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The Metric Completion of Outer Space

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Abstract

We prove that the completion of Outer Space with the Lipschitz metric is homeomorphic to the free splitting complex. We give a new proof of a theorem by Francaviglia and Martino [FMa] that the isometry group of Outer Space is homeomorphic to $\mathrm{Out}(F_n)$ for $n \geq 3$ and equal to $\mathrm{PSL}(2,\mathbb{Z})$ for $n = 2$.

The group of outer automorphisms of a free group of rank n, $\mathrm{Out}(F_n)$ has been studied via its action on several geometric constructions. The purpose of this paper is to relate two of these objects: The complex of free splittings and Outer Space.

The complex of free splittings of F_n, denoted FS_n (and introduced by Hatcher [Hat95] as the sphere complex) is the complex of minimal, simplicial, actions of F_n on simplicial trees with trivial edge stabilizers. An i-simplex in FS_n corresponds to a tree with an $(i+1)$-edge quotient. A face of a simplex corresponds to an equivariant edge collapse. $\mathrm{Out}(F_n)$ acts on FS_n by simplicial automorphisms via pre-composition. Aramayona and Souto [AS] proved that $\mathrm{Out}(F_n)$ is the full group of automorphisms of FS_n. Although we will not use the following result here, one cannot mention the free splitting complex without mentioning the important recent result by Handel and Mosher [HM] that the free splitting complex is Gromov hyperbolic.

Outer Space, defined by Culler and Vogtmann [CV86], is the space of minimal, free and isometric actions of F_n on metric trees. As a set, it is a union of open simplices in FS_n. There is a natural metric defined on $\mathrm{Out}(F_n)$ where $d(X,Y)$ is the maximal amount of stretching any equivariant map from X to Y must apply to the edges of X. $\mathrm{Out}(F_n)$ acts on Outer Space with this metric by isometries. However, it is important to note

that this is not a "proper" metric because it is not symmetric. In fact $\frac{d(X,Y)}{d(Y,X)}$ can be arbitrarily large (see [AKB12] for a general theorem about the asymmetry of Outer Space). Moreover it is not proper in the sense that the set $B(X,r) = \{Y \mid d(X,Y) \leq r\}$ is not compact. One way to fix this is to symmetrize the metric: $d_s(X,Y) = d(X,Y) + d(Y,X)$. It turns out that closed balls in d_s are compact so this provides a resolution to both problems. However, in symmetrizing we lose much of the insight that $\mathcal{X}_n$ provides on the dynamics of the action of $\mathrm{Out}(F_n)$ on F_n. Many applications of the metric are specific for the non-symmetric metric (e.g. [Bes]). Moreover, the symmetric metric is not a geodesic metric, unlike the asymmetric metric. Thus, we might ask, what is the metric completion of $\mathcal{X}_n$ with the asymmetric metric? Our main result is:

Theorem A. *Let $[T]$ be a homothety class in $\partial\mathcal{X}_n$. T is in the completion of $\mathcal{X}_n$ if and only if point orbits in T are not dense and arc stabilizers are trivial.*

We show that the Lipschitz distance can be extended to the completion (allowing the value ∞) and so an isometry of $\mathcal{X}_n$ extends to the completion. We refer to the set of simplicial trees in the completion of Outer Space as the *simplicial completion.*

Theorem B. *The simplicial completion of Outer Space with the Lipschitz topology is homeomorphic to the free splitting complex with the Euclidean topology.*

We also show that the axes topology on the simplicial completion is strictly finer than the Lipschitz topology. Next, we use this theorem to give a new proof of a result of Francaviglia and Martino.

Theorem C. *[FMa] The group of isometries of Outer Space is* $\mathrm{Out}(F_n)$ *if* $n \geq 3$ *and* $PSL(2,\mathbb{Z})$ *if* $n = 2$.

Franaviglia and Martino prove this theorem for $\mathcal{X}_n$ with the symmetric metric. The statement for the asymetric metric follows as an easy corollary. The techniques in this paper only allow us to show the result for the asymmetric metric. However, our proof is relatively light in computations when compared to the original proof. [FMa] apply their theorem to show that certain groups cannot act on Outer Space with no fixed point. For example, one may apply a theorem of Bridson and Wade [BW11] that the image of an irreducible lattice Γ in a higher-rank connected semi-simple Lie group is finite in $\mathrm{Out}(F_n)$ to conclude that Γ cannot act on Outer Space with no

global fixed point.

We begin by outlining a theory for the completion of an asymmetric metric space. In section 2 we give some background material. In section 3 we characterize of the completion points in $\partial\mathcal{X}_n$ and prove Theorem A. In section 4 we discuss the simplicial completion and prove Theorem B. In section 5 we give the new proof of theorem C.

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1 The completion of an asymetric metric space

First we wish to introduce the ingredients needed to complete an asymmetric metric space. For our purposes, an asymmetric metric on a set X is a function $d : X \times X \to \mathbb{R} \cup \{\infty\}$ which satisfies the following properties:

1. $d(x, y) \geq 0$
2. If $d(x, y) = 0$ and $d(y, x) = 0$ then $x = y$.
3. For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

We would like to define the completion in analogy to the completion of a symmetric space as equivalence classes of (appropriately defined) Cauchy sequences. However the interlace of two asymmetric Cauchy sequences need not be a Cauchy sequence. Therefore, it is more natural to construct the completion using what we call "admissible sequences".

Definition 1.1. [Admissible sequences, Cauchy sequences] A sequence $\{x_n\}$ is *Cauchy* if for all ε there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $j > i > N(\varepsilon)$, $d(x_i, x_j) < \varepsilon$.
A sequence $\{x_n\} \subseteq \mathcal{X}_n$ is *admissible* if for all ε there is a natural number $N(\varepsilon)$ such that $\forall n > N(\varepsilon)$ there is a $K(n, \varepsilon)$ such that for all $k > K(n, \varepsilon)$, $d(x_n, x_k) < \varepsilon$.

Remark 1.2.

1. If d is symmetric the two definitions are equivalent.
2. The definitions given here could be phrased as "forwards Cauchy" and "forwards admissible". One may also define "backwards Cauchy" and "backwards admissible" sequences. Here we construct the forward

completion of a space and one may also define the backward completion. In our case, $\mathcal{X}_n$ is "backwards complete" that is, if $\{x_i\}$ is a sequence such that for all ε there exists an $N(\varepsilon)$ so that $d(x_j, x_i) < \varepsilon$ for $N < i < j$ then $\{x_i\}$ converges to a point in $\mathcal{X}_n$. This follows from the compactness of $B'(x, r) = \{y \in \mathcal{X}_n \mid d(y, x) \leq r\}$. The latter fact follows from (a) a bound on the injectivity radius of $y \in B'$ and (b) the fact that the ε thick part of Outer Space is quasi-isometric to the spine of Outer Space.

Proposition 1.3. *Every admissible sequence $\{x_n\}$ has a subsequence which is Cauchy. Moreover we can choose this subsequence so that for all $j < k < m$ we have $d(x_{n_k}, x_{n_m}) < \frac{1}{2^j}$*

Proof. For convenience let us denote $x(n) = x_n$ then the subsequence will be given recursively by $n_1 = N(1)$ and $n_{j+1} = \max\{N\left(\frac{1}{2^{j+1}}\right), K\left(n_j, \frac{1}{2^j}\right)\}$. For all $m > k \geq j$, $n_k > N\left(\frac{1}{2^j}\right)$ and $n_m \geq n_{k+1} \geq K\left(n_k, \frac{1}{2^k}\right) \geq K\left(n_k, \frac{1}{2^j}\right)$ hence $d(x(n_j), x(n_m)) < \frac{1}{2^j} < \varepsilon$. □

Definition 1.4. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. We denote their interlace sequence by $\iota(x_n, y_n) = \{z_n\}$ this is the sequence $z_n = \begin{cases} x_{\frac{n+1}{2}} & n \text{ odd} \\ y_{\frac{n}{2}} & n \text{ even} \end{cases}$. If $\{x_n\}$ and $\{y_n\}$ are admissible, they are equivalent iff their interlace sequence is admissible.

We denote by $\hat{X}$ the set of admissible sequences up to equivalence. Note that admissibility is stable under taking subsequences. Therefore, any equivalence class has a representative as in Proposition 1.3. We extend the distance function as a limit:

Definition 1.5. $\hat{d}([x_n], [y_n]) = c$ if:

1. If $c = \infty$: For all $r > 0$ there is an $N(r) \in \mathbb{N}$ so that for all $n > N(r)$ there is a $K(n, r)$ such that $\forall k > K(n, r)$, $d(x_n, y_k) > r$.
2. If $c < \infty$: For all $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ so that for all $n > N(\varepsilon)$ there is a $K(n, \varepsilon)$ such that $\forall k > K(n, \varepsilon)$, $|d(x_n, y_k) - c| < \varepsilon$.

Lemma 1.6. *$\hat{d}$ is well defined on $\hat{X}$.*

We shall need the following definition and proposition to prove this lemma:

Definition 1.7. A sequence $\{r_i\}_{n=1}^{\infty}$ in $\mathbb{R}$ is *almost monotonically decreasing* if for every ε there is an $N(\varepsilon)$ such that for all $i > N(\varepsilon)$ there is a $K(i,\varepsilon)$ so that: $r_k \leq (1+\varepsilon)r_i$ for $k > K(i,\varepsilon)$.

Proposition 1.8. *If $\{r_i\}$ is almost monotonically decreasing and bounded below then it converges to a limit.*

Proof. First note that $\{r_i\}$ is bounded: for all $k > K(i,1)$ we have $r_k \leq 2r_i$. Thus there is a subsequence r_{i_j} converging to R. R is in fact the limit of r_i. Let $\varepsilon > 0$ and $M(\varepsilon)$ such that for $j > M(\varepsilon)$: $|r_{i_j} - R| < \varepsilon$. Let $k > K(i_M, \varepsilon)$ and choose s so that $i_s > K(k,\varepsilon)$ then

$$r_k \leq r_{i_M}(1+\varepsilon) < (R+\varepsilon)(1+\varepsilon)$$

and

$$R - \varepsilon < r_{i_s} \leq (1+\varepsilon)r_k$$

hence $\frac{R-\varepsilon}{1+\varepsilon} < r_k < (R+\varepsilon)(1+\varepsilon)$ □

Proof of Lemma 1.6. We must show that for every admissible sequences $\{x_n\}$, $\{y_n\}$ either (1) or (2) in Definition 1.5 is satisfied. Let $\varepsilon > 0$ fix $n \in \mathbb{N}$ and consider $a_k = d(x_n, y_k)$, $k \in \mathbb{N}$. $\{a_k\}$ is almost monotonically decreasing. Let $N_1(\varepsilon)$, $j > N_1(\varepsilon)$ and $K_1(j,\varepsilon)$ be the constants from definition 1.1 for $\{y_n\}$, then $a_k = d(x_n, y_k) \leq d(x_n, y_j) + d(y_j, y_k) \leq a_j + \varepsilon$. Thus we know that $\{a_k\}$ converges to a limit c_n. Thus there is a $K_2(n,\varepsilon)$ such that for all $k > K(n,\varepsilon)$, $|d(x_n, x_k) - c_n| < \varepsilon$. We claim that c_n is almost monotonically increasing. Applying definition 1.1 to $\{x_n\}$ we obtain $N_2(\varepsilon)$ so that for $n > N_2(\varepsilon)$ there is a $K_3(n,\varepsilon)$, such that if $j > K(n,\varepsilon)$ we get $d(x_n, y_k) \leq d(x_n, x_j) + d(x_j, y_k) \leq d(x_j, y_k) + \varepsilon$. When $k > \max\{K_2(n,\varepsilon), K_2(j,\varepsilon)\}$ we get that $c_n \leq d(x_n, y_k) + \varepsilon \leq d(x_j, y_k) + 3\varepsilon \leq c_j + 4\varepsilon$. Therefore $\{c_n\}$ is almost monotonically increasing. We now have two cases:

- c_n is bounded and hence converges to a limit c. This implies part 2 of Definition 1.5.
- c_n is unbounded and so $\{x_n\}$ satisfies part 1 of Definition 1.5.

Next we show that $\hat{d}$ is well defined over equivalence classes. Let us denote by $c(x_n, y_n)$ the number c arising from the previous paragraph. Suppose $x_n \sim x'_n$ and $y_n \sim y'_n$. Let $x''_n = \iota(x_n, x'_n)$ and $y''_n = \iota(y_n, y'_n)$, since x_n is a subsequence of x''_n, $c(x_n, y_n) = c(x''_n, y_n)$ by considering the various different subsequences we'll get $c(x_n, y_n) = c(x'_n, y'_n)$. □

Corollary 1.9. *If $\{x_{n_k}\}$ is a subsequence of an admissible sequence $\{x_n\}$ then*

$$\lim_{k\to\infty} d(x_{n_k}, x) = 0 \implies \lim_{n\to\infty} d(x_n, x) = 0$$
$$\lim_{k\to\infty} d(x, x_{n_k}) = 0 \implies \lim_{n\to\infty} d(x, x_n) = 0$$

One way in which $\hat{d}$ is different from d is the separation axioms it satisfies. Even if d satisfies $d(x, y) = 0 \implies x = y$, $\hat{d}$ might not. However,

Proposition 1.10. *If $\hat{d}(\zeta, \xi) = 0$ and $\hat{d}(\xi, \zeta) = 0$ then $\zeta = \xi$.*

Proof. Let $\zeta = [x_n]$ and $\xi = [y_n]$ and define $\{z_n\} = \iota(x_n, y_n)$. Given ε there are $N_x(\varepsilon)$ and $N_y(\varepsilon)$ as in Definition 1.1. Moreover, since $d(\zeta, \xi) = 0$ there is an $N_1(\varepsilon)$ as in Definition 1.5 and since $d(\xi, \zeta) = 0$ there is an $N_2(\varepsilon)$ as in Definition 1.5. Let $N(\varepsilon) = 2\max\{N_x(\varepsilon), N_y(\varepsilon), N_1(\varepsilon), N_2(\varepsilon)\}$ and take $n > N(\varepsilon)$. WLOG assume n is odd and let $n' = \frac{n+1}{2}$. Let $K_x(n', \varepsilon)$ and $K_1(n', \varepsilon)$ be the constants obtained by definitions 1.1 and 1.5 respectively. Define $J(n, \varepsilon) = 2\max\{K_x(n', \varepsilon), K_1(n', \varepsilon)\}$. If $j > J(n, \varepsilon)$ then both $\frac{j+1}{2}, \frac{j}{2} > \max\{K_x(n', \varepsilon), K_1(n', \varepsilon)\}$. Hence, when j is odd let $j' = \frac{j+1}{2}$ then $d(z_n, z_j) = d(x_{n'}, x_{j'}) < \varepsilon$ since $n' > N_x(\varepsilon)$ and $j' > K_x(n', \varepsilon)$ and if j is even let $j' = \frac{j}{2}$ and $d(z_n, z_j) = d(x_{n'}, y_{j'}) < \varepsilon$ since $n' > N_1(\varepsilon)$ and $j' > K_1(n', \varepsilon)$. □

Corollary 1.11. *$(\hat{X}, \hat{d})$ is an asymmetric metric space.*

Proposition 1.12. *$(\hat{X}, \hat{d})$ is complete. Every admissible sequence $\{\xi_n\}$ in $\hat{X}$ has a unique "closest" limit ξ with the properties: $\lim_{n\to\infty} \hat{d}(\xi_n, \xi) = 0$, and for all ζ so that $\lim_{n\to\infty} \hat{d}(\xi_n, \zeta) = 0$ we have $\hat{d}(\xi, \zeta) = 0$*

Proof. Let $\{\xi_n\} \subset \hat{X}$ be an admissible sequence. By switching to a subsequence we may assume that it is as in Proposition 1.3, i.e. for all $m > n > j$, $\hat{d}(\xi_n, \xi_m) < \frac{1}{2^j}$. For each ξ_i we may choose a representative $\{x_{ij}\}_{j=1}^{\infty}$ inductively as follows. Choose $\{x_{1j}\}$ and $\{x_{2j}\}$ as in Proposition 1.3. Since $\hat{d}(\xi_1, \xi_2) < 1$ there is an N such that for all $n > N$ there is a $K(n)$ such that $d(x_{1n}, x_{2k}) < 1$ for all $k \geq K(n)$. We take the subsequence of x_{1j} beginning at N. We take the subsequence of x_{2j} defined by $x_{2,K(n)}$ thus, for the new sequences $d(x_{1j}, x_{2j}) < 1$. Suppose we have chosen representatives x_{ij} for ξ_i for $i \leq I$, so that for all $n < m$ we have $d(x_{in}, x_{im}) < \frac{1}{2^{i-1}}$ and for all $i < I$ and $j \leq m$, $d(x_{ij}, x_{i+1,m}) < \frac{1}{2^{i-1}}$. We modify $\{x_{Ij}\}_{j=1}^{\infty}$ and choose a temporary $\{x_{I+1,j}\}_{j=1}^{\infty}$ as follows. First let $\{x_{I+1,j}\}_{j=1}^{\infty}$ be a sequence as in 1.3. $\hat{d}(\xi_I, \xi_{I+1}) < \frac{1}{2^I}$ implies that there is an $N(\frac{1}{2^I})$ so that for all $n > N$ there is a $K(n) = K(n, \frac{1}{2^I})$ such that $d(x_{I,n}, x_{I+1,k}) < \frac{2}{2^I}$ for $k > K(n)$.

Truncate the first N elements from $x_{I,j}$ and let $\{x_{I+1,K(n)}\}_{n=1}^{\infty}$ be the new sequence representing ξ_{I+1} (which will be truncated in the next step).
This produces sequences $\{x_{i,j}\}_{j=1}^{\infty}$ representing ξ_i such that $d(x_{ij}, x_{ik}) < \frac{1}{2^{i-1}}$ for any $j < k$, and $d(x_{ij}, x_{i+1,j}) < \frac{1}{2^{i-1}}$ for all for i, j. Let $\xi = [\{x_{ii}\}_{i=1}^{\infty}]$ we will show that $\{x_{ii}\}$ is admissible and that $\lim_{n\to\infty} \hat{d}(\xi_n, \xi) = 0$.
$\{x_{ii}\}_{i=1}^{\infty}$ is a Cauchy sequence since $d(x_{ii}, x_{kk}) \leq d(x_{ii}, x_{ik}) + d(x_{ik}, x_{kk}) < \frac{1}{2^{i-2}} + \frac{2}{2^{i-1}} = \frac{1}{2^{i-3}}$ therefore it is admissible. Fix n and $j > n$, for $k > j$ we have $d(x_{nj}, x_{kk}) \leq d(x_{nj}, x_{nk}) + d(x_{nk}, x_{kk}) < \frac{1}{2^{n-3}}$, thus $\lim_{n\to\infty} \hat{d}(\xi_n, \xi) = 0$.
Assume that $\zeta \in \hat{X}$ so that $\lim_{n\to\infty} \hat{d}(\xi_n, \zeta) = 0$. Let $\varepsilon > 0$, then there is an $M(\varepsilon)$ so that for all $m > M(\varepsilon)$, $\hat{d}(\xi_m, \zeta) < \varepsilon$. Choose and admissible $\{z_n\}$ such that $\zeta = [z_n]$. Fix $m > M(\varepsilon)$, then there is an $N(m, \varepsilon)$ (large enough column), so that for all $n > N(m, \varepsilon)$ there is a $K(m, n, \varepsilon)$ such that $d(x_{mn}, z_k) < 2\varepsilon$ for all $k > K(m, n, \varepsilon)$. Let $n = N(m, \varepsilon) + 1$ and $J(m, \varepsilon) = K(m, n, \varepsilon)$ then for $k > J$ we have: $d(x_{mm}, z_k) \leq d(x_{mm}, x_{mn}) + d(x_{mn}, z_k) < \frac{1}{2^{m-1}} + 2\varepsilon$. Hence $\hat{d}(\xi, \zeta) = 0$. Uniqueness follows from Proposition 1.10.

□

Proposition 1.13. *For any complete asymetric metric space (Y, ρ) and any isometric embedding $i : (X, d) \hookrightarrow (Y, \rho)$ there is a unique lift of i to an isometric embedding $j : (\hat{X}, \hat{d}) \hookrightarrow (Y, \rho)$. Consequentally, $\hat{X}$ is the unique complete metric space in which $i(X)$ is dense.*

Proof. If $\{x_n\}$ is admissible and i is an isometric embedding then $\{i(x_n)\}$ is admissible. Since Y is complete there is a unique $y \in Y$ such that $\lim_{n\to\infty} d(i(x_n), y) = 0$ and if $y' \in Y$ also satisfies $\lim_{n\to\infty} d(i(x_n), y') = 0$ then $d(y, y') = 0$. Let $\xi = [x_i]$, if j is an isometric embedding then on the one hand $d(j(\xi), y) = \lim_{n\to\infty} d(i(x_n), y) = 0$ but on the other hand we have $\lim_{n\to\infty} d(i(x_n), j(\xi)) = 0$ hence $d(y, j(\xi)) = 0$ thus $y = j(\xi)$ is the only possible definition. This is an isometric embedding since the distances in $\hat{X}$ and in Y are determined by limits of distances in X. □

Corollary 1.14. *An isometry of X induces an isometry of $\hat{X}$.*

2 The functor from marked graphs to tree actions

2.1 Outer Space in terms of marked graphs

Let R be the wedge of n circles, denote the vertex of R by $*$. Fix a basis $\{x_1, \ldots x_n\}$ of F_n and identify x_i with the edges of R. This gives us an

identification of $\pi_1(*, R)$ with F_n that we will suppress from now on. A point in outer space is an equivalence class of a triple $x = (G, \tau, \ell)$ where G is a graph (a finite 1 dimensional cell complex), $\tau\colon R \to G$ and $\ell\colon E(G) \to (0, 1)$ are maps, and (G, τ, ℓ) satisfy:

1. the valence of $v \in V(G)$ is greater than 2.
2. τ is a homotopy equivalence.
3. $\sum_{e \in E(G)} \ell(e) = 1$.

The equivalence relation is given by: $(G, \tau, \ell) \sim (G', \tau', \ell')$ if there is an isometry $f\colon (G, \ell) \to (G', \ell')$ such that $f \circ \tau$ is freely homotopic to τ'.

We will always identify words in F_n with edge paths in R, note that reduced words are identified with immersed paths in R. Using this identification, an automorphism $\phi : F_n \to F_n$ can be viewed as a simplicial map $\phi : R \to R$. There is a right $\mathrm{Aut}(F_n)$ action on metric marked graphs given by: $x \cdot \phi = (G, \tau \circ \phi, \ell)$. This action is well defined on the equivalence classes, and inner automorphisms act trivially thus the action of $\mathrm{Aut}(F_n)$ on marked graphs descends to an $\mathrm{Out}(F_n)$ action on $\mathcal{X}_n$.

2.2 Outer Space in terms of F_n-trees

An equivalent description of Outer Space is given in terms of minimal free simplicial metric F_n-trees. $\mathcal{X}_n$ is the set of equivalence classes of pairs (X, ρ) where X is a metric tree, and $\rho : F_n \to \mathrm{Isom}(X)$ is a homomorphism and the following conditions are satisfied:

1. The action is by free - if $\rho(g)(p) = p$ for $p \in X$ and $g \in F_n$ then $g = 1$.
2. X is simplicial - for any $1 \neq g \in F_n$, the translation length $l(\rho(g), T) := \inf\{d(x, \rho(g)x) \mid x \in T\}$ is bounded away from zero by a global constant independent of g.
3. The action is minimal - no subtree of X is invariant under the group $\rho(F_n)$.
4. The action is normalized to have unit volume - $X/\rho(F_n)$ is a finite graph whose sum of edges is 1.

Remark 2.1. The first three items imply that $X/\rho(F_n)$ is a finite metric graph. Indeed, by 1,2 the action is properly discontinuous therefore

$p\colon X \to X/\rho(F_n)$ is a covering map. Let $p \in X$ be arbitrary, the orbit of the convex hull of $\{\rho(x_1), \dots \rho(x_n)\} \cdot p$ is an invariant subtree so by minimality, it contains a fundamental domain for the ρ action on X. Hence the quotient is a finite metric graph. Note also that there are no valence 1 vertices since the action is minimal.

The equivalence relation on the collection of F_n actions is: $(X, \rho) \sim (Y, \mu)$ if there is an equivariant isometry $f\colon X \to Y$, i.e. $f^{-1} \circ \mu(g) \circ f(x) = \rho(g)(x)$. In this case (X, ρ), (Y, μ) are often called isometrically conjugate. The action of $\mathrm{Aut}(F_n)$ is given by

$$\phi \in \mathrm{Aut}(F_n), (X, \rho) \cdot \phi = (X, \rho \circ \phi)$$

Clearly, the action is well defined on the equivalence classes. To see that inner automorphisms act trivially, assume $\phi = i_g$ and take $f = \rho(g)\colon X \to X$ then f is an isometry such that $f^{-1} \circ \rho(h) \circ f(x) = \rho(g)^{-1} \circ \rho(h)(\rho(g)(x)) = \rho(ghg^{-1})(x) = \rho \circ i_g(x)$. Therefore the action descends to an action of $\mathrm{Out}(F_n)$ on the isometry classes of trees.

2.3 From marked graphs to trees

Given a marked metric graph (G, τ, ℓ), let (X, p) be the universal cover of G. Choose a point $w \in p^{-1}(\tau(*))$. The choice of w determines a homomorphism $T_w : \pi_1(G, \tau(*)) \to \mathrm{Isom}(X)$ by covering translations. Thus, (G, τ, ℓ) produces a simplicial metric tree X and representation $\rho_w = T_w \circ \tau_* : F_n \to \mathrm{Isom}(T)$. It is clear that (X, ρ_w) is free, and simplicial. Minimality follows from the exclusion of valence 1 vertices.

We study the dependence on the choice of basepoint w. Choosing a different basepoint $z \in p^{-1}(\tau(*))$, we get a new action of $\pi_1(G, \tau(*))$ on X by covering translations $T_z : \pi_1(G, \tau(*)) \to \mathrm{Isom}(X)$. One can check that $T_z(\gamma) = T_w(\delta\gamma\bar{\delta})$ where $\delta = p([w, z])$. If $h \in F_n$ is represented by $\delta \in \pi_1(G, \tau(*))$, then for all $g \in F_n$ (represented by γ), $\rho_z(g) = \rho_w \circ i_h(g)$. Thus, the different choices of a basepoint correspond to conjugating the action by $\rho(h)$. Hence the equivalence class of the tree (X, ρ_w) is independent of w.

We now study the effect of changing the marking. If we had chosen $\tau' : R \to G$ homotopic to τ, via H, consider the path $\delta(t) = H(*, t)$ in G from $\tau'(*)$ to $\tau(*)$. There is an isomorphism $\pi_1(G, \tau'(*)) \to \pi_1(G, \tau(*))$ which is induced by the assignment $\gamma \to \delta\gamma\bar{\delta}$ on the level of paths. Let $w \in p^{-1}(\tau'(*))$ and let z be the terminal point of the lift of δ initiating at w. We have carefully chosen our base points so the interlacing map will be the identity: one can

check that $\rho_z(g) = T_z(\gamma) = T_w(\delta\gamma\bar{\delta}) = \rho_w(g)$.

Lastly, we check that the action of $\mathrm{Out}(F_n)$ on marked graphs translates to the appropriate action on trees. Suppose that $x = (G, \tau, \ell)$ corresponds to (X, ρ) (via a choice of w). Then $x \cdot \phi = (G, \tau \circ \phi, \ell)$. The new representation is $\theta_w(g) = T_w((\tau \circ \phi)_*(g)) = T_w(\tau_*(\phi(g))) = \rho(\phi(g))$. Thus $x \cdot \phi$ corresponds to the tree $(X, \rho \circ \phi) = (X, \rho) \cdot \phi$.

2.4 From trees to graphs

Given a metric free simplicial F_n tree (X, ρ) satisfying our restrictions, we have already commented that $G = X/\rho(F_n)$ is a metric finite graph of unit volume with no valence 1 or valence 2 vertices. Choose a basepoint $w \in X$. Define the map $\tau_w : R \to G$ by mapping the loop x_i to $p[w, \rho(x_i) \cdot w]$ linearly. τ is a homotopy equivalence, and we get a triple (G, τ_w, ℓ).
Let z be a different choice of basepoint, we show that τ_w and τ_z are homotopic. Define $\sigma : I \to X$ to be a path from w to z and $H : R \times I \to X$ the map sending e_i in R to $p[\sigma(t), \rho(x_i)(\sigma(t))]$ linearly. Since $[\sigma(t), \rho(x_i)\sigma(t)]$ changes continuously with t, then H is a homotopy from τ_w to τ_z.
If $f : X \to Y$ is an isometry interlacing the representations ρ and θ, then f descends to an isometry $X/\rho(F_n) \to Y/\theta(F_n)$. The marking μ obtained from $\theta(\{x_1, \ldots, x_n\})f(w)$ is equal to $f \circ \tau$ where τ is the marking obtained from $\rho(\{x_1, \ldots, x_n\})w$.
It is also easy to verify that the action of $\mathrm{Out}(F_n)$ on trees descends to the corresponding action of F_n on graphs.

Remark 2.2. Even though the definition of Outer Space in terms of trees does not require a choice of basepoint such a choice is required to make the correspondence with marked metric graphs explicit.

2.5 Lifting optimal maps

Let $x = (G, \tau, \ell), y = (H, \mu, \ell')$ be two points in $\mathcal{X}_n$ and let $g : G \to H$ a map such that $g \circ \tau$ is homotopic to μ. Such a map is called a *difference in markings*. Let $(X, p), (Y, p')$ be the respective universal covers. Given a choice of basepoints $w \in p^{-1}(\tau(*))$ and $z \in p'^{-1}(\mu(*))$, there is a unique lift of $g \circ p : X \to H$ to a map $\widetilde{g_{wz}} : X \to Y$ such that $\widetilde{g_{wz}}(w) = z$. $g \circ \tau \sim \mu$ implies

$$\widetilde{g_{wz}} \circ \rho^X_w(h) = \rho^Y_z(h) \circ \widetilde{g_{wz}}$$

Thus, a Lipshitz difference in marking $g : G \to H$ and a choice of base points w, z defines a Lipschitz equivariant map $\widetilde{g_{z,w}} : X \to Y$. Conversely, given an equivariant Lipschitz map $h : X \to Y$, it descends to a map $h' : G \to H$ which is a difference in markings.

Let α be a loop in x, we denote by $l(\alpha, x)$ the length of the immersed loop homotopic to α. Let $a \in F_n$ we denote by $l(a, x) = l(\tau(a), x)$, this is equal to the translation length of $\rho(a)$ in X which will be denoted by $l(a, X)$ or $tr(a, X)$. A loop α in x is a *candidate* if it is an embedded circle, an embedded figure 8, or an embedded barbell.

Theorem 2.3 (Definition/Theorem)**.** *[FMb] The function*

$$\begin{aligned} d(x,y) &= \log\inf\{Lip(f) \mid f\colon x \to y \text{ is a Lipshcitz difference in markings }\} \\ &= \log\sup\left\{\frac{l(\gamma,y)}{l(\gamma,x)}\middle|\, \gamma \text{ is a loop in } x\right\} \end{aligned}$$

defines an asymmetric distance on $\mathcal{X}_n$. The supremum is realized by a candidate loop which will be called a realizing candidate*. The infemum is also realized and a realizing map will be called an* optimal *map. For every optimal map f there is a realizing candidate α so that $f(\alpha)$ is immersed. For every realizing candidate α and every difference in marking f, f may be homotoped to f' so that $f'(\alpha)$ is immersed.*

Suppose that β be a candidate loop in x realizing the Lipschitz distance. Let $b \in F_n$ be cyclically reduced so that $\tau(b)$ is homotopic to β. Homotope τ so that $\tau(b) = \beta$. Homotope g so that $g(\beta)$ is an immersed loop. Homotope μ so that $\mu(*) = g(\tau(*))$ and $\mu(b) = g(\beta)$. g is now an optimal Lipschitz map and β is a realizing loop. Lift g to $\widetilde{g_{zw}} : X \to Y$. For $h \in F_n$, $\rho^X_w(h)$ acts as a hyperbolic element on X, denote by $A_X(h)$ its axis in X. Since $g(\beta) = \mu$ and $\tau(*) \in \beta, \mu(*) \in g(\beta)$ we get $\widetilde{g_{wz}}(A_X(b)) = A_Y(b)$. That is, the image of $A_X(b)$ doesn't contain any thorns - valence 1 vertices.

Definition 2.4 (short basis)**.** Choose a maximal forest K_G in G then the oriented edges of $G \setminus K_G$ form a basis of $\pi_1(\tau(*), G)$ this is called the short basis induced by K_G and the orientation O_G.

Given a short basis $\mathcal{B}$ so that $\beta \in \mathcal{B}$. Let $h \in F_n$ so that $\tau(h) = \alpha \neq \beta$ in the short basis, then $g(A_X(h))$ might contain thorns but their lengths are shorter than the lipschitz constant $\mathrm{Lip}(g)$.

Proposition 2.5. *Let $x = (G, \tau, \ell)$ and $y = (H, \mu, \ell')$ be metric marked graphs in $\mathcal{X}_n$. Let $g : x \to y$ an optimal map, and β a candidate realizing*

the Lipschitz distance and $\mathcal{B}$ a short basis containing β. For each choice of base point $w \in X$, there is a basepoint $z \in Y$, such that $\widetilde{g_{wz}} : X \to Y$ the unique lift of g sending w to z satisfies $\widetilde{g_{wz}}(A_X(b)) = A_Y(b)$ where $b \in F_n$ is such that $\beta = g(b)$ and $g(A_X(a))$ is contained in the Lip(g) neighborhood of $A_Y(a)$ for $a \in F_n$ so that $\tau(a) \in \mathcal{B}$.

3 A characterization of the completion points

Recall that the metric completion is given as a quotient space of admissible sequences and that we may represent each equivalence class by a Cauchy sequence. The aim of this section is to characterize the points on $\overline{\mathcal{X}_n}$ that are limits of Cauchy sequences.

We recall some well known facts about the boundary of Outer Space. $\overline{\mathcal{X}_n}$ is the space of minimal very small actions of F_n on $\mathbb{R}$-trees as defined in [CL95]. There is a continuous embedding [CM87] of the space of minimal non-abelian irreducible F_n tree actions into $P\mathbb{R}^{F_n}$ (the translation length function). This embedding induces the axes topology on $\overline{\mathcal{X}_n}$. We give an explicit description of a basis element in the axes topology. A basis element $U(T, P, \varepsilon)$ is parametrized by a very small F_n tree T, a finite subset $P < F_n$, and $\varepsilon > 0$ and is given by $U(T, P, \varepsilon) = \{S \in \overline{\mathcal{X}_n} \mid |l(\alpha, T) - l(\alpha, S)| < \varepsilon \quad \forall \alpha \in P\}$. The Gromov topology on $\mathcal{X}_n$ is generated by the following basis: a basis element $O(T, P, K, \varepsilon)$ is parametrized by T, a compact subset K in T, a finite subset $P < F_n$ and $\varepsilon > 0$. A P-equivariant ε relation R between $K \subset T$ and $K' \subset T'$ is a subset $R \subset K \times K'$ so that the projection of R is surjective on each factor, if $(x, x'), (y, y') \in R$ then $|d(x, x') - d(y, y')| < \varepsilon$ moreover, for all $\alpha \in P$ if $x, gx \in K$ and $(x, x') \in R$ then $gx' \in K'$ and $(gx, gx') \in R$. A basis element $O(T, K, P, \varepsilon)$ is the set of trees S so that there is a compact $K' \subset S$ and a P -equivariant ε -relation $R \subset K \times K'$. Paulin [Pau89] showed that the two topologies are equivalent.

We begin by observing that a Cauchy sequence in $\mathcal{X}_n$ converges in the axes topology without the need to rescale the trees in the sequence. Let $f_{m,k} \colon X_m \to X_k$ be an optimal Lipschitz map. Let $\| \cdot \|_k$ be the translation length function in X_k.

Corollary 3.1. *The F_n-trees X_k converge to an irreducible, minimal, very small $\mathbb{R}$-tree X.*

Proof. For each $a \in F_n$, $l(a, X_i)$ is an almost monotonically decreasing sequence. Hence the translation length functions of X_i converge. □

Corollary 3.2. *For every m the sequence $\{Lipf_{m,k}\}_{k=m+1}^{\infty}$ is almost monotonically decreasing. Hence it converges to some limit L_m and it is bounded by M_m*

Our next goal is to show that for each m there is a map $f_{m,\infty} : X_m \to X$ such that $\mathrm{Lip} f_{m,\infty} \leq \lim_{k\to\infty} \mathrm{Lip} f_{m,k}$. We follow Bestvina's construction in [Bes88] of trees as limits of sequences of representations. Bestvina shows that if $\{\rho_i\}_{i=1}^{\infty}$ is a sequence in $\mathrm{Hom}(G, \mathrm{Isom}(\mathbb{H}^n))/\text{conjugation}$, and there is some $g \in G$ so that the translation distances of $\rho_i(g)$ are unbounded then there is a convergent subsequence of ρ_i to a small action of G on an $\mathbb{R}$-tree. Our setting is a little different because our underlying space is a tree whose topological type changes with i, and because in translation distances are bounded. Nevertheless, the construction goes through with very little modification as follows.

Fix m. For every $k > m$ there is a candidate β_k in x_m that is stretched maximally by $f'_{m,k} \colon x_m \to x_k$. By passing to a subsequence we may assume that it is the same β for all $k > m$. Homotope $\tau_m, \tau_k, f'_{m,k}$ and choose $w_k \in X_k$ for $k \geq m$ as in proposition 2.5 to lift $f'_{m,k}$ to $f_{m,k} \colon X_m \to X_k$ taking w_m to w_k. Choose a short basis $\mathcal{B}'$ of $\pi_1(\tau(*), x_m)$ which contains β. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be the basis of F_n corresponding to this basis of $\pi_1(\tau(*), x_m)$ so that $\tau(b_1) = \beta$.
Let $W^l = \{g \in F_n \mid |g|_{\mathcal{B}} \leq l\}$ and denote by X_k^l the convex hull of $W^l \cdot w_k$. For $l > 0$ and $g \in W^l$,

$$d_{X_k}(w_k, \rho_k(g)w_k) \leq \mathrm{Lip}(f_{m,k}) d(w_m, gw_m) \leq \mathrm{Lip}(f_{m,k}) l \leq M_m l$$

is bounded for all k. A *diagonal* in X_k^l is a path of the form $[\rho(g)w_k, \rho(h)w_k]$. Each diagonal can be covered by $\frac{M_m l}{\varepsilon}$ balls of radius ε. Note that this number is uniform over all k. We now apply Gromov's theorem:

Theorem 3.3. *[Gro81] If $\{A_k\}_{k=k_0}^{\infty}$ is a sequence of compact metric spaces so that for every ε there is an $N(\varepsilon)$ so that A_k may be covered by $N(\varepsilon)$ ε-balls then there is a subsequence A_{k_j} which converges in the Gromov sense to a compact metric space.*

We denote the limit space provided by the theorem T_m^l (a-priori the limit might depend on m).

Claim 1. *T_m^l is a finite tree.*

Proof. This is a repetition of the proof of Lemma 3.5 of [Bes88] for our case. We first prove that for every $a, b \in T_m^l$ for every $0 \leq t \leq D = d_{T_m^l}(a, b)$

there is a unique point $c \in T_m^l$ such that $d_{T_m^l}(a,c) = t$ and $d_{T_m^l}(c,b) = D - t$. To see this, let a_k, b_k be points in X_k^l such that $\lim_{k\to\infty} a_k = a$ and $\lim_{k\to\infty} b_k = b$ then there is a point $c_k \in T_k^l$ such that $d(a_k, c_k) = t_k$ and $d(c_k, b_k) = s_k$ with $\lim t_k = t$ and $\lim s_k = D - t$. c_k has a convergent subsequence to a point $c \in T_m^l$. If c' is another point with $d(a,c') = t$ and $d(c', b) = D - t$ and c_k' a sequence such that $\lim c_k' = c'$ then for large k, $d(a_k, c_k') + d(b_k, c_k') \leq d(a_k, b_k) + \varepsilon$ hence $[a_k, c_k'] \cup [c_k', b_k']$ is a tripod and c_k' is a distance less than ε away form the vertex of the tripod which itself is a distance approximately t form a and $D - t$ from b. The same is true for c_k hence $d(c_k, c_k') < 2\varepsilon$ and $d(c, c') = 0$.
For each $g \in W^l$ let $g \cdot w$ be the limit in T_m^l of the sequence $g \cdot w_k$. Let $H \subset T_m^l$ be the union of all diagonals, i.e. all segments of the form $[g \cdot w, g' \cdot w]$ for $g, g' \in W^l$. We claim that $H = T_m^l$. To see this suppose $x \in T_m^l$ is not covered by a diagonal. Then there is an ε such that $d(g \cdot w, x) + d(x, g' \cdot w) > \varepsilon + d(g \cdot w, g' \cdot w)$ for all $g, g' \in W^l$. Thus, for a large k, there is an $x_k \in X_k^l$ with $d(g \cdot w_k, x_k) + d(x_k, g' \cdot w_k) > \frac{\varepsilon}{2} + d(g \cdot w_k, g' \cdot w_k)$. Hence x_k is not in the convex hull of $W^l \cdot w_k$ which is a contradiction. □

By a diagonal argument we may pass to a subsequence k_j so that $T_{k_j}^l$ converges to T_m^l for every l. Thus, we have $T_m^1 \subset T_m^2 \subset T_m^3 \ldots$. Define $T_m = \cup_{l=1}^{\infty} T_m^l$. Then T_m is a tree. We describe the F_n action: $g \in W^s$, $q \in T_m$ then there is some l such that $q \in T_m^l$ thus there is a sequence $q_k \in X_k^l$ so that $\lim_{k\to\infty} q_k = q$ (in the appropriate space provided by Gromov's theorem) then $\rho_k(g)(q_k) \in X_k^{s+l}$ let $\rho(g)(q) = \lim_{k\to\infty} \rho_k(g)(q_k)$ in T_m^{l+s}. The proof that this limit exists and that this is an isometric action is exactly the same as in [Bes88] page 151.

Claim 2. *T_m is minimal and non-trivial.*

Proof. For all k, $d(w_k, \rho_k(b_1^2)(w_k)) = 2d(w_k, \rho_k(b_1)w_k)$ thus $w = \lim_{k\to\infty} w_k$ is on the axis of $\rho(b_1)$. $\mathrm{Lip}(f_{m,k}) \geq 1$ hence $\|b_1\|_{T_m} \geq \|b_1\|_{X_m} > 0$. Thus b_1 acts freely so no point is fixed by all of F_n. The tree is minimal: If H is an invariant subtree then it must contain the axis of $\rho(b_1)$ in T_m and its orbit under F_n. Since H is connected it must also contain the convex hull of this set. But T_m is precisely the convex hull of $\rho(F_n)w$ hence it is minimal. □

Claim 3. *For every $g \in F_n$: $\|g\| = \lim_{k\to\infty} \|g\|_k$*

Proof. $\|g\| = \lim_{l\to\infty} \dfrac{d(w, g^l w)}{l} = \lim_{l\to\infty} \lim_{k\to\infty} \dfrac{d(w_k, g^l w_k)}{l}$. Note $d(w_k, g^l w_k) \geq l\|g\|_k$ and $d(w_k, g^l w_k) = 2d(w_k, A_k(g)) + l\|g\|_k$. Now $f_{n,m}[A_k(g)]$ is contained

in $N_{\mathrm{Lip}f_{m,k}}(A_m(g))$ hence $d(w_k, A_k(g)) \leq \mathrm{Lip}(f_{m,k})d(w_k, A_k(g)) + \mathrm{Lip}(f_{m,k})$. Therefore, $d(w_k, g^l w_k) \leq 2\mathrm{Lip}(f_{m,k})d(w_m, A_m(g)) + C + l\|g\|_k = D(g) + l\|g\|_k$ where $D(g)$ can be bounded uniformly over l, k. Hence, $\|g\| \leq \lim_{l\to\infty} \frac{1}{l}(l\|g\|_k + D(g)) = \lim_{k\to\infty} \|g\|_k$. □

Corollary 3.4. *There is an equivariant Lipschitz map $h_m : X_m \to T$ with $Lip(h_m) = \lim_{k\to\infty} Lip(f_{m,k})$.*

Proof. Let $f_{m,\infty} : X_m \to T_m$ be the map sending $w_m \to w$ which is equivariant and linear on edges. Clearly $\mathrm{Lip} f_{m,\infty} = \lim_{k\to\infty} f_{m,k}$. The claims above show that T_m is the limit of $\{X_i\}_{i=1}^{\infty}$ in the axes topology. Thus T_m is equivariantly isometric to X. Let $h_m : X_m \to X$ be the composition of $f_{m,\infty}$ with the equivariant isometry then h_m satisfies the claim. □

We now turn to prove a characterization of X. It must have a simplicial part, with unit free volume and trivial segment stabilizers. Consider $[T] \in \partial\mathcal{X}_n$, i.e. T is a minimal $\mathbb{R}$-tree with a very small isometric F_n-action. The quotient space T/F_n is endowed with an appropriate pseudo-metric. Let $G = \widehat{T/F_n}$ be the induced metric space. A priori, G might have inifinitly many vertices and edges. [GL95] showed that G has finitely many vertices. Levitt [Lev94] showed that there are also only finitely many edges, so G is a simplicial finite graph. The universal cover of G, T' is a simplicial F_n tree and there is an equivariant collapsing map $c : T \to T'$ so that the quotient map $T \to T/F_n \to G$ is equal to $p \circ c : T \to T' \to G$. We now want to show that if $T \in \hat{\mathcal{X}}_n$ then the volume of G is 1.

Definition 3.5. If $V \subset T$ then $V = \sqcup\sigma_i$ a finite union of segments with disjoint interiors σ_i. Then the volume of V is the sum of lengths of σ_i. If $h : R \to T$ is an L-Lipschitz map then for each subset $V \subset R$, $vol(h(V)) \leq Lvol(V)$. Let T be an F_n-tree. The volume of T is

$$qvol(T) = \inf\{vol(V) \mid F_n \cdot V = R\}$$

Notice that if $F_n \cdot W$ covers T then the set $H_W = \{g \in F_n \mid gW \cap W \neq \emptyset\}$ generates F_n. Conversely, if H generates F_n and W is a set such that $W \cap b \cdot W \neq \emptyset$ for all $b \in H$ then $F_n \cdot W = T$ (by the minimality of T).

Proposition 3.6. *For every F_n tree T and for every ε there is a finite and connected subtree U, such that the translates of U cover T and $vol(U) \leq qvol(T) + \varepsilon$.*

Proof. Let $c : T \to T'$ with T' simplicial, be an equivariant collapsing map so that each point preimage $c^{-1}(x)$ is a subtree R with an action of $H = Stab_{T'}(x) < F_n$, such that for each $y \in R$, $H \cdot y$ is dense in R. Choose a connected subtree $V \subset T'$ which is the closure of a fundamental domain of T' and a basis $\mathcal{B}$ of F_n such that $bV \cap V \neq \emptyset$ for all $b \in \mathcal{B}$. Lift V to T edge by edge as follows. Lift e_1 to some segment σ_1 in T such that $c(\sigma_1) = e_1$ and $len(\sigma_1, T) \leq l(e_1, T') + \varepsilon$. For e_2 such that $i(e_2) = ter(e_1)$ there is a σ_2' such that $c(\sigma_2') = e_2$ and $l(\sigma_2', T) \leq l(e_2, T') + \varepsilon$. There is an element $h \in stab_{i(e_2)}(T')$ such that $d(i(\sigma_2'), ter(\sigma_1)) < \varepsilon$. Define $\sigma_2 = h\sigma_2'$ and add the segment between σ_1 and σ_2. Continue to lift all of the edges of V to a connected set W with volume $\leq vol(T') + (3n-3)\varepsilon$. Let J be the set of vertices of V let w_v be one of the points of $c^{-1}(v) \cap W$. $Stab_v(T')$ has an action with dense orbits on the componant of $c^{-1}(v)$ containing w. Then by [LL03] there is a basis $\mathcal{B}_v$ of $stab_v(T')$ such that $\sum_{b \in \mathcal{B}_v} d(w, bw) < \varepsilon$. Let U be the union of W and $\cup_{v \in V} \cup_{b \in \mathcal{B}_v} [w_v, bw_v]$. The volume of U is $\leq vol(V) + (3n-3)2\varepsilon + (2n-2)\varepsilon$ it is connected and its translates cover T. □

Proposition 3.7. *If T is a limit of a Cauchy sequence in $\mathcal{X}_n$ then it has unit volume and no non-trivial edge stabilizers.*

Proof. Let $\varepsilon > 0$ and $m > N(\varepsilon)$ so that $\mathrm{Lip}(h_m) < 1 + \varepsilon$. Choose U, a fundamental domain of X_m. Since h_m is equivariant, $F_n \cdot h_m(U) = T$ and $qvol(T) \leq vol(h_m(U)) \leq (1+\varepsilon)vol(U) = 1 + \varepsilon$. Since ε was arbitrary, $qvol(T) \leq 1$.

To show the other inequality: suppose there is a basis $\mathcal{B} \subset F_n$ and U a finite tree in T such that $vol(U) = c < 1$ and for each $b \in \mathcal{B}$, $bU \cap U \neq \emptyset$. Suppose that U is a union of k segments with disjoint interiors. Let $\varepsilon = \frac{1-c}{2k}$ and assume m is large enough so that there is a set $U' \subset X_m$ with a $\mathcal{B}$-invariant ε-relation to U. Each of the k segments of U is approximated by a segment of U' whose length is $\leq$ the length of the corresponding segment in $U + \varepsilon$. Thus $vol(U') < c + \varepsilon k$. Moreover, for all $b \in \mathcal{B}$, $U' \cap bU' \neq \emptyset$, hence U' contains a fundamental domain of X_m thus $qvol(X_m) \leq c + \varepsilon k < 1$ a contradiction.

Lastly, we must show that there are no arcs with non-trivial stabilizers. The idea is that an arc stabilizer will take up a definite part of the volume which would lead to X_m having less than unit volume. Let θ be the length of the smallest edge in G. Choose $\varepsilon < \frac{\theta}{6}$ and U a set whose translates cover T such that $vol(U) < qvol(T) + \varepsilon$. Let $\mathcal{B}$ be a basis such that $bU \cap U \neq \emptyset$ for all $b \in \mathcal{B}$. Suppose U contains a segment ν with non-trivial stabilizer

containing a. $c(\nu)$ is not a point since a segment with a non-trivial stabilizer is not contained in a dense subtree. Thus $l(\nu) > 5\varepsilon$. Let $U' \subset X_m$ be a set with a $\mathcal{B} \cup \{a\}$ ε-relation to U. Then $vol(U') < qvol(T) + 2\varepsilon$. Let $\sigma = [p, q]$ be a segment of length at least $l(\nu) - \varepsilon > 4\varepsilon$ corresponding to ν.
We claim that $len(a\sigma \cap \sigma) > l(\nu) - 3\varepsilon$. The segments $[p, ap]$ and $[q, aq]$ have length bounded above by ε and $d(p, q) \geq l(\nu) - \varepsilon$. Since $l(\sigma) > 3\varepsilon$ then $[p, ap] \cap [q, aq] = \emptyset$. Let $[m, n]$ be the bridge between $[p, ap]$ and $[q, aq]$. Then $len([m, n]) > l(\sigma) - 2\varepsilon > l(\nu) - 3\varepsilon > \varepsilon$ and $[m, n] = \sigma \cap a\sigma$.
Thus we may chop off from U' the segment $\sigma \cap a\sigma$ and still get a set that covers X_m and has volume $< 1 + 2\varepsilon - (l(\nu) - 3\varepsilon) < 1$. This is a contradiction because X_m has unit volume. Hence there are no edges with non-trivial stabilizers. □

A very small F_n tree T gives rise to a graph of actions.

Definition 3.8. [Lev94] A *graph of actions* $\mathcal{G}$ consists of

1. a metric graph G with vertex groups H_v and edge groups H_e and injections $i_e : H_e \to H_v$ when v is the initial point of the oriented edge e.

2. for every vertex v, an action of H_v on an $\mathbb{R}$-tree T_v.

3. for every oriented edge e a point $p_v \in T_v$ which is fixed under the subgroup $i_e(H_e)$.

We can also go back. A graph of actions induces a very small F_n action on an $\mathbb{R}$ tree.

Theorem 3.9. *T is a limit of a Cauchy sequence in $\mathcal{X}_n$ iff T has unit volume and no non-trivial edge stabilizers.*

Proof. Let $\mathcal{G}$ be the Levitt graph of actions of T. Then all edge groups are trivial hence all vertex groups are free factors. Let V be the set of vertices of $\mathcal{G}$ with non-trivial vertex groups. For each $v \in V$ there is a tree R_v in T, invariant under the vertex group H_v and so that $H_v \curvearrowright R_v$ has dense orbits. Levitt and Lustig [LL03] show that for every ε there is a free simplicial tree X'_v with volume $\leq \varepsilon$ and with a $1-$Lipschitz equivariant map onto R_v. Let G_v be the quotient marked graph of X'_v/H_v with volume ε. Let x_ε be a marked graph obtained from $\mathcal{G}$ by attaching G_v at v for every vertex $v \in V$. If X_ε is the free simplicial tree which is the universal cover of x_ε then there is a $1-$Lipschitz map $X_\varepsilon \to T$ and $vol(X_\varepsilon) \leq 1 + (2n - 2)\varepsilon$. By dividing by the volume we get a sequence in $\mathcal{X}_n$ which is a Cauchy sequence and converges to T. □

4 The Simplicial part of the metric completion

One can extend the notion of distance to any two trees in $\hat{\mathcal{X}}_n$ by

$$d(X,Y) = \log \sup \left\{ \frac{l(g,Y)}{l(g,X)} \middle| g \in F_n \right\}$$

Observe that d satisfies the directed triangle inequality.

Proposition 4.1. *For every $X, Y \in \hat{\mathcal{X}}_n$, $\hat{d}(X,Y) = d(X,Y)$ ($\hat{d}$ is defined in Proposition 1.5). Therefore, d is an asymmetric metric on $\hat{\mathcal{X}}_n$.*

Proof. Let $\{X_m\}_{m=1}^{\infty}, \{Y_k\}_{k=1}^{\infty}$ be Cauchy sequences in $\mathcal{X}_n$ such that $X = \lim_{m\to\infty} X_m$ and $Y = \lim_{k\to\infty} Y_k$ in the axes topology. We need to prove:

1. $d(X,Y) = c < \infty$ if and only if for all $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for all $m > N$ there is a $K = K(m,\varepsilon)$ such that $|d(X_m, Y_k) - c| < \varepsilon$ for all $k > K$.

2. $d(X,Y) = \infty$ iff for all r there is an $N(r)$ such that for al $m > N(r)$ there is a $K(m,r)$ such that $d(X_m, Y_k) > r$ for all $k > K$.

In essence, this is true because $d : \hat{\mathcal{X}}_n \times \hat{\mathcal{X}}_n \to \mathbb{R}$ is continuous in the axes topology. We begin by showing that $d(X,Y)$ is coarsely greater than $d(X_m, Y_k)$ for large m, k. By the triangle inequality we have $d(X,Y) \geq d(X_m,Y) - d(X_m,X)$, thus for large enough m, $d(X,Y) \geq d(X_m,Y) - \varepsilon$. Since there exists an equivariant map $X_m \to Y$, $d(X_m,Y) < \infty$. Let $\beta_1, \ldots, \beta_m$ be the list of candidates of X_m. Choose $K(m,\varepsilon)$ large enough so that for all $k > K$: if $l(\beta_i, Y) = 0$ then $l(\beta_i, Y_k) < injrad(X_m)$ and if $l(\beta_i, Y) > 0$ then $|l(\beta_i, Y_k) - l(\beta_i, Y)| < \varepsilon l(\beta_i, Y)$. Let β_k be the candidate realizing the distance $d(X_m, Y_k)$. If β_k is elliptic in Y then $l(\beta_k, Y_k) < l(\beta_k, X_m)$ so it cannot be a realizing candidate. Hence $l(\beta_k, Y_k) \leq (1+\varepsilon) l(\beta_k, Y)$ and therefore $d(X_m, Y_k) \leq \log(1+\varepsilon) + d(X_m, Y) \leq d(X,Y) + \varepsilon + \log(1+\varepsilon)$ for $m > N(\varepsilon)$ and $k > K(m,\varepsilon)$. In conclusion, $d(X,Y) \geq d(X_m, Y_k) + \varepsilon$.

If $d(X,Y) < \infty$ then for all ε there is some β a conjugacy class in F_n such that $\left| \log\left(\frac{l(\beta,Y)}{l(\beta,X)}\right) - d(X,Y) \right| < \varepsilon$. Thus, $l(\beta, X) > 0$ and let $N(\varepsilon)$ be such that for all $m > N(\varepsilon)$, $|l(\beta, X_m) - l(\beta, X)| < \varepsilon l(\beta, X)$. Thus $\frac{l(\beta,Y)}{l(\beta,X_m)} \geq \frac{l(\beta,Y)}{(1-\varepsilon) l(\beta,X)}$ which implies $d(X,Y) \leq d(X_m,Y) + \log(1+\varepsilon) + \varepsilon$ for $m > N(\varepsilon)$. By the triangle inequality, $d(X_m, Y) \leq d(X_m, Y_k) + d(Y_k, Y)$. Thus $d(X,Y) \leq d(X_m, Y_k) + \varepsilon'$ for all $m > N(\varepsilon)$ and $k > K(\varepsilon)$ so that

$d(Y_k, Y) < \varepsilon$.

If $d(X,Y) = \infty$ then either there is some β so that $l(\beta, X) = 0$ and $l(\beta, Y) > 0$, or for all $r > 1$ there is some β in X so that $\frac{l(\beta,Y)}{l(\beta,X)} > 2r$. If the former occurs, then there exist N and K so that for $m > N, k > K$ we have $l(\beta, X_m) < \frac{l(\beta,Y)}{r}$ and $l(\beta, Y_k) \geq (1 - \frac{1}{r})l(\beta, Y)$ thus $\frac{l(\beta,Y_k)}{l(\beta,X_m)} \geq \frac{(1-\frac{1}{r})l(\beta,Y)}{\frac{1}{r}l(\beta,Y)} \geq r - 1$ and $d(X_m, Y_k) > \log(r-1)$. If the latter occurs, then $l(\beta, X), l(\beta, Y) > 0$ and there are N, K large enough so that for all $m > N, k > K$ we have $l(\beta, X_m) \leq (1 + \frac{1}{r})l(\beta, X)$ and $l(\beta, Y_k) \geq (1 - \frac{1}{r})l(\beta, Y)$. Thus $\frac{l(\beta,Y_k)}{l(\beta,X_m)} \geq \frac{(1-\frac{1}{r})l(\beta,Y)}{(1+\frac{1}{r})l(\beta,X)} \geq r$ and $d(X_m, Y_k) > \log r$ for $m > M$ and $k > K$. □

Definition 4.2. A candidate α in a marked metric graph of groups x is an element of the fundamental group which is represented by a reduced path of the following types:

1. an embedded loop
2. an embedded figure 8
3. a barbell
4. a barbell whose bells are single points (whose stabilizers are non-trivial), i.e. a path of the form $e\bar{e}$
5. a barbell which has one proper bell and one collapsed bell. I.e. a path of the form $eu\bar{e}$ where u is an embedded circle (and the vertex which is the collapsed bell has non-trivial vertex group).

Proposition 4.3. *If S is simplicial and $T \in \hat{\mathcal{X}}_n$ then*

$$\begin{aligned} d(S,T) &= \log \max\{st(\alpha) \mid \alpha \text{ a candidate}\} \\ &= \log \min\{Lip(h) \mid h : S \to T \text{ an equivariant Lipschitz map}\} \end{aligned}$$

Proof. We first wish to show that if one of the quantities in the equations is infinite then so is the other. We begin by observing that the first quantity is no smaller than the second. If there is some equivariant Lipschitz map $f : S \to T$ then $\sup\{st(\alpha) \mid \alpha \in F_n\} \leq \mathrm{Lip}(f)$. For any f, $st(\alpha) = \frac{l(\alpha,T)}{l(\alpha,S)} \leq st(\text{maximaly stretched edge}) \leq \mathrm{Lip}(f)$. This shows in particular that if $\sup\{st(\alpha) \mid \alpha \in F_n\} = \infty$ then $\max\{\mathrm{Lip}(h)\} = \infty$.
Now suppose that $\sup\{st(\alpha) \mid \alpha \in F_n\} < \infty$ so in particular, all of the

elliptic elements of S are also elliptic in T. We wish to construct an equivariant map from S to T. Since S is simplicial, one may attach roses at the vertices with non-trivial vertex groups to obtain a graph of groups x and a map $p : x \to S/F_n$ which collapses the attached roses to the vertices. We denote its lift by $c : X \to S$ as well. There is an equivariant map $f : X \to T$ and we must show that it descends to a map $f' : S \to T$. Every attached rose represents an elliptic subgroup in S and therefore must be an elliptic subgroup in T. Let F finite tree which is the closure of a fundamental domain of X. Consider all the edges in F that are lifts of a single attached rose that corresponds to an elliptic subgroup. equivariantly homotope the image of f to a point that is the fixed point of that elliptic subgroup. Now this f descends to a map $f' : S \to T$ such that $f' \circ c = f$. This produced an equivariant Lipschitz map so $\inf\{\mathrm{Lip}(h)\}$ is finite.
Now we may assume that both quantities are finite and the proof of the equality is identical to the case of free simplicial trees in outer space. Let $f : G_S \to G_T$ be an equivariant, linear, Lipschitz map, and $\Delta(f)$ the tension graph - the collection of edges of G_S on which the slope of f is $\mathrm{Lip}(f)$. There is also a train-track structure on Δ induced by f. If a point $p \in \Delta(f)$ has only one gate at v and $H_v = \{1\}$ then one could homotope f so that either the new lipschitz constant is slightly smaller than the original, or the tension graph is smaller. Thus, if f has a minimal Lipschitz constant and the tension graph is minimal, then there are two gates at every vertex with trivial vertex group. This implies that there is a conjugacy class whose geodesic representative in G_S is a candidate loop which takes only legal turns. This conjugacy class will be stretched by $\mathrm{Lip}(f)$ under f. □

Question 4.4. *Does*

$$d(X,Y) = \log\inf\{Lip(h) \mid h : S \to T an\ equivariant\ Lipschitz\ map\}$$

even when X is not simplicial? Clearly, $d(X,Y) \leq \log\inf\{Lip(h)\}$ but the reverse inequality was not obvious to the author.

Proposition 4.5. *If X simplicial and $Y \in \hat{\mathcal{X}}_n$ and $d(X,Y) = 0$ then $X = Y$.*

Proof. There is an equivariant $f : X \to Y$. f is onto (since Y is minimal). Any map can be homotoped without increasing its Lipschitz constant, so that the restriction of the new map on each edge is an immersion or a collapse to one point. Note that if K is a fundamental domain of X then $f(K)$ covers Y. Since $qvol(X) = qvol(Y)$ we have: no edge of X is collapsed by f, no

edge is stretched by f (since $\mathrm{Lip} f = 1$) and no edge is shrunk by f because of the volume equality thus f is an isometry on each edge. Moreover, f is an immersion. If $i(e_1) = i(e_2)$ then no initial subinterval of e_1 is identified to an initial subinterval of another edge e_2. We may assume that $e_1 \in K$. If there is a $g \in F_n$ so that $ge_2' = e_1'$ then g stabilizes $f(e_1') = f(e_2')$ a contradiction. If $g \in K$ such that $ge_2 \in K$ and $ge_2' \cap e_1' = \emptyset$ then the volume of $f(K)$ will be strictly smaller than 1, a contradiction. Thus f is an immersion. f is therefore injective since $f(p) = f(q)$ implies that f does not immerse $[p, q]$. In conclusion, f is in fact an isometric embedding (f takes geodesics to geodesics) and it is onto thus $x = y$ (in particular, y is simplicial). □

We now focus on the subset of simplicial trees in $\hat{\mathcal{X}}_n$ which we denote by $\hat{\mathcal{X}}_n^S$. If $S \in \hat{\mathcal{X}}_n^S$ then $G_S = S/F_n$ is a volume 1 metric graph of groups with trivial edge groups and no valence 1 vertices. These graphs of groups are called free splittings.

Corollary 4.6. *There is a bijection from $\hat{\mathcal{X}}_n^S$ onto the free splitting complex FS_n.*

From now on we denote points in $\hat{\mathcal{X}}_n^S$ by lower case letters: $x, y, \ldots$. Our next goal is to define the Lipschitz and Euclidean topologies on $\hat{\mathcal{X}}_n^S$ and prove that they are equal.

Definition 4.7 (The Lipschitz topology on $\hat{\mathcal{X}}_n^S$)**.** Open sets are generated by $B_{Lip}(x, r) = \{y \mid d(y, x) < r\}$.

Definition 4.8 (The Euclidean topology on FS_n)**.** The collection of Euclidean balls $B_{Euc}(x, \varepsilon)$ for $x \in FS_n$ and $\varepsilon > 0$ generates the Euclidean topology, where $B_{Euc}(x, \varepsilon) = \cup_{x \in \sigma} B_\sigma(x, \varepsilon)$. $B_\sigma(x, \varepsilon)$ is defined as follows: Let $e_1, \ldots, e_J$ denote the edges of the marked graph underlying σ. Let $x_1, \ldots, x_J$ be the coordinates of x in σ. Then $B_\sigma(x, \varepsilon) = \{y \mid \max |x_i - y_i| < \varepsilon\}$.

Remark 4.9. The topology generated by the "outgoing" balls $\{y \mid d(x, y) < r\}$ is different from the Euclidean topology. Consider a point x in the completion so that x/F_n is a single, non-separating edge (a one edge loop). For such x and for all $x \neq y \in \hat{\mathcal{X}}_n^S$, $d(x, y) = \infty$. Hence the only open sets containing x are $\{x\}, \hat{\mathcal{X}}_n^S$. This topology is different from the Euclidean topology.

Lemma 4.10. *$d_{Lip}(\cdot, x)$ is continuous with respect to the Euclidean metric on FS_n.*

Proof. If $y \in B_{Lip}(x,r)$ then $\delta = \varepsilon$(smallest edge length in y). Let $y' \in B_{Euc}(y,\delta)$ then there is a simplex σ so that $y, y' \in \sigma$. Let $y = (y_1, \ldots, y_k), y' = (y'_1, \ldots, y'_k)$ in the σ coordinates, then $|y'_i - y_i| < \delta$.If $y_i > 0$ then $y'_i > 0$ by the choice of δ. Let $\varepsilon' = \frac{\varepsilon}{1-\varepsilon}$ then $1 - \varepsilon' < \frac{y_i}{y'_i} < 1 + \varepsilon'$. Hence, every loop is stretched by at most $1 + \varepsilon'$. Therefore $d(y', x) \leq d(y', y) + d(y, x) \leq \log(1 + \varepsilon') + d(y, x)$ and for small enough ε this is smaller than r. □

Lemma 4.11. *For every simplex σ in FS_n*

1. *For every $x \in \sigma$ and for every τ', a face of σ not containing x there is an $\varepsilon(x, \tau')$ such that $d(\tau', x) > \varepsilon$.*

2. *For all $x \in \sigma$ and for all $r > 0$ there is a $t(x, r)$ such that if $y \in \sigma$ and $d_{Lip}(y, x) < t$ then $d_{Euc}(y, x) < r$.*

Proof. 1. Let $v_1, \ldots, v_k$ be the vertices of τ'. They correspond to collapsing all but one edge of the underlying graph of σ. Let e_i be an edge in G the underlying graph of σ that survives in v_i, let $e \neq e_i$ be an edge in G that survives in x. It is elementary to check that there is always γ a candidate loop in G that contains e but not e_i. Therefore, $d(v_i, x) = \infty$. If z is Euclideanly close to v_i then the distance then $d(z, x)$ is very big. Thus there are Euclidean balls $B_i = B_{\tau'}(v_i, \varepsilon_i)$ so that $d(\tau', x) = \inf\{d(z, x) \mid z \in \overline{\tau' - (\cup B_i)}\}$. This set is compact, hence there is a minimum $\varepsilon(x, \tau')$.

2. $d_{Lip}(\cdot, x)$ is continuous as a function of the parametrization of σ. Moreover, for all points $y \neq x$, $d(y, x) > 0$. $\overline{\sigma - B_\sigma(x, r)}$ is compact so there is a t such that $d(y, x) > t$ for all $y \in \overline{\sigma - B_\sigma(x, r)}$. Thus $d(y, x) < t$ implies $d_{Euc}(y, x) < r$.

□

Lemma 4.12. *For every $x \in \hat{\mathcal{X}}_n^S$ there is an $\varepsilon(x)$ such that if $y \in \hat{\mathcal{X}}_n^S$ such that $d(y, x) < \varepsilon$ then there is a simplex $\sigma \in FS_n$ such that $x, y \in \sigma$.*

Proof. Let x be contained in the interior of the simplex τ. By Lemma 4.11 for any simplex $\sigma \supseteq \tau$ and for any face τ' not containing x there is an $\varepsilon = \varepsilon(x, \tau')$ so that $d(\tau', x) > \varepsilon$.
We show that we can find such an ε independently of τ'. Recall that $\mathrm{Out}(F_n)$ acts on FS_n by simplicial automorphisms thus, for each point z and for each simplex σ, $|\sigma \cap (\mathrm{Out}(F_n) \cdot z)| < \infty$. Moreover, there are finitely many orbits of simplices in FS_n. Now let $x \in FS_n$ and $\phi \in \mathrm{Out}(F_n)$ then $\varepsilon(x, \tau'_1) = \varepsilon(x\phi, \tau'_1\phi)$ for all faces τ'_1 of σ_1 (because ϕ is an isometry). We claim that

the minimum $\varepsilon(x) = \min\{\varepsilon(\tau', x\phi) \mid \phi \in \mathrm{Out}(F_n), x\phi \notin \tau'\}$ is achieved. For each simplex σ' containing x there are finitely many pairs $(x\phi, \tau')$ with τ' a face of σ' disjoint from $x\phi$, and there are finitely many orbits of simplices to take into account so the minimum is taken over a finite set.
We claim that if $y \in \hat{\mathcal{X}}_n^S$ such that $d(y,x) < \infty$ and there is no σ such that $x, y \in \sigma$ then $d(y,x) > \varepsilon(x)$. Let y' be a point in the same simplex as y so that there is a Stallings fold sequence from y' to x' (perturb the edges lengths in y so that the stretch of the edges of the optimal map are all rational). Moreover, we can guarantee that $d(y', x) < d(y,x) + \frac{\varepsilon(x)}{2}$. Let $f : y' \to x$ be an optimal map. We may assume that we perform the folds "at speed 1" as defined in [BF], i.e. fold simultaneously all the individual folds that it is possible to fold at the same time. Let $f' : z \to x$ the last fold in the sequence. Then z and x' are contained in the same simplex. Moreover, $d(y', x) > d(z, x)$. Thus $d(z,x) > \varepsilon(x)$ hence $d(y,x) > \frac{\varepsilon(x)}{2}$.

□

Theorem 4.13. *The Lipschitz topology and the Euclidean topology on $\hat{\mathcal{X}}_n^S$ coincide.*

Proof. That the Euclidean topology is finer than the Lipschitz topology reduces to the fact that a small perturbation of the edge lengths results in a small change in the dilation factor of loops, i.e. the lipschitz distance is continuous with respect to the Euclidean coordinates which is true by Lemma 4.10.

Now Let $B_{Euc}(x, r)$ be a neighborhood in the Euclidean topology. By proposition 4.12, we may choose ε small enough so that $B_{Lip}(x, \varepsilon)$ is contained in the star of x. By lemma 4.11, there is a $t(\sigma)$ so that if $d_{Lip}(y,x) < t$ then $d_{Euc}(y,x) < r$ for $y \in \sigma$. We need to find t that works for all σ containing x. Since the action is simplicial, $\mathrm{Out}(F_n) \cdot x \cap \sigma$ is a finite set. Let t be a constant so that for all $x' \in \mathrm{Out}(F_n) \cdot x \cap \sigma$, if $d_{Lip}(y, x') < t$ then $d^{\sigma}_{Euc}(y,x) < r$. Note that there are only finitely many orbits of simplices, thus we can fix t that works for a representative simplex of every orbit. We claim that for all y so that $d_{Lip}d(y,x) < t$ then $d_{Euc}(y,x) < r$. For $\tau = \phi(\sigma)$, σ an orbit representative, let $x' = \phi^{-1}(x)$, then for all $y \in \tau$ we have $d_{Lip}(y,x) < t \implies d_{Lip}(\phi^{-1}(y), x') < t$ hence $d_{Euc}(\phi^{-1}(y), x') < r$ hence $d_{Euc}(y,x) < r$ (because $\phi \in \mathrm{Out}(F_n)$ is an isometry of the Euclidean metric). □

To complete the picture we have

Lemma 4.14. *The Gromov/Axes topology on $\hat{\mathcal{X}}_n^S$ is strictly finer than the Lipschitz/Euclidean topologies.*

Proof. We must show that for every $T \in \hat{\mathcal{X}}_n^S$ and every $\varepsilon > 0$ there is a neighborhood $U = U(T, K, P, \delta)$ in the Gromov topology so that for each $S \in U \cap \hat{\mathcal{X}}_n^S$, $d(S,T) < \varepsilon$. Let $\mathcal{B}_T$ be a short basis for T, and K the closure of a fundamental domain. Let $S \in U(T, K, \mathcal{B}_T, \delta)$ for δ to be chosen later. Let K' be a δ approximation of K in S then for all $b \in \mathcal{B}$, $bK' \cap K' \neq \emptyset$. Moreover, if $k = |\text{number of edges in } K|$ then $vol(K') \leq 1 + k\delta$. Thus, $vol(K' \cap gK') < k\delta$ for any $g \in F_n$, otherwise we could find a subset K'' of K' with volume < 1 containing a fundamental domain of S. Next we show that if σ is a geodesic in K' which crosses only δ-short edges in K' then $len(\sigma) \leq k\delta + (3n-3)\delta$. Choose some connected fundamental domain U in K'. $vol(U) = 1$ thus $vol(K' - U) < k\delta$. Hence, a $(3n-3)\delta$ long piece of σ is in U, but then some edge orbit is crossed twice in U - a contradiction.

Let $g \in F_n$ and we may assume after conjugation that the axis of g in T intersects K. Let $[x, gx] = \sigma_1 \dots \sigma_m$ be a fundamental domain for the action of g on its axis, split into subsegments passing through translates of K. Let $[x_i, x_{i+1}] = \sigma_i \subseteq g_i K$ and $t_i = g_i^{-1} g_{i+1}$. $t_i \in \mathcal{B}_T^{\pm}$ or t_i is in a stabilizer of a vertex in K and $t_m = g_m^{-1} g \in stab(x)$. Let $z_i = g_i^{-1} x_i$, $y_i = g_{i+1}^{-1} x_{i+1}$ thus $y_i, z_i \in K$ for all i and $g = t_1 \cdots t_m$ where $t_i \in \mathcal{B}^{\pm}$ or $stab(y_i)$. Let $\nu_i = [z_i, y_i]$ and $\nu_i' = [z_i', y_i']$ a δ approximation of ν_i in K'. A path in S from x' to gx' is

$$\nu_1' \cdot [y_0', g_1 z_1'] \cdot g_1 \nu_2' \cdot [g_1 z_1', g_1 y_1'] \cdots g_{m-1} \nu_m' \cdot [g_{m-1} y_m', gx']$$

We show that the cancellation in this path is small, hence the path is close to the geodesic $[x', gx']$. If $t_i \in \mathcal{B}^{\pm}$, $t_i z_i = y_{i-1}$ implies $t_i z_i' \in K'$. Thus, the part that cancels in

$$\nu_{i-1}' \cdot [y_{i-1}', t_i z_i'] \cdot t_i \nu_i'$$

is in $K' \cap t_i K'$ hence it is shorter than $k\delta$.
If t_{i+1} is in $stab(y_i)$ then $[y_i', t_{i+1} z_i']$ contains only δ short edges. Therefore, $[y_i', t_{i+1} z_{i+1}'] \cap \nu_i$ and $[y_i', t_{i+1} z_{i+1}'] \cap t_{i+1} \nu_i'$ are shorter than $k\delta + (3n-3)\delta$. So the cancellation in

$$\nu_i' \cap [y_i', t_{i+1} z_{i+1}'] \cap t_{i+1} \nu_i'$$

is smaller than $M\delta := 4(k\delta + (3n-3)\delta) + 2k\delta$ (the $2k\delta$ coming from $vol(K \cap t_{i+1} K) \leq k\delta$).
Let $\theta =$ length of smallest edge in T and choose $\delta < \frac{\varepsilon}{2(M-1)}\theta$. Then

$$\begin{aligned} d_S(x', gx') &\geq \sum len(\nu_i') - M\delta \geq \sum len(\nu_i) - \delta - M\delta \\ &\geq \sum len(\nu_i)(1 - 2\varepsilon) \geq e^{-\varepsilon} d_T(x, gx) \end{aligned}$$

By the same argument, $[x', gx'] \cup [gx', g^2x'$ has backtracking segments of length bounded above by $M\delta$. So x' is $M\delta$ close to the axis of g in S. Therefore, $l(g,S) \geq d_S(x', gx') - 2M\delta \geq e^{-2\varepsilon} l(g,T)$. This concludes the proof that the axes topology is finer than the Lipschitz topology.

To see that the axes topology is not equal to the Euclidean topology consider the splitting complex for $F_2 =< a, b >$. Let x be the splitting whose graph of groups has one edge representing a and one vertex whose stabilizer is $< b >$. Let σ_i be a simplex whose underlying graph of groups is a wedge of two circles representing $e_i = ab^i, e_i' = b$. Let y be a point in σ_i with $len(a,y) < 1 + \frac{1}{m}$ then the length of $e_i' < \frac{1}{m(i-1)}$. This is not an open set in the Euclidean topology since the lengths of e_i' must get smaller and smaller to stay in the axes $\frac{1}{m}$-ball around x. □

5 The isometries of $\mathcal{X}_n$

Proposition 5.1. *An isometry F of $\mathcal{X}_n$ extends to a map on $\hat{\mathcal{X}}_n$. $\hat{\mathcal{X}}_n^S$ is an invariant subspace and $F|_{\hat{\mathcal{X}}_n^S}$ is a d preserving homeomorphism of $\hat{\mathcal{X}}_n^S$ in the Euclidean topology.*

Proof. By corollary 1.14, F extends to an isometry of $\hat{\mathcal{X}}_n$ with the Lipschitz distance (by proposition 4.1). We claim that $\hat{\mathcal{X}}_n^S$ is invariant under F. The reason is as follows: if T is not simplicial then there is a simplicial T' and a collapsing map $c : T \to T'$. This implies that there is a $T' \neq T$ such that $d(T, T') = 0$. By proposition 4.5 if S is simplicial and $S' \in \hat{\mathcal{X}}_n$ such that $d(S, S') = 0$ then $S' = S$. Thus F preserves $\hat{\mathcal{X}}_n^S$. By Theorem 4.13 the Lipschitz topology is the same as the Euclidean topology. □

Proposition 5.2. *If F is an isometery of $\hat{\mathcal{X}}_n^S$ then it preserves the simplicial structure.*

Proof. Francaviglia and Martino [FMa] show that if ϕ is a homeomorphism on Outer Space then ϕ preserves the simplicial structure. They prove it by induction on the codimension. They consider $\mathcal{X}_n^i$ the i-skeleton of $\mathcal{X}_n$ and show that every $i-1$ simplex is attached to three or more i-simplices. Thus the set of smooth points of $\mathcal{X}_n^i$, i.e. the points which have a neighborhood homeomorphic to $\mathbb{R}^i$ in $\mathcal{X}_n^i$, is the the disjoint union of open i-simplices. Thus i-simplices must be preserved.
For FS_n, it is not true that each $i-1$ simplex is contained in at least 3 i-simplices. There are cases where this is false, let G_σ be the undelying graph of groups related to the simplex σ in FS_n.

1. If G_σ contains a valence 4 vertex then σ is contained in 3 or more simplices (this is Francaviglia and Martino's argument). The reason is that there are 3 ways to blow up the neighborhood of that multi-valence vertex to obtain graphs G_1, G_2, G_3 and edge collapses back to G_σ.

2. G_σ contains a vertex with non-cyclic vertex group H_v. There are infinitely many ways to blow up an edge in G_σ to a loop based at v which represents some basis element in H_v.

3. G_σ contains three or more vertices with non-trivial edge stabilizers. Then there are at least three ways to blow up G_σ.

4. G_σ contains a vertex v with $H_v \neq \{1\}$ and an embedded loop containing v. Then there are infinitely many different graphs and with a single edge collapse to G_σ (see figure 1).

5. G_σ contains a vertex v with $H_v \neq \{1\}$ and a separating edge e with an endpoint at v and $G_\sigma - e = X \cup Y$ with $v \in X$ and $X - \{v\} \neq \emptyset$ then again there are infinitely many different graphs and with a single edge collapse to G_σ (see figure 2).

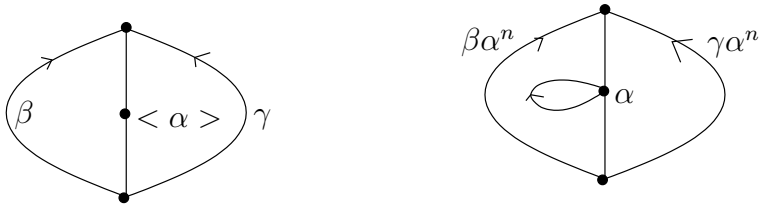

Figure 1: The graph on the left is G_σ of type (4) the graph on the right is G_τ with $\tau \supset \sigma$.

In all of the cases above, σ is contained in three or more simplices of dimension $i+1$. The remaining cases are:

7. G_σ contains a single vertex v with a non-trivial edge group and it is a valence 1 vertex and all other vertices are have valence 3 and trivial vertex groups. In this case σ is contained in a single top dimensional simplex. So it has a neighborhood homeomorphic to a half space in $\mathbb{R}^{3n-5}$.

8. G_σ has exactly two vertices with non-trivial stabilizers v, w and all other vertices have valence 3 and trivial vertex groups. Here, σ is a codimension 2 simplex and it is contained in exactly two codimension 1 simplices.

We start the proof exactly as in the case of $\mathcal{X}_n$. The set $\mathbb{R}^{3n-4}$-smooth points of all of $\mathcal{X}_n$ is the disjoint union of open top dimensional simplices. So a homeomorphism of FS_n preserves the open top dimensional simplices, and the codimension 1 skeleton is invariant. For $\mathcal{X}_n^{3n-5}$ we encounter a problem. Here the set of $3n-5$-smooth points is a larger than the union of open $3n-5$-simplices because it also contains points of codimension 2 simplices of case 8. Note that this is the only dimension in which we encounter a problem. If we somehow knew that the codimension 1 open simplices of FS_n are mapped to themselves then we can proceed to lower dimensional skeleta where there is no problem. Open i-simplices are connected components of the set of i-smooth points in $\mathcal{X}_n^i$ except when i is codimension 1.
Let σ be a codimension 1 simplex. If $\sigma = \overline{\tau_1} \cap \overline{\tau_2}$ where τ_1, τ_2 are top dimensional open simplicies then $\phi(\sigma) = \phi(\overline{\tau_1}) \cap \phi(\overline{\tau_2})$. So $\phi(\sigma)$ is a simplex and by invariance of domain a $3n-5$ simplex.
So we are left with codimension 1 simplices that are contained in a unique top dimensional simplex. We now invoke the hypothesis that F is not just a homeomorphism but an isometry. Let $\sigma \subset \tau$ be the simplices in question, then the image of σ is also contained in a unique top dimensional simplex $\tau' = \overline{F(int\tau)}$. Now $F(int\sigma)$ is contained in the part of $\partial\tau'$ that is not attached to any other top dimensional simplex. Thus $F(int\sigma)$ is contained in the union of closed faces of τ' whose undelying graph has type 7. If $F(int\sigma)$ is not contained in the interior of a codimension 1 simplex then there are points x, y in $int\sigma$ so that $d(F(x), F(y)) = \infty$. But for every $x, y \in int\sigma$, $d(x, y) < \infty$, thus the interior of a codimension 1 face must be mapped into the interior of a codimension 1 face. By applying the same argument to F^{-1} we get that F takes codimension 1 simplices to themselves

Figure 2: The graph on the left is G_σ of type (5) the graph on the right is G_τ with $\tau \supset \sigma$.

and preserves the codimension 2 skeleton. We can continue the induction to get that F is a simplicial map. □

Corollary 5.3. *There is a homomorphism $\phi : Isom(\mathcal{X}_n) \to Aut(FS_n)$.*

Proof. $\hat{G} \circ \hat{F}$ is an isometry of $\hat{\mathcal{X}}_n^S$ that restricts to $G \circ F$ on $\mathcal{X}_n$. By the uniqueness in proposition 1.13, $\hat{G} \circ \hat{F} = \widehat{G \circ F}$. This produces a homomorphism $\mathrm{Isom}(F_n) \to \mathrm{Aut}(FS_n)$. □

Corollary 5.4. *For $n \geq 3$ there is a homomorphism $\phi : Isom(\mathcal{X}_n) \to \mathrm{Out}(F_n)$. For $n = 2$, there is a homomorphism $\phi : Isom(\mathcal{X}_2) \to PSL(2, \mathbb{Z})$.*

Proof. Aramayona and Souto [AS] prove that the automorphisms of FS_n are exactly $\mathrm{Out}(F_n)$ for $n \geq 3$. For $n = 2$, we give the following argument. There is a homomorphism $\psi' : \mathrm{Out}(F_2) \to \mathrm{Aut}(FS_2)$. It is well known that $\mathrm{Out}(F_2) \cong \mathrm{SL}(2, \mathbb{Z})$. It is elementary to check that the kernel of ψ' is generated by $-I$. Thus we get an injective homomorphism $\psi : \mathrm{PSL}(2, \mathbb{Z}) \to \mathrm{Aut}(FS_2)$. Simplices with free faces in FS_2 are precisely the graphs with separating edges, thus an automorphism of FS_2 preserves the non-separating splitting complex which is the Farey complex (graph). It is well known that the automorphism group of the Farey graph is $PSL(2, \mathbb{Z})$ thus ψ is an isomorphism. □

Definition 5.5. Let (G, τ) be a marked graph representing the simplex σ. For any proper subgraph $\emptyset \neq H \subset G$ let σ_H denote the face of σ obtained by collapsing all of the edges of $G \setminus H$.

Proposition 5.6. *Let H be the image of a candidate loop in G. Then for every $x \in int(\sigma)$, $\frac{1}{vol(H)} = d(x, \sigma_H)$. I.e. the lengths of candidate loops in x are determined by the distances $d(x, \tau)$ to faces τ of σ.*

Proof. Let H be the image of a candidate loop α. Define $\lambda = \frac{1}{vol(H)}$. Let $y \in \sigma$ be the point so that if $e \in G - H$ then $len(e, y) = 0$ if $e \in H$ then $len(e, y) = \lambda len(e, x)$. Note that $vol(y) = 1$. The natural map $f : x \to y$ stretching the edges in H by λ and shrinking others to points has $\mathrm{Lip}(f) = \lambda$. Therefore $d(x, y) \leq \log \lambda$ and $d(x, \sigma_H) \leq \log \lambda$. When α is an embedded loop or a figure 8 loop then for any $z \in \sigma_H$, $st(\alpha) = \frac{1}{l(\alpha, x)} = \lambda$. So $d(x, z) \geq \log \lambda$. When α is a barbell loop with embedded loops β, γ and bar δ. If either $l(\beta, z) > \lambda l(\beta, x)$ or $l(\gamma, z) > \lambda l(\gamma, x)$ then $d(x, z) > \log \lambda$. Otherwize, since $l(\beta, z) + l(\gamma, z) + l(\delta, z) = 1$ then $len(\delta, z) > \lambda len(\delta, x)$ hence $l(\alpha, z) = 1 + len(\delta, z) > 1 + \lambda len(\delta, x) = \lambda len(\alpha, x)$. Hence $d(x, z) > \log \lambda$. □

Proposition 5.7. *If F, G induce the same action on the free splitting complex then they induce the same isometry.*

Proof. It is enough to show that if F is an isometry of $\mathcal{X}_n$ such that that $\phi(F) = id$ then F is the identity on $\mathcal{X}_n$. $F(\sigma) = \sigma$ for all simplices $\sigma \in \hat{\mathcal{X}}_n^S$. Hence, for all τ faces of σ, $d(x, \tau) = d(F(x), F(\tau)) = d(F(x), \tau)$. By proposition 5.6, the lengths of any candidate loops are the same in both x and $F(x)$. Since the distance is the maximal stretch of candidate loops from x to $F(x)$ then $d(x, F(x)) = 0$ and $d(F(x), x) = 0$ therefore $F(x) = x$ by Proposition 1.10. □

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