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On minimising a portfolio's shortfall probability

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Abstract

We obtain a lower asymptotic bound on the decay rate of the probability of a portfolio's underperformance against a benchmark over a large time horizon. It is assumed that the prices of the securities are governed by geometric Brownian motions with the coefficients depending on an economic factor, possibly nonlinearly. The bound is tight so that there exists a portfolio that optimises the decay rate. That portfolio is also risk-sensitive optimal.

1 Introduction

The problem of optimising the probability of underperformance of a financial portfolio against a benchmark over an infinite time horizon by using large deviation asymptotics has been studied by a number of authors. Discrete time setups were considered in Stutzer [30, 31, 32, 33]. Continuous-time models were treated in Hata, Nagai, and Sheu [12] and Nagai [25]. The methods of those papers used duality considerations and relied on connection with risk sensitive control. In Puhalskii [29], a direct approach was proposed and implemented, albeit for a simpler model. Whereas Hata, Nagai, and Sheu [12] and Nagai [25] assumed that the prices of the securities were influenced by an economic factor through the equation coefficients, the results of Puhalskii [29] concerned the case of constant coefficients in the associated geometric Brownian motion equations. The purpose of this paper is an extension of the methods of Puhalskii [29] to the model with the economic factor. Under general hypotheses, we obtain an asymptotic lower bound on the scaled by the length of the time period logarithmic probability of underperformance. The bound is tight in the sense that there exists a portfolio that attains it. We show, in addition, that the portfolio is risk-sensitive optimal.

Our results extend those available in the literature in a number of ways. Unlike Hata, Nagai, and Sheu [12], who assume that the coefficients in the equations for the security prices are affine functions of the economic factor and that the economic factor is represented by a Gaussian process, we tackle a general setup with the coefficients being nonlinear. Furthermore, Hata, Nagai, and Sheu [12] optimise the probability of underperformance assuming

that the risk-free interest rate does not depend on the economic factor and that no benchmark is involved. (Interestingly enough, the presence of the benchmark lends regularity to the problem.) As a matter of fact, we look at a model similar to that studied in Nagai [25], who allows nonlinearities in the coefficients. But, although that paper obtains a portfolio that is presumably optimal, it falls short of proving that fact. Besides, the author assumes that the benchmark is not volatile and a number of other conditions are required which narrow the scope of the contribution. The risk-sensitive framework has been studied more extensively, see Fleming and Sheu [10], Kuroda and Nagai [17], Davis and Lleo [8] and references therein. Those results are confined to the case of affine coefficients. A nonlinear model is looked at in Nagai [24] but fairly restrictive assumptions are required for their main result (Theorem 4.1) to hold. Thus, to the best of our knowledge, this paper is the first one to produce provably asymptotically optimal portfolios on an infinite time horizon for the fully nonlinear model in the shortfall probability setup. For risk-sensitive optimisation, our assumptions seem to be weaker and the range of models treated broader than in the available literature. A similar contention applies to optimising the shortfall probability for the case of affine coefficients. Besides, in our opinion, the methods employed in this paper are less involved and more direct than those used previously.

The advances in this paper are due to some novel approaches. The line of attack thus far has been to carry out the analysis of a Hamilton–Jacobi–Bellman equation on a finite time horizon in order to find an optimal control, first, and, afterwards, let the length of time tend to infinity. In this paper, a different viewpoint is adopted. By a change of variables we are able to cast the setup as a large deviation problem for two coupled diffusions with time scale separation. The economic factor can be assumed to "live in fast time" whereas the portfolio price is associated with a process that "lives in slow time". This insight enables us to take advantage of the methods developed for such diffusions in Liptser [20] and Puhalskii [28]. In particular, the empirical measure of the factor process plays a pivotal role in our study.

This is how this paper is organised. In section 2, the model under consideration is formulated, main results are stated and relation to earlier contributions is discussed in more detail. Section 3 contains auxiliary results needed for the proofs and the main results are proved in Section 4.

2 A model description and main results

We consider a portfolio consisting of n risky securities priced $S_t^1, \dots, S_t^n$ at time t and a safe security of price S_t^0 at time t. We assume that, for $i = 1, 2, \dots, n$,

$$\frac{dS_t^i}{S_t^i} = a^i(X_t)\,dt + b^i(X_t)^T\,dW_t\,,$$

and that

$$\frac{dS_t^0}{S_t^0} = r(X_t)\,dt\,,$$

where X_t represents an economic factor. It is governed by the equation

$$dX_t = \theta(X_t)\,dt + \sigma(X_t)\,dW_t\,. \tag{2.1}$$

In the equations above, the $a^i(x)$ are real-valued functions, the $b^i(x)$ are $\mathbb{R}^k$-valued functions, $\theta(x)$ is an $\mathbb{R}^l$-valued function, $\sigma(x)$ is an $l \times k$-matrix, W_t is a k-dimensional standard Wiener process, and $S_0^i > 0$, ^{T} is used to denote the transpose of a matrix or a vector. Accordingly, the process $X = (X_t\,, t \in \mathbb{R}_+)$ is l-dimensional.

Benchmark $Y = (Y_t\,, t \in \mathbb{R}_+)$ follows an equation similar to those for the risky securities:

$$\frac{dY_t}{Y_t} = \alpha(X_t)\,dt + \beta(X_t)^T\,dW_t,$$

where $\alpha(x)$ is an $\mathbb{R}$-valued function, $\beta(x)$ is an $\mathbb{R}^k$-valued function, and $Y_0 > 0$.

All processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is assumed, furthermore, that the processes $S^i = (S_t^i\,, t \in \mathbb{R}_+)$, X, and $Y = (Y_t\,, t \in \mathbb{R}_+)$ are adapted to filtration $\mathbf{F} = (\mathcal{F}_t\,, t \in \mathbb{R}_+)$ and that $W = (W_t\,, t \in \mathbb{R}_+)$ is an $\mathbf{F}$-Wiener process. We note that assuming that the S_t^i, X_t, and Y_t are driven by the same Wiener process is not a loss of generality.

We introduce more notation and assumptions. We let $a(x)$ denote the n-vector with entries $a^1(x), \ldots, a^n(x)$, let $b(x)$ denote the $n \times k$ matrix with rows $b^1(x)^T, \ldots, b^n(x)^T$ and let $\mathbf{1}$ denote the n-vector with unit entries. The matrices $b(x)b(x)^T$ and $\sigma(x)\sigma(x)^T$ are assumed to be uniformly positive definite and bounded. The functions $a(x)$, $r(x)$, $\theta(x)$, $\alpha(x)$, $b(x)$, $\sigma(x)$, and $\beta(x)$ are assumed to be continuously differentiable and the function is $\sigma(x)\sigma(x)^T$ is assumed to be twice continuously differentiable. The function $|\beta(x)|^2$ is assumed to be bounded and bounded away from zero. (We will also indicate how the results change if the benchmark "is not volatile" meaning that $\beta(x) = 0$.) The following "linear growth" condition is assumed: for some $K > 0$ and all $x \in \mathbb{R}^l$,

$$|a(x)| + |r(x)| + |\alpha(x)| + |\theta(x)| \le K(1 + |x|)\,. \tag{2.2}$$

Under those hypotheses, the processes S^i, X, and Y are well defined, see, e.g., Chapter 5 in Karatzas and Shreve [16].

The investor holds l_t^i shares of risky security i and l_t^0 shares of the safe security at time t, so the total wealth is given by $Z_t = \sum_{i=1}^n l_t^i S_t^i + l_t^0 S_t^0$. Portfolio $\pi_t = (\pi_t^1, \ldots, \pi_t^n)^T$ specifies the proportions of the total wealth invested in the risky securities so that, for $i = 1, 2, \ldots, n$, $l_t^i S_t^i = \pi_t^i Z_t$. The processes $\pi^i = (\pi_t^i\,, t \in \mathbb{R}_+)$ are assumed to be $\mathcal{B} \otimes \mathcal{F}_t$-progressively measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}_+$, and such that $\int_0^t {\pi_s^i}^2\,ds < \infty\,, t \in \mathbb{R}_+$, a.s. We do not impose any other restrictions on the magnitudes of the π_t^i so that unlimited borrowing and shortselling are allowed. Let

$$L_t^\pi = \frac{1}{t}\,\ln\Big(\frac{Z_t}{Y_t}\Big)\,.$$

Given $q \in \mathbb{R}$, the objective is to minimise $\liminf_{t\to\infty}(1/t)\ln \mathbf{P}(L_t^\pi \le q)$ over all portfolios $\pi = (\pi_t\,, t \in \mathbb{R}_+)$ and identify portfolios for which the bound is attained.

Since the amount of wealth invested in the safe security is $(1 - \sum_{i=1}^n \pi_t^i) Z_t$, in a standard fashion by using the self financing condition, cf. Nagai [25], we obtain that

$$\frac{dZ_t}{Z_t} = \sum_{i=1}^n \pi_t^i\,\frac{dS_t^i}{S_t^i} + \big(1 - \sum_{i=1}^n \pi_t^i\big)\,\frac{dS_t^0}{S_t^0}\,.$$

Assuming that $Z_0 = Y_0$ and letting $c(x) = b(x)b(x)^T$, we have by Itô's lemma that, cf. Nagai [25] and Pham [27],

$$L_t^\pi = \frac{1}{t}\int_0^t \big(\pi_s^T a(X_s) + (1-\pi_s^T\mathbf{1})r(X_s) - \frac{1}{2}\,\pi_s^T c(X_s)\pi_s - \alpha(X_s) + \frac{1}{2}\,|\beta(X_s)|^2\big)\,ds + \frac{1}{t}\int_0^t \big(b(X_s)^T\pi_s - \beta(X_s)\big)^T\,dW_s\,. \quad (2.3)$$

Let, for $u \in \mathbb{R}^n$ and $x \in \mathbb{R}^l$,

$$M(u,x) = u^T(a(x) - r(x)\mathbf{1}) - \frac{1}{2}\,u^T c(x)u + r(x) - \alpha(x) + \frac{1}{2}\,|\beta(x)|^2 \quad (2.4a)$$

and

$$N(u,x) = b(x)^T u - \beta(x)\,. \quad (2.4b)$$

A change of variables brings equation (2.3) to the form

$$L_t^\pi = \int_0^1 M(\pi_{ts}, X_{ts})\,ds + \frac{1}{\sqrt{t}}\int_0^1 N(\pi_{ts}, X_{ts})^T\,dW_s^t\,, \quad (2.5)$$

where $W_s^t = W_{ts}/\sqrt{t}$. We note that $W^t = (W_s^t,\, s \in [0,1])$ is a Wiener process relative to $\mathbf{F}^t = (\mathcal{F}_{ts},\, s \in [0,1])$. The righthand side of (2.5) can be viewed as a diffusion process with a small diffusion coefficient that "lives in slow time" represented by the variable s, whereas in X and π "time" is accelerated by a factor of t. Similar setups have been considered in Liptser [20] and Puhalskii [28]. Those papers show that in order to study the large deviation properties of the "slow" process it is convenient to work with the pair that comprises the slow process and the empirical measure of the fast process. For the equations (2.1) and (2.5), this means working with the pair (L_t^π, μ^t), where $\mu^t = (\mu^t(ds,dx))$ represents the empirical process of $(X_{ts}\,, s \in [0,1])$, which is defined by the relation

$$\mu^t([0,s],\Gamma) = \int_0^s \chi_\Gamma(X_{t\tilde{s}})\,d\tilde{s}\,,$$

with Γ representing an arbitrary Borel subset of $\mathbb{R}^l$ and with $\chi_\Gamma(x)$ representing the indicator function of the set Γ. Letting $\pi_s^t = \pi_{ts}$ and $X_s^t = X_{ts}$ in (2.5) obtains that

$$L_t^\pi = \int_0^1 M(\pi_s^t, X_s^t)\,ds + \frac{1}{\sqrt{t}}\int_0^1 N(\pi_s^t, X_s^t)^T\,dW_s^t = \int_0^1\int_{\mathbb{R}^l} M(\pi_s^t, x)\,\mu^t(ds,dx) + \frac{1}{\sqrt{t}}\int_0^1 N(\pi_s^t, X_s^t)^T\,dW_s^t\,. \quad (2.6)$$

We note that both $X^t = (X^t_s, s \in [0,1])$ and $\pi^t = (\pi^t_s, s \in [0,1])$ are $\mathbf{F}^t$-adapted.

Since, by (2.1) and Itô's lemma, for a twice continuously differentiable function f on $\mathbb{R}^l$, with ∇f and $\nabla^2 f$ denoting the gradient and the Hessian of f, respectively, and with tr standing for the trace of a square matrix,

$$f(X_t) = f(X_0) + \int_0^t \nabla f(X_s)^T \theta(X_s)\, ds + \frac{1}{2}\int_0^t \mathrm{tr}\left(\sigma(X_s)\sigma(X_s)^T \nabla^2 f(X_s)\right) ds + \int_0^t \nabla f(X_s)^T \sigma(X_s)\, dW_s \quad (2.7)$$

and since the process $\left(\exp\left(\int_0^t (-\lambda N(\pi_s, X_s) + \sigma(X_s)^T \nabla f(X_s))^T\, dW_s - (1/2)\int_0^t |-\lambda N(\pi_s, X_s) + \sigma(X_s)^T \nabla f(X_s)|^2\, ds\right), t \in \mathbb{R}_+\right)$ is a local martingale, by (2.6) and (2.7),

$$\mathbf{E}\exp\Big(-t\lambda L^\pi_t - t\int_0^1 -\lambda M(\pi^t_s, X^t_s)\, ds + f(X^t_1) - f(X_0) - t\int_0^1 \nabla f(X^t_s)^T\, \theta(X^t_s)\, ds - \frac{t}{2}\int_0^1 \mathrm{tr}\left(\sigma(X^t_s)\sigma(X^t_s)^T\, \nabla^2 f(X^t_s)\right) ds - \frac{t}{2}\int_0^1 |-\lambda N(\pi^t_s, X^t_s) + \sigma(X^t_s)^T \nabla f(X^t_s)|^2\, ds\Big) \le 1\,. \quad (2.8)$$

Intuitively, if we assume that equality prevails in (2.8), which would be the case under certain integrability conditions on π_s and $\nabla f(X_s)$, then, for $\lambda \ge 0$, L^π_t is "maximised" under the constraint (2.8) by minimising the integrals over π^t_s, i.e., by choosing $\pi^t_s = u(X^t_s)$ with $u(x)$ attaining $\inf_{u\in\mathbb{R}^n}\left(-\lambda M(u,x) + (1/2)\,|-\lambda N(u,x) + \sigma(x)^T \nabla f(x)|^2\right)$. For that portfolio,

$$\mathbf{E}\chi_{\{L^\pi_t \le q\}} \exp\Big(-t\lambda q + f(X^t_1) - f(X_0) - t\int_0^1 \int_{\mathbb{R}^l} \Big(\inf_{u\in\mathbb{R}^n}\Big(-\lambda M(u,x) + \frac{1}{2}\,|-\lambda N(u,x) + \sigma(x)^T\nabla f(x)|^2\Big) + \nabla f(x)^T\, \theta(x) + \frac{1}{2}\,\mathrm{tr}\left(\sigma(x)\sigma(x)^T\, \nabla^2 f(x)\right)\Big)\mu^t(ds, dx)\Big) \le 1\,.$$

Consequently,

$$\frac{1}{t}\,\ln \mathbf{E}\chi_{\{L^\pi_t \le q\}} \exp(f(X^t_1) - f(X_0)) \le \lambda q + \sup_{\nu\in\mathcal{P}} \int_{\mathbb{R}^l} \Big(\inf_{u\in\mathbb{R}^n}\Big(-\lambda M(u,x) + \frac{1}{2}\,|-\lambda N(u,x) + \sigma(x)^T\nabla f(x)|^2\Big) + \nabla f(x)^T\, \theta(x) + \frac{1}{2}\,\mathrm{tr}\left(\sigma(x)\sigma(x)^T\, \nabla^2 f(x)\right)\Big)\nu(dx)\Big)\,, \quad (2.9)$$

where $\mathcal{P}$ represents the set of probability measures ν on $\mathbb{R}^l$ such that $\int_{\mathbb{R}^l} |x|^2\, \nu(dx) < \infty$. The "best" upper bound on the normalised logarithmic shortfall probability $(1/t)\ln \mathbf{P}(L^\pi_t \le q)$ is obtained by minimising the righthand side over $\lambda \ge 0$ and a suitable collection of functions f so that the optimal portfolio should be apparently associated with λ and f that minimise the

righthand side of (2.9). The main results of the paper bear out that intuition. Furthermore, we show that the upper bound for that particular portfolio choice furnishes a lower bound for all portfolios.

Before we state the main results, more conditions are in order. We assume the stability condition that there exist bounded Borel-measurable function $\Phi(x)$ with values in the set of $l \times n$-matrices and positive definite symmetric $l \times l$-matrix Ψ such that

$$\limsup_{|x|\to\infty} \big(\theta(x) - \Phi(x)(a(x) - r(x)\mathbf{1})\big)^T \frac{\Psi x}{|x|^2} < 0\,. \tag{2.10}$$

(If $\Phi(x) = 0$ and Ψ is the $l\times l$ identity matrix, then one recovers Has'minskii's drift condition.)

The following nondegeneracy condition is also needed. Let I_k represent the $k\times k$-identity matrix and let $Q_1(x) = I_k - b(x)^T c(x)^{-1} b(x)$. The matrix $Q_1(x)$ represents the orthogonal projection operator onto the null space of $b(x)$ in $\mathbb{R}^k$. We will assume that

(N)
1. The matrix $\sigma(x)Q_1(x)\sigma(x)^T$ is uniformly positive definite.
2. The quantity $\beta(x)^T Q_2(x)\beta(x)$ is bounded away from zero, where $Q_2(x) = Q_1(x)\big(I_k - \sigma(x)^T(\sigma(x)Q_1(x)\sigma(x)^T)^{-1}\sigma(x)\big)Q_1(x)$.

This condition admits the following geometric interpretation. The matrix $\sigma(x)Q_1(x)\sigma(x)^T$ is uniformly positive definite if and only if the ranges of $\sigma(x)^T$ and $b(x)^T$ do not have common nontrivial subspaces and, in addition, arbitrary nonzero vectors from those respective ranges are at angles bounded away from zero if and only if the matrix $c(x) - b(x)\sigma(x)^T(\sigma(x)\sigma(x)^T)^{-1}\sigma(x)b(x)^T$ is uniformly positive definite. Also, $\beta(x)^T Q_2(x)\beta(x)$ is bounded away from zero if and only if the projection of $\beta(x)$ onto the null space of $b(x)$ is of length bounded away from zero and is at angles bounded away from zero to all nonzero vectors from the projection of the range of $\sigma(x)^T$ onto that null space. Under part 1 of condition (N), we have that $k \geq n+l$ and the rows of the matrices $\sigma(x)$ and $b(x)$ are linearly independent. Part 2 of condition (N) implies that $\beta(x)$ does no belong to the range of $b(x)^T$. Available literature on the subject does not utilise condition (N).

We will also assume that $|x^T \mathrm{div}\,(\sigma(x)\sigma(x)^T)|$ is a bounded function, the divergence of a square matrix being evaluated rowwise, and that the initial condition X_0 is such that $\mathbf{E}e^{\gamma|X_0|^2} < \infty$ for some $\gamma > 0$.

Let $\mathbb{P}$ represent the set of probability densities $m = (m(x)\,, x \in \mathbb{R}^l)$ such that $\int_{\mathbb{R}^l} |x|^2\, m(x)\, dx < \infty$. Let $\mathbb{C}^2$, $\mathbb{C}^2_0$, $\mathbb{C}^1$, and $\mathbb{C}^1_\ell$ represent the set of real-valued twice continuously differentiable functions on $\mathbb{R}^l$, the set of real-valued compactly supported twice continuously differentiable functions on $\mathbb{R}^l$, the set of real-valued continuously differentiable functions on $\mathbb{R}^l$, and the set of real-valued continuously differentiable functions on $\mathbb{R}^l$ whose gradients satisfy the linear growth condition, respectively. For a $\mathbb{C}^2_0$-function f, density $m \in \mathbb{P}$ and $\lambda \geq 0$, we let

$$\begin{aligned} G(\lambda, f, m) = \int_{\mathbb{R}^l} \Big(-\lambda \sup_{u\in\mathbb{R}^n} \big(M(u,x) - \frac{1}{2}\,\lambda|N(u,x)|^2 + \nabla f(x)^T \sigma(x) N(u,x)\big) + \nabla f(x)^T\, \theta(x) \\ + \frac{1}{2}\,|\sigma(x)^T \nabla f(x)|^2 + \frac{1}{2}\,\mathrm{tr}\,\big(\sigma(x)\sigma(x)^T \nabla^2 f(x)\big)\Big)\, m(x)\, dx\,. \end{aligned} \tag{2.11}$$

Let us define, for $\lambda \geq 0$,

$$F(\lambda) = \sup_{m \in \mathbb{P}} \inf_{f \in \mathbb{C}_0^2} G(\lambda, f, m)\,. \tag{2.12}$$

We show in Lemma 3.4 below that the function $F(\lambda)$ is strictly convex and differentiable.

Let

$$J_q^s = \sup_{\lambda \geq 0}(-\lambda q - F(\lambda))\,. \tag{2.13}$$

The following theorem furnishes the asymptotic bound on the shortfall probability.

Theorem 2.1. *For arbitrary portfolio* $\pi = (\pi_t\,, t \in \mathbb{R}_+)$,

$$\liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(L_t^\pi \leq q) \geq -J_q^s\,.$$

We produce now an asymptotically optimal portfolio. As shown in Lemma 3.4 below, $F(\lambda)$ converges to ∞ as $\lambda \to \infty$, so the supremum in (2.13) is attained, at unique $\hat{\lambda}$. Furthermore, by Lemma 3.5 below, the function $\lambda q + G(\lambda, f, m)$, being convex in (λ, f) and concave in m, has saddle point $((\hat{\lambda}, \hat{f}), \hat{m})$ in $\mathbb{R}_+ \times (\mathbb{C}_\ell^1 \cap \mathbb{C}^2) \times \hat{\mathbb{P}}$, with $\hat{\lambda}$, $\nabla\hat{f}$, and $\hat{m}$ being specified uniquely. Also, $\hat{m}$ can be assumed to be bounded, positive and continuously differentiable.

By Lemma 3.5 once again, $(\hat{f}, \hat{m})$ satisfies the equations

$$\begin{aligned} -\hat{\lambda}\big(M(\hat{u}(x), x) - \frac{1}{2}\,\hat{\lambda}|N(\hat{u}(x), x)|^2 + \nabla\hat{f}(x)^T\sigma(x)N(\hat{u}(x), x)\big) \\ + \nabla\hat{f}(x)^T\theta(x) + \frac{1}{2}\,|\sigma(x)^T\nabla\hat{f}(x)|^2 + \frac{1}{2}\,\mathrm{tr}\,(\sigma(x)\sigma(x)^T\,\nabla^2\hat{f}(x)) = F(\hat{\lambda}) \end{aligned} \tag{2.14a}$$

and

$$\begin{aligned} \int_{\mathbb{R}^l} \big(\nabla h(x)^T(-\hat{\lambda}\sigma(x)N(\hat{u}(x), x) + \theta(x) + \sigma(x)\sigma(x)^T\nabla\hat{f}(x)) \\ + \frac{1}{2}\,\mathrm{tr}\,(\sigma(x)\sigma(x)^T\,\nabla^2 h(x))\big)\,\hat{m}(x)\,dx = 0\,, \end{aligned} \tag{2.14b}$$

with $\hat{u}(x)$ being given by the u that attains supremum in (2.11) for $\lambda = \hat{\lambda}$ and $f = \hat{f}$ so that

$$\hat{u}(x) = \frac{1}{1+\hat{\lambda}}\,c(x)^{-1}\big(a(x) - r(x)\mathbf{1} + \hat{\lambda}b(x)\beta(x) + b(x)\sigma(x)^T\nabla\hat{f}(x)\big) \tag{2.15}$$

and with (2.14a) and (2.14b) holding for all $x \in \mathbb{R}^l$ and for all $h \in \mathbb{C}_0^2$, respectively. In effect, (2.14a) and (2.14b) represent Euler–Lagrange equations for $G(\hat{\lambda}, f, m)$ at $(\hat{f}, \hat{m})$. Equation (2.14a) is known as an ergodic Bellman equation, see Fleming and Sheu [10], Kaise and Sheu [15], Hata, Nagai, and Sheu [12], and equation (2.14b) signifies that $\hat{m}$ is the invariant density of a certain diffusion. Kaise and Sheu [15], see also Ichihara [13], develop an elegant theory of the ergodic Bellman equation which is essential for our study.

We define

$$\hat{\pi}_t = \hat{u}(X_t)\,.$$

Theorem 2.2. *The portfolio* $\hat{\pi} = (\hat{\pi}_t\,, t \in \mathbb{R}_+)$ *attains the lower bound in Theorem 2.1:*

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(L_t^{\hat{\pi}} \le q) = -J_q^s\,.$$

(By Lemma 3.3 below, we also have that $F(\lambda) = \inf_{f\in\mathbb{C}_b^2} \sup_{m\in\mathbb{P}} G(\lambda, f, m)$, where $\mathbb{C}_b^2$ represents the set of twice continuously differentiable functions on $\mathbb{R}^l$ with bounded second derivatives. Thus, the assertion of Theorem 2.2 is consistent with the intuition provided after (2.9).)

The following theorem shows that, for suitable q, the portfolio $\hat{\pi}$ is also risk-sensitive optimal. For $\lambda > 0$, we let $\pi(\lambda)$ represent the portfolio $\hat{\pi}$ that is associated with $q = -F'(\lambda)$, the latter function representing the derivative of F.

Theorem 2.3. *Let* $\lambda > 0$. *For any portfolio* $\pi = (\pi_t\,, t \in \mathbb{R}_+)$,

$$\liminf_{t\to\infty} \frac{1}{t} \ln \mathbf{E} e^{-\lambda t L_t^{\pi}} \ge F(\lambda)\,.$$

Furthermore, if X_0 *is bounded a.s., then*

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{E} e^{-\lambda t L_t^{\pi(\lambda)}} = F(\lambda)\,.$$

The assertions of Theorems 2.1, 2.2, and 2.3 also hold in the case where $\beta(x) = 0$, provided $\inf_{x\in\mathbb{R}^l}(r(x) - \alpha(x)) < q$, with an optimal portfolio being defined similarly. If $\inf_{x\in\mathbb{R}^l}(r(x) - \alpha(x)) \ge q$, then investing in the safe security only is obviously optimal.

For the case where $\beta(x) = 0$, the portfolio in (2.15) appears in Theorem 2.5 in Nagai [25], who obtains the limit in Theorem 2.2 under a number of additional hypotheses (see (2.19), (2.21), (2.25) there) and the requirement that $0 < q < -F'(0+)$, thus ruling out the possibility that $\hat{\lambda} = 0$. The stability condition (2.10) is assumed to hold with $\Phi(x) = \sigma(x)b(x)^T c(x)^{-1}$ and $\Psi(x)$ being the $l \times l$-identity matrix. Theorem 2.4 in Nagai [25] establishes a similar limit for a non time-homogeneous portfolio. No analogue of Theorem 2.1 is obtained. Theorem 4.1 in Nagai [24] which obtains optimality of $\pi(\lambda)$ in the vein of Theorem 2.3, requires not only that $\beta(x) = 0$ but also that a number of other conditions hold including a certain growth condition which involves the solution of the ergodic Bellman equation. No representation (2.12) is obtained.

In the linear Gaussian case, finding the optimal portfolio $\hat{\pi}$ can be reduced to solving an algebraic Riccati equation. Let us assume that the functions $a(x)$, $r(x)$, $\alpha(x)$ and $\theta(x)$ are affine functions of x and that the diffusion coefficients are constant. More specifically,

$$\begin{aligned} a(x) &= A_1 x + a_2\,, \\ r(x) &= r_1^T x + r_2\,, \\ \alpha(x) &= \alpha_1^T x + \alpha_2\,, \\ \theta(x) &= \Theta_1 x + \theta_2 \end{aligned}$$

and

$$b(x) = b,\ \beta(x) = \beta,\ \sigma(x) = \sigma\,,$$

where $A_1 \in \mathbb{R}^{n\times l}$, $a_2 \in \mathbb{R}^n$, $r_1 \in \mathbb{R}^l$, $r_2 \in \mathbb{R}$, $\alpha_1 \in \mathbb{R}^l$, $\alpha_2 \in \mathbb{R}$, $\Theta_1 \in \mathbb{R}^{l\times l}$, $\theta_1 \in \mathbb{R}^l$, b is an $n \times k$-matrix such that the matrix bb^T is positive definite, β is a non-zero k-vector, and σ is an $l \times k$-matrix such that the matrix $\sigma\sigma^T$ is positive definite. Condition (2.10) is fulfilled provided the pair $(A_1 - \mathbf{1}r_1^T, \Theta_1)$ is detectable, i.e., there exists $l \times n$–matrix Φ such that the matrix $\Theta_1 - \Phi(A_1 - \mathbf{1}r_1^T)$ is stable, for, in that case, there exists symmetric positive definite $l \times l$–matrix Ψ such that $(\Theta_1 - \Phi(A_1 - \mathbf{1}r_1^T))^T\Psi + \Psi(\Theta_1 - \Phi(A_1 - \mathbf{1}r_1^T)) = -I_l$, see, e.g., p.252 in Bellman [4] or Theorem 8.7.2 on p.270 in Lancaster [19], so one can take $\Phi(x) = \Phi$, where I_l stands for the $l \times l$ identity matrix.

We introduce

$$\begin{aligned} A &= \Theta_1 - \frac{\hat\lambda}{1+\hat\lambda}\,\sigma b^T c^{-1}(A_1 - \mathbf{1}r_1^T), \\ B &= \sigma\sigma^T - \frac{\hat\lambda}{1+\hat\lambda}\,\sigma b^T c^{-1} b\sigma^T\,, \\ C &= (A_1 - \mathbf{1}r_1^T)^T c^{-1}(A_1 - \mathbf{1}r_1^T)\,. \end{aligned}$$

Let us suppose that there exists symmetric matrix $\hat P_1$ that satisfies the algebraic Riccati equation

$$A^T\hat P_1 + \hat P_1 A - \hat P_1 B\hat P_1 + \frac{\hat\lambda}{1+\hat\lambda}\,C = 0\,.$$

Conditions for the existence of solutions can be found in Kuroda and Nagai [17], Willems [36], Wonham [37], Fleming and Sheu [10]. For instance, if $\Theta_1 - \sigma b^T c^{-1}(A_1 - \mathbf{1}r_1^T)$ is a stable matrix, then there exists a nonnegative definite symmetric matrix that satisfies the equation, see Proposition 2.2 in Kuroda and Nagai [17]. Furthermore, by Proposition 2.3 in Hata, Nagai, and Sheu [12], the matrix

$$D = -\frac{\hat\lambda}{1+\hat\lambda}\,\sigma b^T c^{-1}(A_1 - \mathbf{1}r_1^T) + \Theta_1 + \big(\sigma\sigma^T - \frac{\hat\lambda}{1+\hat\lambda}\,\sigma b^T c^{-1} b\sigma^T\big)\hat P_1$$

is stable. With D being stable, the equation

$$D^T\hat p_2 - \frac{\hat\lambda}{1+\hat\lambda}\,(A_1 - \mathbf{1}r_1^T + b\sigma^T\hat P_1)^T c^{-1}(a_2 - r_2\mathbf{1} + \hat\lambda b\beta) - \hat\lambda(r_1 - \alpha_1 - \hat P_1\sigma\beta) + \hat P_1\theta_2 = 0$$

has a unique solution for $\hat p_2$. The function $\hat f(x) = x^T\hat P_1 x/2 + \hat p_2 x$ solves the ergodic Bellman equation (2.14a), where

$$\hat u(x) = \frac{1}{1+\hat\lambda}\,c^{-1}\big(A_1 - \mathbf{1}r_1^T + b\sigma^T\hat P_1)x + \frac{1}{1+\hat\lambda}\,c^{-1}\big(a_2 - r_2\mathbf{1} + \hat\lambda b\beta + b\sigma^T\hat p_2\big) \qquad (2.17)$$

so that the optimal control is linear in the economic factor. By (2.14b), $\hat m$ is the invariant density of the Ornstein-Uhlenbeck diffusion

$$dY_t = DY_t\,dt + \big(-\frac{\hat\lambda}{1+\hat\lambda}\,\sigma b^T c^{-1}(a_2 - r_2\mathbf{1} + \hat\lambda b\beta + b\sigma^T\hat p_2) + \hat\lambda\sigma\beta + \sigma\sigma^T\hat p_2 + \theta_2\big)\,dt + \sigma\,dW_t$$

and

$$F(\hat{\lambda}) = -\frac{1}{2}\,\frac{\hat{\lambda}}{1+\hat{\lambda}}\,\|a_2 - r_2\mathbf{1} + \hat{\lambda} b\beta + b\sigma^T\hat{p}_2\|^2_{c^{-1}} - \hat{\lambda}(r_2 - \alpha_2 + \frac{1}{2}\,|\beta|^2 - \beta^T\sigma^T\hat{p}_2)$$
$$+ \frac{1}{2}\,\hat{\lambda}^2|\beta|^2 + \frac{1}{2}\,\hat{p}_2^T\sigma\sigma^T\hat{p}_2 + \hat{p}_2^T\theta_2 + \frac{1}{2}\,\mathrm{tr}\,(\sigma\sigma^T\,\hat{P}_1)\,.$$

The portfolio in (2.17) is obtained and its optimality is proved in Hata, Nagai, and Sheu [12] (see (2.39) and Theorem 2.2 there). They also obtain an asymptotically optimal non time homogeneous portfolio. For the optimality of the time homogeneous portfolio, those authors, who assume that $r_1 = 0$, in addition to requiring that the matrix $\Theta_1 - \sigma b^T c^{-1} A_1$ be stable, need that (Θ_1, σ) be controllable, that the matrix $P_1\sigma b^T c^{-1} b\sigma^T P_1 - A_1^T c^{-1} A_1$ be negative definite, and that $q < -F'(0+)$. Our results do away with those restrictions as well as capture the case of nonzero r_1, α_1, α_2, and β. The portfolio of Theorem 2.3 for the linear Gaussian case and with no benchmark was obtained in Fleming and Sheu [10] (with no optimality proof) and in Kuroda and Nagai [17] (with an optimality proof) and in Davis and Lleo [8] for the benchmark case. Kuroda and Nagai [17] and Davis and Lleo [8] assume conditions that are similar to those needed in Hata, Nagai, and Sheu [12] to assert the optimality of the time homogeneous portfolio.

3 Technical preliminaries

In this section, we lay the groundwork for the proofs of the main results.

Let, given $x \in \mathbb{R}^l$, $\lambda \geq 0$, and $y \in \mathbb{R}^l$,

$$\hat{H}(x;\lambda,y) = -\lambda \sup_{u\in\mathbb{R}^n}\big(M(u,x) - \frac{1}{2}\,\lambda|N(u,x)|^2 + y^T\sigma(x)N(u,x)\big) + y^T\,\theta(x) + \frac{1}{2}\,|\sigma(x)^T y|^2\,. \tag{3.1}$$

One can thus write (2.11) more succintly as

$$G(\lambda, f, m) = \int_{\mathbb{R}^l}\big(\hat{H}(x;\lambda,\nabla f(x)) + \frac{1}{2}\,\mathrm{tr}\,\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big)\big)\,m(x)\,dx\,. \tag{3.2}$$

We note that

$$\sup_{u\in\mathbb{R}^n}\big(M(u,x) - \frac{1}{2}\,\lambda|N(u,x)|^2 + y^T\sigma(x)N(u,x)\big)$$
$$= \frac{1}{2}\,\frac{1}{1+\lambda}\,\|a(x) - r(x)\mathbf{1} + \lambda b(x)\beta(x) + b(x)\sigma(x)^T y\|^2_{c(x)^{-1}}$$
$$- \frac{1}{2}\,\lambda|\beta(x)|^2 + r(x) - \alpha(x) + \frac{1}{2}\,|\beta(x)|^2 - \beta(x)^T\sigma(x)^T g \tag{3.3}$$

where we use the notational convention that, for $z \in \mathbb{R}^d$ and positive definite symmetric

$d \times d$-matrix V, $\|z\|_V^2 = z^T V z$, so that

$$\hat{H}(x;\lambda,y) = \frac{1}{2}\, y^T\big(\sigma(x)\sigma(x)^T - \frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} b(x)\sigma(x)^T\big) y$$
$$+ \big(-\frac{\lambda}{1+\lambda}\,(a(x) - r(x)\mathbf{1} + \lambda b(x)\beta(x))^T c(x)^{-1} b(x)\sigma(x)^T + \lambda\beta(x)^T\sigma(x)^T + \theta(x)^T\big) y$$
$$- \frac{\lambda}{2(1+\lambda)}\,\|a(x) - r(x)\mathbf{1} + \lambda b(x)\beta(x)\|^2_{c(x)^{-1}} - \lambda(r(x) - \alpha(x) + \frac{1}{2}\,|\beta(x)|^2) + \frac{1}{2}\,\lambda^2|\beta(x)|^2\,. \tag{3.4}$$

We recall, see, e.g., Aubin and Ekeland [3] or Bonnans and Shapiro [7], that a function $h : \mathbb{T} \to \mathbb{R}$, with $\mathbb{T}$ representing a topological space, is inf–compact (respectively, sup–compact) if the sets $\{x \in \mathbb{T} : h(x) \le \delta\}$ (respectively, the sets $\{x \in \mathbb{T} : h(x) \ge \delta\}$) are compact for all $\delta \in \mathbb{R}$. We endow the set $\mathcal{P}$ of probability measures ν on $\mathbb{R}^l$ such that $\int_{\mathbb{R}^l} |x|^2\,\nu(dx) < \infty$ with the Kantorovich–Rubinstein distance

$$d_1(\mu,\nu) = \sup\{|\int_{\mathbb{R}^l} g(x)\,\mu(dx) - \int_{\mathbb{R}^l} g(x)\,\nu(dx)| :\ \frac{|g(x)-g(y)|}{|x-y|} \le 1 \text{ for all } x \ne y\}\,.$$

Convergence with respect to d_1 is equivalent to weak convergence coupled with the convergence of first moments, see, e.g., Villani [35].

We introduce

$$H(x;\lambda,f) = \hat{H}(x;\lambda,\nabla f(x)) + \frac{1}{2}\,\mathrm{tr}\,\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big) \tag{3.5}$$

so that

$$G(\lambda,f,m) = \int_{\mathbb{R}^l} H(x;\lambda,f)\,m(x)\,dx\,. \tag{3.6}$$

For $\kappa > 0$, let $\overline{f}_\kappa(x) = \kappa|\Psi x|^2$ and let $\mathcal{A}_\kappa$ represent the convex hull of $\mathbb{C}_0^2$ and of the function $\overline{f}_\kappa$.

Lemma 3.1. *For all $\kappa > 0$ small enough, the function $\int_{\mathbb{R}^l} H(x;\lambda,\overline{f}_\kappa)\,\nu(dx)$ is* sup*–compact in $\nu \in \mathcal{P}$ for the Kantorovich–Rubinstein distance d_1. The function $\inf_{f\in\mathbb{C}_0^2}\int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$ is* sup*–compact in ν. Furthermore, the set $\cup_{\{\lambda:\,|\lambda-\overline{\lambda}|\le\overline{\lambda}/2\}}\{\nu \in \mathcal{P} :\ \inf_{f\in\mathbb{C}_0^2}\int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \ge \delta\}$ is relatively compact, where $\overline{\lambda} > 0$.*

Proof. By (3.4),

$$\hat{H}(x;\lambda,\nabla\overline{f}_\kappa(x)) = \frac{\kappa^2}{2}\,x^T\Psi\big(\sigma(x)\sigma(x)^T - \frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} b(x)\sigma(x)^T\big)\Psi x$$
$$+ \kappa\big(\theta(x) - \Phi(x)(a(x) - r(x)\mathbf{1})\big)^T\Psi x$$
$$+ \kappa\big(\big(-\frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} + \Phi(x)\big)\,(a(x) - r(x)\mathbf{1})\big)^T\Psi x - \frac{\lambda}{2(1+\lambda)}\,\|a(x) - r(x)\mathbf{1}\|^2_{c(x)^{-1}}$$
$$+ \kappa\big(-\frac{\lambda^2}{1+\lambda}\,\beta(x)^T b(x)^T c(x)^{-1} b(x)\sigma(x)^T + \lambda\beta(x)^T\sigma(x)^T\big)\Psi x$$

$$- \frac{\lambda}{2(1+\lambda)} \left(2\lambda(a(x) - r(x)\mathbf{1})^T c(x)^{-1} b(x)\beta(x) + \lambda^2 \|b(x)\beta(x)\|^2_{c(x)^{-1}}\right)$$
$$- \lambda(r(x) - \alpha(x) + \frac{1}{2}\,|\beta(x)|^2) + \frac{1}{2}\,\lambda^2 |\beta(x)|^2\,. \quad (3.7)$$

Since $\Phi(x)$ is a bounded function, by the Cauchy inequality there exists $K_1 > 0$ such that, for all $\epsilon > 0$,

$$\left(\left(-\frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} + \Phi(x)\right)(a(x) - r(x)\mathbf{1})\right)^T \Psi x \le \frac{1}{2\epsilon^2}\,\|a(x) - r(x)\mathbf{1}\|^2_{c(x)^{-1}} + K_1\,\frac{\epsilon^2}{2}|\Psi x|^2\,.$$

By condition (2.10), if κ and ϵ are small enough, then

$$\limsup_{|x|\to\infty} \frac{1}{|x|^2}\Big(\kappa K_1\,\frac{\epsilon^2}{2}|\Psi x|^2 + \frac{\kappa^2}{2}\,x^T\Psi\big(\sigma(x)\sigma(x)^T - \frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} b(x)\sigma(x)^T\big)\Psi x$$
$$+ \kappa\big(\theta(x) - \Phi(x)(a(x) - r(x)\mathbf{1})\big)^T \Psi x\Big) < 0\,.$$

Finally, given ϵ, κ can be chosen such that

$$\frac{\kappa}{2\epsilon^2}\,\|a(x) - r(x)\mathbf{1}\|^2_{c(x)^{-1}} - \frac{\lambda}{2(1+\lambda)}\,\|a(x) - r(x)\mathbf{1}\|^2_{c(x)^{-1}} \le 0\,.$$

Putting everything together and noting that the terms on the lower three lines of (3.7) grow at most linearly with $|x|$, we conclude that, provided κ is small enough, for suitable K_2 and $K_3 > 0$,

$$H(x;\lambda,\overline{f}) \le K_2 - K_3|x|^2\,. \quad (3.8)$$

Therefore, given $\delta \in \mathbb{R}$, $\sup_{\nu\in\Gamma_\delta} \int_{\mathbb{R}^l} |x|^2\,\nu(dx) < \infty$, where $\Gamma_\delta = \big\{\nu : \int_{\mathbb{R}^l} H(x;\lambda,\overline{f})\,\nu(dx) \ge \delta\big\}$. In addition, by $H(x;\lambda,\overline{f}_\kappa)$ being continuous in x, $\int_{\mathbb{R}^l} H(x;\lambda,\overline{f}_\kappa)\,\nu(dx)$ is an upper semicontinuous function of ν, so Γ_δ is a closed set. Thus, by Prohorov's theorem, Γ_δ is compact.

If $f \in \mathcal{A}_\kappa$, then by (3.4), (3.5), and (3.8), the function $\int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$ is upper semicontinuous in ν. Since $\overline{f}_\kappa \in \mathcal{A}_\kappa$, we obtain that $\inf_{f\in\mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$ is sup–compact. Since $\inf_{f\in\mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) = \inf_{f\in\mathbb{C}^2_0} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$, the latter function is sup–compact.

A closer examination of the reasoning that led to (3.8) reveals that there exist $\overline{K}_2$ and $\overline{K}_3 > 0$ such that $H(x;\lambda,\overline{f}_\kappa) \le \overline{K}_2 - \overline{K}_3|x|^2$ if $|\lambda - \overline{\lambda}| \le \overline{\lambda}/2$. Therefore,

$$\bigcup_{\{\lambda:\,|\lambda-\overline{\lambda}|\le\overline{\lambda}/2\}} \{\nu\in\mathcal{P} :\, \inf_{f\in\mathbb{C}^2_0} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \ge \delta\}$$
$$\subset \bigcup_{\{\lambda:\,|\lambda-\overline{\lambda}|\le\overline{\lambda}/2\}} \{\nu\in\mathcal{P} : \int_{\mathbb{R}^l} H(x;\lambda,\overline{f}_\kappa)\,\nu(dx) \ge \delta\} \subset \{\nu\in\mathcal{P} :\, \overline{K}_3 \int_{\mathbb{R}^l} |x|^2\,\nu(dx) \le \overline{K}_2 - \delta\}\,.$$

□

We let $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$ represent the Hilbert space of $\mathbb{R}^l$-valued functions $h(x)$ on $\mathbb{R}^l$ that are square integrable with respect to $m(x)\,dx$ equipped with the norm $\big(\int_{\mathbb{R}^l} |h(x)|^2\, m(x)\,dx\big)^{1/2}$ and we let $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$ represent the closure in $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$ of the set of gradients of $\mathbb{C}_0^1$-functions, with $\mathbb{C}_0^1$ denoting the set of real-valued compactly supported continuously differentiable functions on $\mathbb{R}^l$. We will use the notation ∇f for the elements of $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$, although those functions might not be proper gradients. Let $\hat{\mathbb{P}}$ represent the set of probability densities such that $m \in \mathbb{P}$, $m \in \mathbb{W}_{\rm loc}^{1,1}(\mathbb{R}^l)$, and $\sqrt{m} \in \mathbb{W}^{1,2}(\mathbb{R}^l)$, where $\mathbb{W}$ is used for denoting a Sobolev space, see, e.g., Adams and Fournier [1]. We note that $\hat{\mathbb{P}}$ is a convex subset of $\mathbb{P}$.

Lemma 3.2. *If, for $\nu \in \mathcal{P}$, $\inf_{f\in\mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) > -\infty$, then ν admits density which belongs to $\hat{\mathbb{P}}$.*

Proof. The reasoning follows that of Puhalskii [28], cf. Lemma 6.1, Lemma 6.4, and Theorem 6.1 there. If there exists $\kappa \in \mathbb{R}$ such that $\int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \geq \kappa$ for all $f \in \mathbb{C}_0^2$, then by (3.5), for arbitrary $\delta > 0$,

$$-\delta \int_{\mathbb{R}^l} \frac{1}{2} \operatorname{tr}\big(\sigma(x)\sigma(x)^T \nabla^2 f(x)\big)\,\nu(dx) \geq \kappa - \int_{\mathbb{R}^l} \hat{H}(x;\lambda,-\delta\nabla f(x))\,\nu(dx)\,.$$

On letting

$$\delta = \kappa^{1/2}\Big(\int_{\mathbb{R}^l} \nabla f(x)^T\big(\sigma(x)\sigma(x)^T - \frac{\lambda}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1} b(x)\sigma(x)^T\big)\nabla f(x)\,\nu(dx)\Big)^{-1/2},$$

we obtain with the aid of (3.4) that there exists constant $K > 0$ such that, for all $f \in \mathbb{C}_0^2$,

$$\int_{\mathbb{R}^l} \operatorname{tr}\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big)\;\nu(dx) \leq K\Big(\int_{\mathbb{R}^l} |\nabla f(x)|^2\,\nu(dx)\Big)^{1/2}.$$

It follows that the lefthand side extends to a linear functional on $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$, hence, by the Riesz representation theorem, there exists $h \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, \nu(dx))$ such that

$$\int_{\mathbb{R}^l} \operatorname{tr}\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big)\;\nu(dx) = \int_{\mathbb{R}^l} h(x)^T\nabla f(x)\,\nu(dx) \tag{3.9}$$

and $\int_{\mathbb{R}^l} |h(x)|^2 \nu(dx) \leq K$. Theorem 2.1 in Bogachev, Krylov, and Röckner [5] implies that the measure $\nu(dx)$ has density $m(x)$ with respect to Lebesgue measure which belongs to $L_{\rm loc}^{\vartheta}(\mathbb{R}^l)$ for all $\vartheta < l/(l-1)$. It follows that, for arbitrary open ball S in $\mathbb{R}^l$, there exists $M > 0$ such that for all $h \in \mathbb{C}_0^2(S)$,

$$\Big|\int_S \operatorname{tr}\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big)\,m(x)\,dx\Big| \leq M\Big(\int_S |\nabla h(x)|^{2\vartheta/(\vartheta-1)}\,dx\Big)^{(\vartheta-1)/(2\vartheta)}.$$

By Theorem 6.1 in Agmon [2], the density m belongs to $\mathbb{W}_{\rm loc}^{1,\varrho}(S)$ for all $\varrho < 2l/(2l-1)$. Furthermore, $h(x) = -\nabla m(x)/m(x)$ so that $\sqrt{m} \in \mathbb{W}^{1,2}(\mathbb{R}^l)$. □

Remark 3.1. Essentially, (3.9) signifies that one can integrate by parts on the lefthand side, so $m(x)$ has to be weakly differentiable.

Lemma 3.3. *Given λ, there exists $(f^\lambda, m^\lambda) \in (\mathbb{C}^1_\ell \cap \mathbb{C}^2) \times \hat{\mathbb{P}}$ that is a saddle point of $G(\lambda, f, m)$ so that*

$$\inf_{f\in\mathbb{C}^1_\ell\cap\mathbb{C}^2} \sup_{m\in\hat{\mathbb{P}}} G(\lambda,f,m) = \inf_{f\in\mathbb{C}^2_b} \sup_{m\in\mathbb{P}} G(\lambda,f,m) = \sup_{m\in\mathbb{P}} \inf_{f\in\mathbb{C}^2_0} G(\lambda,f,m)\,. \tag{3.10}$$

The function f^λ satisfies the ergodic Bellman equation

$$\begin{aligned} -\lambda\big(M(u^\lambda(x),x) - \frac{1}{2}\,\lambda|N(u^\lambda(x),x)|^2 + \nabla f^\lambda(x)^T\sigma(x)N(u^\lambda(x),x)\big) \\ + \nabla f^\lambda(x)^T\theta(x) + \frac{1}{2}\,|\sigma(x)^T\nabla f^\lambda(x)|^2 + \frac{1}{2}\,tr\,(\sigma(x)\sigma(x)^T\,\nabla^2 f^\lambda(x)) = F(\lambda), \end{aligned}$$

for all $x \in \mathbb{R}^l$, where

$$u^\lambda(x) = \frac{1}{1+\lambda}\,c(x)^{-1}\big(a(x) - r(x)\mathbf{1} + \lambda b(x)\beta(x) + b(x)\sigma(x)^T\nabla f^\lambda(x)\big)\,, \tag{3.11}$$

and $m^\lambda(x)$ is the invariant density of a diffusion:

$$\begin{aligned} \int_{\mathbb{R}^l} \big(\nabla h(x)^T(-\lambda\sigma(x)N(u^\lambda(x),x) + \theta(x) + \sigma(x)\sigma(x)^T\nabla f^\lambda(x)) \\ + \frac{1}{2}\,tr\,(\sigma(x)\sigma(x)^T\,\nabla^2 h(x))\big)\,m^\lambda(x)\,dx = 0\,, \end{aligned} \tag{3.12}$$

for all $h \in \mathbb{C}^2_0$. The functions $\nabla f^\lambda(x)$ and $m^\lambda(x)$ are specified uniquely almost everywhere with respect to Lebesgue measure. The density $m^\lambda(x)$ is positive, bounded and is of class $\mathbb{C}^1$.

Proof. Since $\int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$ is an upper semicontinuous and concave function of $\nu \in \mathcal{P}$, for all $f \in \mathcal{A}_\kappa$, is convex in $f \in \mathcal{A}_\kappa$, and $\int_{\mathbb{R}^l} H(x;\lambda,\overline{f}_\kappa)\,\nu(dx)$ is sup–compact in ν, an application of Theorem 7 on p.319 in Aubin and Ekeland [3] yields

$$\begin{aligned} \sup_{\nu\in\mathcal{P}} \inf_{f\in\mathbb{C}^2_b} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) &= \sup_{\nu\in\mathcal{P}} \inf_{f\in\mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \\ &= \inf_{f\in\mathcal{A}_\kappa} \sup_{\nu\in\mathcal{P}} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \geq \inf_{f\in\mathbb{C}^2_b} \sup_{\nu\in\mathcal{P}} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)\,, \end{aligned}$$

the supremum on the leftmost side being attained at some ν^λ. It follows that

$$\sup_{\nu\in\mathcal{P}} \inf_{f\in\mathbb{C}^2_b} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) = \inf_{f\in\mathbb{C}^2_b} \sup_{\nu\in\mathcal{P}} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)\,.$$

By Lemma 3.2,

$$\sup_{\nu\in\mathcal{P}} \inf_{f\in\mathbb{C}^2_b} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) = \sup_{m\in\mathbb{P}} \inf_{f\in\mathbb{C}^2_0} \int_{\mathbb{R}^l} H(x;\lambda,f)\,m(x)\,dx$$

and $\nu^\lambda(dx) = m^\lambda(x)\,dx$, where $m^\lambda \in \hat{\mathbb{P}}$, and, by an approximation argument,

$$\sup_{\nu\in\mathcal{P}} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) = \sup_{m\in\mathbb{P}} \int_{\mathbb{R}^l} H(x;\lambda,f)\,m(x)\,dx\,.$$

We obtain that

$$\inf_{f\in\mathbb{C}_b^2}\sup_{m\in\mathbb{P}} G(\lambda,f,m) = \sup_{m\in\mathbb{P}}\inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m) = \inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m^\lambda)\,.$$

Therefore, on applying Lemma 3.2,

$$\inf_{f\in\mathbb{C}_\ell^1\cap\mathbb{C}^2}\sup_{m\in\hat{\mathbb{P}}} G(\lambda,f,m) \le \inf_{f\in\mathbb{C}_b^2}\sup_{m\in\mathbb{P}} G(\lambda,f,m) = \sup_{m\in\mathbb{P}}\inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m) = \sup_{m\in\hat{\mathbb{P}}}\inf_{f\in\mathbb{C}_\ell^1\cap\mathbb{C}^2} G(\lambda,f,m)\,.$$

The leftmost side not being less than the rightmost side obtains (3.10).

Since the quantity on the rightmost side of (3.10) is $F(\lambda)$, maximisation in the middle term, with the use of (3.6), yields the representation

$$F(\lambda) = \inf_{f\in\mathbb{C}_b^2}\sup_{x\in\mathbb{R}^l} H(x;\lambda,f)\,. \tag{3.13}$$

(The righthand side is finite: take $f = \overline{f}_\kappa$ as in Lemma 3.1.) Applying the reasoning on pp.289–294 in Kaise and Sheu [15], one can see that, for arbitrary $\epsilon > 0$, there exists $\mathbb{C}^2$-function f_ϵ such that, for all $x \in \mathbb{R}^l$, $H(x;\lambda,f_\epsilon) = F(\lambda)+\epsilon$. Considering that some details are omitted in Kaise and Sheu [15], we give an outline of the proof, following the lead of Ichihara [13]. By the definition of the infimum, there exists function $f_1 \in \mathbb{C}_b^2$ such that $H(x;\lambda,f_1) \le F(\lambda)+\epsilon$ for all x. Given an open ball S, centred at the origin, by Theorem 6.14 on p.107 in Gilbarg and Trudinger [11], there exists $\mathbb{C}^2$–solution f_2 of the linear elliptic boundary value problem $H(x;\lambda,f) - (1/2)|\sigma(x)\nabla f(x)|^2 = F(\lambda)+\epsilon$ when $x \in S$ and $f(x) = \overline{f}_\kappa(x)$ when $x \in \partial S$, with ∂S standing for the boundary of S. Therefore, $H(x;\lambda,f_2) \ge F(\lambda)+\epsilon$ in S. By Theorem 8.4 on p.302 of Chapter 4 in Ladyzhenskaya and Uraltseva [18], for any ball S' contained in S and centred at the origin, there exists $\mathbb{C}^2$–solution $f_{S'}$ to the boundary value problem $H(x;\lambda,f) = F(\lambda)+\epsilon$ in S' and $f(x) = \overline{f}_\kappa(x)$ on $\partial S'$. Since $f_{S'}$ solves the boundary value problem $(1/2)\mathrm{tr}\,(\sigma(x)\sigma(x)^T\nabla^2 f(x)) = -H(x;\lambda,\nabla f_{S'}(x)) + F(\lambda)+\epsilon$ when $x \in S'$ and $f(x) = f_{S'}(x)$ when $x \in \partial S'$, we have by Theorem 6.17 on p.109 of Gilbarg and Trudinger [11] that $f_{S'}(x)$ is thrice continuously differentiable. Letting the radius of S' (and that of S) go to infinity, we have, by p.294 in Kaise and Sheu [15], see also Proposition 3.2 in Ichihara [13], that the $f_{S'}$ converge in $\mathbb{C}_{\mathrm{loc}}(\mathbb{R}^l)\cap\mathbb{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^l)$ to f_ϵ which is a weak solution to $H(x;\lambda,f) = F(\lambda)+\epsilon$. Furthermore, by Lemma 2.4 in Kaise and Sheu [15], the $W^{1,\infty}(S'')$–norms of the $f_{S'}$ are uniformly bounded over balls S' for any fixed ball S'' contained in the S'. Therefore, f_ϵ belongs to $\mathbb{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^l)$. By Theorem 6.4 on p.284 of Ladyzhenskaya and Uraltseva [18], f_ϵ is thrice continuously differentiable.

As in Theorem 4.2 in Kaise and Sheu [15], by using the gradient bound in Lemma 2.4 there, we have that the f_ϵ converge along a subsequence uniformly on compact sets as $\epsilon \to 0$ to a $\mathbb{C}^2$–solution of $H(x;\lambda,f) = F(\lambda)$. That solution, which we denote by f^λ, delivers the infimum on the leftmost side of (3.10) and satisfies the Bellman equation, with the gradient $\nabla f^\lambda(x)$ obeying the linear growth condition, see Remark 2.5 in Kaise and Sheu [15].

The pair (f^λ, m^λ) is a saddle point by Proposition 2.156 on p.104 in Bonnans and Shapiro [7] or by Propostion 1.2 on p.167 in Ekeland and Temam [9]. Equation (3.12) expresses the requirement of the directional derivative of $G(\lambda, f, m)$ with respect to f in the direction h being equal to zero at (f^λ, m^λ), cf. Proposition 1.6 on p.169 in Ekeland and Temam [9]. In some more detail, either by Theorem 4.13 on p.273, or by Theorem 4.17 on p.276 in Bonnans and Shapiro [7], or by a direct calculation, the function $\sup_{u\in\mathbb{R}^n}\big(M(u,x) - \lambda|N(u,x)|^2/2 + y^T\sigma(x)N(u,x)\big)$, where $y \in \mathbb{R}^l$, is differentiable in y with the directional derivative at $y = \nabla f(x)$ in direction $\tilde{y} \in \mathbb{R}^l$ being $(\sigma(x)N(\tilde{u}(x),x))^T\tilde{y}$ and with $\tilde{u}(x)$ representing the point at which $\sup_{u\in\mathbb{R}^n}\big(M(u,x) - \lambda|N(u,x)|^2/2 + \nabla f(x)^T\sigma(x)N(u,x)\big)$ is attained. It follows by (2.11) and dominated convergence that $G(\lambda, f, m)$ has a directional derivative in f in direction $h \in \mathbb{C}_0^2$ given by

$$\int\limits_{\mathbb{R}^l}\Big(\nabla h(x)^T\big(-\lambda(\sigma(x)N(\tilde{u}(x),x)) + \theta(x) + \sigma(x)\sigma(x)\nabla f(x)\big) + \frac{1}{2}\operatorname{tr}\big(\sigma(x)\sigma(x)^T\nabla^2 h(x)\big)\Big)\, m(x)\,dx\,.$$

The function ∇f^λ is specified uniquely because $G(\lambda, f, m)$ is a strictly convex function of ∇f, cf. Proposition 1.5 on p.169 in Ekeland and Temam [9], and the density m^λ is specified uniquely because, by $\nabla f^\lambda(x)$ having at most linear growth in x and by Example 1.7.11 in Bogachev, Krylov, and Röckner [6], (3.12) is uniquely solvable for m^λ. The function $m^\lambda(x)$ is positive, bounded and is of class $\mathbb{C}^1$ by Corollaries 2.10 and 2.11 in Bogachev, Krylov, and Röckner [5] and Agmon [2], see also Theorem 4.1(ii) and p.413 in Metafune, Pallardi, and Rhandi [23].

□

Remark 3.2. By Theorem 6.4 on p.284 of Ladyzhenskaya and Uraltseva [18], f^λ is thrice continuously differentiable. Furthermore, by (3.13), $F(\lambda)$ is the smallest Λ such that there exists $\mathbb{C}^2$–function f that satisfies the equation $H(x;\lambda,f) = \Lambda$, for all $x \in \mathbb{R}^l$. An application of the results of Keise and Sheu [15] and Ichihara [13], yields an alternative proof of (3.12).

If $m \in \hat{\mathbb{P}}$, then one can integrate by parts in (2.11) to obtain that

$$G(\lambda, f, m) = \hat{G}(\lambda, \nabla f, m)\,, \tag{3.14}$$

where

$$\hat{G}(\lambda, \nabla f, m) = \int\limits_{\mathbb{R}^l}\Big(\hat{H}(x;\lambda,\nabla f(x)) - \frac{1}{2}\,\nabla f(x)^T\,\frac{\operatorname{div}\left(\sigma(x)\sigma(x)^T\,m(x)\right)}{m(x)}\Big)\,m(x)\,dx\,. \tag{3.15}$$

Since the righthand side is well defined for $f \in \mathbb{C}^1_\ell$, we use (3.14) to extend the definition of $G(\lambda, f, m)$ to include those functions f. It is noteworthy that if $f \in \mathbb{C}^1_\ell \cap \mathbb{C}^2$, then

$$\lim_{R\to\infty}\int\limits_{x\in\mathbb{R}^l:\,|x|\le R}\frac{1}{2}\operatorname{tr}\left(\sigma(x)\sigma(x)^T\nabla^2 f(x)\right)\,m(x)\,dx = \int\limits_{\mathbb{R}^l} -\frac{1}{2}\,\nabla f(x)^T\operatorname{div}\left(\sigma(x)\sigma(x)^T\,m(x)\right)dx\,.$$

By (3.15), (2.12), (2.11), and by the set of gradients of functions from $\mathbb{C}_0^2$ being dense in $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$,

$$F(\lambda) = \sup_{m\in\hat{\mathbb{P}}} \inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)} \hat{G}(\lambda, \nabla f, m)\,. \tag{3.16}$$

Since the matrix $\sigma(x)\sigma(x)^T - (\lambda/(1+\lambda))\sigma(x)b(x)^T c(x)^{-1}b(x)\sigma(x)^T$ is uniformly positive definite by condition (N), by (3.15), (3.1), and (3.4), $\hat{G}(\lambda, \nabla f, m)$ tends to infinity as the $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$-norm of ∇f tends to infinity. Therefore, the infimum with respect to ∇f in (3.16) is attained at a unique point, see, e.g., Proposition 1.2 on p.35 in Ekeland and Temam [9]. Since, by Lemma 3.2 and (3.14),

$$\begin{aligned}\sup_{m\in\hat{\mathbb{P}}} \inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)} \hat{G}(\lambda, \nabla f, m) &= \sup_{\nu\in\mathcal{P}} \inf_{f\in\mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \\ &= \sup_{\nu\in\mathcal{P}} \inf_{f\in\mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) \end{aligned} \tag{3.17}$$

and $\inf_{f\in\mathcal{A}_\kappa} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)$ is sup–compact in ν by Lemma 3.1, the supremum in (3.16) is attained too. Besides, by (f^λ, m^λ) being the only saddle point of $G(\lambda, f, m)$, the supremum on the righthand side of (3.17) is attained at unique ν which is ν^λ.

Lemma 3.4. *The function $\hat{H}(x;\lambda,y)$ is strictly convex in $(\lambda, y) \in \mathbb{R}_+ \times \mathbb{R}^l$. Given $m \in \hat{\mathbb{P}}$, the function $\hat{G}(\lambda, \nabla f, m)$ is strictly convex in $(\lambda, \nabla f) \in \mathbb{R}_+ \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$. The function $G(\lambda, f, m)$ is convex in $(\lambda, f) \in \mathbb{R}_+ \times \mathbb{C}_b^2$. The function $F(\lambda)$ is strictly convex and is differentiable, provided $\lambda > 0$, and $\inf_{f\in\mathbb{C}_0^2} G(\lambda, f, m)$ and $F(\lambda)$ tend to ∞ quadratically, as $\lambda \to \infty$. Besides,*

$$F'(\lambda+) = \int_{\mathbb{R}^l} \big(-M(u^\lambda(x), x) + \lambda|N(u^\lambda(x), x)|^2 - \nabla f^\lambda(x)^T \sigma(x) N(u^\lambda(x), x)\big) m^\lambda(x)\,dx\,.$$

Proof. The Hessian matrix of $\hat{H}(x;\lambda,y)$ with respect to (λ, y) is given by

$$\begin{aligned}
\hat{H}_{yy}(x;\lambda,y) &= \frac{1}{1+\lambda}\,\sigma(x)b(x)^T c(x)^{-1}b(x)\sigma(x)^T + \sigma(x)Q_1(x)\sigma(x)^T\,,\\
\hat{H}_{\lambda\lambda}(x;\lambda,y) &= \frac{1}{(1+\lambda)^3}\,\|a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T y - b(x)\beta(x)\|^2_{c(x)^{-1}} + \beta(x)^T Q_1(x)\beta(x)\,,\\
\hat{H}_{\lambda y}(x;\lambda,y) &= -\frac{1}{(1+\lambda)^2}\,\big(a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T y - b(x)\beta(x)\big)^T c(x)^{-1}b(x)\sigma(x)^T\\
&\quad + \beta(x)^T Q_1(x)\sigma(x)^T\,.
\end{aligned}$$

We show that it is positive definite. More specifically, we prove that for all $\tau \in \mathbb{R}$ and $z \in \mathbb{R}^l$ such that $z \neq 0$,

$$\tau^2\hat{H}_{\lambda\lambda}(x;\lambda,y) + z^T\hat{H}_{yy}(x;\lambda,y)z + 2\tau\hat{H}_{\lambda y}(x;\lambda,y)z > 0\,.$$

Since $\hat{H}_{yy}(x;\lambda,y)$ is a positive definite matrix by condition (N), the latter inequality holds when $\tau = 0$. Assuming $\tau \neq 0$, we need to show that

$$\hat{H}_{\lambda\lambda}(x;\lambda,y) + z^T \hat{H}_{yy}(x;\lambda,y)z + 2\hat{H}_{\lambda y}(x;\lambda,y)z > 0\,. \tag{3.18}$$

Let, for $d_1 = (v_1(x), w_1(x))$ and $d_2 = (v_2(x), w_2(x))$, where $v_1(x) \in \mathbb{R}^n$, $w_1(x) \in \mathbb{R}^k$, $v_2(x) \in \mathbb{R}^n$, $w_2(x) \in \mathbb{R}^k$, and $x \in \mathbb{R}^l$, the inner product be defined by $d_1 \cdot d_2 = v_1(x)^T c(x)^{-1} v_2(x) + w_1(x)^T w_2(x)$. By the Cauchy-Schwarz inequality, applied to $d_1 = \big((1+\lambda)^{-3/2}(a(x) - r(x)\mathbf{1} + b(x)\sigma(x)^T y - b(x)\beta(x)), Q_1(x)\beta(x)\big)$ and $d_2 = ((1+\lambda)^{-1/2} b(x)\sigma(x)^T z, Q_1(x)\sigma(x)^T z)$, we have that

$$(\hat{H}_{\lambda y}(x;\lambda,y)z)^2 < z^T \hat{H}_{gg}(x;\lambda,y) z \hat{H}_{\lambda\lambda}(x;\lambda,y)\,,$$

with the inequality being strict because, by condition (N), $Q_1(x)\beta(x)$ is not a scalar multiple of $Q_1(x)\sigma(x)^T z$. Thus, (3.18) holds, so the function $\hat{H}(x;\lambda,y)$ is strictly convex in (λ, y) on $\mathbb{R}_+ \times \mathbb{R}^l$ for all $x \in \mathbb{R}^l$. By (3.15), $\hat{G}(\lambda, \nabla f, m)$ is strictly convex in $(\lambda, \nabla f)$, provided $m \in \hat{\mathbb{P}}$, and by (3.2), $G(\lambda, f, m)$ is convex in (λ, f). Since the infimum and supremum in (3.16) are attained, $F(\lambda)$ is strictly convex and finite-valued, so it is continuous.

We address the differentiability of $F(\lambda)$. By Theorem 4.13 on p.273 in Bonnans and Shapiro [7] and dominated convergence, we can conclude on recalling (3.1) and (3.15), that if $m \in \hat{\mathbb{P}}$ and $\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$, then $\hat{G}(\lambda, \nabla f, m)$ is differentiable in λ at $\lambda > 0$ with the derivative being equal to $\int_{\mathbb{R}^l} \big(-M(\tilde{u}(x), x) + \lambda |N(\tilde{u}(x), x)|^2 - \nabla f(x)^T \sigma(x) N(\tilde{u}(x), x)\big) m(x)\,dx$ and with $\tilde{u}(x)$ being defined in analogy with (3.11) as the point at which $\sup_{u\in\mathbb{R}^n} \big(M(u,x) - \lambda |N(u,x)|^2/2 + \nabla f(x)^T \sigma(x) N(u,x)\big)$ is attained. Furthermore, $\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)} \hat{G}(\lambda, \nabla f, m)$ is attained at $\nabla \tilde{f}$ such that the Fréchet derivative of $\hat{G}(\lambda, \nabla f, m)$ with respect to ∇f equals zero, so, cf. (3.11) and (3.12),

$$\int_{\mathbb{R}^l} \Big(\nabla h(x)^T \big(-\lambda \sigma(x) \big(b(x)^T \frac{1}{1+\lambda}\, c(x)^{-1} \big(a(x) - r(x)\mathbf{1} + \lambda b(x)\beta(x) \big) - \beta(x) \big) + \theta(x) \\ - \frac{1}{2} \frac{\operatorname{div}(\sigma(x)\sigma(x)^T m(x))}{m(x)} + (\sigma(x)\sigma(x)^T - \frac{\lambda}{1+\lambda}\, \sigma(x) b(x)^T c(x)^{-1} b(x) \sigma(x)^T) \nabla \tilde{f}(x) \big) \Big) m(x)\,dx = 0\,, \tag{3.19}$$

for all $h \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$. The mapping from $\mathbb{R}_+ \times \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$ to the space of linear functionals on $\mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$ that associates with $(\lambda, \nabla f)$ the linear functional defined by the lefthand side of (3.19) for $\nabla h \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$, with ∇f as $\nabla \tilde{f}$, is continuously differentiable. Since the matrix $\sigma(x)\sigma(x)^T - (\lambda/(1+\lambda))\,\sigma(x) b(x)^T c^{-1} b(x) \sigma(x)^T$ is uniformly positive definite, by the Riesz representation theorem, the partial derivative with respect to ∇f is a linear homeomorphism. By the implicit function theorem, $\nabla \tilde{f}$ is differentiable in λ. On recalling that the Fréchet derivative of $\hat{G}(\lambda, \nabla \tilde{f}, m)$ equals zero at $\nabla \tilde{f}$, we obtain by the chain rule that $\hat{G}(\lambda, \nabla \tilde{f}, m)$ is differentiable in λ with the full λ-derivative being equal to its partial λ-derivative evaluated at $(\lambda, \nabla \tilde{f}, m)$, i.e., $\int_{\mathbb{R}^l} \big(-M(\tilde{u}(x), x) + \lambda |N(\tilde{u}(x), x)|^2 - \nabla \tilde{f}(x)^T \sigma(x) N(\tilde{u}(x), x)\big) m(x)\,dx$. By (3.14), $\inf_{\nabla f \in \mathbb{L}_0^{1,2}(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)} \hat{G}(\lambda, \nabla f, m) = \inf_{f \in \mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x;\lambda,f)\, m(x)\,dx$, so the latter function is differentiable in λ, with the same derivative.

Given $\overline{\lambda} > 0$, since F is continuous, if λ is close enough to $\overline{\lambda}$, then

$$\sup_{\nu\in\mathcal{P}} \inf_{f\in\mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx) = \sup_{\nu\in\mathcal{P}_{\overline{\lambda}}} \inf_{f\in\mathbb{C}_0^2} \int_{\mathbb{R}^l} H(x;\lambda,f)\,\nu(dx)\,, \tag{3.20}$$

where $\mathcal{P}_{\overline{\lambda}} = \cup_{\{\check{\lambda}:\,|\check{\lambda}-\overline{\lambda}|\le\overline{\lambda}/2\}}\{\nu\in\mathcal{P}:\ \inf_{f\in\mathbb{C}_0^2}\int_{\mathbb{R}^l} H(x;\check{\lambda},f)\,\nu(dx) \ge F(\overline{\lambda})-1\}$. Since $\mathcal{P}_{\overline{\lambda}} \subset \hat{\mathbb{P}}$ by Lemma 3.2, the function in the supremum on the right of (3.20) is differentiable in λ for $\nu\in\mathcal{P}_{\overline{\lambda}}$. It is also convex in λ and upper semicontinuous in ν. By Lemma 3.1, the set $\mathcal{P}_{\overline{\lambda}}$ is relatively compact. In addition, $\nu(dx) = m^{\overline{\lambda}}(x)\,dx$ is the only point at which the supremum on the lefthand side of (3.20) is attained when $\lambda = \overline{\lambda}$. Theorem 3 on p.201 in Ioffe and Tihomirov [14] enables us to conclude that the righthand side of (3.20) is differentiable in λ, with the derivative at $\overline{\lambda}$ being equal to $\int_{\mathbb{R}^l}\big(-M(u^{\overline{\lambda}}(x),x) + \lambda|N(u^{\overline{\lambda}}(x),x)|^2 - \nabla f^{\overline{\lambda}}(x)^T\sigma(x)N(u^{\overline{\lambda}}(x),x)\big)m^{\overline{\lambda}}(x)\,dx$. By (3.16) and (3.17), this is also true of $F(\lambda)$. The expression for $F'(0+)$ is argued similarly.

By (3.4), as $\lambda\to\infty$,

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\inf_{g\in\mathbb{R}^l}\big(\hat{H}(x;\lambda,g) - \frac{1}{2}\,g^T\,\frac{\operatorname{div}\,(\sigma(x)\sigma(x)^T m(x))}{m(x)}\big)$$
$$= \frac{1}{2}\,\beta(x)^T Q_1(x)\beta(x) - \frac{1}{2}\,\|\sigma(x)Q_1(x)\beta(x)\|^2_{(\sigma(x)Q_1(x)\sigma(x)^T)^{-1}}\,.$$

The latter quantity being positive by condition (N) implies, by (3.14) and Fatou's lemma, that

$$\liminf_{\lambda\to\infty}\frac{1}{\lambda^2}\inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m) > 0\,.$$

so that, by (2.12), $\liminf_{\lambda\to\infty} F(\lambda)/\lambda^2 > 0$. □

Remark 3.3. If $\beta(x) = 0$, then part 2 of Condition (N) does not hold but the proof still goes through except for the last property in that $F(\lambda)/\lambda^2$ tends to zero as $\lambda\to\infty$. In fact,

$$\liminf_{\lambda\to\infty}\frac{1}{\lambda}\inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m) \ge -\int_{\mathbb{R}^l}(r(x)-\alpha(x))m(x)\,dx\,,$$

so that

$$\liminf_{\lambda\to\infty}\frac{F(\lambda)}{\lambda} \ge -\inf_{x\in\mathbb{R}^l}(r(x)-\alpha(x))\,.$$

Consequently, if $\inf_{x\in\mathbb{R}^l}(r(x)-\alpha(x)) < q$, then $-\lambda q - F(\lambda)$ tends to $-\infty$ as $\lambda\to\infty$, so $\sup_{\lambda\ge0}(-\lambda q - F(\lambda))$ is attained. That might not be the case if $\inf_{x\in\mathbb{R}^l}(r(x)-\alpha(x)) \ge q$. For instance, if the functions $a(x)$, $r(x)$, $b(x)$, and $\sigma(x)$ are constant, $\alpha(x) = 0$, and q is small enough, then the derivative of $-\lambda q - F(\lambda)$ is positive for all λ.

According to Lemma 3.4, $\sup_{\lambda\ge0}(-\lambda q - F(\lambda))$ is attained at unique λ which we denote by $\hat{\lambda}$. Let, given $\lambda\ge0$, $f\in\mathbb{C}^2$, $m\in\mathbb{P}$, and $v = (v(x),\,x\in\mathbb{R}^l)\in\mathbb{LG}$, the latter set

representing the set of measurable functions on $\mathbb{R}^l$ growing no faster than linearly,

$$\breve{H}(x;\lambda,f,v) = -\lambda M(v(x),x) + \frac{1}{2}\,|-\lambda N(v(x),x) + \sigma(x)^T\nabla f(x)|^2 \\ + \nabla f(x)^T\,\theta(x) + \frac{1}{2}\,\mathrm{tr}\,\big(\sigma(x)\sigma(x)^T\nabla^2 f(x)\big)\,. \quad (3.21)$$

As with $G(\lambda,f,m)$, integration by parts allows one to define $\int_{\mathbb{R}^l}\breve{H}(x;\lambda,f,v)m(x)\,dx$ when $f\in\mathbb{C}^1_\ell\cap\mathbb{C}^2$ and $m\in\hat{\mathbb{P}}$.

Lemma 3.5. *The function $\lambda q+G(\lambda,f,m)$ has saddle point $((\hat\lambda,\hat f),\hat m)\in\mathbb{R}_+\times(\mathbb{C}^1_\ell\cap\mathbb{C}^2)\times\hat{\mathbb{P}}$, with $\hat\lambda$, $\nabla\hat f$, and $\hat m$ being specified uniquely. The density $\hat m$ is positive, bounded, and is of class $\mathbb{C}^1$. Equations* (2.14a), (2.14b), *and* (2.15) *hold,*

$$\int_{\mathbb{R}^l}\big(M(\hat u(x),x) - \hat\lambda|N(\hat u(x),x)|^2 + \nabla\hat f(x)^T\sigma(x)N(\hat u(x),x)\big)\,\hat m(x)\,dx \le q \quad (3.22)$$

and, provided $\hat\lambda>0$,

$$\int_{\mathbb{R}^l}\big(M(\hat u(x),x) - \hat\lambda|N(\hat u(x),x)|^2 + \nabla\hat f(x)^T\sigma(x)N(\hat u(x),x)\big)\,\hat m(x)\,dx = q\,, \quad (3.23)$$

with $\hat u$ being defined by (2.15).

In addition,

$$\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,\hat u)\nu(dx) = F(\hat\lambda)\,. \quad (3.24)$$

Proof. We let $\hat f = f^{\hat\lambda}$ and $\hat m = m^{\hat\lambda}$. Equations (2.14a), (2.14b), and (2.15) hold by Lemma 3.3. Equation (3.22) represents the condition that $-q - F'(\hat\lambda+)\le 0$ if one recalls the expression for $F'(\lambda+)$ in Lemma 3.4. Since $G(\lambda,f,m)$ is convex in (λ,f) and is concave in m, those equations imply that $((\hat\lambda,\hat f),\hat m)$ is a saddle point of $\lambda q+G(\lambda,f,m)$, cf. Proposition 1.7 on p.170 in Ekeland and Temam [9]. Equation (3.23) is an obvious consequence.

Let $\mathbb{LG}_0$ represent the subset of $\mathbb{LG}$ of continuous functions of compact support. Assuming that $\kappa>0$ is small enough, we have, in analogy with the proof of Lemma 3.3, that $\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,v)\nu(dx)$ is an upper semicontinuous and concave function of $\nu\in\mathcal{P}$, for all $f\in\mathcal{A}_\kappa$ and $v\in\mathbb{LG}_0$, is convex in $(f,v)\in\mathcal{A}_\kappa\times\mathbb{LG}_0$, and $\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,\overline{f}_\kappa,v)\nu(dx)$ is sup–compact in ν for $v\in\mathbb{LG}_0$. An application of Theorem 7 on p.319 in Aubin and Ekeland [3], with the use of Lemma 3.2, yields, given $v\in\mathbb{LG}_0$,

$$\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,v)\nu(dx) = \sup_{\nu\in\mathcal{P}}\inf_{f\in\mathcal{A}_\kappa}\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,v)\nu(dx) \\ = \sup_{\nu\in\mathcal{P}}\inf_{f\in\mathbb{C}^2_b}\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,v)\nu(dx) \le \inf_{f\in\mathbb{C}^2_b}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat\lambda,f,v)\nu(dx)\,.$$

Hence,

$$\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)=\inf_{f\in\mathbb{C}_b^2}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\,. \tag{3.25}$$

Similarly,

$$\begin{aligned}\inf_{v\in\mathbb{LG}_0}\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)&=\sup_{\nu\in\mathcal{P}}\inf_{f\in\mathcal{A}_\kappa}\inf_{v\in\mathbb{LG}_0}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\\ =\sup_{\nu\in\mathcal{P}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\inf_{v\in\mathbb{LG}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)&=\sup_{\nu\in\mathcal{P}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\int_{\mathbb{R}^l}H(x;\hat{\lambda},f)\nu(dx)\\ &=\sup_{m\in\hat{\mathbb{P}}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}G(\lambda,f,m)=F(\hat{\lambda})\,.\end{aligned}$$

Since

$$\sup_{\nu\in\mathcal{P}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\inf_{v\in\mathbb{LG}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\le\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\inf_{v\in\mathbb{LG}}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\,,$$

we obtain that

$$\begin{aligned}\inf_{v\in\mathbb{LG}_0}\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)&=\inf_{v\in\mathbb{LG}}\inf_{f\in\mathcal{A}_\kappa}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\\ =\inf_{v\in\mathbb{LG}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\sup_{\nu\in\mathcal{P}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)&=\sup_{\nu\in\mathcal{P}}\inf_{f\in\mathbb{C}^2\cap\mathbb{C}^1_\ell}\inf_{v\in\mathbb{LG}}\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)\,.\end{aligned} \tag{3.26}$$

According to (2.14a), (2.14b), and (2.15), $((\hat{f},\hat{u}),\hat{m}(x)\,dx)$ is a saddle point of $\int_{\mathbb{R}^l}\breve{H}(x;\hat{\lambda},f,v)\nu(dx)$ as a function of $((f,v),\nu)\in((\mathbb{C}^2\cap\mathbb{C}^1_\ell)\times\mathbb{LG})\times\mathcal{P}$, cf. the proof of Proposition 1.6 on p.169 in Ekeland and Temam [9]. Therefore, accounting for (3.25), infimum over v in the second term of (3.26) is attained at $\hat{u}$, cf. Proposition 1.2 on p.167 in Ekeland and Temam [9], yielding (3.24). □

Remark 3.4. It is noteworthy that $\kappa>0$ on the lefthand side of (3.24) can be chosen arbitrarily small.

4 Proofs of the main results

Proof of Theorem 2.1 . By the function $-\lambda q-G(\lambda,f,m)$ having a saddle point and the definition of $F(\lambda)$, it suffices to prove that, for arbitrary $m\in\hat{\mathbb{P}}$,

$$\liminf_{t\to\infty}\frac{1}{t}\ln\mathbf{P}(L_t^\pi\le q)\ge-\sup_{\lambda\ge0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m)\big)\,. \tag{4.1}$$

We shall first consider the case where the density $m(x)$ is such that

$$m(x)=L\,|x|^{-k} \tag{4.2}$$

for all $|x|$ great enough, where $L > 0$ and $k > 0$, the latter value being great enough. By Lemma 3.4, the supremum on the right of (4.1) is attained at some $\tilde\lambda$. Let $\nabla\tilde f$ represent the function that attains the infimum in $\inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\hat G(\lambda,\nabla f,m) = \inf_{f\in\mathbb{C}_0^2} G(\lambda,f,m)$ for $\lambda=\tilde\lambda$. As we have seen, cf. (3.19), it is determined by the equation, for $h\in\mathbb{C}_0^2$,

$$\int\limits_{\mathbb{R}^l}\nabla h(x)^T\big(\sigma(x)\sigma(x)^T - \frac{\tilde\lambda}{1+\tilde\lambda}\,\sigma(x)b(x)^Tc(x)^{-1}b(x)\sigma(x)^T\big)\nabla\tilde f(x)m(x)\,dx$$
$$+\int\limits_{\mathbb{R}^l}\nabla h(x)^T\big(-\frac{\tilde\lambda}{1+\tilde\lambda}\,\sigma(x)b(x)^Tc(x)^{-1}(a(x)-r(x)\mathbf{1}+\tilde\lambda b(x)\beta(x))+\tilde\lambda\sigma(x)\beta(x)+\theta(x)$$
$$-\frac{\mathrm{div}(\sigma(x)\sigma(x)^Tm(x))}{2m(x)}\big)\,m(x)\,dx=0\,, \quad (4.3)$$

where $\tilde u(x)$ represents the point u at which the supremum on the righthand side of (2.11) is attained for $\lambda=\tilde\lambda$ and $\nabla f(x)=\nabla\tilde f(x)$. By Theorem 1 in Pardoux and Veretennikov [26], if k is great enough, then $\tilde f$ can be assumed to be an element of $\mathbb{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^l)$ for all $p>1$ and to satisfy the equation

$$\mathrm{tr}\big(\big(\sigma(x)\sigma(x)^T - \frac{\tilde\lambda}{1+\tilde\lambda}\,\sigma(x)b(x)^Tc(x)^{-1}b(x)\sigma(x)^T\big)\nabla^2\tilde f(x)\big)$$
$$+\frac{1}{m(x)}\,\mathrm{div}\big(\big(\sigma(x)\sigma(x)^T - \frac{\tilde\lambda}{1+\tilde\lambda}\,\sigma(x)b(x)^Tc(x)^{-1}b(x)\sigma(x)^T\big)m(x)\big)^T\nabla\tilde f(x)$$
$$=-\frac{1}{m(x)}\,\mathrm{div}\,\big(\big(-\frac{\tilde\lambda}{1+\tilde\lambda}\,\sigma(x)b(x)^Tc(x)^{-1}(a(x)-r(x)\mathbf{1}+\tilde\lambda b(x)\beta(x))-\beta(x)+\theta(x)$$
$$-\frac{\mathrm{div}(\sigma(x)\sigma(x)^Tm(x))}{2m(x)}\big)\,m(x)\big)\,. \quad (4.4)$$

By the divergence theorem, the integral of the righthand side against $m(x)\,dx$ over $\mathbb{R}^l$ equals zero, provided k is great enough, thus ensuring condition (A_f) in Pardoux and Veretennikov [26]. Condition (A_b) there holds for $\alpha=-1$. If k is great enough, then (4.4) satisfies the hypotheses of part (ii) of Theorem 2 in Pardoux and Veretennikov [26] with $\alpha=-1$ and $\beta=1$ (in fact, any $\beta>0$ would do). By the assertion of that part, $\nabla\tilde f(x)$ has at most linear growth at infinity.

Since $-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m)$ attains a local maximum in λ at $\tilde\lambda$, in analogy with (3.22) and (3.23),

$$\int\limits_{\mathbb{R}^l}\big(M(\tilde u(x),x)-\tilde\lambda|N(\tilde u(x),x)|^2+N(\tilde u(x),x)^T\sigma(x)^T\nabla\tilde f(x)\big)m(x)\,dx\le q\,, \quad (4.5)$$

with equality holding provided $\tilde\lambda>0$. Therefore,

$$-\tilde\lambda q-G(\tilde\lambda,\tilde f,m)=-\tilde\lambda\int\limits_{\mathbb{R}^l}\big(M(\tilde u(x),x)-\tilde\lambda|N(\tilde u(x),x)|^2+N(\tilde u(x),x)^T\sigma(x)^T\nabla\tilde f(x)\big)m(x)\,dx$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^l} \big(-\tilde{\lambda} M(\tilde{u}(x),x) + \frac{1}{2}\,\tilde{\lambda}^2 |N(\tilde{u}(x),x)|^2 - \tilde{\lambda} N(\tilde{u}(x),x)^T \sigma(x)^T \nabla \tilde{f}(x) \\
&+ \frac{1}{2}\,|\sigma(x)^T \nabla \tilde{f}(x)|^2 + \nabla \tilde{f}(x)^T \theta(x) + \frac{1}{2}\,\mathrm{tr}\,(\sigma(x)\sigma(x)^T \nabla^2 \tilde{f}(x)\,)\big) m(x)\,dx \\
&= \int_{\mathbb{R}^l} \frac{1}{2}\,\tilde{\lambda}^2 |N(\tilde{u}(x),x)|^2 m(x)\,dx - \int_{\mathbb{R}^l} \big(\frac{1}{2}\,|\sigma(x)^T \nabla \tilde{f}(x)|^2 + \nabla \tilde{f}(x)^T \theta(x) \\
&\qquad\qquad + \frac{1}{2}\,\mathrm{tr}\,(\sigma(x)\sigma(x)^T \nabla^2 \tilde{f}(x)\,)\big) m(x)\,dx\,. \quad (4.6)
\end{aligned}
$$

Let us note that (3.19) can also be written as, cf. (3.12),

$$
\begin{aligned}
&\int_{\mathbb{R}^l} \big(\nabla h(x)^T (-\tilde{\lambda}\sigma(x) N(\tilde{u}(x),x) + \theta(x) + \sigma(x)\sigma(x)^T \nabla \tilde{f}(x)) \\
&\qquad\qquad + \frac{1}{2}\,\mathrm{tr}\,(\sigma(x)\sigma(x)^T\, \nabla^2 h(x))\big)\, m(x)\,dx = 0\,. \quad (4.7)
\end{aligned}
$$

Integration by parts in (4.7) coupled with the facts that $|\nabla \tilde{f}(x)|$ grows at most linearly with $|x|$, that $\tilde{u}(x)$ is a linear function of $\nabla \tilde{f}(x)$, that $\int_{\mathbb{R}^l} |x|^2\, m(x)\,dx < \infty$, and that $\int_{\mathbb{R}^l} |\nabla m(x)|^2/m(x)\,dx < \infty$, shows that (4.7) holds with $\tilde{f}(x)$ as $h(x)$. Substitution on the rightmost side of (4.6) yields

$$
-\tilde{\lambda} q - G(\tilde{\lambda}, \tilde{f}, m) = \frac{1}{2} \int_{\mathbb{R}^l} |-\tilde{\lambda} N(\tilde{u}(x),x) + \sigma(x)^T \nabla \tilde{f}(x)|^2 m(x)\,dx\,. \quad (4.8)
$$

Let $W_s^{t,*}$ for $s \in [0,1]$ and measure $\mathbf{P}^{t,*}$ be defined by the respective equalities

$$
W_s^{t,*} = W_s^t - \sqrt{t} \int_0^s (-\tilde{\lambda} N(\tilde{u}(X_{\tilde{s}}^t), X_{\tilde{s}}^t) + \sigma(X_{\tilde{s}}^t)^T \nabla \tilde{f}(X_{\tilde{s}}^t))\,d\tilde{s} \quad (4.9)
$$

and

$$
\begin{aligned}
\frac{d\mathbf{P}^{t,*}}{d\mathbf{P}} = \exp\big(\sqrt{t} \int_0^1 &(-\tilde{\lambda} N(\tilde{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \tilde{f}(X_s^t))^T\, dW_s^t \\
&- \frac{t}{2} \int_0^1 |-\tilde{\lambda} N(\tilde{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \tilde{f}(X_s^t)|^2\, ds\big)\,. \quad (4.10)
\end{aligned}
$$

A multidimensional extension of Theorem 4.7 on p.137 in Liptser and Shiryayev [21], which is proved similarly, obtains that, given $t > 0$, there exists $\gamma' > 0$ such that $\sup_{s \le t} \mathbf{E} e^{\gamma' |X_s|^2} < \infty$. By Example 3 on pp.220,221 in Liptser and Shiryayev [21] and the linear growth condition on $\nabla \tilde{f}(x)$, the expectation of the righthand side of (4.10) with respect to $\mathbf{P}$ equals unity. Therefore, $\mathbf{P}^{t,*}$ is a valid probability measure and the process $(W_s^{t,*},\ s \in [0,1])$ is a standard

Wiener process under $\mathbf{P}^{t,*}$, see Lemma 6.4 on p.216 in Liptser and Shiryayev [21] and Theorem 5.1 on p.191 in Karatzas and Shreve [16].

By (2.4b) and the analogue of (2.15),

$$a(x) - r(x)\mathbf{1} + b(x)(-\tilde{\lambda}N(\tilde{u}(x), x) + \sigma(x)^T \nabla \tilde{f}(x)) = c(x)\tilde{u}(x)\,.$$

It follows that

$$\begin{aligned}
&\int_0^1 M(\pi_s^t, X_s^t)\,ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T\,dW_s^t = \int_0^1 M(\pi_s^t, X_s^t)\,ds \\
&\quad + \int_0^1 N(\pi_s^t, X_s^t)^T(-\tilde{\lambda}N(\tilde{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \tilde{f}(X_s^t))\,ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t, X_s^t)^T\,dW_s^{t,*} \\
&= \frac{1}{t}\,\ln \mathcal{E}_s + \int_0^1 M(\tilde{u}(X_s^t), X_s^t)\,ds + \int_0^1 N(\tilde{u}(X_s^t), X_s^t)^T(-\tilde{\lambda}N(\tilde{u}(X_s^t), X_s^t) + \sigma(X_s^t)^T \nabla \tilde{f}(X_s^t))\,ds \\
&\qquad + \frac{1}{\sqrt{t}} \int_0^1 N(\tilde{u}(X_s^t), X_s^t)^T\,dW_s^{t,*}\,, \quad (4.11)
\end{aligned}$$

where $\mathcal{E}_s$ represents the stochastic exponential defined by

$$\mathcal{E}_s = \exp\big(\sqrt{t} \int_0^s (\pi_{\tilde{s}}^t - \tilde{u}(X_{\tilde{s}}^t))^T b(X_{\tilde{s}}^t)\,dW_{\tilde{s}}^{t,*} - \frac{t}{2} \int_0^s \|\pi_{\tilde{s}}^t - \tilde{u}(X_{\tilde{s}}^t)\|^2_{c(X_{\tilde{s}}^t)} d\tilde{s}\big)\,.$$

In addition, by (2.1) and (4.9),

$$dX_s^t \;=\; t\,\theta(X_s^t)\,ds \,+\, t\,\sigma(X_s^t)\,\big(-\tilde{\lambda}N(\tilde{u}(X_s^t), X_s^t) \,+\, \sigma(X_s^t)^T \nabla \tilde{f}(X_s^t)\big)\,ds \,+\, \sqrt{t}\sigma(X_s^t) dW_s^{t,*}\,.$$

We obtain that $X_s^t = \tilde{X}_{st}^t$, where the latter process is given by the equation

$$d\tilde{X}_s^t = \theta(\tilde{X}_s^t)\,ds + \sigma(\tilde{X}_s^t)\,\big(-\tilde{\lambda}N(\tilde{u}(\tilde{X}_s^t), \tilde{X}_s^t) + \sigma(\tilde{X}_s^t)^T \nabla \tilde{f}(\tilde{X}_s^t)\big)\,ds + \sigma(\tilde{X}_s^t) d\tilde{W}_s^t\,, \tilde{X}_0^t = X_0\,, \quad (4.12)$$

$(\tilde{W}_s^t)$ being a standard Wiener process under $\mathbf{P}^{t,*}$. Furthermore, (4.7) implies that

$$\begin{aligned}
&\int_{\mathbb{R}^l} \nabla h(x)^T(-\tilde{\lambda}\sigma(x)N(\tilde{u}(x), x) + \theta(x) + \sigma(x)\sigma(x)^T \nabla \tilde{f}(x))m(x)\,dx \\
&\qquad = \int_{\mathbb{R}^l} \frac{1}{2}\,\nabla h(x)^T\,\mathrm{div}(\sigma(x)\sigma(x)^T\,m(x))\,dx\,. \quad (4.13)
\end{aligned}$$

Since, for k great enough, $-\tilde{\lambda}\sigma(x)N(\tilde{u}(x), x) + \theta(x) + \sigma(x)\sigma(x)^T \nabla \tilde{f}(x) \in \mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$, $\nabla m(x)/m(x) \in \mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\,dx)$, and $\sigma(x)\sigma(x)^T$ and $\mathrm{div}(\sigma(x)\sigma(x)^T)$ are both bounded,

(4.13) extends to $\mathbb{C}^1(\mathbb{R}^l)$-functions $h(x)$ such that $\int_{\mathbb{R}^l}\bigl(h(x)^2+|\nabla h(x)|^2\bigr)\,m(x)\,dx<\infty$. For given $\kappa>0$, $\delta>0$, and $x\in\mathbb{R}^l$, we let $h(y)=|y|^2e^{-\delta[(|y-x|^2/\kappa-1)^+]^2}$. By dominated convergence,

$$\begin{aligned}&\lim_{\delta\to\infty}\int_{\mathbb{R}^l}\nabla h(y)^T(-\tilde\lambda\sigma(y)N(\tilde u(y),y)+\theta(y)+\sigma(y)\sigma(y)^T\nabla\tilde f(y))m(y)\,dy\\&=2\int_{y\in\mathbb{R}^l:\,|y-x|<\kappa}y^T(-\tilde\lambda\sigma(y)N(\tilde u(y),y)+\theta(y)+\sigma(y)\sigma(y)^T\nabla\tilde f(y))m(y)\,dx\end{aligned}$$

and

$$\lim_{\delta\to\infty}\int_{\mathbb{R}^l}\frac{1}{2}\,\nabla h(y)^T\,\mathrm{div}(\sigma(y)\sigma(y)^T\,m(y))\,dy=\int_{y\in\mathbb{R}^l:\,|y-x|<\kappa}y^T\mathrm{div}(\sigma(y)\sigma(y)^T\,m(y))\,dy\,.$$

Dividing the righthand sides by the volume of the ball of radius κ centred at x, letting $\kappa\to0$, and accounting for (4.2) yields the identity

$$x^T(-\tilde\lambda\sigma(x)N(\tilde u(x),x)+\theta(x)+\sigma(x)\sigma(x)^T\nabla\tilde f(x))=x^T\,\frac{\mathrm{div}(\sigma(x)\sigma(x)^T\,m(x))}{2m(x)}\,.\tag{4.14}$$

Since, by (4.2), $\nabla m(x)/(2m(x))=-kx/(2|x|^2)$, provided $|x|$ is great enough, $\sigma(x)\sigma(x)^T$ is uniformly positive definite, and $|x^T\mathrm{div}\,(\sigma(x)\sigma(x)^T)|$ is bounded, we have by (4.14) that, if k is great enough, then

$$\limsup_{|x|\to\infty}\frac{x^T}{|x|}\,(-\tilde\lambda\sigma(x)N(\tilde u(x),x)+\theta(x)+\sigma(x)\sigma(x)^T\nabla\tilde f(x))<0\,.$$

Therefore, the distribution of $\tilde X^t_s$ converges in variation to a unique stationary measure as $s\to\infty$, see Theorem 2 in Veretennikov [34], Proposition 1 in Pardoux and Veretennikov [26]. By (4.7) and (4.12), the stationary measure is $m(x)\,dx$. Hence, $\nu^t(dx)\to m(x)\,dx$ weakly in $\mathbf{P}^{t,*}$-probability, where $\nu^t(dx)=\mu^t([0,1],dx)$. In addition, if $|g(x)|\le L(1+|x|^2)$, then

$$\lim_{t\to\infty}\mathbf{P}^{t,*}\Bigl(\Bigl|\int_{\mathbb{R}^l}g(x)\nu^t(dx)-\int_{\mathbb{R}^l}g(x)m(x)\,dx\Bigr|>\epsilon\Bigr)=0\,,\tag{4.15}$$

which follows by Lemma 1 in Veretennikov [34], provided k is great enough.

By (4.10) and (4.11),

$$\begin{aligned}\mathbf{P}\big(L_t^\pi < q+3\delta\big) &= \mathbf{E}^{t,*}\chi_{\{\int_0^1 M(\pi_s^t,X_s^t)\,ds+\frac{1}{\sqrt{t}}\int_0^1 N(\pi_s^t,X_s^t)^T\,dW_s^t<q+3\delta\}}\\ &\qquad \exp\big(-\sqrt{t}\int_0^1(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t))^T\,dW_s^{t,*}\\ &\qquad -\frac{t}{2}\int_0^1|-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t)|^2\,ds\big)\\ &\geq \mathbf{E}^{t,*}\chi_{\{\frac{1}{t}\ln\mathcal{E}_1<\delta\}}\,\chi_{\{\frac{1}{\sqrt{t}}|\int_0^1 N(\tilde u(X_s^t),X_s^t)^T\,dW_s^{t,*}|<\delta\}}\,\chi_{\{\int_0^1 M(\tilde u(X_s^t),X_s^t)\,ds}\\ &\qquad +\int_0^1 N(\tilde u(X_s^t),X_s^t)^T(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t))\,ds<q+\delta\}}\\ &\qquad \chi_{\{\frac{1}{\sqrt{t}}|\int_0^1(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t))^T\,dW_s^{t,*}|<\delta\}}\\ &\chi_{\{\int_0^1|-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t)|^2\,ds-\int_{\mathbb{R}^l}|-\tilde\lambda N(\tilde u(x),x)+\sigma(x)^T\nabla\tilde f(x)|^2m(x)\,dx<2\delta\}}\\ &\qquad \exp\big(-2\delta t-\frac{t}{2}\int_{\mathbb{R}^l}|-\tilde\lambda N(\tilde u(x),x)+\sigma(x)^T\nabla\tilde f(x)|^2m(x)\,dx\big)\,. \quad (4.16)\end{aligned}$$

By Markov's inequality, $\mathbf{P}^{t,*}\big(\ln\mathcal{E}_1/t\geq\delta\big)\leq e^{-\delta t}\,\mathbf{E}^{t,*}\mathcal{E}_1\leq e^{-\delta t}$. Hence,

$$\lim_{t\to\infty}\mathbf{P}^{t,*}\big(\frac{1}{t}\ln\mathcal{E}_1<\delta\big)=1\,.$$

By (4.15) and the linear growth condition on $\nabla\tilde f(x)$,

$$\begin{aligned}\lim_{t\to\infty}\mathbf{P}^{t,*}\big(\big|\int_0^1|-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t)|^2\,ds\\ -\int_{\mathbb{R}^l}|-\tilde\lambda N(\tilde u(x),x)+\sigma(x)^T\nabla\tilde f(x)|^2m(x)\,dx\big|<2\delta\big)=1\,.\end{aligned}$$

Since, for $\eta>0$, by the Lénglart–Rebolledo inequality, see Theorem 3 on p.66 in Liptser

and Shiryayev [22],

$$\mathbf{P}^{t,*}\big(|\frac{1}{\sqrt{t}}\int_0^1(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(x)^T\nabla\tilde f(X_s^t))\,dW_s^{t,*}|\ge\delta\big)$$

$$\le\frac{\eta}{\delta^2}+\mathbf{P}^{t,*}\big(\int_0^1|-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(x)^T\nabla\tilde f(X_s^t)|^2\,ds\ge\eta t\big)\,,$$

we conclude that

$$\lim_{t\to\infty}\mathbf{P}^{t,*}\big(\frac{1}{\sqrt{t}}\,|\int_0^1(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t))\,dW_s^{t,*}|<\delta\big)=1\,.$$

Similarly,

$$\lim_{t\to\infty}\mathbf{P}^{t,*}\big(\frac{1}{\sqrt{t}}\,|\int_0^1 N(\tilde u(X_s^t),X_s^t)^T\,dW_s^{t,*}|<\delta\big)=1\,.$$

By (4.5) and (4.15),

$$\lim_{t\to\infty}\mathbf{P}^{t,*}\big(\int_0^1\big(M(\tilde u(X_s^t),X_s^t)+N(\tilde u(X_s^t),X_s^t)^T(-\tilde\lambda N(\tilde u(X_s^t),X_s^t)+\sigma(X_s^t)^T\nabla\tilde f(X_s^t))\big)\,ds$$

$$<q+\delta\big)=1\,.$$

Recalling (4.16) obtains that

$$\liminf_{t\to\infty}\frac{1}{t}\,\ln\mathbf{P}\big(L_t^\pi\le q\big)\ge-\frac{1}{2}\int_{\mathbb{R}^l}|-\tilde\lambda N(\tilde u(x),x)+\sigma(x)^T\nabla\tilde f(x)|^2m(x)\,dx\big)$$

so, (4.1) follows by (4.8).

Let us suppose that (4.2) does not hold. We introduce, for $\rho>0$,

$$m_\rho(x)=c_\rho\big(m(x)\eta^2\big(\frac{|x|}{\rho}\big)+|x|^{-k}\big(1-\eta^2\big(\frac{|x|}{\rho}\big)\big)\big)\,,$$

where $\eta(y)$ is a smooth nonincreasing function on $\mathbb{R}_+$ such that $\eta(y)=1$ for $y\in[0,1]$ and $\eta(y)=0$ for $y\ge2$.

Since $m_\rho(x)$ satisfies condition (4.2), we have that if k is great enough, then

$$\liminf_{t\to\infty}\frac{1}{t}\ln\mathbf{P}(L_t^\pi\le q)\ge-\sup_{\lambda\ge0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m_\rho)\big)\,.$$

The proof of (4.1) concludes by proving that the latter quantity converges to $-\sup_{\lambda\ge0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m)\big)$ as $\rho\to\infty$. Since $G(\lambda,f,m_\rho)\to G(\lambda,f,m)$, we have that

$$\liminf_{\rho\to\infty}\sup_{\lambda\ge0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m_\rho)\big)\ge\sup_{\lambda\ge0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m)\big)\,.$$

We prove the opposite inequality that

$$\limsup_{\rho\to\infty}\sup_{\lambda\geq 0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m_\rho)\big)\leq\sup_{\lambda\geq 0}\big(-\lambda q-\inf_{f\in\mathbb{C}_0^2}G(\lambda,f,m)\big)\,,$$

which is equivalent to the inequality

$$\limsup_{\rho\to\infty}\sup_{\lambda\geq 0}\big(-\lambda q-\inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\hat{G}(\lambda,\nabla f,m_\rho)\big)$$
$$\leq\sup_{\lambda\geq 0}\big(-\lambda q-\inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\hat{G}(\lambda,\nabla f,m)\big)\,. \quad (4.17)$$

On using representation (3.15) and (3.1),

$$\sup_{\lambda\geq 0}\big(-\lambda q-\inf_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\hat{G}(\lambda,\nabla f,m_\rho)\big)\leq J_\rho^1+J_\rho^2\,,$$

where, letting η' denote the derivative of η,

$$J_\rho^1=\sup_{\lambda\geq 0}\sup_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\big(-\lambda q-\int_{\mathbb{R}^l}\big(\big(\hat{H}(x;\lambda,\nabla f(x))-\nabla f(x)^T\frac{\mathrm{div}\,(\sigma(x)\sigma(x)^T\,m(x))}{2m(x)}\big)$$
$$c_\rho\eta^2\big(\frac{|x|}{\rho}\big)-\nabla f(x)^T\sigma(x)\sigma(x)^T\,c_\rho\frac{x}{\rho|x|}\,\eta'\big(\frac{|x|}{\rho}\big)\eta\big(\frac{|x|}{\rho}\big)\big)m(x)dx\big)$$

and

$$J_\rho^2=\sup_{\lambda\geq 0}\sup_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\big(-\int_{\mathbb{R}^l}\big(\hat{H}(x;\lambda,\nabla f(x))c_\rho\big(1-\eta^2\big(\frac{|x|}{\rho}\big)\big)$$
$$+\nabla f(x)^T\sigma(x)\sigma(x)^T\,c_\rho\,\frac{x}{\rho|x|}\,\eta'\big(\frac{|x|}{\rho}\big)\eta\big(\frac{|x|}{\rho}\big)$$
$$-\frac{1}{2}\,\nabla f(x)^T\big(-\sigma(x)\sigma(x)^T\,k\,\frac{x}{|x|^2}+\mathrm{div}\,(\sigma(x)\sigma(x)^T)\big)c_\rho\,\big(1-\eta^2\big(\frac{|x|}{\rho}\big)\big)\big)|x|^{-k}dx\big)\,. \quad (4.18)$$

Given $\kappa\in(0,1)$, we write, on recalling (3.1) and (3.3),

$$J_\rho^1\;\leq\;\sup_{\lambda\geq 0}\sup_{\nabla f\in\mathbb{L}_0^{1,2}(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)}\big(-\lambda q\;-\int_{\mathbb{R}^l}T_\kappa(x;\lambda,\nabla f(x))c_\rho\eta^2\big(\frac{|x|}{\rho}\big)\,m(x)\,dx\big)\;+\;J_\rho^3\,, \quad (4.19)$$

where

$$T_\kappa(x;\lambda,y)=-\frac{1}{2}\,\frac{\lambda}{1+\lambda}\,\|a(x)-r(x)\mathbf{1}+\lambda b(x)\beta(x)+b(x)\sigma(x)^Ty\|^2_{c(x)^{-1}}+\frac{1}{2}\,\lambda^2|\beta(x)|^2+y^T\,\theta(x)$$
$$-\lambda(r(x)-\alpha(x)+\frac{1}{2}\,|\beta(x)|^2-\beta(x)^T\sigma(x)^Ty)+(1-\kappa)\,\frac{1}{2}\,|\sigma(x)^Ty|^2-\frac{1}{2}\,y^T\frac{\mathrm{div}\,(\sigma(x)\sigma(x)^T\,m(x))}{m(x)}$$

and

$$J^3_\rho = \int\limits_{\mathbb{R}^l} \sup_{y\in\mathbb{R}^l} \big(-\kappa\frac{1}{2}\,|\sigma(x)^T y|^2 c_\rho \eta^2\big(\frac{|x|}{\rho}\big) + y^T\sigma(x)\sigma(x)^T\, c_\rho \frac{x}{\rho|x|}\,\eta'\big(\frac{|x|}{\rho}\big)\eta\big(\frac{|x|}{\rho}\big)\big) m(x)dx\,. \tag{4.20}$$

We will prove that

$$\begin{aligned}\limsup_{\rho\to\infty}\sup_{\lambda\ge 0}\sup_{\nabla f\in \mathbb{L}^{1,2}_0(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)} \big(-\lambda q - \int\limits_{\mathbb{R}^l} T_\kappa(x;\lambda,\nabla f(x)) c_\rho\eta^2\big(\frac{|x|}{\rho}\big)\, m(x)\,dx\big)\\ = \sup_{\lambda\ge 0}\sup_{\nabla f\in \mathbb{L}^{1,2}_0(\mathbb{R}^l,\mathbb{R}^l,m(x)\,dx)} \big(-\lambda q - \int\limits_{\mathbb{R}^l} T_\kappa(x;\lambda,\nabla f(x))\, m(x)\,dx\big) \qquad (4.21a)\end{aligned}$$

and

$$\lim_{\rho\to\infty} J^2_\rho = \lim_{\rho\to\infty} J^3_\rho = 0\,. \tag{4.21b}$$

By the proof of Lemma 3.4 and continuity considerations, the Hessian matrix of $T_\kappa(x;\lambda,y)$ with respect to (λ,y) is positive definite, provided κ is small enough, so the function in the supremum on the righthand side of (4.19) is strictly convex. Let the supremum be attained at $(\tilde\lambda_\rho,\nabla\tilde f_\rho)$. We note that, since $\lambda\ge 0$, for $y\in\mathbb{R}^l$,

$$\begin{aligned} T_\kappa(x;\lambda,y) \ge &-\frac{1}{2}\,\|a(x)-r(x)\mathbf{1}+\lambda b(x)\beta(x)+b(x)\sigma(x)^T y\|^2_{c(x)^{-1}} + \frac{1}{2}\,\lambda^2|\beta(x)|^2\\
-\lambda(r(x)-\alpha(x)+&\frac{1}{2}\,|\beta(x)|^2-\beta(x)^T\sigma(x)^T y)+y^T\theta(x)+(1-\kappa)\,\frac{1}{2}\,|\sigma(x)^T y|^2-\frac{1}{2}\,y^T\frac{\operatorname{div}(\sigma(x)\sigma(x)^T m(x))}{m(x)}\\
=\;&\frac{1}{2}\,y^T(\sigma(x)Q_1(x)\sigma(x)^T-\kappa\sigma(x)\sigma(x)^T)y+\big(-(a(x)-r(x)\mathbf{1}+\lambda b(x)\beta(x))^T c(x)^{-1}b(x)\sigma(x)^T\\
&+\lambda\beta(x)^T\sigma(x)^T+\theta(x)^T-\frac{1}{2}\,\frac{\operatorname{div}(\sigma(x)\sigma(x)^T m(x))^T}{m(x)}\big)y-\;\frac{1}{2}\,\|a(x)-r(x)\mathbf{1}\|^2_{c(x)^{-1}}\\
-(a(x)-r(x)\mathbf{1})^T c(x)^{-1}&\lambda b(x)\beta(x)+\frac{1}{2}\,\lambda^2\|\beta(x)\|^2_{Q_1(x)}-\lambda(r(x)-\alpha(x)+\frac{1}{2}\,|\beta(x)|^2)\,. \qquad (4.22)\end{aligned}$$

Since, by condition (N), the matrix $\sigma(x)Q_1(x)\sigma(x)^T$ is uniformly positive definite, the matrix $\sigma(x)Q_1(x)\sigma(x)^T-\kappa\sigma(x)\sigma(x)^T$ is uniformly positive definite if κ is small enough, so the minimum over $y\in\mathbb{R}^l$ of the rightmost side of (4.22) equals

$$\begin{aligned}&\frac{1}{2}\,\lambda^2\big(\|\beta(x)\|^2_{Q_1(x)}-\|\sigma(x)Q_1(x)\beta(x)\|^2_{(\sigma(x)Q_1(x)\sigma(x)^T-\kappa\sigma(x)\sigma(x)^T)^{-1}}\big)\\
-\frac{1}{2}\,\|\big(-(a(x)-r(x)\mathbf{1})^T c(x)^{-1}&b(x)\sigma(x)^T+\theta(x)^T-\frac{1}{2}\,\frac{\operatorname{div}(\sigma(x)\sigma(x)^T m(x))^T}{m(x)}\big)^T\|^2_{(\sigma(x)Q_1(x)\sigma(x)^T-\kappa\sigma(x)\sigma(x)^T)^{-1}}\\
&-\frac{1}{2}\,\|a(x)-r(x)\mathbf{1}\|^2_{c(x)^{-1}}-\lambda\beta(x)^T Q_1(x)\sigma(x)^T(\sigma(x)Q_1(x)\sigma(x)^T-\kappa\sigma(x)\sigma(x)^T)^{-1}\\
&\big(-(a(x)-r(x)\mathbf{1})^T c(x)^{-1}b(x)\sigma(x)^T+\theta(x)^T-\frac{1}{2}\,\frac{\operatorname{div}(\sigma(x)\sigma(x)^T m(x))^T}{m(x)}\big)\\
&-\lambda(a(x)-r(x)\mathbf{1})^T c(x)^{-1}b(x)\beta(x)-\lambda(r(x)-\alpha(x)+\frac{1}{2}\,|\beta(x)|^2)\,. \qquad (4.23)\end{aligned}$$

Since $\|\beta(x)\|^2_{Q_1(x)} - \|\sigma(x)Q_1(x)\beta(x)\|^2_{(\sigma(x)Q_1(x)\sigma(x)^T)^{-1}} = \|\beta(x)\|^2_{Q_2(x)}$ and, by condition (N), the latter quantity is bounded away from zero, we have, by continuity, that if $\kappa > 0$ is small enough, then $\|\beta(x)\|^2_{Q_1(x)} - \|\sigma(x)Q_1(x)\beta(x)\|^2_{(\sigma(x)Q_1(x)\sigma(x)^T - \kappa\sigma(x)\sigma(x)^T)^{-1}}$ is bounded away from zero. Therefore, uniformly over all ρ great enough and uniformly over f, the integral in (4.19) tends to infinity superlinearly in λ as $\lambda \to \infty$, which implies, by (4.22) and the $\tilde{\lambda}_\rho$ being points of maximum in (4.19), that the set of the $\tilde{\lambda}_\rho$ is bounded. Likewise, since the matrix $\sigma(x)Q_1(x)\sigma(x)^T - \kappa\sigma(x)\sigma(x)^T$ is uniformly positive definite if $\kappa > 0$ is small enough, by (4.22), the set of the $\int_{\mathbb{R}^l} |\nabla \tilde{f}_\rho|^2 \, \eta^2(|x|/\rho)\, m(x)\, dx$ is bounded in ρ. Let $\tilde{\lambda}$ represent a subsequential limit of the $\tilde{\lambda}_\rho$ and let $\nabla \tilde{f}$ represent a weak subsequential limit of the $\nabla \tilde{f}_\rho \, \eta_\rho(x)$ in $\mathbb{L}^2(\mathbb{R}^l, \mathbb{R}^l, m(x)\, dx)$. Then, along subsequence ρ' that attains the lim sup in (4.17), by (4.22) and (4.23),

$$\limsup_{\rho' \to \infty} \big(-\tilde{\lambda}_{\rho'} q - \int_{\mathbb{R}^l} T_\kappa(x; \tilde{\lambda}_{\rho'}, \nabla \tilde{f}_{\rho'}(x)) c_{\rho'} \eta^2\big(\frac{|x|}{\rho'}\big) m(x) dx \big) \le \tilde{\lambda} q - \int_{\mathbb{R}^l} T_\kappa(x; \tilde{\lambda}, \nabla \tilde{f}(x)) m(x) dx \, .$$

The proof of (4.21a) concludes by noting that $\nabla \tilde{f} \in \mathbb{L}^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x)\, dx)$, which is established as in Puhalskii [28]. The expression in the supremum on the righthand side of (4.20) being equal to $(1/\kappa)(c_\rho/\rho^2)(\eta'(|x|/\rho))^2$ implies the limit of the J^3_ρ in (4.21b). The limit of the J^2_ρ in (4.21b) follows by a similar reasoning: one pushes the suprema inside the integral in (4.18), notices that the integration can be carried out over x such that $|x| \ge \rho$ and applies Lebesgue's dominated convergence theorem.

In order to complete the proof of (4.17), it remains to note that the limit of the righthand sides of (4.21a) as $\kappa \downarrow 0$ equals the righthand side of (4.17), which is argued as follows. The functions in the supremum in (4.21a) are monotonically decreasing with κ and are continuous functions of $\lambda \in \mathbb{R}$ and $\nabla f \in \mathbb{L}^{1,2}_0(\mathbb{R}^l, \mathbb{R}^l, m(x)\, dx)$. Furthermore, the suprema can be taken over the same compact set of $(\lambda, \nabla f)$ which is proved similarly to the reasoning used when proving (4.21a). Dini's lemma then implies the convergence of the suprema. □

Remark 4.1. The change of measure in (4.10) is inspired by Puhalskii [28]. The idea of using a stochastic exponential to "absorb" control, as in (4.11), is borrowed from Hata, Nagai, and Sheu [12]. The choice of $m(x)$ in (4.2) is dictated by the necessity to use the results of Pardoux and Veretennikov [26] in order to be able to assert that $\nabla \tilde{f}$ is of linear growth. The ergodicity requirement then stipulates that $|x^T \mathrm{div}(\sigma(x)\sigma(x)^T)|$ be bounded.

Remark 4.2. If $\beta(x) = 0$, then part 2 of condition (N) does not hold. Nevertheless, the convergences in (4.21a) and (4.21b) are maintained unless $q = \int_{\mathbb{R}^l} (r(x) - \alpha(x)) m(x)\, dx$, with the limit in (4.21a) being equal to ∞ if $q < \int_{\mathbb{R}^l} (r(x) - \alpha(x)) m(x)\, dx$, and the righthand side of (4.21a) converges, as $\kappa \downarrow 0$, to the righthand side of (4.17), so the theorem holds for that case too.

Proof of Theorem 2.2. We prove that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbf{P}(L^{\hat{\pi}}_t \le q) \le -J^{\mathrm{s}}_q \, . \tag{4.24}$$

The proof was outlined in the Introduction. Since $\hat{\lambda} \geq 0$ so that $e^{-\hat{\lambda} t L_t^{\hat{\pi}}} \geq e^{-\hat{\lambda} q t} \chi_{\{L_t^{\hat{\pi}} \leq q\}}$ and $\hat{\pi}_s^t = \hat{u}(X_s^t)$, by (2.8), on recalling (3.21), for a $\mathbb{C}_b^2$-function f,

$$\frac{1}{t} \ln \mathbf{E} \chi_{\{L_t^{\hat{\pi}} \leq q\}} e^{f(X_t) - f(X_0)} \leq \hat{\lambda} q + \sup_{\nu \in \mathcal{P}} \int\limits_{\mathbb{R}^l} \breve{H}(x; \hat{\lambda}, f, \hat{u}) \nu(dx) \,.$$

Assuming, moreover, that $f \in \mathcal{A}_\kappa$, where $\kappa > 0$, and recalling (3.24) in Lemma 3.5 and Remark 3.4, we have by the functions from $\mathcal{A}_\kappa$ being bounded from below and by their not exceeding $\kappa |\Psi x|^2$ by more than a constant, that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbf{E} \chi_{\{L_t^{\hat{\pi}} \leq q\}} \, e^{-\kappa |\Psi X_0|^2} \leq \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int\limits_{\mathbb{R}^l} \breve{H}(x; \hat{\lambda}, f, \hat{u}) \, \nu(dx) = \hat{\lambda} q + F(\hat{\lambda}) \,. \tag{4.25}$$

By the reverse Hölder inequality, for $\epsilon > 0$,

$$\mathbf{E} \chi_{\{L_t^{\hat{\pi}} \leq q\}} \, e^{-\kappa |\Psi X_0|^2} \geq \mathbf{P}(L_t^{\hat{\pi}} \leq q)^{1+\epsilon} \big(\mathbf{E} e^{\kappa |\Psi X_0|^2 / \epsilon} \big)^{-\epsilon} .$$

Letting $\kappa \to 0$ and $\epsilon \to 0$ in (4.25) obtains (4.24). □

Proof of Theorem 2.3. Since $\mathbf{E} e^{-\lambda t L_t^\pi} \geq e^{-\lambda q t} \mathbf{P}(L_t^\pi \leq q)$ provided $\lambda \geq 0$, Theorem 2.1 implies that

$$\liminf_{t \to \infty} \frac{1}{t} \ln \mathbf{E} e^{-\lambda t L_t^\pi} \geq \sup_{q \in \mathbb{R}} \big(-\lambda q - J_q^{\mathrm{s}} \big) = F(\lambda) \,,$$

with the latter equality holding because by (2.13) J_q^{s} is the Legendre–Fenchel transform of $\tilde{F}(\lambda)$ that equals $F(-\lambda)$ when $\lambda \leq 0$ and equals infinity when $\lambda > 0$.

Now, suppose that q is such that $-\lambda q - F(\lambda) = J_q^{\mathrm{s}}$ and $\pi(\lambda)_t = \hat{u}(X_t)$, with the latter control being associated with q. Similarly to (2.8), for $f \in \mathcal{A}_\kappa$,

$$\mathbf{E} \exp\big(-\lambda t L_t^{\pi(\lambda)} + f(X_t) - f(X_0) - t \int\limits_{\mathbb{R}^l} \breve{H}(x; \lambda, f, \hat{u}) \, \nu^t(dx) \big) \leq 1 \,.$$

In analogy with (4.25), by Lemma 3.5 and X_0 being bounded,

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbf{E} e^{-\lambda t L_t^{\pi(\lambda)}} \leq \inf_{f \in \mathcal{A}_\kappa} \sup_{\nu \in \mathcal{P}} \int\limits_{\mathbb{R}^l} \breve{H}(x; \lambda, f, \hat{u}) \, \nu(dx) = F(\lambda) \,.$$

□

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