ell})'(x)$ in (4.9) leads to the formula of (ii). Proof of Lemma 4.1 is complete. □

The asymptotic form of $V_{\rm d}U_{\rm a}$ given in Lemma 4.1 may also be expressed as

$$\lim U_{\rm a}(c_n)V_{\rm d}(c_n)/n \longrightarrow \kappa \text{ or } 1/2 \quad \text{according as } \alpha < 2 \text{ or } \alpha = 2 \tag{4.11}$$

(if $\alpha = 2$ the sequence (c_n) is determined by $c_n^2/L(c_n) \sim n$, where $L(x) = \int_{-x}^x t^2 dF(t)$) and known if $0 < \rho < 1$ [3, Eq(15)], [20, Eq(15, 31)] apart from the explicit expression of the limit value.

Lemma 4.2. *Suppose $\rho > 0$. Then for any $\varepsilon > 0$, there exists a constant $M > 0$ such that for all sufficiently large R,*

$$E_x\big[V_{\rm d}(S_{N(R)}); Z(R) \geq MR,\, \Lambda_R\big] \leq \varepsilon V_{\rm d}(x) \quad (0 \leq x < R),$$

and, if $\alpha\rho = 1$,

$$\lim_{R\to\infty} \sup_{0\leq x<R} \frac{1}{V_{\rm d}(x)} E_x\big[V_{\rm d}(S_{N(R)}); Z(R) \geq \varepsilon R, \Lambda_R\big] = 0.$$

Proof. As in the proof of Lemma 3.4 but with $V_{\rm d}(x) \sim x^{\alpha(1-\rho)}/\hat{\ell}(x)$ in place of $V_{\rm d}(x) \sim 1/\ell_\sharp(x)$, one deduces that if $\rho > 0$, for all sufficiently large R,

$$\sum_{w\geq R+MR} p(w-z)V_{\rm d}(w) \leq C_* \frac{V_{\rm d}(R)H(R)}{M^{\alpha\rho}} \quad (0 \leq x < R) \tag{4.12}$$

with some constant C_* for which one can take $2/\rho$. Using Lemmas 2.1 and 4.1(i) in turn one then obtains that for any $M > 0$

$$\frac{1}{V_{\rm d}(x)} E_x\big[V_{\rm d}(S_{N(R)}); Z(R) \geq \varepsilon R, \Lambda_R\big] \leq C_* \frac{U_{\rm a}(R)V_{\rm d}(R)H(R)}{M^{\alpha\rho}} = C_* \frac{\kappa + o(1)}{M^{\alpha\rho}}, \tag{4.13}$$

so that the first half of the lemma is ensured by taking

$$M = [2C_*\kappa/\varepsilon]^{1/)\alpha\rho)}. \tag{4.14}$$

If $\alpha\rho = 1$, we have either $\kappa = 0$ or $1 < \alpha < 2$ with $p = 0$ and the second half follows from (4.13), for if $p = 0$, $H(x)$ can be replaced by $o(H(x))$. □

Lemma 4.3. *Suppose $\rho > 0$. Then for some constant $\theta > 0$, $P_x(\Lambda_R) \geq \theta V_{\rm d}(x)/V_{\rm d}(R)$ for $0 \leq x < R$ and for R large enough. If $\alpha \leq 1$, then θ can be chosen so as to approach unity as $\rho \to 1$, and for any $\delta > 0$, for $x \geq \delta R$,*

$$P_x(\Lambda_R) \to 1 \quad \textit{as} \quad R \to \infty \quad \textit{and} \quad \rho \to 1 \quad \textit{in this order.}$$

Proof. Because of the identity (2.2) (due to the optional stopping theorem) Lemma 4.2 shows that for any $\varepsilon > 0$ we can choose a constant $M > 0$ so that

$$(1-\varepsilon)V_{\rm d}(x) \leq E_x\big[V_{\rm d}(S_{N(R)}); Z(R) < MR \,\big|\, \Lambda_R\big] P_x(\Lambda_R).$$

Since the conditional expectation on the RHS is less than $V_{\rm d}(R+MR) \sim (1+M)^{\alpha\hat\rho}V_{\rm d}(R)$ we have the lower bound of $P_x(\Lambda_R)$ with $\theta = (1-\varepsilon)(1+M)^{-\alpha\hat\rho}$. Noting that M may be determined by (4.14) and that as $\rho \to 1$ (possible only when $\alpha \le 1$), $\theta \to 1-\varepsilon$, hence $\theta \to 1$ for ε may be arbitrarily small. Since one also has $V_{\rm d}(x)/V_{\rm d}(R) \sim (x/R)^{\alpha\hat\rho} \to 1$ (uniformly for $\delta R < x < R$) one can concludes the second half of the lemma. □

Lemma 4.4. *Suppose $0 < (\alpha \vee 1)\rho < 1$. Then for any $\delta < 1$, uniformly for $0 \le x < \delta R$, as $R \to \infty$ and $\varepsilon \downarrow 0$ (interchangeably)*

$$P_x\big[Z(R) \le \varepsilon R \,\big|\, \Lambda_R\big] \to 0.$$

Proof. By the restriction as to $(\alpha \vee 1)\rho$ it follows that $\alpha < 2$ and

(a) $p > 0$ so that $1-F$ is regularly varying;
(b) $P[Z > x]$ is regularly varying with index $\in (0,1)$; (4.15)
(c) $V_{\rm d}(x)/V_{\rm d}(R) \le CP_x(\Lambda_R)$ and for each $\varepsilon > 0$, $\inf_{R\ge1}\inf_{x\ge\varepsilon R} P_x(\Lambda_R) > 0$.

The reasoning for (a) is already mentioned, while (b) and (c) are due to (4.2) and Lemma 4.3, respectively. According to [6, Section XIV.3] (b) implies that $P[Z(r) < \varepsilon r] \to 0$ as $\varepsilon \to 0$ uniformly for $r \ge 1$. This in particular shows the asserted convergence of the lemma to be valid for $x \ge \frac14 R$, therein $P_x(\Lambda_R)$ being bounded away from zero by (c).

For $x < \frac14 R$, take a constant $0 < h < 1/4$ arbitrarily and let $\mathcal{E}_{\varepsilon,R}$ stand for the event $\{(1-h)R \le S_{\sigma[R/4,\infty)} < (1+\varepsilon)R\}$. It holds that

$$\Lambda_R \cap \{Z(R) < \varepsilon R\} \subset \Lambda_{R/4} \cap \big[\{S_{\sigma[R/4,\infty)} < (1-h)R, Z(R) < \varepsilon R\} \cup \mathcal{E}_{\varepsilon,R}\big], \tag{4.16}$$

and

$$P_x(\Lambda_{R/4} \cap \mathcal{E}_{\varepsilon,R}) \le \sum_{0\le y<R/4} g_\Omega(x,y) P[(1-h)R - y \le X < (1+\varepsilon)R - y]. \tag{4.17}$$

Put

$$\omega_R(h,\varepsilon) = \sup_{y<R/4} \frac{P[(1-h)R - y \le X < (1+\varepsilon)R - y]}{P[X \ge (1-h)R - y]}.$$

One sees that $\omega_R(h,\varepsilon) \le \sup_{z<R/2} P[z \le X < z + 2(h+\varepsilon)z]/P[X \ge z]$, hence the regular variation of $1-F$ entails that $\omega_R(h,\varepsilon) \to 0$ as $h+\varepsilon \to 0$ uniformly for $R \ge 1$. Since in (4.17) the probability under the summation sign is less than $\omega_R(h,\varepsilon)[1-F(R/2)]$, by Lemmas 2.1 and 4.1 the sum is evaluated to be at most a constant multiple of

$$\omega_R(h,\varepsilon)\big[1-F(\tfrac12 R)\big]V_{\rm d}(x)U_{\rm a}(\tfrac12 R) \le \omega_R(h,\varepsilon)\frac{V_{\rm d}(x)}{V_{\rm d}(\frac12 R)}\{\kappa + o(1)\}.$$

By (4.15c) and regular variation of $V_{\rm d}$, $V_{\rm d}(x)V_{\rm d}(\frac12 R) \le CP_x(\Lambda_R)$. Hence

$$P_x[\Lambda_{R/4} \cap \mathcal{E}_{\varepsilon,R} \,\big|\, \Lambda_R\big] \le C_2\,\omega_R(h,\varepsilon) \quad \text{for} \quad 0 \le x < R/4. \tag{4.18}$$

Now by (4.16), $P_x[Z(R) \le \varepsilon R \,|\, \Lambda_R]$ is less than

$$P_x[\Lambda_{R/4} \cap \mathcal{E}_{\varepsilon,R} \,\big|\, \Lambda_R\big] + \sup_{R/4\le y<(1-h)R} P_y[Z(R) < \varepsilon R]\frac{P_x(\Lambda_{R/4})}{P_x(\Lambda_R)}.$$

By (4.15c) again, $P_x(\Lambda_{R/4})/P_x(\Lambda_R) = 1/P_x[\Lambda_R \,|\, \Lambda_{R/4}] < C$. By (4.18), for any $\varepsilon_0 > 0$, one can choose $h > 0$ so that the first term is less than ε_0 for all $0 < \varepsilon < h$ and all sufficiently large R, while for h thus chosen the second term is less than ε_0 for ε small enough and all R large enough. Thus the convergence asserted of the lemma follows. □

Lemma 4.5. *Let $\alpha \le 1$. For any $\delta < 1$,*

$$\sup_{0\le x<\delta R} \frac{P_x(\Lambda_R)V_{\rm d}(R)}{V_{\rm d}(x)} \to 0 \quad \text{as } R\to\infty \text{ and } \rho\to 0 \text{ in this order;}$$

if $\rho = 0$, the above supremum approaches zero as $R \to \infty$.

Proof. The case $\rho = 0$ is a special case of Lemma 3.9. The proof of the rest is made by examining that of Lemma 3.9. Let $\alpha \le 1$. Suppose $0 < \rho < 1$. Since for $\frac{1}{3}R \le x < \delta R$, $V_{\rm d}(x)/V_{\rm d}(R)$ is bounded away from zero and $P_x(\Lambda_R) \to 0$ as $R \to \infty$, $\rho \to 0$ in this order because of the dual of the last assertion of Lemma 4.3 we have only to consider the case $x < \frac{1}{3}R$. Putting $J_1(R)$ and $J_2(R)$ as in the proof of Lemma 3.9 so that $P_x(\Lambda_R) = J_1 + J_2$. Using Lemma 4.1(i) we have

$$J_1(R)\,\frac{V_{\rm d}(R)}{V_{\rm d}(x)} \le V_{\rm d}(R)U_{\rm a}(R)\big[1-F(R')\big] \le 4p\kappa \tag{4.19}$$

for R large enough. As before we see that as $R \to \infty$ and $\rho \to 0$ in turn,

$$J_2(R), \frac{V_{\rm d}(R)}{V_{\rm d}(x)} \le \frac{V_{\rm d}(R)}{V_{\rm d}(R')} \sum_{z=R'}^{R''} P_x\big[S_{[R',\infty)} = z \,\big|\, \Lambda_{R'}\big] P_z(\Lambda_R) \to 0,$$

for, in this limit, $P_z(\Lambda_R) \to 0$ uniformly for $R' \le z \le R''$. Since $p\kappa \to 0$ as $\rho \to 0$, this together with (4.19) shows the first half of the lemma. □

Proof of Proposition 1. Let $0 < (\alpha \vee 1)\rho < 1$. The asserted lower bound of $P_x(\Lambda_R)$ asserted in (i) follows from Lemma 4.3. For proof of the upper bound let $1/2 < \delta < 1$. Lemma 4.4 entails

$$P_x\big[Z(R) > \varepsilon R \,\big|\, \Lambda_R\big] \ge 1/2 \qquad (x < \delta R) \tag{4.20}$$

for some $\varepsilon > 0$. Writing the equality (2.2) as

$$\frac{V_{\rm d}(x)}{P_x(\Lambda_R)} = \sum_{y=R}^{\infty} P_x[S_{N(R)} = y \,|\, \Lambda_R] V_{\rm d}(y),$$

one deduces from (4.20) that the sum above is larger than $[V_{\rm d}(R) + V_{\rm d}((1+\varepsilon)R)]/2$. By (4.4) $\limsup V_{\rm d}((1+\varepsilon)R)/V_{\rm d}(R) = (1+\varepsilon)^{\alpha\hat\rho} > 1$, for $(\alpha \vee 1)\rho < 1$ entails $\hat\rho > 0$. Hence $P_x(\Lambda_R) \le \big(2/(1+(1+\varepsilon)^{\alpha\hat\rho}]\big)V_{\rm d}(x)/V_{\rm d}(R)$. This verifies the upper bound.

(ii) follows from Lemmas 4.3 and 4.5. □

5 Appendix

(A) Here we present three lemmas involving s.v. functions, of which the first, second and third ones are used in the proofs of Lemmas 3.1, 3.2 and 3.6, respectively.

Lemma 5.1. *Let α be a positive constant, $\ell(t), t \ge 0$ a s.v. function, $u(t)$ and $G(t)$ positive measurable functions of $t \ge 0$. Suppose that $\int_0^x u(t)dt \sim x^\alpha/\alpha\ell(x)$ $(x \to \infty)$, $G(t)$ is non-increasing and both u and $1/\ell$ are locally bounded, and put*

$$\tilde u(t) = \frac{t^{\alpha-1}}{\ell(t)}, \quad h_\sharp(x) = \int_x^\infty \tilde u(t)G(t)dt,$$

$$h(x)=\int_0^\infty u(t)G(x+t)dt,\quad \textit{and}\quad \tilde{h}(x)=\int_0^\infty \tilde{u}(t)G(x+t)dt,$$

If one of $h(1),\tilde{h}(1)$ *and* $h_\sharp(1)$ *is finite, then so are the other two and*

(i) *if either* h *or* $\tilde{h}$ *is regularly varying with index* >-1*, then* $h(x)\sim\tilde{h}(x)$;

(ii) *if either* $\tilde{h}$ *or* $h_\sharp$ *is s.v., then* $h(x)\sim h_\sharp(x)$.

Proof. We may suppose ℓ is positive and continuous. Integrating by parts verifies that $\int_0^x\tilde{u}(t)G(t)\asymp\int_0^x u(t)G(t)dt$ so that $h(1),\tilde{h}(1)$ and $h_\sharp(1)$ are finite if either of them is finite.

Note that $h(1)<\infty$ entails $G(t)=o(t^{-\alpha}\ell(t))$, so that $G(t)\int_0^t u(s)ds\to 0$ $((t\to\infty)$. Put $U(x)=\int_0^t u(s)ds$ and $\tilde{U}(t)=\int_0^t\tilde{u}(s)ds$. Then, interchanging the order of integration and integrating by parts in turn one deduces

$$\int_0^x h(t)dt=\int_0^\infty u(t)dt\int_t^{x+t}G(s)ds=\int_0^\infty[G(t)-G(x+t)]U(t)dt,$$

and similarly

$$\int_0^x\tilde{h}(t)dt=\int_0^\infty[G(t)-G(x+t)]\tilde{U}(t)dt.$$

Since G is monotone and $U(t)\sim\tilde{U}(t)$, these identities together show

$$\int_0^x h(s)ds\sim\int_0^x\tilde{h}(s)ds,$$

provided $\int_0^x h(s)ds$ or $\int_0^x\tilde{h}(s)ds$ diverges to infinity. It therefore follows that if one of h or $\tilde{h}$ is regularly varying with index >-1, then so is the other owing to the monotone density theorem [2], entailing $\tilde{h}(x)\sim\ell_\sharp(x)$. Thus (i) is verified.

For proof of (ii), first suppose that $\tilde{h}$ is s.v. Take $M>2$ large. Then we see

$$\tilde{h}(x)\ge\int_0^{(M-1)x}\tilde{u}(t)G(x+t)dt=(M-1)^\alpha\frac{x^\alpha}{\alpha\ell(x)}G(Mx)\{1+o(1)\}$$

and

$$\tilde{h}(Mx)=\Big(\int_0^x+\int_x^\infty\Big)\tilde{u}(t)G(Mx+t)dt=\frac{x^\alpha}{\alpha\ell(x)}G(Mx)+\ell_\sharp(x).$$

By the equivalence $\tilde{h}(Mx)\sim\tilde{(}x)$, these inequalities lead to $h_\sharp(x)\ge\tilde{h}(x)\{1+O(M^{-\alpha})+o(1)\}$. Since M can be made arbitrarily large, one concludes $\tilde{h}(x)\le h_\sharp(x)\{1+o(1)\}$.

The reverse inequality is verified as follows. If $\alpha\ge1$, then taking $\varepsilon\in(0,1)$ so small that for $0<s<1$, $(1-\varepsilon s)^{\alpha-1}>1-\alpha\varepsilon$, one deduces

$$\tilde{h}(\varepsilon x)\ge\int_x^\infty\frac{(t-\varepsilon x)^{\alpha-1}}{\ell(t-\varepsilon x)}G(t)dt\ge(1-\alpha\varepsilon)h_\sharp(x)\{1+o(1)\},$$

which shows $\tilde{h}(x)\ge h_\sharp(x)\{1+o(1)\}$. Thus $\tilde{h}(x)\sim h_\sharp(x)$ if $\alpha\ge1$. Let $\alpha<1$. We may suppose ℓ to be normalized so that for all sufficiently large x, $\tilde{u}(t)\ge\tilde{u}(x+t)$ $(t\ge0)$, hence $\tilde{h}(x)\ge h_\sharp(x)$, so $\tilde{h}(x)\sim h_\sharp(x)$. □

Lemma 5.2. *Let* $V(x)$ *and* $\mu(x)$ *be positive functions on* $x\ge0$*. If* μ *is non-increasing,* $\limsup\mu(\lambda x)/\mu(x)<1$ *for some* $\lambda>1$ *and* V *is non-decreasing and s.v., then* $\int_1^\infty\mu(t)V(t)<\infty$ *and*

$$\int_x^\infty\mu(t)dV(t)=o\big(V(x)\mu(x)\big)\quad(x\to\infty).$$

Proof. We may suppose that V is continuous and there exists constants $\delta < 1$ and x_0 such that $\mu(\lambda x) < \delta\mu(x)$ for $x \geq x_0$. Let $x > x_0$. It then follows that

$$\int_x^\infty \mu(t)dV(t) = \int_{n=0}^\infty \int_{\lambda^n x}^{\lambda^{n+1}x} \mu(t)dV(t) \leq \mu(x)\sum_{n=0}^\infty \delta^n[V(\lambda^{n+1}x) - V(\lambda^n x)]. \tag{5.1}$$

Take $\varepsilon > 0$ so that $(1+\varepsilon)\delta < 1$. Then for all sufficiently large x, $V(\lambda x)/V(x) \leq 1+\varepsilon$ so that $V(\lambda^n x)/V(x) < (1+\varepsilon)^n$, hence $\sum_{n=0}^\infty \delta^n V(\lambda^n x) \leq CV(x)$, which shows that the last member in (5.1) is $o(\mu(x)V(x))$, for $V(\lambda^{n+1}x) - V(\lambda^2 x) = o((V(\lambda^n x))$ uniformly in n. □

Lemma 5.3. *Let $v(t)$ and $G(t)$ be measurable functions on $t \geq 0$ such that $V(t) := \int_0^t v(s)ds$ is s.v. at infinity, G is non-increasing and $\int_1^\infty v(t)G(t)dt < \infty$. Then as $t \to \infty$*

$$\frac{\int_t^\infty v(s)G(s)ds}{V(t)G(t) + \int_t^\infty V(s)G(s)ds/s} \longrightarrow 0.$$

Proof. Let $\tilde{V}$ be a normalized version of V: $\tilde{V}(t) \sim V(t)$, V is differentiable and $\tilde{v}(t) := \tilde{V}'(t) = o(V(t)/t)$. We show that

$$\int_t^\infty v(s)G(s)ds = \int_t^\infty \tilde{v}(s)G(s)ds\{1+o(1)\} + o\big(V(t)G(t)\big), \tag{5.2}$$

which implies the asserted result. Put $D(t) = \int_t^\infty v(s)G(s)ds$, $E(t) = -\int_t^\infty V(s)G(s)ds$. Then $E(t) = V(t)G(t) + D(t)$. Since $E(t) \sim -\int_t^\infty \tilde{V}(s)dG(s)$, this entails

$$\begin{aligned} D(t) &= -\tilde{V}(t)G(t)\{1+o(1)\} - \int_t^\infty \tilde{V}(s)dG(s) + o(E(t)) \\ &= \int_t^\infty \tilde{v}(s)G(s)ds + o\big(V(t)G(t)\big); + o(E(t)). \end{aligned}$$

Since $o(E(t)) = o(V(t)G(t)) + o(D(t))$, we obtain (5.2).

(B) Let F be transient so that we have the Green kernel $G(x) := \sum_{n=0}^\infty P[S_n = x] < \infty$. For $y \geq 0, x \in \mathbb{Z}$,

$$G(y-x) - g_\Omega(x,y) = \sum_{w=1}^\infty P_x[S_T = -w]G(y+w). \tag{5.3}$$

According to the Feller-Orey renewal theorem [6, Section XI.9], $\lim_{|x|\to\infty} G(x) = 0$ (under $E|X| < \infty$), showing that the RHS above tends to zero as $y \to \infty$ (uniformly in $x \in \mathbb{Z}$), in particular $\lim g_\Omega(x,x) = G(0) = 1/P[\sigma_{\{0\}} = \infty]$. It also follows that $P_x[\sigma_{\{0\}} < \infty] = G(-x)/G(0) \to 0$.

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