arXiv:1306.4232v1 [math.OA] 18 Jun 2013

A VON NEUMANN ALGEBRA SPLITTING OF THE DOUBLE ACTION OF A DISCRETE GROUP G ON ITS PROFINITE COMPLETION, BY LEFT AND RIGHT MULTIPLICATION, AND THE RAMANUJAN PETERSSON CONJECTURES

FLORIN RĂDULESCU*

DIPARTIMENTO DI MATEMATICA
UNIVERSITA DEGLI STUDI DI ROMA "TOR VERGATA"

ABSTRACT. Let G be a discrete countable group, and let Γ be an almost normal subgroup. For $\sigma \in G$, let Γ_σ be the group $\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$ of finite index in Γ. Let K be the profinite completion of group Γ and let $\mathcal{S}$ be the corresponding profinite completion of G. Assume $G^{\rm op}$ is the group G with opposite multiplication. Let $\mathcal{M}$ be the reduced, von Neumann algebra, crossed product $L(G \rtimes L^\infty(\mathcal{S}, \nu))$, with respect Haar measure on $\mathcal{S}$ (assumed to be bivariant). Let $\rho_{\Gamma\backslash G}$ be the right regular representation of $G^{\rm op}$ on $l^2(\Gamma\backslash G)$. Assume that there exists a (projective) unitary representation π of G on $l^2(\Gamma)$, extending the left regular representation. In the presence of a cocycle, all crossed products algebras are skewed.

Let $P = L(\Gamma \rtimes L^\infty(K, \mu))$ be the von Neumann algebra, reduced crossed product. Then there exists a representation α_g of G into the automorphism group of P (unique up to cocycle perturbation), such that, $G^{\rm op}$ equivariantly, we have that $\mathcal{M} \cong P \otimes B(l^2(\Gamma\backslash G))$, where on the let side of the isomorphism above, the group $G^{\rm op}$ acts trivially on $L(G)$ and by right translations, on $\mathcal{S}$. On the right side of the above isomorphism we have the action of $G^{\rm op}$ defined by $g \to \alpha_g \otimes {\rm Ad}\rho_{\Gamma/G}(g)$, $g \in G$. We use the canonical, von Neumann conditional expectation (see e.g. [Sa]), $E_{L(\Gamma)}^{L(\Gamma \rtimes L^\infty(K,\mu))}$ from $L(\Gamma \rtimes L^\infty(K, \mu))$ onto $L(\Gamma)$; it follows that the Hecke algebra representation on $L(\Gamma)$ associated to π, G, Γ as in [Ra] is the same, up to unitary equivalence, as the representation associating to a double coset $\Gamma\sigma\Gamma$ in G, the completely positive map $\Psi_{\Gamma\sigma\Gamma}(x) = E_{L(\Gamma)}^{L(\Gamma \rtimes L^\infty(K,\mu))}(\alpha_\sigma(x))$ for x in $L(\Gamma)$, $\sigma \in G$. In particular the representation $\alpha_g, g \in G$ of G (with α_γ leaving $L(\Gamma)$ fixed, for γ in Γ), into the automorphism group of $L(\Gamma \rtimes L^\infty(K, \mu))$, which determinines the Hecke algebra representation, is a factor of the representation of $G^{\rm op}$ on $L(G \rtimes L^\infty(\mathcal{S}, \nu))$.

* Member of the Institute of Mathematics "S. Stoilow" of the Romanian Academy

INTRODUCTION AND DEFINITIONS

Let G be a discrete group and let Γ be an almost normal subgroup. For σ in G let $\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$ be the subgroup of finite index of Γ associated to σ.

Let $\mathcal{G}$ be the downward directed lattice of finite index subgroups of Γ, generated by the subgroups Γ_σ, $\sigma \in G$. Let K be the profinite completion of Γ, and let $\mathcal{S}$ be the corresponding locally profinite completion of G, obtained by translations of K. We assume for simplicity that the corresponding Haar measures μ, ν on K and respectively $\mathcal{S}$ are bivariant with respect the left and right convolutions.

We use the canonical von Neumann algebra construction, of the reduced crossed product with respect an invariant measure, to construct the von Neumann algebras $\mathcal{M} = L(G \rtimes L^\infty(\mathcal{S}, \nu))$ and $P = L(\Gamma \rtimes L^\infty(K, \mu))$. (If a group cocycle is present in the unitary representation considered bellow, as in ([Ra1], see also [Ra3]), then we consider the skew version of the crossed product).

For simplicity we assume that the groups G, Γ have infinite, non-trivial, conjugacy classes, so that the corresponding reduced, von Neumann, crossed product algebras are factors with infinite traces (see e.g. [Sa]). We let $G^{\rm op}$ be the group corresponding to G, with the opposite multiplication.

Then the right product action of $G^{\rm op}$ on $\mathcal{S}$ and hence the convolution by $G^{\rm op}$ on $L^\infty(\mathcal{S}, \nu)$, induces an action $(\beta_g)_{g\in G}$ defined on $G^{\rm op}$ with values into the automorphism group $\mathrm{Aut}(L(G \rtimes L^\infty(\mathcal{S}, \nu)))$ of the von Neumann algebra $\mathcal{M}$. Here $G^{\rm op}$ acts trivially (identical) on $L(G)$. We also assume that the actions of G, Γ on $\mathcal{S}$ and respectively on $K = \overline{\Gamma}$, is ergodical, and consequently the action is outer ([Co]).

Let $l^\infty(\Gamma\backslash G)$, $l^\infty(G/\Gamma)$ be the algebras of bounded, left, and respectively right invariant functions, on cosets of G by Γ. Then, these algebras may be identified with the subalgebras of $L^\infty(S, \nu)$, generated by translations of the characteristic function of the characteristic functions of the set K, viewed as a subset of $\mathcal{S}$.

We denote by $l^2(\Gamma\backslash G)$, $l^2(G/\Gamma)$ the corresponding Hilbert spaces, and denote by $\rho_{\Gamma/G}$, $\lambda_{G/\Gamma}$, the corresponding unitary representations of G.

Our main assumption is the existence of a representation $\pi : G \to \mathcal{U}(H_\pi)$ such that $\pi|_\Gamma$ is unitarely equivalent to the left regular representation of Γ.

Note that the case of a projective, unitary representation will require to take the skewed, left regular representation (see e.g. [Ra 1], or the N. Ozawa's notes [Ra 3]).

The more general case when $\dim_{\{\pi(\Gamma)\}''} H_\pi$ is just finite, instead of being equal to 1 as above, (see [(GHJ)] for the definition of the corresponding dimension function) may be treated similarly using the results [Ra 2]; this case will be defered to a subsequent paper.

In [Ra 1] we constructed, using the matrix coefficients of the representation π, a representation $t : \mathcal{H} \to L(G)$, such that $t([\Gamma\sigma\Gamma])$ belongs to $l^2(\Gamma\sigma\Gamma) \cap L(G)$ (thus the support of $t([\Gamma\sigma\Gamma])$ is contained in the closure of the span of $\Gamma\sigma\Gamma$), forr any double coset $[\Gamma\sigma\Gamma]$. Here $\mathcal{H} \subseteq B(l^2(\Gamma\backslash G))$ is the reduced von Neumann, Hecke algebra associated to $\Gamma \subseteq G$ (see e.g. [BC]).

We proved that in fact the representation t extends to a larger representation of the operator system $\mathcal{SO} = \mathcal{SO}(\Gamma, G) = \mathbb{C}(G/\Gamma) \underset{\mathbb{C}(\Gamma\backslash G/\Gamma)}{\otimes} \mathbb{C}(\Gamma \setminus G)$.

Here $\mathbb{C}(\Gamma \setminus G/G)$ is the Hecke $\mathbb{C}$ $*$-algebra of double cosets, and $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$ acts canonically on left and right cosets by right (and respectively left multiplication). Then $\mathcal{SO}$ is canonically identified to $\mathbb{C}(\sigma_1\Gamma\sigma_2 \mid \sigma_1, \sigma_2 \in G\}/\mathcal{J}$, where $\mathcal{J}$ is the subspace generated by the identities $\sum_j [\sigma_1^j \Gamma \sigma_2^j] = \sum_s [\mu_1^s \Gamma \mu_2^s]$, if the following equality of disjoint unions holds true: $\bigcup_j \sigma_1^j \Gamma \sigma_2^j = \bigcup_s \mu_1^s \sigma \mu_2^s$.

We have then a $*$-representation $t : \mathcal{SO} \to L(G)$.

We identify $[\sigma\Gamma] \otimes [\Gamma]$ with $[\sigma\Gamma]$, then $t(\sigma\Gamma)^* = t(\Gamma\sigma^{-1})$, and

$$t(\sigma_1\Gamma)t(\Gamma\sigma_2) = t(\sigma_1\Gamma\sigma_2), \sigma_1, \sigma_2 \in G.$$

We proved in [Ra 1], see also [Ra 2], [Ra 3] that the existence of the unitary representation π is equivalent to the existence of a representation with the above properties of the system $\mathcal{SO}$. The precise formula, for the representation t, is as follows: for A a subset of G, 1 a trace vector in H_π for Γ, we define

$$t(A) = \sum_{\theta \in A} \overline{\langle \pi(\theta)1, 1\rangle}\theta.$$

The reason for taking the conjugate is that (as shown in [Ra 3]) the representation t naturaly lives in the commutant $\pi(\Gamma)'$ which is isomorphic to $R(G)$, the von Neumann algebra of right convolutors. To identify with $L(G)$ we need to take the conjugate. This also works when the representation is projective ([Ra 1], [Ra 3]).

In this paper we prove that an equivalent characterization of the above property is that the following holds true: There exists a $G^{\rm op}$ - equivariant matrix unit:

$$(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G} = (\chi_{\Gamma\sigma_1} t^{\Gamma\sigma_1\sigma_2^{-1}\Gamma}\chi_{\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}.$$

The G - equivariance holds true with respect to the $G^{\rm op}$ unitary representation $\rho_{\Gamma\backslash G}$ of $G^{\rm op}$ into $l^2(\Gamma\setminus G)$.

Using this matrix unit it clearly follows that the von Neumann algebra $\mathcal{M}$ is $G^{\rm op}$ equivariantly isomorphic to $\chi_K\mathcal{M}\chi_K\otimes B(l^2(\Gamma\backslash G))$ with respect to an induced action of $G^{\rm op}$ in the form $\alpha_g\otimes {\rm Ad}\rho_{\Gamma\backslash G}(g)$, $g\in G^{\rm op}$. Moreover it is clear that $\chi_K(L(G\rtimes L^\infty(\mathcal{S},\nu)))\chi_K$ is in fact isomorphic to $L(\Gamma\rtimes L^\infty(K,\mu))$ (the unit of the last algebra being identified to χ_K).

On the other hand, the action of ${\rm Ad}\pi(\sigma)$ on $B(H_\pi)$ clearly restricts to the upward directed, union of commutants $\bigcup_{\Gamma_{\sigma_1}}\pi(\Gamma_{\sigma_1})'$. Since all these commutant von Neumann algebras algebras are ${\rm II}_1$, factors, it follows that, after renormalizing the trace, the actions ${\rm Ad}\,\pi(\sigma)$, $\sigma\in G$, induce an action of G onto the ${\rm II}_1$ factor completion of the upward directed inductive limit, $\bigcup_{\{e\}\leftarrow\Gamma_{\sigma_1},\Gamma_{\sigma_1}\in\mathcal{G}}\pi(\Gamma_{\sigma_1})'$. This latest algebra is obviously isomorphic to the simultaneous Jones' basic construction for all the inclusions $\Gamma_\sigma\subseteq\Gamma$, $\sigma\in\Gamma$.

But this simultaneous Jones' basic construction is clearly isomorphic to $R(\Gamma\rtimes L^\infty(K,\mu))$, which, by taking the conjugate, is $L(\Gamma\rtimes L^\infty(K,\mu))$. Note that this is the same von Neumann algebra as the one considered in ([OP]), where in the end of the paper, the unicity of the Cartan subalgebra $L^\infty(K,\mu)$ was proved.

We will prove that the action α_g into the automorphism group of the type ${\rm II}_1$ factor $L(\Gamma\rtimes L^\infty(K,\mu))$ that we are constructing in Theorem 1 (and which gives the factorization) coincides with the latest action induced by ${\rm Ad}\,\pi(\sigma)$, $\sigma\in G$. Consequently, it will follow that the Hecke operators representation induced by the representation α_g on $L(\Gamma\rtimes L^\infty(K,\mu))$ (having as Γ - fixed vectors the algebra $L(\Gamma)$) is the same as the Hecke algebra representation in [Ra 1].

Thus the Hecke algebra representation constructed in [Ra 1], is in fact induced by an action of $G^{\rm op}$ on $L(\Gamma\rtimes L^\infty(K,\mu))$, which in turn is a tensor product factor of the action of $G^{\rm op}$ on $L(G\rtimes L^\infty(\mathcal{S},\nu))$.

Moreover, unicity of the G - equivariant matrix units up to unitary perturbation proves that the action $(\alpha_g)_{g\in G}$ is unique up to cocycle perturbation.

1. MAIN RESULTS

Using the previous introduced definitions we prove the main results of the paper, namely we prove the tensor splitting of the representation of $G^{\rm op}$ on $L(G \rtimes L^\infty(\mathcal{S}, \nu))$.

Theorem. *Assume that $\Gamma \subseteq G$ is a pair consisting of a discrete group G and an almost normal subgroup Γ, both assumed to be with infinite conjugacy classes. Let $\mathcal{G}$ be the downward directed class of subgroups of Γ, generated by $\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$, $\sigma \in G$, and let (K, μ) be the profinite completion of Γ and let $(\mathcal{S}, \nu)$ be the corresponding profinite completion of G, obtained by translating K. Here ν, μ are the corresponding Haar measures.*

We assume that for all $\sigma \in G$ the subgroups Γ_σ, $\Gamma_{\sigma^{-1}}$ have equal indices. Hence the Haar measures on $\mathcal{S}$, K are bivariant and hence $\mathcal{M} = L(G \rtimes L^\infty(\mathcal{S}, \nu))$ and $N = L(\Gamma \rtimes L^\infty(K, \mu))$ are type II_∞ (respectively II_1), factors. We also assume that G acts ergodically on $\mathcal{S}$. Then the following are equivalent

1) There exists a (projective) unitary representation $\pi : G \to H_\pi$ such that $\dim_{\{\pi(\Gamma)\}''}(H_\pi) = 1$ (see [GHJ] for notations; in the case a cocycle is present in the unitary representation, then in the definitions of $\mathcal{M}$ and N we take the skew crossed product algebras)). This hypothesis also implies that $[\Gamma : \Gamma_\sigma] = [\Gamma, \Gamma_{\sigma^{-1}}]$ for all $\sigma \in G$. By [Kr] this also implies that $[\Gamma\sigma\Gamma] = [\Gamma\sigma^{-1}\Gamma]$ for σ in G.

2) There exists a $*$ - algebra representation of the operator system $\mathcal{SO}$ introduced in the first section, $t : \mathcal{SO} \to L(G)$ such that $t([\Gamma]) = \mathrm{Id}$, $t(\Gamma\sigma) \in L(G) \cap l^2(\Gamma\sigma)$.

3) There exists a unitary representation of $G^{\rm op}$ into the von Neumann algebra $\chi_K L((G \times G^{\rm op}) \rtimes L^\infty(\mathcal{S}, \mu))\chi_K$, denoted $\sigma \to u(\sigma)$, $\sigma \in G$, such that $u(\sigma)$ is of the form $\chi_K(t^{\Gamma\sigma\Gamma} \otimes \sigma^{-1})\chi_K$ for a selfadjoint element $t^{\Gamma\sigma\Gamma}$ in $L(G) \cap l^2(\Gamma\sigma\Gamma)$.

It will automatically follow that $[\Gamma\sigma\Gamma] \to t^{\Gamma\sigma\Gamma}$ extends to a $*$ representation of the Hecke algebra $\mathbb{C}(\Gamma \setminus G/\Gamma)$.

4) Let $\beta_g : G \to \mathrm{Aut}(\mathcal{M})$ be the canonical representation of $G^{\rm op}$ into the automorphism group of $\mathcal{M}$, leaving $L(G)$ invariant and acting by right translation on $L^\infty(\mathcal{S}, \nu)$. There exists a $G^{\rm op}$ - equivariant matrix unit $(v_{\Gamma\sigma_1, \Gamma\sigma_2})_{\Gamma\sigma_1, \Gamma\sigma_2 \in \Gamma\setminus G}$ such that $v_{\Gamma\sigma, \Gamma\sigma} = \chi_{\overline{\Gamma\sigma}} \in L^\infty(\mathcal{S}, \nu)$, for σ in G. By $G^{\rm op}$ equivariance, we understand that for all $g \in G^{\rm op}$ and all $\Gamma\sigma_1, \Gamma\sigma_2 \in \Gamma \setminus G$ we have that

$$\beta_g(v_{\Gamma\sigma_1, \Gamma\sigma_2}) = v_{\Gamma\sigma_1 g, \Gamma\sigma_2 g}.$$

In particular it follows that the factor $\mathcal{M}$ is $G^{\rm op}$ equivariantly isomorphic to $\chi_K\mathcal{M}\chi_K \otimes B(l^2(\Gamma \setminus G))$.

Thus if the equivalent conditions in the statement hold true, then, we obtain (using the $G^{\rm op}$ equivariant matrix unit) a canonical action $G^{\rm op} \ni g \to \alpha_g \in {\rm Aut}(\chi_K\mathcal{M}\chi_K)$ Then $G^{\rm op}$ acts on $\chi_K\mathcal{M}\chi_K \otimes B(l^2(\Gamma \setminus G))$ by the automorphism $\alpha_g \otimes {\rm Ad}\rho_{\Gamma\setminus G}(g)$, $g \in G^{\rm op}$.

Note the II_1 factor $P = \chi_K\mathcal{M}\chi_K$ is isomorphic to $L(\Gamma \rtimes L^\infty(K, \mu))$, and note that $\alpha|_{\Gamma^{\rm op}}$ acts as the identy on $L(\Gamma) \subseteq P$. Thus, in this representation, $L(\Gamma)$ is consequently the set of fixed points for the action of $\Gamma^{\rm op}$ on P.

Note that by construction the diagonal algebra $l^\infty(\Gamma \setminus G)$ in the type I algebra $B(l^2(\Gamma \setminus G))$ asociated to the $G^{\rm op}$ equivariant matrix unit, coincides with the diagonal algebra $l^\infty(\Gamma \setminus G)$ on $L^\infty(\mathcal{S}, \nu)$ spanned by the cosets $\chi_{\overline{\Gamma_\sigma}}$, $\sigma \in G$.

Finally, the automorphism group $\alpha_g, g \in G^{\rm op}$ is unique up to cocycle cojugacy.

If the above properties hold true then the $*$ algebra representation $\mathbb{C}(\Gamma \setminus G/\Gamma) \ni [\Gamma\sigma\Gamma] \to \Psi_{[\Gamma\sigma\Gamma]}$ with values in to the completely positive maps on $\pi(\Gamma)' \cong R(\Gamma) \cong L(\Gamma)$, constructed in [Ra 1], is unitarely equivalent to the completely positive maps $\Psi_{\Gamma\sigma\Gamma}$ on $L(\Gamma)$ defined by the formula $L(\Gamma) \ni x \to E_{L(\Gamma)}^{L(\Gamma\rtimes L^\infty(K,\mu))}(\alpha_g(x))$ for $g \in \Gamma\sigma\Gamma$ (since α_ν is the identy on $L(\Gamma)$ it follows that the definition of $\Phi_{\Gamma\sigma\Gamma}$ is independent of the choice of g in $\Gamma\sigma\Gamma$).

In particular the properties of the Hecke operators associated to the (projective) unitary representation π constructed in [Ra 1] are depend on the ergodic poperties of the $G \times G^{\rm op}$ action on $\mathcal{S}$.

Before proving the theorem, we note that, as proved in [Ra 1], property (2) implies that with $t^{\Gamma\sigma\Gamma} = \sum_i t(\Gamma\sigma s_i) = \sum_j t(v_j\sigma\Gamma)$ where s_i, v_i are the corresponding coset representatives, for $\sigma \in G$, we have that the map $[\Gamma\sigma\Gamma] \to \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma} \otimes t^{\Gamma\sigma\Gamma})\chi_{\overline{\Gamma}}$ with values in $\chi_{\overline{\Gamma}}(L^{\rm op}(G \times G^{\rm op}) \rtimes L^\infty(\mathcal{S}, \nu))\chi_{\overline{\Gamma}}$ is a von Neumann algebras isomorphism preserving the trace. In fact, we have a groupoid action of $G \times G^{\rm op}$ on K, and the above maps extend by linearity to a $*$ - algebra morphism from the Hecke algebra $\mathbb{C}(\Gamma \setminus G/\Gamma)$ into $C^*_{\rm red}((G \times G^{\rm op}) \rtimes K)$. This representation is trace preserving, and hence extends to the reduced C^* - Hecke algebra.

Proof. The equivalence of (1) and (2) has been proved in [Ra1].

To prove that (4) implies (2) we proceed as follows. For σ in G let $X^{\Gamma\sigma\Gamma} = \sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma} v_{\Gamma\sigma_1,\Gamma\sigma_2}$, where the sum runs over all cosets $\Gamma\sigma_1, \Gamma\sigma_2$ in $\Gamma \setminus G$ such that $\Gamma\sigma_1\sigma_2^{-1}\Gamma = [\Gamma\sigma\Gamma]$.

By the $G^{\rm op}$ equivariance of the matrix unit $(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2}$ we have that $\beta_g(v_{\Gamma\sigma_1,\Gamma\sigma_2}) = v_{\Gamma\sigma_1 g,\Gamma\sigma_2 g}$, $g \in G$. Hence $\beta_g(X^{\Gamma\sigma\Gamma}) = X^{\Gamma\sigma\Gamma}$ for all $g \in G^{\rm op}$, σ in G.

Since $G^{\rm op}$ acts ergodically on $\mathcal{S}$, it follows that $X^{\Gamma\sigma\Gamma}$ belongs to $L(G)$ which is the set $\mathcal{M}^G$ of fixed points by the action of G on $\mathcal{M}$.

Moreover by definition, and since $v_{\Gamma\sigma,\Gamma\sigma}$ is equal to $\chi_{\Gamma\sigma}$ it follows that $v_{\Gamma\sigma_1,\Gamma\sigma_2}$ is a partial isometry from $\chi_{\Gamma\sigma_2}$ to $\chi_{\Gamma\sigma_1}$.

The formula defining $X^{\Gamma\sigma\Gamma}$ implies that if $\Gamma\sigma_1, \Gamma\sigma_2$ are so that $\Gamma\sigma_1\sigma_2^{-1}\Gamma = [\Gamma\sigma\Gamma]$ then $\chi_{\Gamma\sigma_1} X^{\Gamma\sigma\Gamma} \chi_{\sigma_2\Gamma} = v_{\Gamma\sigma_1,\Gamma\sigma_2}$, $\sigma \in G$. We may write:

$$\chi_{\Gamma\alpha} X^{\Gamma\sigma\Gamma} \chi_{\Gamma\beta} = \delta_{[\Gamma\alpha\beta^{-1}\Gamma],[\Gamma\sigma\Gamma]} v_{\Gamma\alpha,\Gamma\beta}$$

for $\alpha, \beta, \sigma \in G$. Here, δ is the Kronecker symbol.

Obviously for $\theta \in G$, $\chi_{\Gamma\sigma_1}\theta\chi_{\Gamma\sigma_2} \neq 0$ if and only if there exist γ_1, γ_2 in Γ such that $\theta\gamma_2\sigma_2 = \gamma_1\sigma_1$ and thus if and only if θ belongs to $\Gamma\sigma_1\sigma_2^{-1}\Gamma$.

Thus necessary $X^{\Gamma\sigma\Gamma}$ belongs to $L(G) \cap l^2(\Gamma\sigma\Gamma)$.

To see that $X^{\Gamma\sigma\Gamma}$ is bounded, observe that formally the operators $X^{\Gamma\sigma\Gamma}$, $\sigma \in G$, which are apriori affiliated to $L(G)$, have the property that the mapping $[\Gamma\sigma\Gamma] \to X^{\Gamma\sigma\Gamma}$ extends to a $*$ - algebra representation of $\mathbb{C}(\Gamma \setminus G/G)$, because of the way the cosets are combining themselves in the products.

Moreover, this representation is trace preserving (with respect to the trace τ on $L(G)$).

But then the above map extends to $\mathcal{H}_{\rm red} = C^*_{\rm red}(G\backslash G/\Gamma)$ the reduced C^* - Hecke algebra, and hence the elements $t^{\Gamma\sigma\Gamma} = X^{\Gamma\sigma\Gamma}$, $\sigma \in G$, are bounded.

The property that

$$\chi_{\Gamma\alpha} t^{\Gamma\alpha\beta^{-1}\Gamma} \chi_{\Gamma\beta} t^{\Gamma\beta\gamma^{-1}\Gamma} \chi_{\Gamma\gamma} = \chi_{\Gamma\alpha} t^{\Gamma\alpha\gamma^{-1}\Gamma} \chi_{\Gamma\alpha}$$

implies, when moving in the left side member, the characteristic function $\chi_{\Gamma\beta}$ to the right, a series of identities $\sum t^{A_i} t^{B_i} = \sum t^{C_j}$ (see [Ra4]). Here, for A a coset (of some subgroup group Γ_σ on Γ) we denote by t^A the sum $\sum_{\theta\in A} t(\theta)\theta$, where $t(\theta)$ are the coefficients in $t^{\Gamma\sigma\Gamma} = X^{\Gamma\sigma\Gamma}$.

But these identities, when summed up, are the identities

$$t^{\sigma_1\Gamma} t^{\Gamma\sigma_2} = t^{\sigma_1\Gamma\sigma_2}, \sigma_1, \sigma_2 \in G,$$

which are exactly the sufficient conditions that imply that the map $[\sigma\Gamma] \to t^{\sigma\Gamma}$, $\sigma \in G$ extends to a representation of $\mathcal{SO}$ (as in property (2) in the statement).

To prove that (1) implies (3) we use the following. We identify the commutant algebra $\{\pi(\Gamma)\}'$ with $R(\Gamma)$, and the Jones simultaneuous basic construction for the subfactors $\{\pi(\Gamma\sigma)\}'' \subseteq \{\pi(\Gamma)\}''$ has the term at ∞ the type II_1 factor $\mathcal{A}_\infty$, the inductive limit of the II_1 factors $\{\pi(\Gamma_\sigma)\}' = \{\pi(\Gamma)\}' \bigvee \{e_{\Gamma_\sigma}\}''$, where e_{Γ_σ} are the Jones projection corresponding to the subfactors $\{\pi(\Gamma_\sigma)\}'' \subseteq \{\pi(\Gamma)\}''$, $\sigma \in G$. Then $\mathcal{A}_\infty$ is clearly identified with $R(\Gamma \rtimes K)$, where K is the profinite completion of Γ, with respect to the subgroups Γ_σ, and the isomorphism is realized by having the Jones's projection e_{Γ_σ} identified with the characteristic function $\chi_{\overline{\Gamma_\sigma}} \in C(K)$, (here the closure of Γ_σ is in the profinite completion).

Note that the Jones's construction can be extend to a pair of subfactors of same index $N_0, N_1 \subseteq M$, if $\theta : N_0 \to N_1$ is a given isomorphism.

In this case we define $W_\theta : L^2(N_0) \to L^2(N_1)$ by $W_\theta(n_0) = \theta(n_0)$, $n_0 \in N_0$. Viewed as an element of $B(L^2(M,\tau))$, W_θ is a partial isometry from the image of the projection e_{N_0} onto the image of the projection e_{N_1}, where e_{N_0}, e_{N_1} are the corresponding Jones projections onto the subfactors N_0, N_1. We develop an analogy with the Jones construction of the first step in the basic construction ([Jo]): $\langle M, e_N\rangle = Me_NM \cong N' \subseteq B(L^2(M,\tau)$, for a subfactor $N \subseteq M$. We consider instead of N', the space intertwiners $\text{Int}_\theta(N_0, N_1) = \{\alpha \in B(L^2(M,\tau)) \mid \alpha n_0 = \theta(n_0)\alpha, \text{for } n_0 \in N_0\}$.

Then Int_θ is isomorphic to the space $MW_\theta M$ and the multiplication corresponds to the product $MW_\theta M \times MW_\theta M$ goes onto $MW_\theta MW_\theta M$.

In particular, if we consider a family of such subfactors $N_0^S \overset{\theta^S}{\to} N_1^S$, we obtain a von Neumann algebra (viewed as a space of intertwiners), containing all the W_{θ_S}, and the algebra M.

In the case of the subgroups $\Gamma_\sigma \subseteq \Gamma$, $\sigma \in G$, letting the isomorphism θ_σ from $L(\Gamma_{\sigma^{-1}}) \to L(\Gamma_\sigma)$ be the conjugation by σ, we obtain by taking inductive limit a type II, factor B_∞ which is clearly the reduced von Neumann algebra $\chi_{\overline{\Gamma}}(L((G \times G^{\text{op}}) \rtimes L^\infty(\mathcal{S}, \nu)))\chi_{\overline{\Gamma}}$.

Here we switch from right convolutors to left convolutors, by taking the conjugate linear map. We denote W_{θ_σ} by W_σ. In the above representation for B_∞, W_σ is identified with $(\sigma \otimes \sigma^{-1})e_{\Gamma_{\sigma^{-1}}} = e_{\Gamma_\sigma}(\sigma \otimes \sigma^{-1})$.

By letting $\sigma \otimes \sigma^{-1}$ act as a partial isomorphism, we have a groupoid action of $G \times G^{\text{op}}$ on K. Using this partial (groupoid action) we identify $\mathcal{B}_\infty = L((G \times G^{\text{op}}) \rtimes L^\infty(K, \mu))$ with unit identified to $\chi_{\overline{\Gamma}}$ (here we are using the $G \times G^{\text{op}}$ groupoid invariant measure on K).

Then $\pi(\sigma) \in \mathrm{Int}_{\theta_\sigma}(\pi(\Gamma_{\sigma^{-1}}), \pi(\Gamma_\sigma))$ and through the above identification of the space of intertwiners, the intertwiner $\pi(\sigma), \sigma \in G$, is identified to $\sum t_i^* W_{\theta_\sigma} s_i$, where $s_i \in L(\Gamma) = \{\pi(\Gamma)\}''$ is a Pimsner Popa basis for $L(\Gamma_\sigma) \subseteq L(\Gamma)$, (e. g. a system of coset representatives), and $t_i = \pi(\sigma)s_i$.

Thus we get that $\pi(\sigma)$ in the identification of $\mathcal{B}_\infty$ with $L((G \times G^{\mathrm{op}}) \rtimes K)$ is the element $u(\sigma) = \sum_i (t_i^* \otimes 1)(\sigma \otimes \sigma^{-1}) e_{\Gamma_{\sigma^{-1}}}(s_i \otimes 1)$ which is clearly equal to

$$\sum_i \chi_{\overline{\Gamma}}(t_i^* \otimes 1)(\sigma \otimes \sigma^{-1})(s_i \otimes 1)\chi_{\overline{\Gamma}} =$$

$$= \sum_i \chi_{\overline{\Gamma}}(t_i^* \sigma s_i) \otimes \sigma^{-1} \chi_{\overline{\Gamma}}.$$

In this expresentation, we denote for $\sigma \in G$, by $t(\Gamma\sigma\Gamma) \in L(G)$, the sum $= \sum_i t_i^* \sigma s_i$. Note that this is exactly the Hecke algebra representation considered in [Ra 1], and note that we may write

$$u(\sigma) = \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma} \otimes 1)(1 \otimes \sigma^{-1})\chi_{\overline{\Gamma}},\ \sigma \in G.$$

This is a representation of G^{op} into

$$L((G \times G) \rtimes L^\infty(K, \mu)) = \chi_{\overline{\Gamma}} L((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{S}, \nu))\chi_{\overline{\Gamma}},$$

since $\pi(\sigma)$, $\sigma \in G$, was a unitary (eventually projective) representation of G.

To prove that (3) implies (4) we note that we can write $u(\sigma) = (v_{\Gamma,\Gamma\sigma} \otimes 1) \otimes (1 \otimes \sigma^{-1})$.

Hence $v_{\Gamma,\Gamma\sigma} = \chi_\Gamma(t^{\Gamma\sigma\Gamma})\chi_{\Gamma\sigma}$ is an isometry, in $L(G \rtimes L^\infty(\mathcal{S}, \nu))$ from χ_Γ, onto $\chi_{\Gamma\sigma}$.

We define $v_{\Gamma\sigma_1,\Gamma\sigma_2} = \beta_{\sigma_1}(v_{\Gamma,\Gamma\sigma_2\sigma_1^{-1}})$ which is thus equal to

$$\chi_{\Gamma\sigma_1} t^{\Gamma\sigma_1\sigma_2^{-1}\Gamma} \chi_{\Gamma\sigma_2},$$

which is a partial isometry from $\chi_{\Gamma\sigma_2}$ onto $\chi_{\Gamma\sigma_1}$.

The property that $u(\sigma)$, $\sigma \in G$ is a representation of G, translates into the fact that $(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_i \in \Gamma\backslash G}$ is a matrix unit, which by construction is G^{op} equivariant (where G^{op} acts on $L(G \times L^\infty(\mathcal{S}, \nu))$) by leaving $L(G)$ invariant, and by right translations on $L^\infty(\mathcal{S}, \nu)$.

The construction of the splitting

$$L(G \times L^\infty((\mathcal{S}, \nu)) = P \otimes B(l^2(\Gamma \setminus G)),$$

in a G^{op} equivariant way, is now straightforward. We let

$$P = \chi_{\overline{\Gamma}}(L(G \rtimes L^\infty(\mathcal{S}, \nu)))\chi_{\overline{\Gamma}} = L(\Gamma \rtimes L^\infty(K, \mu)),$$

(with unit identified to $\chi_{\overline{\Gamma}}$) and, for $p \in P$ we define,

$$\alpha_g(p) = v_{\Gamma,\Gamma_g}\beta_g(p)v_{\Gamma g,\Gamma},$$

for $p \in \chi_\Gamma \mathcal{M}\chi_\Gamma$, so that $\beta_g(p)$ belongs to $\chi_{\Gamma g}\mathcal{M}\chi_{\Gamma g}$, for all p in P.

Since $G^{\rm op}$ acts trivially on G, it follows that β_g acts trivially on $L(\Gamma)$, and hence β_γ acts trivially on $L(\Gamma)$ for γ in Γ. Indeed, in this case, for $x \in L(\Gamma) \subseteq \chi_{\overline{\Gamma}}L(\Gamma \rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}}$, $\gamma \in \Gamma^{\rm op}$, we obtain that $\alpha_\gamma(x) = v_{\Gamma,\Gamma}\beta_\gamma(x)v_{\Gamma,\Gamma} = x$.

Thus α_g is a G - automorphism representation into the automorphism group of the type II_1 factor $L(\Gamma \rtimes L^\infty(K,\mu))$ such that $\alpha|_\Gamma$ acts identically on $L(\Gamma)$.

Note that in this case the formula for $\alpha_g(p)$ is $v_{\Gamma,\Gamma g}\beta_g(p)v_{\Gamma g,\Gamma}$.

Moreover recall that the formula for $u(\sigma)$ is

$$\chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma}\otimes 1)(1\otimes\sigma^{-1})\chi_{\overline{\Gamma}} = \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma}\otimes 1)\chi_{\overline{\Gamma\sigma}}(1\otimes\sigma^{-1}) = (v_{\Gamma,\Gamma\sigma}\otimes 1)(1\otimes\sigma^{-1}).$$

Thus we see that for $x = \chi_{\overline{\Gamma}}x\chi_{\overline{\Gamma}}$ in $L(\Gamma \rtimes L^\infty(K,\mu))$ which is identified to $L(\Gamma \rtimes L^\infty(K,\mu)) \otimes 1$, we have that $u(g)xu(g)^*$ is equal to $u(g)(x \otimes 1)u(g)^* = (v_{\Gamma,\Gamma\sigma}\otimes 1)\chi_{\Gamma\sigma}(1\otimes\sigma^{-1})[(\chi_{\overline{\Gamma}}x\chi_{\overline{\Gamma}})\otimes 1](1\otimes\sigma)(v_{\Gamma\sigma,\Gamma}\otimes 1)$ and this is thus equal to

$$(v_{\Gamma,\Gamma\sigma}\beta_\sigma(x)v_{\Gamma\sigma,\Gamma})\otimes 1.$$

Thus $u(\sigma) \in \chi_{\overline{\Gamma}}(C(G \times G \rtimes L^\infty(\mathcal{S})))\chi_{\overline{\Gamma}}$ normalizes P (which is identified to $P \otimes 1$ and $\alpha_\sigma(x) = {\rm Ad}u(\sigma)(x)$, $x \in P, \sigma \in G$.

For x in $L(\Gamma) \subseteq \chi_{\overline{\Gamma}}L(\Gamma \rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}}$, the formula for $\alpha_\sigma(x)$ becomes $\chi_{\overline{\Gamma}}t^{\Gamma\sigma\Gamma}\chi_{\Gamma\sigma}x\chi_{\Gamma\sigma}t^{\Gamma\sigma\Gamma}\chi_{\overline{\Gamma}}$. Note that last expression depends, in fact, only on the coset $\Gamma\sigma \in \Gamma \setminus G$.

In particular, for $x \in L(\Gamma) \subseteq \chi_{\overline{\Gamma}}L(\Gamma \rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}}$, we obtain that

$$E_{L(\Gamma)}^{L(\Gamma\rtimes L^\infty(K,\mu))}(\alpha_\sigma(x)) = \sum_i \chi_{\overline{\Gamma}}t^{\Gamma\sigma\Gamma}\chi_{\Gamma\sigma s_i}x\chi_{\Gamma\sigma s_i}t^{\Gamma\sigma\Gamma}\chi_{\overline{\Gamma}}.$$

Here the family s_i are the corresponding coset representatives. The right hand term is clearly $\chi_{\overline{\Gamma}}t^{\Gamma\sigma\Gamma}xt^{\Gamma\sigma\Gamma}\chi_{\overline{\Gamma}}$ which is further equal to the map $\Psi_{\Gamma\sigma\Gamma}(x)$, $x \in L(\Gamma)$, constructed in [Ra 1] and associated to the representation π in point (1) of the equivalences.

Although the following implication is not needed we note that to prove that (3) implies (1) it is sufficient to use the formula $\pi(\sigma)(x) = E_{L(\Gamma)}^{L(G)}(t^{\Gamma\sigma\Gamma}x\sigma^{-1})$ for x in $l^2(\Gamma)$. □

Remark. *With the notation from the statement of the theorem, for $g \in G$, the automorphism α_g will map $s_i\chi_{\overline{\Gamma_{g^{-1}}}}L^\infty(K,\mu)$ into $t_i\chi_{\overline{\Gamma}_g}L^\infty(K,\mu)$, for a suitable choice of left coset representatives s_i for $\Gamma_{g^{-1}}$ in Γ. Here, the elements t_i are a Pimsner Popa basis ([PP]) for $L(\Gamma_g)$ in $L(G)$: they are the images of*

the elements s_i through the representation π. The automorphism α_g acts as conjugation by g from $\chi_{\overline{\Gamma_{g^{-1}}}}L^\infty(K,\mu)$ onto $\chi_{\overline{\Gamma}_g}L^\infty(K,\mu)$. We conjecture that this conditions should uniquely determine the representation $\alpha_g, g \in G^{\rm op}$, up to conjugation by an element in the unitary group of $L(\Gamma)$, by analogy with the results in the last chapter of the paper by N. Ozawa and S. Popa ([OP]).

We address now the question of unicity. We prove that up to unitary conjugacy the group $(\alpha_g)_{g\in G}$ of automorphisms of $P = L(\Gamma \rtimes L^\infty(K,\mu))$ is uniquely determined. We have:

Proposition. *Given (α_g) constructed as above, in the presence of the properties (1) - (4), any other $(\widetilde{\alpha}_g)_g$ obtained from similar data is of the form $\widetilde{\alpha}_g = {\rm Ad}u_g\alpha_g$, where $u_g \in \mathcal{U}(P)$ is a 1-cocycle of G, with respect to α_g, $g \in G$ with values in the unitaries of P (that is $u_{g_1g_2} = u_{g_1}\alpha_g(u_{g_2})$, $g_1, g_2 \in G$).*

Proof. Indeed assume $(\alpha_g)_{g\in G}$ is given.

Then $\mathcal{M} = L(G \rtimes L^\infty(\mathcal{S},\nu))$ has the structure $P \otimes B(l^2(\Gamma \setminus G))$ with $P = L(\Gamma \rtimes L^\infty(K,\mu))$ and the canonical action of $G^{\rm op}$ onto $\mathcal{M}$ becomes $\alpha_g \otimes {\rm Ad}\rho_{\Gamma/G}(g)$ for $g \in G$.

Elements in $\mathcal{M}$ could thus be described as infinite matrices

$$(p_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G},$$

where the entries are in the algebra P.

Any other $G^{\rm op}$ - equivariant matrix unit will be of the form

$$(u(\Gamma\sigma_1)u(\Gamma\sigma_2)^*)_{\Gamma\sigma_1,\Gamma\sigma_2},$$

where $u(\Gamma\sigma)$, $\Gamma\sigma \in \Gamma \setminus G$ are unitaries in P.

Note that we are using here the facthat the diagonal algebra

$$D = l^\infty(\Gamma \setminus G) \subseteq B(l^2(\Gamma \setminus G))$$

is independent of the choice of equivariant matrix unit.

We may also describe $(u(\Gamma\sigma))_{\Gamma\sigma}$ as a unitary

$$w = \sum u(\Gamma\sigma) \otimes \chi_{\overline{\Gamma\sigma}}$$

in $D' \cap L(G \times L^\infty(\mathcal{S},\nu))$. The new matrix unit

$$\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2} = u(\Gamma\sigma_1)^*u(\Gamma\sigma_2) \otimes v_{\Gamma\sigma_1,\Gamma\sigma_2}$$

(where $v_{\Gamma\sigma_1,\Gamma\sigma_2}$ is the original matrix unit for α_g) has also to be G - equivariant.

Denote $B = \{v_{\Gamma\sigma_1,\Gamma\sigma_2}\}''$, $\widetilde{B} = \{v_{\Gamma\sigma_1\Gamma\sigma_2}\}''$ and let

$$X^{\Gamma\sigma\Gamma} = \sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma} v_{\Gamma\sigma_1,\Gamma\sigma_2}, \widetilde{X}^{\Gamma\sigma\Gamma} = \sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma} \widetilde{v}_{\Gamma\sigma_1,\sigma_2\Gamma}$$

for every double coset $\Gamma\sigma\Gamma$ in G. These are the elements constructed in the proof of the implication (3) $\Rightarrow$ (1) in the proof of the theorem.

Then both applications mapping $[\Gamma\sigma\Gamma] \to X^{\Gamma\sigma\Gamma}$, $[\Gamma\sigma\Gamma] \to \widetilde{X}^{\Gamma\sigma\Gamma}$, ($[\Gamma\sigma\Gamma]$ a double coset), extend to representations of the algebra $\mathbb{C}(\Gamma \setminus G/\Gamma)$ (and in fact exend to the reduced Hecke C^* - algebra with valus into the algebra $L(G)$).

We then have $wX^{\Gamma\sigma\Gamma}w^* = \widetilde{X}^{\Gamma\sigma\Gamma}$, for all $[\Gamma\sigma\Gamma]$, and $wBw^* = \widetilde{B}$.

By the G - invariance it follows that $\beta_g(w)$ has the same properties as w, and thus $\beta_g(w)^*w$ belongs to $B' = P \otimes I$.

But then $c(g) = \beta_g(w)^*w$ is a 1-cocycle with respect to α_g with values in P, for the group G. Moreover the action $\widetilde{\alpha}_g$ of G into the automorphism of P, associated to the matrix unit $(\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2})$ is then $\widetilde{\alpha}_g = \mathrm{Ad}c(g)\alpha_g$, $g \in G$.

Conjecture. Recall that $D = l^\infty(\Gamma \setminus G) \subseteq L^2(G \rtimes L^\infty(\mathcal{S}, \nu))$.

We conjecture that vanishing in the 1 - group cohomology $H^1_\alpha(G, D')$ for a cocycle c with values in $P \cong P \otimes 1 = B' \subseteq D'$, is equivalent to vanishing of the cocycle c in $H^1_\alpha(G, P)$.

If then would follow that there exists a unitary p in P, such that $c(g) = \beta^*_g(p)p$, for g in G.

Thus $\beta^*_g(w)w = \beta_g(p^*)p$ for g in G and hence$\beta_g(wp^*) = wp^*$ for all g in G.

Thus wp^* belongs to $L(G)$. Since both w, p belongs to D' it follows that $wp^* \in L(G) \cap (l^\infty(\Gamma \setminus G))'$ which is $L(\Gamma)$.

Consequently $w = xp$ for some unitary x in $L(\Gamma)$. Hence $wBw^* = x\widetilde{B}x^*$, and hence we may assume that w belongs to $L(\Gamma)$. Thus $\widetilde{X}^{\Gamma\sigma\Gamma} = xX^{\Gamma\sigma\Gamma}x^*$ and hence $\widetilde{\alpha}_g$ depends from α_g by a perturbation with an element in $L(\Gamma)$.

Consequently the associated representations of the Hecke algebra, $[\Gamma\sigma\Gamma] \to \Psi_{\Gamma\sigma\Gamma}, \widetilde{\Psi}_{\Gamma\sigma\Gamma}$ are distinguished only by conjugating with a unitary in $L(\Gamma)$ (in particular they have the same spectrum).

Thus if the conjecture holds true, then $(\alpha_g)_g$ is unique up to a perturbation by a unitary in $L(\Gamma)$ and thus the Hecke algebra maps $\Psi_{[\Gamma\sigma\Gamma]} = E^{L(\Gamma\times G)}_{L(\Gamma)}(\alpha_g(\cdot))$ are uniquely determined by the factorization of $L(G \rtimes L^\infty(\mathcal{S}, \nu))$,

up to a unitary in $L(\Gamma)$. Moreover it would follow that the splitting in Theorem 1 is unique up to conjugation by a unitary in $L(\Gamma)$.

Remark. In the case $G = PGL_2(\mathbb{Z}[\frac{1}{p}])$, $\Gamma = PSL_2(\mathbb{Z})$, p a prime number, the space of cosets has a homogenuous tree structure. Then to get a G - invariant matrix unit in $L(G \rtimes L^\infty(\mathcal{S}, \nu))$ as in the Theorem 1 it is sufficient to find $X = X^*$ in $L(G) \cap l^2(\Gamma\sigma_p\Gamma)$, such that $\chi_{\overline{\Gamma}} X \chi_{\overline{\Gamma\sigma_p}}$ is an isometry from $\chi_{\overline{\Gamma}}$ onto $\chi_{\overline{\Gamma\sigma_p}}$. Here, for $n \in \mathbb{N}$, we define

$$\sigma_{p^n} = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}.$$

In addition (to have compatibility with the adjoint operation) we impose that $\chi_\Gamma X \chi_{\overline{\Gamma\sigma_p}}$ to be the adjoint of $\beta_{(\sigma_p s_i)}(X)$ (if $\Gamma\sigma_p^{-1} = \Gamma\sigma_p s_i$). Also X has to be invariant under the action of $\beta_\gamma, \gamma \in \Gamma_{\sigma_p}$. We may now take $v_{\Gamma\sigma_1, \Gamma\sigma_2} = \beta_{\sigma_1}(\chi_{\overline{\Gamma}} X \chi_{\overline{\Gamma\sigma_p}})$ if $\Gamma\sigma_1\sigma_2^{-1}\Gamma = [\Gamma\sigma_p\Gamma]$, and the tree structure implies that we may define a G - equivariant matrix unit by defining for $\Gamma\sigma_1\sigma_n^{-1}\Gamma = [\Gamma\sigma_{p^n}\Gamma]$

$$v_{\Gamma\sigma_0, \Gamma\sigma_n} = \prod_i v_{\Gamma\sigma_i\Gamma\sigma_{i+1}},$$

where $\Gamma\sigma_i\Gamma\sigma_{i+1}^{-1}\Gamma = [\Gamma\sigma_p\Gamma]$, and $(\Gamma\sigma_i)$ are the edges in the path on the tree connecting $\Gamma\sigma_0$ to $\Gamma\sigma_n$. □

The factorization result in Theorem 1 could enable one to find directly the action α_g from the ergodical action of $G \times G^{\mathrm{op}}$ on $L^\infty(\mathcal{S}, \nu)$. Note that in the new representation $L(G \rtimes L^\infty(\mathcal{S}, \nu))$ as the tensor $P \otimes B(l^2(\Gamma \setminus G))$, the algebra $L(\Gamma) \subseteq L(G)$ in the new description is representad as the diagonal algebra $\sum(\alpha_g(x) \otimes \chi_{\Gamma g})_{\Gamma g \in \Gamma\setminus G}$, $x \in L(\Gamma)$ (Note that $\alpha_g(x)$ for x in $L(\Gamma)$ depends only on Γg).

The Hecke algebra operators compute the difference between the two representations of the algebra $L(\Gamma)$.

Remark. Let $\widetilde{\mathcal{L}(\Gamma)}$ be the new copy of $L(\Gamma)$ in the representation $\mathcal{M} = L(\Gamma \rtimes L^\infty(K, \mu)) \otimes B(l^2(\Gamma \setminus G))$.

The fixed points under the action of Γ^{op} under the new representation are the crossed product of $L(\Gamma \rtimes [C(K, B(l^2(\Gamma \setminus G)))^{\Gamma^{\mathrm{op}}})]$.

Here Γ acts on $[C(K, Bl^2(\Gamma \setminus G))]^{\Gamma^{\mathrm{op}}})$ by right translations on K, and by $\mathrm{Ad}\rho_{\Gamma\setminus G}$ on $B(l^2(\Gamma/G))$. In particular the crossed product algebra

$$L(\Gamma \rtimes [C(K, B(l^2(\Gamma \setminus G)))^{\Gamma^{\mathrm{op}}})],$$

contains $L(\Gamma) \otimes B(l^2(\Gamma \setminus G))^{\mathrm{Ad}\rho_{\Gamma/G}}$. It is obviously that $B(l^2(\Gamma \setminus G))^{\mathrm{Ad}\rho_{\Gamma/G}}$ is the algebra generated by the Hecke algebra acting on the right and by the diagonal algebra $l^\infty(\Gamma \setminus G/\Gamma)$ identified with the algebra generated by the characteristic functions of the corresponding cosets.

In the original representation of $\mathcal{M} = L(G \rtimes L^\infty(\mathcal{S}, \nu))$, the Γ^{op} fixed points are $L(G \rtimes l^\infty(G \setminus \Gamma))$ (where $l^\infty(\Gamma/G)$ is identified as a subalgebra of $L^\infty(\mathcal{S}, \nu)$, by identifying the coset $\chi_{\Gamma\sigma}$ with the characteristic function of$\chi_{K\sigma}$, $\sigma \in G$).

Then $\widetilde{L(\Gamma)}$ the new representation of $L(\Gamma)$, sits in $\{\mathcal{H}, l^\infty(\Gamma \setminus G)\}' \subseteq L^\infty(G \rtimes l^\infty(\Gamma \setminus G))$, where $\mathcal{H}$ is identified with the subalgebra of $L(G)$ generated by the elements $t^{\Gamma\sigma\Gamma} = \sum_{\Gamma\sigma_1, \sigma_2^{-1}\Gamma = [\Gamma\sigma\Gamma]} v_{\Gamma\sigma_1, \Gamma\sigma_2}$, for a double coset $[\Gamma\sigma\Gamma]$, $\sigma \in G$, which determine a representation of the Hecke algebra.

Remark. To determine the behaviour of Hecke operators we have to determine $E_{\widetilde{L(\Gamma)}}(\alpha_g(x))$, $x \in L(\Gamma)$, which means we have to compute the matrix coefficients $\tau(\alpha_g(x)x)$ for $x \in \widetilde{L(\Gamma)}$, where τ is the trace on the tye $II_!$ factor $L(\Gamma \rtimes L^\infty(K, \mu))$, identified as a subalgebra of the type II_∞ factor $L(G \rtimes L^\infty(\mathcal{S}, \nu))$.

APPENDIX

Given a structure of Hilbert spaces (see Section 1 in [Ra1] for notations) of Γ_σ - invariant vectors $\sigma \in G$, for an action of G on V, $H^{\Gamma_\sigma} \subseteq V^{\Gamma_\sigma}$, such that $\pi(\sigma)$, which originally maps $H^{\Gamma_{\sigma^{-1}}}$ into H^{Γ_σ}, we introduced in [Ra1] the hypothesis that this extends to a unitary representation π^{ad} of G on the isometric inductive limit H^{ad} of the Hilbert spaces H^{Γ_σ}.

The Hecke operators $T^{\Gamma\sigma\Gamma}$ were proved to be obtained in the diagram

$$\begin{array}{ccc} H^{\Gamma_{\sigma^{-1}}} & \xrightarrow{\sigma} & H^{\Gamma_\sigma} \\ & \nwarrow^{\subseteq} \quad \swarrow & \\ & H^\Gamma & \end{array} \qquad [\Gamma : \Gamma\sigma] \quad \mathrm{proj.}$$

Then for $v \in H^\Gamma$, $\langle T^{\Gamma\sigma\Gamma} v, v\rangle$ is $[\Gamma : \Gamma_\sigma]\langle \pi^{\mathrm{ad}}(\sigma)v, v\rangle$, $\sigma \in G$, and hence the matrix coefficients of the Hecke operators are computed by rescaling, from the matrix coefficients $\langle \pi^{\mathrm{ad}}(g)v, v\rangle$.

If $v \in H^\Gamma$, these matrix coefficients are cosntant on double cosets.

Thus to determine the continuity of the Hecke algebraa with respect the $C^*_{\mathrm{red}}(\mathcal{H})$ topology it is sufficient to determine if the matrix coefficients $\langle \pi^{\mathrm{ad}}(g)v, v\rangle$ are contained in the corresponding representation of the Hecke algebra.

In the standard representation we have for all $\sigma \in G$ that $H_{st}^{\Gamma^\sigma} = l^\infty(\Gamma_\sigma/G)$ and hence we have clearly that $H^{\mathrm{ad}}_{\mathrm{st}} = L^2(\mathcal{S}, \nu)$, and π^{ad}_{st} is nothing else then the regular representation of G on $H^{\mathrm{ad}}\mathrm{st}$.

Thus continuity of the Hecke algebra representation on H^Γ, is equivalent to the fact that π^{ad} is contained in the left regular representation of G on the profinite completion.

Note that in the case $G = PGL_2(\mathbb{Q}[\frac{1}{p}])$, (see [Lu]), all irreducible representation, having Γ - invariant vector extend to $PGL_2(\mathbb{Q}_p)$, and the representation in the left regular representation are the principal series while all others (corresponding to discrete laplacian eigenvalues $(2\sqrt{p}, p+1)$) are in the complementary series.

Observation. The unitary representation $\mathrm{Ad}\rho_{\Gamma\backslash G}$ on the Hilbert Schmidt operators $\mathcal{C}_2(B(l^2(\Gamma \setminus G)))$ is weakly contained in the representation of the principal series.

Proof. Indeed the matrix coefficients in this case are linear combinations of $\sum \lambda_i\lambda_j\chi_{\sigma_i\Gamma\sigma_j^{-1}}(g)$ as a function on G, and these are weak limits of positive definite function from the left regular representation of G on $\varprojlim_{\Gamma_\sigma} l^2(\Gamma_\sigma \setminus G)$ which are limits of positive definite functions of the form

$$\sum \lambda_i\lambda_j\chi_{\sigma_i^{-1}\Gamma_0\sigma_j}$$

for a fixed subgroup Γ_σ of G. □

Note that the matrix coefficients on Γ - invariant vectors, for representations in the complementary series are of the form $\sum c^\lambda(n)\chi_{\Gamma\sigma_{p^n}\Gamma}$ where c^λ defines a character of the Hecke algebra, for eigenvalue outside $2\sqrt{p}$.

We conclude by noting that the action of G^{op} on $L^2(M) = L^2(L(G \rtimes L^\infty(\mathcal{S}, \nu)))$ is clearly weakly contained in the left regular representation.

Then we have a splitting, with $P = L(\Gamma \rtimes L^\infty(K, \mu))$

$$L^2(M) = L^2(P) \otimes \mathcal{C}_2(l^2(\Gamma \setminus G))$$

in which the action of $G^{\rm op}$ is $\alpha_g \otimes {\rm Ad}\rho_{\Gamma\backslash G}$. Because of the previous observation, and since the left regular representation absorbs any other representation, more specific information, coming from the specific tensor algebra decomposition in Theorem 1, is needed about the splitting, to conclude that the matrix coefficients $\tau_P(\alpha_g(x)x^*)$, for x in $L^2(P) \setminus \mathbb{C}1$, are continuous with respect to left regular representation.

Acknowledgment. We are deeply indebted to Ryszard Nest anf Stephan Vaes for several comments on this paper.

REFERENCES

[BC] J.-B. Bost and A. Connes; *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory.* Selecta Math. (N.S.) **1** (1995), 411–457.

[Co] Connes, A. Outer conjugacy of automorphisms of factors. Symposia Mathematica, Vol. XX (Convegno sulle Algebre C* e loro Applicazioni in Fisica Teorica, Convegno sulla Teoria degli Operatori Indice e Teoria K, INDAM, Rome, 1975), pp. 149-159. Academic Press, London, 1976.

[GHJ] F.M. Goodman, P. de la Harpe and V.F.R. Jones; *Coxeter graphs and towers of algebras*. Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989. x+288 pp.

[Jo] V. F. R. Jones Index for subfactors, Inventiones mathematicae 1983, Volume 72, Issue 1, pp 1-25

[Kr] A. Krieg; *Hecke algebras.* Mem. Amer. Math. Soc. **87** (1990), no. 435, x+158 pp.

[Lu] A. Lubotzky, Discrete groups, expanding graphs and invariant measures. With an appendix by Jonathan D. Rogawski. Reprint of the 1994 edition. Modern BirkhŁuser Classics. BirkhŁuser Verlag, Basel, 2010. iii+192 pp. ISBN:

[OP] N. Ozawa, S. Popa, Sorin On a class of II1 factors with at most one Cartan subalgebra. Ann. of Math. (2) 172 (2010), no. 1, 713749.

[PP] M. Pimsner, S. Popa, Entropy and index for subfactors. Ann. Sci. cole Norm. Sup. (4) 19 (1986), no. 1, 57106

[Ra1] F. Radulescu, Type $\rm II_1$ von Neumann representations for Hecke operators on Maass forms and Ramanujan-Petersson conjecture, Preprint

[Ra2] F. Radulescu, Conditional expectations, traces, angles between spaces and Representations of the Hecke algebras, Preprint

[Ra3] F. Radulescu, Free group factors and Hecke operators,is a note taken by N. Ozawa, preprint

[Ra4] F. Radulescu,A universal, non-commutative C*-algebra associated to the Hecke algebra of double cosets

[Sa] S. Sakai, C^*-algebras and W^*-algebras. Reprint of the 1971 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1998. xii+256 pp.