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# KMS STATES ON NICA-TOEPLITZ ALGEBRAS OF PRODUCT SYSTEMS

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Abstract. We investigate existence of KMS states of Fowler's Nica-Toeplitz algebra $\mathcal{NT}(X)$ associated to a compactly aligned product system $X$ over a semigroup $P$ of Hilbert bimodules in terms of restrictions to the core algebra which satisfy appropriate scaling conditions. If $(G, P)$ is a lattice ordered group and $X$ is a product system of finite type over $P$ satisfying certain coherence properties, we construct many $\mathrm{KMS}_\beta$ states of $\mathcal{NT}(X)$ associated to a scalar dynamics. Our results were motivated by, and generalize results of Laca and Raeburn obtained for the Toeplitz algebra of the affine semigroup over the natural numbers.

## 1. Introduction

$\mathrm{KMS}_\beta$ states for a quasi-free dynamics on the Toeplitz algebra associated to a right-Hilbert bimodule over a $C^*$-algebra have been constructed in many contexts by different authors. A unified approach that moreover greatly generalized earlier specific constructions was obtained by Laca and Neshveyev in [12].

Recently $C^*$-algebras associated with rings and exhibiting an interesting structure of KMS states have been discovered. In [6], Cuntz associated $C^*$-algebras to the affine semigroup over the natural numbers and to the ring of integers, and proved in both cases existence of a single $\mathrm{KMS}_\beta$ state at (inverse) temperature $\beta = 1$ for a natural dynamics. Cuntz's $C^*$-algebra $\mathcal{Q}_\mathbb{N}$ of the affine semigroup over the natural numbers is purely infinite and simple, and its reminiscence of a boundary construction prompted Laca and Raeburn to find a Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of the affine semigroup over the natural numbers with a much richer structure of KMS states for a natural dynamics, [14]. Laca and Raeburn proved that $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is a quasi-lattice ordered group in the sense of Nica [17], and using results on the associated Nica spectrum established that indeed $\mathcal{Q}_\mathbb{N}$ is a boundary quotient of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$.

Our goal in the present work is to initiate the study of KMS states of the Nica-Toeplitz algebra of a product system over a semigroup of Hilbert bimodules. Our motivation comes from the fact that $\mathcal{Q}_\mathbb{N}$ and several extensions of it were shown to be modeled as Cuntz-Pimsner and respectively Toeplitz-type $C^*$-algebras associated to

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product systems over the semigroup $\mathbb{N}^\times$ of Hilbert bimodules. In [3], the authors prove that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is the Nica-Toeplitz algebra of a product system over $\mathbb{N}^\times$, and identify subquotients $\mathcal{T}_{add}$ and $\mathcal{T}_{mult}$ that descend onto $\mathcal{Q}_\mathbb{N}$, both of which can also be described as Nica-Toeplitz algebras of product systems over $\mathbb{N}^\times$ of Hilbert bimodules. Moreover, Brownlowe, an Huef, Laca and Raeburn analyze KMS-states of the intermediary subquotients by using the original analysis from [14]. Both $\mathcal{T}_{add}$ and $\mathcal{Q}_\mathbb{N}$ were realized by different methods in [9] as algebras associated to a product system, and in [24] the algebra $\mathcal{Q}_\mathbb{N}$ is shown to be associated to a product system. Since the product system structure of these algebras arises from fairly natural endomorphisms and corresponding transfer operators, it seems worthwhile to investigate the problem of whether KMS-states can be obtained systematically for $C^*$-algebras associated to product systems over semigroups of Hilbert bimodules.

An analysis of the KMS states of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ based on a crossed product approach was carried out in [13], and showed that the unique $\mathrm{KMS}_\beta$ state in the critical interval $1 \leq \beta \leq 2$ has type $\mathrm{III}_1$. The classification of KMS states for $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ was generalized by Cuntz, Deninger and Laca to Toeplitz-like $C^*$-algebras associated to the ring of integers in a number field, [7].

Given a cancellative semigroup $P$ with identity, a product system $X$ over $P$ of Hilbert bimodules over a $C^*$-algebra $A$ is a family of right-Hilbert $A$–$A$-bimodules $X_s$ for $s \in P$ which forms a semigroup compatible with the Hilbert bimodule structure of the $X_s$'s, and the Toeplitz algebra $\mathcal{T}(X)$ is universal for Toeplitz representations of the $X_s$'s which respect the semigroup multiplication, see Fowler [8]. When $(G, P)$ is a quasi-lattice ordered group and $X$ is a compactly aligned product system, Fowler showed that a quotient of $\mathcal{T}(X)$ which encodes the Nica-covariant representations of $X$ is the appropriate object of study. We follow [3] and denote this quotient by $\mathcal{NT}(X)$ and refer to it as the *Nica-Toeplitz algebra of* $X$.

In the case of a single right-Hilbert $A$–$A$-bimodule $X$, Laca and Neshveyev [12] constructed $\mathrm{KMS}_\beta$ states of $\mathcal{T}(X)$ for $\beta \in (0, \infty)$ from certain traces of the coefficient algebra $A$ by means of state extensions to the fixed point algebra associated to the canonical gauge-action which satisfy a scaling-type condition. Composition with the canonical conditional expectation then gives rise to a state of the Toeplitz algebra which fulfills the $\mathrm{KMS}_\beta$ condition.

The algebra $\mathcal{NT}(X)$ carries a coaction of $G$, and admits a conditional expectation onto its fixed-point algebra, or *core*, $\mathcal{F}$. We aim to follow Laca-Neshveyev and look for certain states of $\mathcal{F}$ in order to get KMS states of $\mathcal{NT}(X)$. For ground states, it is possible in great generality to identify a necessary and sufficient condition on states of $\mathcal{F}$ which correspond to ground states of $\mathcal{NT}(X)$. However, for $\beta \in (0, \infty)$, only a necessary condition may be obtained in the greatest generality. In either case, it is desirable to reduce the problem further and characterize KMS states in terms of states or tracial states of $A$.

Since the guiding examples we have in mind are the algebras $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and $\mathcal{T}_{add}$, both viewed as Nica-Toeplitz algebras of product systems over $(\mathbb{Q}_+^*, \mathbb{N}^\times)$, we will assume that $(G, P)$ is a lattice-ordered group and that each bimodule $X_s$ for $s \in P$ has a finite orthonormal basis as a right Hilbert $A$-module such that certain compatibility conditions which reflect the multiplication in the semigroup are satisfied; we call such $X$ a *product*

*system of finite type*, see section 3.2. In this setting we identify a scaling condition on a tracial state $\phi$ of $\mathcal{F}$ which is sufficient for the composition of $\phi$ with the conditional expectation to produce a $\mathrm{KMS}_\beta$ state of $\mathcal{NT}(X)$, see Theorem 3.8. Since the scaling condition involves scaling by isometries in $\mathcal{NT}(X)$ (in the case at hand arising from the elements in the orthonormal bases for the $X_s$) this result is similar in spirit to [11, Theorem 12].

The next question we address is how to decide when a state of $\mathcal{F}$ is tracial and satisfies the scaling condition. Under some fairly natural hypotheses on the number of elements in the orthonormal bases, we show in Theorem 4.6 that a state of $\mathcal{F}$ which restricts to a tracial state on $A$ and satisfies the scaling condition on elements of the image of $A$ is a trace of $\mathcal{F}$. Under a further condition of convergence of a certain series in an interval $(\beta_c, \infty)$, we prove in Theorem 4.10 that for $\beta > \beta_c$ every tracial state of $A$ gives rise to a state of $\mathcal{F}$ that satisfies the hypotheses of Theorem 4.6, and hence gives rise to a $\mathrm{KMS}_\beta$ state. If every $\mathrm{KMS}_\beta$ state satisfies a reconstruction-type formula in the spirit of the similar formula found by Laca an Raeburn in [14, §10], then we can show that all $\mathrm{KMS}_\beta$ states arise by the recipe of Theorem 4.10 from traces of $A$.

Our methods of construction of KMS states reflect the properties of the elements in the orthonormal bases for the bimodules, so we think they are very natural in this setup. They also explain the origin of some of the computations and resulting formulas from [14] upon viewing the $C^*$-algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ as a $\mathcal{NT}(X)$, cf. [3].

The situation for ground states is easier, mirroring the similar case in [12] and [14]. The set of ground states is in affine bijection with the collection of states of $A$. We describe briefly which ground states are $\mathrm{KMS}_\infty$ states.

After a section of preliminaries, we start our analysis of KMS states in section 3, and here we first make some considerations valid for arbitrary compactly aligned product systems over quasi-lattice ordered pairs, after which we introduce product systems of finite type, see Definition 3.5. Theorem 3.8 identifies a scaling condition on tracial states of $\mathcal{F}$ which give rise to $\mathrm{KMS}_\beta$ states. In section 4 we prove the reduction of the scaling condition to traces of the image of $A$ in $\mathcal{F}$, see Theorem 4.6, and in Theorem 4.10 we construct $\mathrm{KMS}_\beta$ states from tracial states on $A$ and show surjectivity of this procedure in the presence of a reconstruction formula for $\mathrm{KMS}_\beta$ states. In the last section we describe further properties of the core $\mathcal{F}$ and discuss very briefly connections to the construction of a $\mathrm{KMS}_\beta$ state for $\beta \in \mathbb{R}$ in terms of states extended from a commutative subalgebra $\mathcal{A}$ of $\mathcal{F}$.

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## 2. Preliminaries

2.1. **Product systems of Hilbert bimodules.** Let $A$ be a $C^*$-algebra and $X$ be a complex vector space with a right action of $A$. Suppose that there is an $A$-valued inner product $\langle \cdot, \cdot \rangle_A$ on $X$ which is conjugate linear in the first variable and linear in the second variable, and satisfies

(1) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$,
(2) $\langle \xi, \eta \cdot a \rangle_A = \langle \xi, \eta \rangle_A \, a$,
(3) $\langle \xi, \xi \rangle_A \geq 0$ and $\langle \xi, \xi \rangle_A = 0 \iff \xi = 0$,

for $\xi, \eta \in X$ and $a \in A$. Then $X$ becomes a right Hilbert $A$-module when it is complete with respect to the norm given by $\|\xi\| := \|\langle \xi, \xi \rangle_A\|^{\frac{1}{2}}$ for $\xi \in X$.

A linear map $T : X \to X$ is said to be adjointable if there is a map $T^* : X \to X$ such that $\langle T\xi, \zeta \rangle_A = \langle \xi, T^*\zeta \rangle_A$ for all $\xi, \eta \in X$. The set $\mathcal{L}(X)$ of all adjointable operators on $X$ endowed with the operator norm is a $C^*$-algebra. The rank-one operator $\theta_{\xi,\eta}$ defined on $X$ as

$$\theta_{\xi,\eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle_A \quad \text{for } \xi, \eta, \zeta \in X, \tag{2.1}$$

is adjointable and we have $\theta^*_{\xi,\eta} = \theta_{\eta,\xi}$. Then $\mathcal{K}(X) = \overline{\operatorname{span}}\,\{\theta_{\xi,\eta} \mid \xi, \eta \in X\}$ is the ideal of (generalized) compact operators in $\mathcal{L}(X)$.

If $\varphi : A \to \mathcal{L}(X)$ is a $*$-homomorphism, then $\varphi$ induces a left action of $A$ on a right Hilbert $A$-module $X$ given by $a \cdot \xi = \phi(a)\xi$, for $a \in A$ and $\xi \in X$. Then $X$ becomes a right-Hilbert $A$–$A$-bimodule. The standard bimodule ${}_AA_A$ is equipped with $\langle a, b \rangle_A = a^*b$, and the right and left actions are simply given by right and left multiplication in $A$, respectively.

For right-Hilbert $A$–$A$-bimodules $X$ and $Y$, the (balanced) tensor product $X \otimes_A Y$ becomes a right-Hilbert $A$–$A$-bimodule with the natural right action, the left action implemented by the homomorphism $A \ni a \mapsto \varphi(a) \otimes_A I_Y \in \mathcal{L}(X \otimes_A Y)$, and the $A$-valued inner product given by

$$\langle \xi_1 \otimes_A \eta_1, \xi_2 \otimes_A \eta_2 \rangle_A = \langle \langle \xi_2, \xi_1 \rangle_A \cdot \eta_1, \eta_2 \rangle_A, \tag{2.2}$$

for $\xi_i \in X$ and $\eta_i \in Y$.

Let $P$ be a multiplicative semigroup with identity $e$ and $A$ a unital $C^*$-algebra. For each $p \in P$ let $X_p$ be a complex vector space. Then the disjoint union $X := \bigsqcup_{p \in P} X_p$ is a *product system* over $P$ if the following conditions hold:

- (P1) For each $p \in P \setminus \{e\}$, $X_p$ is a right-Hilbert $A$–$A$-bimodule.
- (P2) $X_e$ equals the standard bimodule ${}_AA_A$.
- (P3) $X$ is a semigroup such that $\xi\eta \in X_{pq}$ for $\xi \in X_p$ and $\eta \in X_q$, and for $p, q \in P \setminus \{e\}$, this product extends to an isomorphism $F^{p,q} : X_p \otimes_A X_q \to X_{pq}$ of Hilbert $A$-$A$ bimodules. If $p$ or $q$ equals $e$ then the corresponding product in $X$ is induced by the $A$-$A$ bimodule structure on the fibers.

Let $I_q$ be the identity operator in $\mathcal{L}(X_q)$ for every $q$. The product system $X$ is *associative* provided that

$$F^{st,r}(F^{s,t} \otimes_A I_r) = F^{s,tr}(I_s \otimes_A F^{t,r}) \tag{2.3}$$

for all $s, t, r \in P$, see e.g. [21] or [23].

**Remark 2.1.** For $p \in P$, the multiplication on $X$ induces maps $F^{p,e} : X_p \otimes_A X_e \to X_p$ and $F^{e,p} : X_e \otimes_A X_p \to X_p$ by multiplication $F^{p,e}(\xi \otimes a) = \xi\, a$ and $F^{e,p}(a \otimes \xi) = a\, \xi$ for $a \in A$ and $\xi \in X_p$. Note that $F^{p,e}$ is automatically an isomorphism, but $F^{e,p}$ may not be. The latter map is an isomorphism whenever $\overline{\varphi(A)X_p} = X_p$, in which case $X_p$ is called essential, see [8]. If $A$ is unital and $\varphi(1)\xi = \xi$ for all $\xi \in X_p$ then $X_p$ is essential.

We denote by $\langle \cdot, \cdot \rangle_p$ the $A$-valued inner product on $X_p$, by $\rho_p$ the right action of $A$ on $X_p$, and by $\varphi_p$ the homomorphism from $A$ into $\mathcal{L}(X_p)$. Due to associativity of the multiplication on $X$, we have $\varphi_{pq}(a)(\xi\eta) = (\varphi_p(a)\xi)\eta$ for all $\xi \in X_p$, $\eta \in X_q$, and $a \in A$.

For each pair $p, q \in P \setminus e$, the isomorphism $F^{p,q} : X_p \otimes_A X_q \to X_{pq}$ allows us to define a $*$-homomorphism $i_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ as

$$i_p^{pq}(S) = F^{p,q}(S \otimes_A I_q)(F^{p,q})^*, \tag{2.4}$$

for $S \in \mathcal{L}(X_p)$. When $p = e$, the homomorphism $i_e^q$ defined on $\mathcal{L}(A) = A$ is given simply by $i_e^q(a) = \varphi_q(a)$ for $a \in A$. Also, $i_p^p = I_p$ for all $p \in P$.

Many interesting product systems arise over semigroups equipped with additional structures. In [17], $(G, P)$ is called a quasi-lattice ordered group if (i) $G$ is a discrete group, (ii) $P$ is a subsemigroup of $G$ with $P \bigcap P^{-1} = \{e\}$, (iii) with respect to the order $p \preceq q \;\Leftrightarrow\; p^{-1}q \in P$, every two elements $p, q \in G$ which have a common upper bound in $P$ have a least upper bound $p \vee q \in P$. If this is the case we write $p \vee q < \infty$, otherwise we write $p \vee q = \infty$.

Assuming $X$ is a product system over $P$ with $(G, P)$ a quasi-lattice ordered group, there naturally arises a certain property related to compactness. A product system $X = \sqcup_{p\in P} X_p$ is called *compactly aligned*, [8], if for all $p, q \in P$ with $p \vee q < \infty$ and $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have

$$i_p^{p\vee q}(S) i_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q}). \tag{2.5}$$

2.2. **$C^*$-algebras associated to product systems.** Let $P$ be a semigroup with identity, $A$ be a unital $C^*$-algebra, and $X = \sqcup_{p\in P} X_p$ be a product system of right-Hilbert $A$–$A$-bimodules over $P$.

Let $C$ be a $C^*$-algebra. A mapping $\psi : X \longrightarrow C$ is said to be a Toeplitz representation of $X$ if the following conditions hold:

(T1) for each $p \in P \setminus \{e\}$, $\psi_p := \psi \restriction_{X_p}$ is linear,
(T2) $\psi_e : A \longrightarrow C$ is a $C^*$-homomorphism,
(T3) $\psi_p(\xi)\psi_q(\eta) = \psi_{pq}(\xi\eta)$ for $\xi \in X_p$ and $\eta \in X_q$,
(T4) $\psi_p(\xi)^*\psi_p(\eta) = \psi_e(\langle \xi, \eta \rangle_p)$ for $\xi, \eta \in X_p$.

As shown in [19], for each $p \in P$ then there exists a corresponding $*$-homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \longrightarrow C$ such that

$$\psi^{(p)}(\theta_{\xi,\eta}) = \psi_p(\xi)\psi_p(\eta)^*, \text{ for } \xi, \eta \in X_p. \tag{2.6}$$

Assume $(G, P)$ is a quasi-lattice ordered group and $X$ is compactly aligned. In [8], a Toeplitz representation $\psi$ of $X$ is said to be Nica covariant if

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(i_p^{p\vee q}(S) i_q^{p\vee q}(T)), & \text{if } p \vee q < \infty \\ 0\,, & \text{otherwise} \end{cases} \tag{2.7}$$

for $S \in \mathcal{K}(X_p)$, $T \in \mathcal{K}(X_q)$, and $p, q \in P$.

**Definition 2.2.** ([8, 4, 3]) Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. The Nica-Toeplitz algebra $\mathcal{NT}(X)$ is the $C^*$-algebra generated by a universal Nica covariant Toeplitz representation $i_X$ of $X$.

Fix a compactly aligned product system of right-Hilbert $A$–$A$-bimodules $X$ over a semigroup $P$ in a quasi-lattice ordered group $(G, P)$. Let $i_X$ be the universal Nica covariant Toeplitz representation of $X$ and denote by $i_s$ the restriction of $i_X$ to $X_s$ for $s \in P$. Recall that $\mathcal{NT}(X)$ is spanned by $i_s(\xi)i_r(\eta)^*$ for $\xi \in X_s, \eta \in X_r$, and there is a

gauge coaction $\delta$ of $G$ such that $\delta(i_s(\xi)) = i_s(\xi) \otimes s$, cf. [8, 22, 4]. The *core* of $\mathcal{NT}(X)$ is the $C^*$-subalgebra $\mathcal{F}$ spanned by the monomials $i_s(\xi)i_s(\eta)^*$ for $\xi, \eta \in X_s$ and $s \in P$. Then

$$\mathcal{F} = \overline{\operatorname{span}}\,\{i^{(s)}(T) \mid s \in P,\, T \in \mathcal{K}(X_s)\}, \tag{2.8}$$

see e.g. equation (3.4) in [4]. Let $\Phi^\delta$ be the conditional expectation from $\mathcal{NT}(X)$ onto $\mathcal{F}$ given by

$$\Phi^\delta(i_s(\xi)i_r(\eta)^*) = \begin{cases} i_s(\xi)i_s(\eta)^* & \text{if } s = r \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

for $r, s \in P$. For a finite subset $F$ of $P$ which is $\vee$-closed, [4, Lemma 3.6] says that

$$B_F = \{\sum_{s\in F} i^{(s)}(T_s) \mid T_s \in \mathcal{K}(X_s)\}$$

is a $C^*$-subalgebra of $\mathcal{F}$, and we have $\mathcal{F} = \overline{\bigcup_F B_F}$. We write $B_s$ when $F = \{s\}$.

2.3. **The Fock representation and Nica covariance.** Let $X$ be a product system over $P$ of right-Hilbert $A$–$A$-bimodules and let $l : X \to \mathcal{L}(F(X))$ be the Fock representation of $X$ constructed in [8, page 340]. We use the notation of [23] to describe $l$: the restriction of $l$ to $X_s$ is given by $l_s(\xi)\eta = F^{s,r}(\xi \otimes_A \eta)$ if $\xi \in X_s$ and $\eta \in X_r$ for $s, r \in P$. The adjoint acts by

$$l_s(\eta)^*\zeta = \begin{cases} \varphi_{s^{-1}r}(\langle \eta, \zeta' \rangle_s)\zeta'' & \text{if } r \in sP \text{ and } \zeta = F^{s,s^{-1}r}(\zeta' \otimes_A \zeta'') \\ 0 & \text{if } r \notin sP. \end{cases}$$

Let $\xi, \eta \in X_s$ for $s \in P$. It follows that $l_s(\xi)l_s(\eta)^*\zeta = 0$ if $\zeta \in X_r$ with $r \notin sP$. If $r \in sP$ then we have

$$\begin{aligned} l_s(\xi)l_s(\eta)^*\zeta &= F^{s,s^{-1}r}(\xi \otimes_A \varphi_{s^{-1}r}(\langle \eta, \zeta' \rangle_s)\zeta'') \\ &= F^{s,s^{-1}r}(\theta_{\xi,\eta}(\zeta') \otimes_A \zeta'') \\ &= F^{s,s^{-1}r}(\theta_{\xi,\eta} \otimes_A I_{s^{-1}r})(\zeta' \otimes_A \zeta'') \\ &= i_s^r(\theta_{\xi,\eta})\zeta. \end{aligned}$$

It was asserted in [22, §4] that when $(G, P)$ is quasi-lattice ordered and $X$ is compactly aligned, Fowler had proved that $l$ is Nica covariant as in (2.7). However, although one can use [8, Propositions 5.6 and 5.9] to deduce this claim, there is no such explicit result in [8]. It is an instructive exercise to see how the quasi-lattice ordered property of $(G, P)$ almost imposes the condition on $X$ that makes it compactly aligned, and consequently makes $l$ Nica covariant as a representation into the $C^*$-algebra $\mathcal{L}(F(X))$. Indeed, let $\theta_{\xi,\eta} \in \mathcal{K}(X_s)$ and $\theta_{z,w} \in \mathcal{K}(X_r)$ for $s, r \in P$. What can be said of the element $K_{s,r} := l^{(s)}(\theta_{\xi,\eta})l^{(r)}(\theta_{z,w})$ in $\mathcal{L}(F(X))$? If $\zeta \in X_q$ then $K_{s,r}\zeta = 0$ unless $q \in rP \cap sP$ or, equivalently, $s \vee r < \infty$ and $q \in (s \vee r)P$. Thus for $\zeta \in X_{s\vee r}$ we have $K_{s,r}\zeta = i_s^{s\vee r}(\theta_{\xi,\eta})i_r^{s\vee r}(\theta_{z,w})\zeta$. Now, if $i_s^{s\vee r}(\theta_{\xi,\eta})i_r^{s\vee r}(\theta_{z,w}) = \theta_{x,y}$ for some $x, y \in X_{s\vee r}$, it follows that $K_{s,r}\zeta = l^{(s\vee r)}(\theta_{x,y})\zeta$. By linearity and continuity,

$$l^{(s)}(\theta_{\xi,\eta})l^{(r)}(\theta_{z,w}) = l^{(s\vee r)}(i_s^{s\vee r}(\theta_{\xi,\eta})i_r^{s\vee r}(\theta_{z,w}))$$

whenever $i_s^{s\vee r}(\theta_{\xi,\eta})i_r^{s\vee r}(\theta_{z,w}) \in \mathcal{K}(X_{s\vee r})$. So if $X$ is compactly aligned in Fowler's sense, i.e. if $i_s^{s\vee r}(S)i_r^{s\vee r}(T) \in \mathcal{K}(X_{s\vee r})$ whenever $S \in \mathcal{K}(X_s)$ and $T \in \mathcal{K}(X_r)$, then $l$ is Nica covariant, i.e.

$$l^{(s)}(S)l^{(r)}(T) = l^{(s\vee r)}(i_s^{s\vee r}(S)i_r^{s\vee r}(T)). \tag{2.10}$$

Also when $s \vee r = \infty$ we have (2.10) because then both terms are 0.

By the universal property of $\mathcal{NT}(X)$ there is a homomorphism

$$l_* : \mathcal{NT}(X) \to \mathcal{L}(F(X)) \tag{2.11}$$

such that $l_*(i_m(\xi)) = l_m(\xi)$ for all $m \in P$ and $\xi \in X_m$.

## 3. KMS states on the Nica-Toeplitz algebra of product systems

KMS states on the Toeplitz algebra associated to a single bimodule were studied in many contexts, and a general unified approach was obtained in [12].

In the present paper we aim to analyze KMS states in the context of product systems of right-Hilbert bimodules. We begin by introducing a certain type of dynamics $\sigma_t$, $t \in \mathbb{R}$, on the algebra $\mathcal{NT}(X)$ for an arbitrary compactly aligned product system $X$ over $P$ in case $(G, P)$ is a quasi-lattice ordered group in Nica's sense. Our construction is analogous to quasi-free dynamics on Cuntz-Pimsner algebras considered in [25] and [12].

Later we introduce a class of compactly aligned product systems of finite type over a lattice semigroup $P$ and analyze KMS states corresponding to certain natural dynamics. The characterizations we obtain of the $\mathrm{KMS}_\beta$, the $\mathrm{KMS}_\infty$ and the ground states in terms of certain states of the core $\mathcal{F}$ of $\mathcal{NT}(X)$ are very much in the spirit of Laca's work [11], see also [13]. In Laca's setting, the pair $(G, S)$ is lattice-ordered, and the $\mathrm{KMS}_\beta$ states and the ground states of $C \rtimes_\alpha S$ for a natural dynamics were characterized in terms of certain states of $C$.

### 3.1. The case of compactly aligned product systems over a quasi-lattice ordered group.

**Proposition 3.1.** *Let $(G, P)$ be a quasi-lattice ordered pair and $X$ a compactly aligned product system over $P$ of right-Hilbert $A$–$A$-bimodules. Assume that for each $m \in P$ there is a strongly continuous one-parameter group $t \to U_t^{(m)}$ in $\mathcal{L}(X_m)$, $t \in \mathbb{R}$, such that $U_t^{(m)}$ is unitary for all $t$, $U_0^{(m)} = I_m$, and*

$$U_t^{(m)}(\phi_m(a)\xi) = \phi_m(a)(U_t^{(m)}\xi), \tag{3.1}$$

*for all $a \in A$, $m \in P$, $\xi \in X_m$, and $t \in \mathbb{R}$. If, in addition, the isomorphisms $F^{m,r} : X_m \otimes_A X_r \to X_{mr}$ satisfy*

$$F^{m,r} \circ (U_t^{(m)} \otimes_A U_t^{(r)}) = U_t^{(mr)} \circ F^{m,r} \tag{3.2}$$

*for all $m, r \in P$, then there is a one-parameter group of automorphisms $t \to \sigma_t \in \mathrm{Aut}(\mathcal{NT}(X))$ such that*

$$\sigma_t(i_e(a)) = i_e(a) \quad \text{and} \quad \sigma_t(i_m(\xi)) = i_m(U_t^{(m)}\xi) \tag{3.3}$$

*for all $a \in A$, $\xi \in X_m$ and $m \in P$.*

*Proof.* Roughly, (3.2) says that the one-parameter unitary group is compatible with the product system, and ensures that the one-parameter groups $\{U_t^{(m)}\}_{t\in\mathbb{R},m\in P}$ combine to give a dynamics on $\mathcal{NT}(X)$. To see this, we define a new representation $\psi$ of $X$ in $\mathcal{NT}(X)$ by $\psi_1(a) = i_e(a)$ for $c \in A$ and $\psi_m(\xi) = i_m(U_t^{(m)}\xi)$ for $\xi \in X_m$ and $m \in P$. Condition (3.1) shows that $(\psi_1, \psi_m)$ is a Toeplitz representation of $X_m$ for all $m \in P$, and condition (3.2) implies that $\psi_{mr}(\xi\eta) = \psi_m(\xi)\psi_r(\eta)$ for all $\xi \in X_m$, $\eta \in X_r$, $m, r \in P$.

We next prove that $\psi$ is Nica covariant, from which by applying the universal property of $\mathcal{NT}(X)$ we deduce the existence of $*$-homomorphisms $\sigma_t$, $t \in \mathbb{R}$, as postulated in (3.3).

Note first that $\psi^{(m)}(\theta_{\xi,\eta}) = i^{(m)}(\theta_{U_t^{(m)}\xi,U_t^{(m)}\eta}) = i^{(m)}(U_t^{(m)}\theta_{\xi,\eta}U_t^{(m)*})$ for all $m \in P$ and rank-one generalized compacts $\theta_{\xi,\eta}$ in $\mathcal{K}(X_m)$. By continuity of all maps involved we therefore have

$$\psi^{(m)}(S) = i^{(m)}(U_t^{(m)}SU_t^{(m)*}) \tag{3.4}$$

for all $S \in \mathcal{K}(X_m)$ and $m \in P$.

Let $S \in \mathcal{L}(X_m)$ and $T \in \mathcal{L}(X_n)$ for $m, n \in P$. We aim to prove that

$$U_t^{(m\vee n)}(i_m^{m\vee n}(S)i_n^{m\vee n}(T))U_t^{(m\vee n)*} = i_m^{m\vee n}(U_t^{(m)}SU_t^{(m)*})i_n^{m\vee n}(U_t^{(n)}TU_t^{(n)*}). \tag{3.5}$$

For this it suffices to show that

$$U_t^{(m\vee n)}i_m^{m\vee n}(S) = i_m^{m\vee n}(U_t^{(m)}SU_t^{(m)*})U_t^{(m\vee n)}. \tag{3.6}$$

Write $n' = m^{-1}(m \vee n)$. The left-hand side of (3.6) can be transformed as follows:

$$\begin{aligned} U_t^{(m\vee n)}i_m^{m\vee n}(S) &= U_t^{(m\vee n)}F^{m,n'}(S \otimes_A I_{n'})(F^{m,n'})^* \\ &= F^{m,n'}(U_t^{(m)} \otimes_A U_t^{(n')})(S \otimes_A I_{n'})(F^{m,n'})^* \ \{\text{by (3.2)}\} \\ &= F^{m,n'}(U_t^{(m)}S \otimes_A U_t^{(n')})(F^{m,n'})^* \\ &= F^{m,n'}(U_t^{(m)}SU_t^{(m)*} \otimes_A I_{n'})(U_t^{(m)} \otimes U_t^{(n')})(F^{m,n'})^*; \end{aligned} \tag{3.7}$$

now, inserting a factor $(F^{m,n'})^*(F^{m,n'})$ after $(U_t^{(m)}SU_t^{(m)*} \otimes_A I_{n'})$, grouping terms and using (3.2) once more shows that (3.7) equals $i_m^{m\vee n}(U_t^{(m)}SU_t^{(m)*})U_t^{(m\vee n)}$, as claimed in (3.6). Taking adjoints in (3.6) and replacing $S^*$ with $T$ proves (3.5). Now, if $S \in \mathcal{K}(X_m)$ and $T \in \mathcal{K}(X_n)$, we have $U_t^{(m)}SU_t^{(m)*} \in \mathcal{K}(X_m)$ and $U_t^{(n)}TU_t^{(n)*} \in \mathcal{K}(X_n)$. Further, since $X$ is compactly aligned, we also have $i_m^{m\vee n}(S)i_n^{m\vee n}(T) \in \mathcal{K}(X_{m\vee n})$ as well as

$$i_m^{m\vee n}(U_t^{(m)}SU_t^{(m)*})i_n^{m\vee n}(U_t^{(n)}TU_t^{(n)*}) \in \mathcal{K}(X_{m\vee n})$$

for $m \vee n < \infty$. Thus, still assuming $m \vee n < \infty$, we use Nica covariance of $i_X$ to deduce that

$$\begin{aligned} \psi^{(m)}(S)\psi^{(n)}(T) &= i^{(m)}(U_t^{(m)}SU_t^{(m)*})i^{(n)}(U_t^{(n)}TU_t^{(n)*}) \text{ by (3.4)} \\ &= i^{(m\vee n)}(i_m^{m\vee n}(U_t^{(m)}SU_t^{(m)*})i_n^{m\vee n}(U_t^{(n)}TU_t^{(n)*})) \\ &= i^{(m\vee n)}(U_t^{(m\vee n)}(i_m^{m\vee n}(S)i_n^{m\vee n}(T))U_t^{(m\vee n)*}) \text{ by (3.5)} \\ &= \psi^{(m\vee n)}(i_m^{m\vee n}(S)i_n^{m\vee n}(T)) \text{ by (3.4)}; \end{aligned}$$

this and the fact that for $m \vee n = \infty$ we have $i_m^{m\vee n}(S)i_n^{m\vee n}(T) = 0$ prove the required Nica covariance of $\psi$. Consequently, each $\sigma_t$ is a $*$-homomorphism satisfying (3.3). But

(3.3) immediately implies that $\sigma_0 = \mathrm{id}$ and $\sigma_{t+t'} = \sigma_t\sigma_{t'}$, and thus $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{NT}(X))$ is the required one-parameter automorphism group. □

We next recall the notions of $\mathrm{KMS}_\beta$ state, ground state and $\mathrm{KMS}_\infty$ state. Nowadays one often employs definitions of ground state and $\mathrm{KMS}_\beta$-state which are different from, although of course equivalent to, the more classical ones in [2] and [18], and we refer to the discussion in the beginning of Section 7 in [14] for an explanatory presentation and comparison.

Given a $C^*$-algebra $C$ and a homomorphism (a dynamics) $\sigma : \mathbb{R} \to \mathrm{Aut}(C)$, an element $c \in C$ is called *analytic* provided that $t \mapsto \sigma_t(c)$ extends to an entire function on $\mathbb{C}$. The set of analytic elements is dense in $C$, see [18, §8.12]. For $\beta \in (0, \infty)$, a *$KMS_\beta$- state* of $(C, \sigma)$ is a state $\omega$ of $C$ which satisfies the $\mathrm{KMS}_\beta$ condition

$$\omega(cd) = \omega(d\sigma_{i\beta}(c)) \tag{3.8}$$

for all $c, d$ analytic in $C$. It is known that it suffices to have (3.8) satisfied for a subset of analytic elements of $C$ which spans a dense subspace of $C$, [2, Proposition 8.12.3]. A state $\omega$ of $C$ is a ground state of $(C, \sigma)$ if for every $c, d$ analytic in $C$, the entire function $z \mapsto \omega(c\sigma_z(d))$ is bounded on the upper-half plane. Again, it is known that it suffices to have boundedness for a set of elements which spans a dense subspace of the analytic elements. More recently, the notion of $\mathrm{KMS}_\infty$-state was coined down in [5] and refers to states which are, by definition, weak$^*$-limits of $\mathrm{KMS}_\beta$-states as $\beta$ runs over a net $\beta_i \to \infty$.

Fix a quasi-lattice ordered group $(G, P)$ and a compactly aligned product system $X$ over $P$. Suppose that $N : G \to (0, \infty)$ is a multiplicative homomorphism. For every $r \in P$ and $t \in \mathbb{R}$ define $U_t^{(r)}\xi = N(r)^{it}\xi$ in $\mathcal{L}(X_r)$. Since $N$ is multiplicative, the family $U_t^{(r)}$ for $r \in P$, $t \in \mathbb{R}$ satisfies the conditions of Proposition 3.1. Hence there is a dynamics $\sigma^N$ on $\mathcal{NT}(X)$ such that

$$\sigma_t^N(i_e(a)) = i_e(a) \text{ and } \sigma_t^N(i_r(\xi)) = N(r)^{it} i_r(\xi) \tag{3.9}$$

for all $a \in A$, $\xi \in X_r$, $r \in P$. A routine proof of the following lemma is omitted.

**Lemma 3.2.** *The spanning elements $i_s(\xi)i_r(\eta)^*$ of $\mathcal{NT}(X)$ are $\sigma^N$-analytic, for all $\xi \in X_s, \eta \in X_r$ and $s, r \in P$.*

We aim to establish analogues of [11, Theorem 12]. As we shall see, the degree of sharpness of the results in our case depends on assumptions on the product system $X$, so we separate the characterizations of $\mathrm{KMS}_\beta$ states, ground states, and $\mathrm{KMS}_\infty$ states.

In one direction, for an arbitrary $\mathcal{NT}(X)$, a dynamics $\sigma^N$ where $N$ is an injective homomorphism, and for every $0 < \beta < \infty$, $\mathrm{KMS}_\beta$ states are lifted from tracial states of $\mathcal{F}$ with a scaling property, as shown in the next result. The non-trivial converse will be proved in Theorem 3.8, under additional hypotheses on $X$.

**Proposition 3.3.** *Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. Let $\sigma^N$ be the dynamics on $\mathcal{NT}(X)$ arising from an injective homomorphism $N : G \to (0, \infty)$. Let $0 < \beta < \infty$. If $\omega$ is a $KMS_\beta$ state of $\mathcal{NT}(X)$, then $\omega$ factors through $\Phi^\delta$ to give a tracial state $\phi$ on $\mathcal{F}$ that satisfies the scaling identity*

$$\phi(i_s(\xi)yi_r(\eta)^*) = \delta_{s,r}N(s)^{-\beta}\phi(y\langle\eta, \xi\rangle_s) \tag{3.10}$$

*for all $\xi \in X_s$, $\eta \in X_r$, $s, r \in P$ and $y \in \mathcal{F}$.*

*Proof.* Since all elements of the core $\mathcal{F}$ are fixed by $\sigma^N$, it follows that the restriction of $\omega$ to $\mathcal{F}$ is a trace, [2].

Now let $i_s(\xi)i_r(\eta)^*$ be a spanning element of $\mathcal{NT}(X)$ with $\xi \in X_s$, $\eta \in X_r$, $r, s \in P$, and note that by twice applying the $\mathrm{KMS}_\beta$ condition we get

$$\omega(i_s(\xi)i_r(\eta)^*) = N(sr^{-1})^{-\beta}\omega(i_s(\xi)i_r(\eta)^*).$$

Since $N$ is injective, $\omega(i_s(\xi)i_r(\eta)^*) = 0$ unless $s = r$. In other words, $\omega$ factors through $\Phi^\delta$ to give a trace $\phi$ on $\mathcal{F}$ such that $\phi \circ \Phi^\delta = \omega$. That $\phi$ satisfies the scaling identity (3.10) is immediate from the $\mathrm{KMS}_\beta$ condition. □

We next characterize ground states.

**Theorem 3.4.** *Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. Let $\sigma^N$ be the dynamics on $\mathcal{NT}(X)$ arising from a homomorphism $N : G \to (0, \infty)$ such that $N(r) \geq 1$ for $r \in P$ with equality only when $r = e$. Then $\phi \mapsto \phi \circ \Phi^\delta$ is an affine isomorphism from the states of $\mathcal{F}$ such that $\phi|_{B_s} = 0$ for all $s \in P \setminus \{e\}$ onto the ground states of $\mathcal{NT}(X)$.*

*Proof.* Let $\phi$ be a state of $\mathcal{F}$ which is zero on $B_s$ for $s > e$. Set $\omega := \phi \circ \Phi^\delta$. Let $y$ be arbitrary and $y' = i_s(\xi)i_r(\eta)^*$ an analytic element in $\mathcal{NT}(X)$. We must show that the function $F(z) := \omega(y\sigma_z^N(y'))$ is bounded on the upper-half plane. Since $F(z) = N(sr^{-1})^{iz}\omega(yy')$, this function is bounded on the upper-half plane in case $r = e$ because $N(s) \geq 1$. If $r > e$, an application of the Cauchy-Schwarz inequality (where we let $y_1 = yi_s(\xi)$) gives

$$|\omega(yy')| \leq \omega(y_1y_1^*)^{1/2}\omega(i_r(\eta)i_r(\eta)^*)^{1/2}.$$

Since $i_r(\eta)i_r(\eta)^* \in B_r$, the assumption on $\phi$ implies that $\omega(yy')$, and hence $F(z)$ vanish, proving the requested boundedness.

Conversely, if $\omega$ is a ground state of $\mathcal{NT}(X)$, then boundedness on the upper-half plane of the function $z \mapsto \omega(y\sigma_z^N(y'))$ for arbitrary $y$ and $y' = i_r(\eta)^*$ with $r > e$ forces $\omega(yy')$ to be 0. Hence $\omega$ vanishes on $B_r$ for any $r > e$, as claimed.

The correspondence $\phi \mapsto \phi \circ \Phi^\delta$ is an affine map by the same argument as in the proof of [11, Theorem 12]. □

3.2. **Product systems of finite type.** Throughout this section, $A$ is a unital $C^*$-algebra and $(G, P)$ denotes a lattice group as in [11, Definition 1]. Thus $P$ is an abelian cancellative semigroup with identity $e$, $G = PP^{-1}$ is the Groethendieck enveloping group of $P$, and we assume $P \cap P^{-1} = \{e\}$ and further that every pair of elements $s, r \in G$ has a (unique) least common upper bound $s \vee r \in G$ with respect to the partial order $g \leq h \iff g^{-1}h \in P$. Note that for $s, r \in P$, the element $(s \vee r)^{-1}sr$ is their greatest lower bound; we denote it $s \wedge r$. The properties of $s \vee r$ and $s \wedge r$ that are relevant to our analysis are contained in [11, Lemma 2].

**Definition 3.5.** Let $(G, P)$ be a lattice group. Let $X$ be a product system over $P$ of right-Hilbert $A$–$A$-bimodules. We say that $X$ is *of finite type* if for every $s \in P$ there are $N_s \in \mathbb{N}$ and elements $\{\mathbb{1}_0^s, \ldots, \mathbb{1}_{N_s-1}^s\} \in X_s$ such that:

(1) $X_s = \left\{\sum_{j=0}^{N_s-1} \varphi_s(a_j)\mathbb{1}_j^s : a_j \in A, j = 0, \ldots, N_s - 1\right\}$,

(2) $X_s = \Big\{\sum_{j=0}^{N_s-1} \rho_s(b_j)\mathbb{1}_j^s : b_j \in A, j = 0, \dots, N_s - 1\Big\}$,
(3) $\langle \mathbb{1}_j^s, \mathbb{1}_k^s\rangle_s = \delta_{j,k}$ for $s \in P$, $j, k \in \{0, \dots, N_s - 1\}$, and
(4) for every $s, r \in P$ there is a map

$$\mathfrak{m}_{s,r} : \{0, \dots, N_s - 1\} \times \{0, \dots, N_r - 1\} \to \{0, \dots, N_{sr} - 1\}$$

such that

$$F^{s,r}(\mathbb{1}_j^s \otimes_A \mathbb{1}_k^r) = \mathbb{1}^{sr}_{\mathfrak{m}_{s,r}(j,k)} \tag{3.11}$$

for $j \in \{0, \dots, N_s - 1\}$ and $k \in \{0, \dots, N_r - 1\}$.

For $j \in \{0, \dots, N_s-1\}$, $k \in \{0, \dots, N_r-1\}$ we often write $j \cdot k$ for the element $\mathfrak{m}_{s,r}(j, k)$ in $\{0, \dots, N_{sr} - 1\}$.

In case there is $Z \in A$ such that $\mathbb{1}_j^s = \varphi_s(Z^j)\mathbb{1}_0^s$ for all $j \in \{0, \dots, N_s - 1\}$ and all $s \in P$, we say that $X$ is *singly generated*. We then write $\mathbb{1}_s := \mathbb{1}_0^s$.

Conditions (2) and (3) say that the right Hilbert $A$-module $X_s$ has an orthonormal basis $\{\mathbb{1}_0^s, \dots, \mathbb{1}_{N_s-1}^s\}$, condition (1) then expresses the fact that this basis also generates $X_s$ as a left $A$-module, and (4) says that these bases for the modules $X_s$, $s \in P$, are coherent with respect to the multiplication in the product system. We note that the tensor product of the orthonormal bases $\{\mathbb{1}_j^s\}_{j=0,\dots,N_s-1}$ for $X_s$ and $\{\mathbb{1}_k^r\}_{k=0,\dots,N_r-1}$ for $X_r$ will be an orthonormal basis for $X_s \otimes_A X_r$ when $X_r$ is essential, see [16, Proof of Proposition 4.2]. Thus, if $X_r$ are essential for all $r \in P$, the maps $\mathfrak{m}_{s,r}$ are bijective for all $r, s \in P$.

Conditions (2) and (3) in Definition 3.5 imply that each $\xi \in X_s$ has a unique representation, known as the reconstruction formula, given by

$$\xi = \sum_{j=0}^{N_s-1} \mathbb{1}_j^s \cdot \langle \mathbb{1}_j^s, \xi\rangle. \tag{3.12}$$

Note that when $X$ is a product system of finite type, $\{\theta_{\mathbb{1}_j^s,\mathbb{1}_j^s} : j = 0, \dots, N_s - 1\}$ is a family of mutually orthogonal self-adjoint projections in $\mathcal{K}(X_s)$ for every $s \in P$. Equation (3.12) implies that

$$I_s := \sum_{j=0}^{N_s-1} \theta_{\mathbb{1}_j^s,\mathbb{1}_j^s} \tag{3.13}$$

for every $s \in P$. Hence $I_s \in \mathcal{K}(X_s)$ for every $s \in P$, so $\mathcal{L}(X_s) = \mathcal{K}(X_s)$, showing that the left action is by compact operators in every fibre. By [8, Proposition 5.8], a product system $X$ of finite type is therefore compactly aligned.

**Example 3.6.** Let $X$ be the product system over $\mathbb{N}^\times$ with fibers isomorphic to the Toeplitz algebra $\mathcal{T}$ from [3, §6]. The right action is implemented by an action $\beta : \mathbb{N}^\times \to \mathrm{End}(\mathcal{T})$, and the inner products are defined via an action of transfer operators $K$ for $\beta$. One can verify that $X$ is associative. By [3, Proposition 6.3], $X$ is of finite type with $N_m = m$ for $m \in \mathbb{N}^\times$. In fact, $X$ is even singly generated, where $Z = S$ is the generating non-unitary isometry in $\mathcal{T}$.

**Example 3.7.** Let $X$ be the product system over $\mathbb{N}^\times$ with fibers isomorphic to $C(\mathbb{T})$ from [3, §5] and [9]. The right action in each $X_n = C(\mathbb{T})$ is implemented by the endomorphism $\alpha_n(f) : z \mapsto f(z^n)$ of $C(\mathbb{T})$ for $n \in \mathbb{N}^\times$. The inner product is given by $\langle f, g\rangle_n = L_n(f^*g)$ for the transfer operator naturally associated with $\alpha_n$. A routine calculation shows that $X$ is associative. That $X$ is singly generated, with $Z$ the identity function on $\mathbb{T}$, follows from [3] (see the proof of Theorem 5.2 therein) or [9].

The following result is the promised converse to Proposition 3.3.

**Theorem 3.8.** *Let $(G, P)$ be a lattice group and $X$ a compactly aligned product system of finite type over $P$. Let $\sigma^N$ be the dynamics on $\mathcal{NT}(X)$ arising from an injective homomorphism $N : G \to (0, \infty)$. Let $0 < \beta < \infty$.*

*If $\phi$ is a tracial state of $\mathcal{F}$ such that*

$$\phi(i_s(\mathbb{1}^s_j)yi_r(\mathbb{1}^r_l)^*) = \delta_{s,r}\delta_{j,l}N(s)^{-\beta}\phi(y) \tag{3.14}$$

*for all $y \in \mathcal{F}$, $s, r \in P$ and $j, l \in \{0, \dots, N_s - 1\}$, then $\omega := \phi \circ \Phi^\delta$ is a KMS$_\beta$ state of $\mathcal{NT}(X)$.*

*Proof.* Let $\phi$ be a tracial state on $\mathcal{F}$ satisfying the scaling identity (3.14). Let $\omega = \phi \circ \Phi^\delta$. It suffices to verify the KMS$_\beta$ condition for $\omega$ on analytic elements $y_1 = i_s(\varphi_s(a)\mathbb{1}^s_l)i_r(\varphi_r(b)\mathbb{1}^r_k)^*$ and $y_2 = i_g(\varphi_g(a')\mathbb{1}^g_n)i_h(\varphi_h(b')\mathbb{1}^h_m)^*$ from the spanning set of $\mathcal{NT}(X)$, where $l, k = 0, \dots, N_s - 1$ and $n, m = 0, \dots, N_r - 1$. We must prove that

$$\omega(y_1y_2) = N(sr^{-1})^{-\beta}\omega(y_2y_1). \tag{3.15}$$

By [8, Proposition 5.10], the element $i_r(\varphi_r(b)\mathbb{1}^r_k)^*i_g(\varphi_g(a')\mathbb{1}^g_n)$ can be approximated from the span of elements of the form $i_{r^{-1}(r\vee g)}(\xi)i_{g^{-1}(r\vee g)}(\eta)^*$, and so the definition of $\Phi^\delta$ in (2.9) implies that $\omega(y_1y_2) = 0$ unless $sr^{-1}(r \vee g)(g^{-1}(r \vee g)h)^{-1} = e$, or equivalently, unless $sr^{-1}gh^{-1} = e$ in $G$.

Thus we assume $sg = rh$, and we therefore have $y_1y_2 \in \mathcal{F}$. The scaling identity (3.14) implies that

$$N(r)^{-\beta}\omega(y_1y_2)\delta_{i,j} = \phi(i_r(\mathbb{1}^r_i)y_1y_2i_r(\mathbb{1}^r_j)^*). \tag{3.16}$$

Next use property (1) in Definition 3.5 to write

$$i_r(\mathbb{1}^r_i)i_e(a) = \sum_{\nu=0}^{N_r-1} i_r(\varphi_r(a_\nu)\mathbb{1}^r_\nu) = \sum_{\nu=0}^{N_r-1} i_e(a_\nu)i_r(\mathbb{1}^r_\nu) \text{ and}$$
$$i_r(\mathbb{1}^r_j)i_e(b') = \sum_{\mu=0}^{N_r-1} i_r(\varphi_\mu(b_\mu)\mathbb{1}^r_\mu) = \sum_{\mu=0}^{N_r-1} i_e(b_\mu)i_r(\mathbb{1}^r_\mu).$$

Then the term under $\phi$ in the right-hand side of (3.16) is a product of

$$i_r(\mathbb{1}^r_i)y_1 = \sum_{\nu=0}^{N_r-1} i_e(a_\nu)i_{rs}(\mathbb{1}^{rs}_{\nu\cdot l})i_r(\mathbb{1}^r_k)^*i_e(b)^* \tag{3.17}$$

and

$$y_2i_r(\mathbb{1}^r_j)^* = i_e(a')i_g(\mathbb{1}^g_n)\left(\sum_{\mu=0}^{N_r-1} i_e(b_\mu)i_{rh}(\mathbb{1}^{rh}_{\mu\cdot m})\right)^*. \tag{3.18}$$

Fix $\alpha = 0, \dots, N_s - 1$. Inserting $i_s(\mathbb{1}^s_\alpha)^* i_s(\mathbb{1}^s_\alpha) = \langle \mathbb{1}^s_\alpha, \mathbb{1}^s_\alpha \rangle_s = 1$ between $y_1$ and $y_2$ and using the fact that $rh = sg$ we conclude that $i_r(\mathbb{1}^r_i) y_1 y_2 i_r(\mathbb{1}^r_j)^*$ can be split as the product of two terms in $\mathcal{F}$, one belonging to $B_{rs}$ and the other to $B_{sg}$. Applying the trace property of $\phi$ on $\mathcal{F}$ and the scaling identity (3.14) with $\mathbb{1}^s_\alpha$ shows, via (3.17) and (3.18), that the right-hand side of (3.16) is equal to

(3.19)
$$N(s)^{-\beta}\phi\left( i_e(a') i_g(\mathbb{1}^g_n) \left( \sum_{\mu=0}^{N_r-1} i_e(b_\mu) i_{rh}(\mathbb{1}^{rh}_{\mu\cdot m}) \right)^* \left( \sum_{\nu=0}^{N_r-1} i_e(a_\nu) i_{rs}(\mathbb{1}^{rs}_{\nu\cdot l}) \right) i_r(\mathbb{1}^r_k)^* i_e(b)^* \right).$$

Now we compute the middle product of linear combinations in $X_{rh}$ and $X_{rs}$ under $\phi$. We have

$$\begin{aligned}
&\left( \sum_{\mu=0}^{N_r-1} i_e(b_\mu) i_{rh}(\mathbb{1}^{rh}_{\mu\cdot m}) \right)^* \left( \sum_{\nu=0}^{N_r-1} i_e(a_\nu) i_{rs}(\mathbb{1}^{rs}_{\nu\cdot l}) \right) \\
&= \left( \sum_{\mu=0}^{N_r-1} i_e(b_\mu) i_r(\mathbb{1}^r_\mu) i_h(\mathbb{1}^h_m) \right)^* \left( \sum_{\nu=0}^{N_r-1} i_e(a_\nu) i_r(\mathbb{1}^r_\nu) i_s(\mathbb{1}^s_l) \right) \\
&= i_h(\mathbb{1}^h_m)^* \left( \sum_{\mu=0}^{N_r-1} i_e(b_\mu) i_r(\mathbb{1}^r_\mu) \right)^* \left( \sum_{\nu=0}^{N_r-1} i_e(a_\nu) i_r(\mathbb{1}^r_\nu) \right) i_s(\mathbb{1}^s_l) \\
&= i_h(\mathbb{1}^h_m)^* (i_r(\mathbb{1}^r_j) i_e(b'))^* i_r(\mathbb{1}^r_i) i_e(a) i_s(\mathbb{1}^s_l) \\
&= i_h(\mathbb{1}^h_m)^* i_e(b')^* i_r(\mathbb{1}^r_j)^* i_r(\mathbb{1}^r_i) i_e(a) i_s(\mathbb{1}^s_l). \qquad (3.20)
\end{aligned}$$

By combining (3.19) with (3.20) we get $\phi(i_r(\mathbb{1}^r_i) y_1 y_2 i_r(\mathbb{1}^r_j)^*) = N(s)^{-\beta}\phi(y_2 i_e(\langle \mathbb{1}^r_j, \mathbb{1}^r_i \rangle_r) y_1)$. Hence (3.16) becomes $N(r)^{-\beta}\omega(y_1 y_2)\delta_{i,j} = N(s)^{-\beta}\phi(y_2 i_e(\langle \mathbb{1}^r_j, \mathbb{1}^r_i \rangle_r) y_1)$. Thus both terms are 0 when $i \neq j$, and with $i = j$ we have

$$\omega(y_1 y_2) = N(r)^\beta \phi(i_r(\mathbb{1}^r_i) y_1 y_2 i_r(\mathbb{1}^r_i)^*) = N(sr^{-1})^{-\beta}\omega(y_2 y_1),$$

which is exactly claim (3.15). This finishes the proof of the theorem. □

Since the trace property is preserved under weak*-limits we obtain the following necessary condition on a ground state to be a $\mathrm{KMS}_\infty$ state.

**Corollary 3.9.** *Assume the hypotheses of Theorem 3.8, and suppose $N(s) > 1$ for $s > e$. If $\omega$ is a $\mathrm{KMS}_\infty$ state of $\mathcal{NT}(X)$, then $\omega$ restricts to a tracial state of $\mathcal{F}$.*

## 4. From traces on $A$ to $\mathrm{KMS}_\beta$ states on $\mathcal{NT}(X)$

It would be helpful to simplify the characterization of $\mathrm{KMS}_\beta$ states on $\mathcal{NT}(X)$ given by the requirements in Theorem 3.8. In this section we begin by improving this characterization in two steps: first, under the assumption that all maps $\mathfrak{m}_{s,r}$ are injective, we push the scaling condition (3.14) down to a scaling on elements in $i_e(A) = B_e$ from $\mathcal{F}$ and second, under further conditions on the $\mathfrak{m}_{s,r}$ we shall relax the requirement that $\phi$ must be a tracial state of $\mathcal{F}$ to just asking for it to hold on $B_e$. The second part of the section contains our constructions of ground states and $\mathrm{KMS}_\beta$ states above a certain value of $\beta$ from states on $A$.

4.1. **A simplification of the scaling condition for KMS$_\beta$ states.**

**Lemma 4.1.** *Let $X$ be a compactly aligned product system over $P$ with the coefficient algebra $A$. Let $\omega$ be a state on $\mathcal{NT}(X)$ and $\phi$ be its restriction to the core subalgebra $\mathcal{F}$.*

*(a) If $\omega$ is a KMS$_\beta$ state ($\beta > 0$) for the dynamics $\sigma^N$ arising from an injective homomorphism $N$ as in (3.9), then for $\xi \in X_s$, $\eta \in X_r$ for some $s, r \in P$ we have*

$$\omega(i_s(\xi)i_r(\eta)^*) = \delta_{s,r}N(s)^{-\beta}\omega(i_e(\langle \eta, \xi\rangle_s)). \tag{4.1}$$

*Thus the restriction map $\omega \mapsto \omega\restriction_{i_e(A)}$ from the KMS$_\beta$ states on $\mathcal{NT}(X)$ to traces on $i_e(A)$ is injective.*

*(b) In addition, assume that $X$ is of finite type and $\mathfrak{m}_{r,s}$ is injective for all $s, r \in P$. Then $\omega$ is a KMS$_\beta$ state ($\beta > 0$) for the dynamics $\sigma^N$ if and only if $\phi$ is a tracial state on $\mathcal{F}$ such that $\omega = \phi \circ \Phi^\delta$ and*

$$\phi(i_s(\mathbb{1}^s_j)i_e(a)i_s(\mathbb{1}^s_j)^*) = N(s)^{-\beta}\phi(i_e(a)) \tag{4.2}$$

*for all $s \in P$, $j = 0, \dots, N_s - 1$, $a \in A$.*

*Proof.* (a) Equality (4.1) follows immediately from Proposition 3.3. It shows that a KMS$_\beta$ state is uniquely determined by its values on the coefficient algebra $A$.

(b) Assume $\phi$ is a tracial state on $\mathcal{F}$ which satisfies (4.2). We will show that $\phi$ satisfies (3.14) for all $y \in \mathcal{F}$. First, for $s \in P$, $i, m \in \{0, \dots, N_s - 1\}$ and $y \in \mathcal{F}$ we have

$$\phi(i_s(\mathbb{1}^s_i)yi_s(\mathbb{1}^s_m)^*) = \phi(i_s(\mathbb{1}^s_m)i_s(\mathbb{1}^s_m)^*i_s(\mathbb{1}^s_i)yi_s(\mathbb{1}^s_m)^*) = 0$$

if $i \neq m$, by condition (3) of Definition 3.5. Thus for $y = i_r(\mathbb{1}^r_j)i_e(a)(i_r(\mathbb{1}^r_k)i_e(b))^*$ in $\mathcal{F}$, where $j, k \in \{0, \dots, N_r - 1\}$ and $a, b \in A$, we have

$$\begin{aligned}
\phi(i_s(\mathbb{1}^s_i)yi_s(\mathbb{1}^s_m)^*) &= \delta_{i,m}\phi(i_s(\mathbb{1}^s_i)i_r(\mathbb{1}^r_j)i_e(ab^*)(i_s(\mathbb{1}^s_i)i_r(\mathbb{1}^r_k))^*) \\
&= \delta_{i,m}\phi(i_{sr}(\mathbb{1}^{sr}_{i\cdot j})i_e(ab^*)i_{sr}(\mathbb{1}^{sr}_{i\cdot k})^*) \\
&= \delta_{i,m}\delta_{i\cdot j, i\cdot k}N(sr)^{-\beta}\phi(i_e(ab^*)) \text{ by (4.2)} \\
&= \delta_{i,m}\delta_{j,k}N(sr)^{-\beta}\phi(i_e(ab^*)) \text{ by injectivity of } \mathfrak{m}_{s,r} \\
&= \delta_{i,m}N(s)^{-\beta}\phi(i_r(\mathbb{1}^r_j)i_e(ab^*)i_r(\mathbb{1}^r_k)^*) \text{ by (4.2)};
\end{aligned}$$

the last term is $\delta_{i,m}N(s)^{-\beta}\phi(y)$, which proves (3.14) for all such $y$. Since an arbitrary spanning element $y$ in $\mathcal{F}$ is a linear combination of elements $i_r(\mathbb{1}^r_j)i_e(a)(i_r(\mathbb{1}^r_k)i_e(b))^*$ by Definition 3.5(2), the scaling condition (3.14) is valid for all $y \in \mathcal{F}$. Theorem 3.8 implies therefore that $\omega$ is a KMS$_\beta$ state.

The reverse implication is an immediate consequence of Theorem 3.8. □

**Lemma 4.2.** *Let $X$ be of finite type and let $\phi$ be a functional on $\mathcal{NT}(X)$. If $\phi$ satifies*

$$\phi(i_e(c)i_s(\mathbb{1}^s_j)i_s(\mathbb{1}^s_k)^*i_e(d)^*) = N(s)^{-\beta}\phi(i_e(\langle \varphi_s(d)\mathbb{1}^s_k, \varphi_s(c)\mathbb{1}^s_j\rangle)) \tag{4.3}$$

*for all $c, d \in A$, $s \in P$ and $j, k \in \{0, \dots, N_s - 1\}$, then $\phi$ satisfies*

$$\phi(i_s(\mathbb{1}^s_j)i_e(a)i_s(\mathbb{1}^s_l)^*) = \delta_{j,l}N(s)^{-\beta}\phi(i_e(a)) \tag{4.4}$$

*for all $s \in P$, $j, l = 0, \dots, N_s - 1$ and $a \in A$.*

*Conversely, if $\phi$ satisfies (4.4) and $\phi \circ i_e$ is a trace on $A$, then $\phi$ satisfies (4.3).*

*Proof.* Assume (4.3). For $a \in A$, $s \in P$ and $j, l = 0, \dots, N_s - 1$, write $\rho_s(a)\mathbb{1}_j^s = \sum_{h=0}^{N_s-1} \varphi_s(a_h)\mathbb{1}_h^s$. Then

$$\begin{aligned}\phi(i_s(\mathbb{1}_j^s)i_e(a)i_s(\mathbb{1}_l^s)^*) &= N(s)^{-\beta}\sum_h \phi(i_e(\langle \mathbb{1}_l^s, \varphi_s(a_h)\mathbb{1}_h^s\rangle))\\ &= N(s)^{-\beta}\phi(i_e(\langle \mathbb{1}_l^s, \rho_s(a)\mathbb{1}_j^s\rangle))\\ &= \delta_{j,l}N(s)^{-\beta}\phi(i_e(a)),\end{aligned}$$

as claimed in (4.4).

If $\phi$ restricts to a trace on $i_e(A)$ and satisfies (4.4), let $y = i_s(\varphi_s(c)\mathbb{1}_j^s)i_s(\varphi_s(d)\mathbb{1}_k^s)^* \in \mathcal{F}$, and express

$$\varphi_s(c)\mathbb{1}_j^s = \sum_{h=0}^{N_s-1} \rho_s(c'_h)\mathbb{1}_h^s \text{ and } \varphi_s(d)\mathbb{1}_k^s = \sum_{i=0}^{N_s-1} \rho_s(d'_i)\mathbb{1}_i^s.$$

Then $\phi(y) = \sum_h \sum_i \phi\big(i_s(\mathbb{1}_h^s)i_e(c'_h(d'_i)^*)i_s(\mathbb{1}_i^s)^*\big) = N(s)^{-\beta}\sum_{h,i}\delta_{h,i}\phi\big(i_e(c'_h(d'_i)^*)\big)$, so $\phi(y) = N(s)^{-\beta}\phi(\sum_h(i_e((d'_h)^*c'_h)))$ because $\phi \circ i_e$ is a trace. This last term is, by the choice of $c'_h$ and $d'_h$, equal to $N(s)^{-\beta}\phi(i_e(\langle\varphi_s(d)\mathbb{1}_k^s, \varphi_s(c)\mathbb{1}_j^s\rangle))$, giving (4.3). □

**Remark 4.3.** Under the hypothesis of part (b) of Lemma 4.1, the condition

$$\phi(i_e(a)i_s(\mathbb{1}_j^s)i_s(\mathbb{1}_k^s)^*i_e(b^*)) = N(s)^{-\beta}\phi(i_e(\langle \mathbb{1}_k^s, \varphi_s(b^*a)\mathbb{1}_j^s\rangle)$$

for all $s \in P$, $j, k \in \{0, \dots, N_s - 1\}$, $a, b \in A$, is similar to [14, equation (8.2)] in the case of the product system over $\mathbb{N}^\times$ from Example 3.6.

Given a tracial state on $i_e(A)$ satisfying the scaling identity (4.4), formula (4.1) may be used to extend it to a state on $\mathcal{NT}(X)$. The question remains if such a state is tracial on $\mathcal{F}$. We examine this issue next. We thank N. Stammeier for indicating to us a problem in the formulation of the next definition in an earlier version of the paper.

**Definition 4.4.** Let $X$ be an associative product system of finite type over $P$. The functions $\mathfrak{m}_{s,r}$ *respect co-prime pairs* if the following condition holds: for all $p, q \in P$ such that $p \wedge q = e$, all $j, g, h \in \{0, \dots, N_p - 1\}$ and all $l, m, n \in \{0, \dots, N_q - 1\}$ we have

$$\mathfrak{m}_{p,q}(j, m) = \mathfrak{m}_{q,p}(l, g) \text{ and } \mathfrak{m}_{p,q}(j, n) = \mathfrak{m}_{q,p}(l, h) \Rightarrow m = n \text{ and } g = h. \tag{4.5}$$

**Remark 4.5.** For the product systems from Examples 3.6 and 3.7 the maps $\mathfrak{m}_{s,r}$ respect co-prime pairs in the sense of (4.5). We only show this in the case of Example 3.6 because the argument is similar for the second example. We have $P = \mathbb{N}^\times$ and $N_m = m$ for $m \in \mathbb{N}^\times$. Suppose that $p, q$ are co-prime integers. Since the multiplication of $X_p$ with $X_q$ is implemented by applying the endomorphism that raises the generating isometry $S$ to the power $p$, it follows that $\mathbb{1}_{j\cdot k}^{pq} = \mathbb{1}_{j+pk}^{pq}$ for all $j \in \{0, \dots, p-1\}$ and $k \in \{0, \dots, q-1\}$. Assume $\mathfrak{m}_{p,q}(j, m) = \mathfrak{m}_{q,p}(l, g)$ and $\mathfrak{m}_{p,q}(j, n) = \mathfrak{m}_{q,p}(l, h)$, where $j, g, h \in \{0, \dots, p-1\}$ and $l, m, n \in \{0, \dots, q-1\}$. Then $j - l = qh - pn = qg - pm$, and therefore $q(h - g) = p(n - m)$. Then necessarily $h = g$ and $n = m$, as required.

**Theorem 4.6.** *Let $X$ be an associative product system of finite type over $P$ such that $\mathfrak{m}_{s,r}$ are bijective for all $s, r \in P$ and respect co-prime pairs.*

*If $\phi$ is a state of $\mathcal{F}$ such that $\phi|_{i_e(A)}$ is a trace such that for some $0 < \beta < \infty$ we have*

$$\phi(i_s(\mathbb{1}_j^s)i_e(a)i_s(\mathbb{1}_l^s)^*) = \delta_{j,l}N(s)^{-\beta}\phi(i_e(a)) \tag{4.6}$$

*for all $a \in A$, $s \in P$, $j, l = 0, \dots, N_s - 1$, then $\phi$ is a trace on $\mathcal{F}$. In particular, $\phi \circ \Phi^\delta$ is a KMS$_\beta$-state of $\mathcal{NT}(X)$.*

The proof of this theorem will rely on some preparation. First we introduce some notation. For $s, r$ in $P$, if $\mathfrak{m}_{s,r}$ is bijective, then every $k = 0, \dots, N_{sr} - 1$ has a unique decomposition $k = k(s) \cdot k(r)$ in $\{0, \dots, N_s - 1\} \times \{0, \dots, N_r - 1\}$, and we must have

$$N_{sr} = N_s N_r \text{ for all } s, r \in P.$$

For $s, r$ in $P$ let

$$s' = r^{-1}(s \vee r) \text{ and } r' = s^{-1}(s \vee r),$$

and note then that $s \vee r = sr' = rs'$ as well as $(s \wedge r)s' = s$ and $(s \wedge r)r' = r$, with $s \wedge r$ denoting the least upper bound of $s$ and $r$.

As noticed in [9], if the product system $X$ is such that $I_s \in \mathcal{K}(X_s)$ for all $s \in P$, then [8, Proposition 5.10] proves the stronger statement that every product of the form $i_s(\xi)^* i_r(\eta)$ in $\mathcal{NT}(X)$ is a linear combination (rather than a limit of linear combinations) of elements of the form $i_e(a) i_{s^{-1}(s\vee r)}(\xi') i_{r^{-1}(s\vee r)}(\eta')^* i_e(b)$ for appropriate $a, b \in A$ and $\xi' \in X_{s^{-1}(s\vee r)}$, $\eta' \in X_{r^{-1}(s\vee r)}$. The next results makes this decomposition explicit in the case of $X$ of finite type with bijective maps counting the elements in the bases.

**Lemma 4.7.** *Let $X$ be a product system of finite type over $P$ such that $\mathfrak{m}_{s,r}$ is bijective, for all $s, r \in P$. For any $\xi \in X_s$ and $\eta \in X_r$, where $s, r \in P$, we have*

$$i_s(\xi)^* i_r(\eta) = \sum_{i=0}^{N_{s\vee r}-1} i_e(\langle \xi, \mathbb{1}^s_{i(s)} \rangle_s) i_{r'}(\mathbb{1}^{r'}_{i(r')}) i_{s'}(\mathbb{1}^{s'}_{i(s')})^* i_e(\langle \eta, \mathbb{1}^r_{i(r)} \rangle_r)^*. \tag{4.7}$$

*Proof.* Writing $i_s(\xi)^* i_r(\eta) = i_s(\xi)^* i^{(s\vee r)}(I_{s\vee r}) i_r(\eta)$, and using (3.13) and the properties of the multiplication in $X$ gives (4.7). □

*Proof of Theorem 4.6.* Let $\phi$ be a state of $\mathcal{F}$ such that $\phi \circ i_e$ is a tracial state and (4.6) is satisfied. Let $y_1 = i_s(\mathbb{1}^s_j) i_e(ab^*) i_s(\mathbb{1}^s_k)^*$ and $y_2 = i_r(\mathbb{1}^r_m) i_e(cd^*) i_r(\mathbb{1}^r_n)^*$ be spanning elements in $\mathcal{F}$, where $a, b, c, d \in A$, $s, r \in P$, $j, k \in \{0, \dots, N_s - 1\}$ and $m, n \in \{0, \dots, N_r - 1\}$. To prove that $\phi$ is a trace on $\mathcal{F}$, it suffices to show that

$$\phi(y_1 y_2) = \phi(y_2 y_1). \tag{4.8}$$

Using (4.7), we have

$$\begin{aligned}
&i_s(\rho_s(ba^*)\mathbb{1}^s_k)^* i_r(\rho_r(cd^*)\mathbb{1}^r_m) \\
&\quad = \sum_{i=0}^{N_{s\vee r}-1} i_e(\langle \rho_s(ba^*)\mathbb{1}^s_k, \mathbb{1}^s_{i(s)} \rangle_s) i_{r'}(\mathbb{1}^{r'}_{i(r')}) i_{s'}(\mathbb{1}^{s'}_{i(s')})^* i_e(\langle \rho_r(cd^*)\mathbb{1}^r_m, \mathbb{1}^r_{i(r)} \rangle_r)^* \\
&\quad = \sum_{i=0}^{N_{s\vee r}-1} i_e(ab^* \langle \mathbb{1}^s_k, \mathbb{1}^s_{i(s)} \rangle_s) i_{r'}(\mathbb{1}^{r'}_{i(r')}) i_{s'}(\mathbb{1}^{s'}_{i(s')})^* i_e(dc^* \langle \mathbb{1}^r_m, \mathbb{1}^r_{i(r)} \rangle_r)^* \\
&\quad = i_e(ab^*) i_{r'}(\mathbb{1}^{r'}_{k'}) i_{s'}(\mathbb{1}^{s'}_{m'})^* i_e(dc^*)^* \text{ by Definition 3.5(3),}
\end{aligned}$$

where $k'$ is the unique element in $\{0, \dots, N_{r'} - 1\}$ and $m'$ the unique element in $\{0, \dots, N_{s'} - 1\}$ such that $k \cdot k' = m \cdot m'$ in $\{0, \dots, N_{s\vee r} - 1\}$. It follows that

$$y_1 y_2 = i_s(\mathbb{1}^s_j) i_e(ab^*) i_{r'}(\mathbb{1}^{r'}_{k'}) i_{s'}(\mathbb{1}^{s'}_{m'})^* i_e(dc^*)^* (i_r(\mathbb{1}^r_n))^*.$$

Now invoke Definition 3.5(2) to write

$$\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'} = \sum_{h=0}^{N_{r'}-1} \rho_{r'}(e_h)\mathbb{1}^{r'}_h$$

and

$$\varphi_{s'}(dc^*)\mathbb{1}^{s'}_{m'} = \sum_{i=0}^{N_{s'}-1} \rho_{s'}(f_i)\mathbb{1}^{s'}_i.$$

Then by regrouping terms in $y_1y_2$ we have

$$\begin{aligned}\phi(y_1y_2) &= \sum_{h=0}^{N_{r'}-1}\sum_{i=0}^{N_{s'}-1} \phi\big(i_{s\vee r}(\mathbb{1}^{s\vee r}_{j\cdot h})i_e(e_hf_i^*)i_{s\vee r}(\mathbb{1}^{s\vee r}_{n\cdot i})^*\big)\\ &= N(s\vee r)^{-\beta}\sum_{h=0}^{N_{r'}-1}\sum_{i=0}^{N_{s'}-1}\delta_{j\cdot h,n\cdot i}\phi(i_e(e_hf_i^*)) \text{ by (4.6)}\\ &= N(s\vee r)^{-\beta}\phi\big(i_e(\langle\mathbb{1}^{r'}_h,\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'}\rangle_{r'}\langle\mathbb{1}^{s'}_{m'},\varphi_{s'}(cd^*)\mathbb{1}^{s'}_i\rangle_{s'})\big), \qquad (4.9)\end{aligned}$$

where $h\in\{0,\dots,N_{r'}-1\}$ and $i\in\{0,\dots,N_{s'}-1\}$ are uniquely determined such that $\mathfrak{m}_{s,r'}(j,h)=\mathfrak{m}_{r,s'}(n,i)$ in $\{0,\dots,N_{s\vee r}-1\}$.

On the other hand, we can also invoke Definition 3.5(1) to write

$$\rho_s(ab^*)\mathbb{1}^s_j = \sum_{g=0}^{N_s-1}\varphi_s(u_g)\mathbb{1}^s_g \text{ and } \rho_r(dc^*)\mathbb{1}^r_n = \sum_{h=0}^{N_r-1}\varphi_r(w_h)\mathbb{1}^r_h.$$

Hence $y_1y_2=\sum_g\sum_h i_e(u_g)i_{s\vee r}(\mathbb{1}^{s\vee r}_{g\cdot k'})i_{s\vee r}(\mathbb{1}^{s\vee r}_{h\cdot m'})^*i_e(w_h)^*$. Since $\phi\circ i_e$ is a trace, $\phi$ satisfies (4.3), and so

$$\begin{aligned}\phi(y_1y_2) &= \sum_g\sum_h N(s\vee r)^{-\beta}\phi(i_e(\langle\varphi_{s\vee r}(w_h)\mathbb{1}^{s\vee r}_{h\cdot m'},\varphi_{s\vee r}(u_g)\mathbb{1}^{s\vee r}_{g\cdot k'}\rangle))\\ &= N(s\vee r)^{-\beta}\phi(i_e(\langle F^{r,s'}(\mathbb{1}^r_n\otimes_A\varphi_{s'}(dc^*)\mathbb{1}^{s'}_{m'}),F^{s,r'}(\mathbb{1}^s_j\otimes_A\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'})\rangle)). \qquad (4.10)\end{aligned}$$

Writing $\mathbb{1}^s_j=F^{s\wedge r,s'}(\mathbb{1}^{s\wedge r}_{j(s\wedge r)}\otimes\mathbb{1}^{s'}_{j(s')})$ and $\mathbb{1}^r_n=F^{s\wedge r,r'}(\mathbb{1}^{s\wedge r}_{n(s\wedge r)}\otimes\mathbb{1}^{r'}_{n(r')})$, using the associativity to decompose

$$F^{(s\wedge r)r',s'}(F^{s\wedge r,r'}\otimes_A I_{s'}) = F^{s\wedge r,r's'}(I_{s\wedge r}\otimes_A F^{r',s'}),$$

and using the definition of the inner product in $X_{s'r'}=X_{r's'}$, the term under $\phi$ in (4.10) is

$$\begin{aligned}&\phi(i_e(\langle F^{r,s'}(\mathbb{1}^r_n\otimes_A\varphi_{s'}(dc^*)\mathbb{1}^{s'}_{m'}),F^{s,r'}(\mathbb{1}^s_j\otimes_a\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'})\rangle))\\ &\quad=\phi\big(i_e(\langle F^{r',s'}(\mathbb{1}^{r'}_{n(r')}\otimes\varphi_{s'}(dc^*)\mathbb{1}^{s'}_{m'}),\varphi_{s'r'}(\langle\mathbb{1}^{s\wedge r}_{n(s\wedge r)},\mathbb{1}^{s\wedge r}_{j(s\wedge r)}\rangle)F^{s',r'}(\mathbb{1}^{s'}_{j(s')}\otimes\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'})\rangle)\big).\end{aligned}$$

Thus Definition 3.5(3) implies that $\phi(y_1y_2)=0$ unless the equality $n(s\wedge r)=j(s\wedge r)$ holds, in which case

$$\phi(y_1y_2)=N(s\vee r)^{-\beta}\phi\big(i_e(\langle F^{r',s'}(\mathbb{1}^{r'}_{n(r')}\otimes\varphi_{s'}(dc^*)\mathbb{1}^{s'}_{m'}),F^{s',r'}(\mathbb{1}^{s'}_{j(s')}\otimes\varphi_{r'}(ab^*)\mathbb{1}^{r'}_{k'})\rangle)\big). \qquad (4.11)$$

Similarly, if we let $n', g \in \{0, \dots, N_{s'} - 1\}$ and $j', l \in \{0, \dots, N_{r'} - 1\}$ be the uniquely determined elements such that $n \cdot n' = j \cdot j'$ and $m \cdot g = k \cdot l$ in $\{0, \dots, N_{s\vee r} - 1\}$, then by employing Definition 3.5(2) we have

$$\phi(y_2y_1) = N(s \vee r)^{-\beta}\phi\big(i_e(\langle \mathbb{1}_g^{s'}, \varphi_{s'}(cd^*)\mathbb{1}_{n'}^{s'}\rangle_{s'}\langle \mathbb{1}_{j'}^{r'}, \varphi_{r'}(ab^*)\mathbb{1}_l^{r'}\rangle_{r'})\big).$$

Since $\phi \circ i_e$ is a trace on $A$, we can rewrite this as

$$\phi(y_2y_1) = N(s \vee r)^{-\beta}\phi\big(i_e(\langle \mathbb{1}_{j'}^{r'}, \varphi_{r'}(ab^*)\mathbb{1}_l^{r'}\rangle_{r'}\langle \mathbb{1}_g^{s'}, \varphi_{s'}(cd^*)\mathbb{1}_{n'}^{s'}\rangle_{s'})\big) \tag{4.12}$$

On the other hand, by employing Definition 3.5(1) we obtain

$$\phi(y_2y_1) = N(s \vee r)^{-\beta}\phi\big(i_e(\langle F^{s',r'}(\mathbb{1}_{k(s')}^{s'} \otimes \varphi_{r'}(dc^*)\mathbb{1}_{j'}^{r'}), F^{r',s'}(\mathbb{1}_{m(r')}^{r'} \otimes \varphi_{s'}(ab^*)\mathbb{1}_{n'}^{s'})\rangle)\big)$$

if $k(s \wedge r) = m(s \wedge r)$, and $\phi(y_2y_1) = 0$ otherwise.

**Case 1.** $k(s\wedge r) = m(s\wedge r)$ and $n(s\wedge r) = j(s\wedge r)$. Then the equalities $k\cdot k' = m\cdot m'$ and $j \cdot j' = n \cdot n'$ rewrite as

$$\begin{aligned} k(s \wedge r) \cdot k(s') \cdot k' &= m(s \wedge r) \cdot m(r') \cdot m' \text{ and} \\ j(s \wedge r) \cdot j(s') \cdot j' &= n(s \wedge r) \cdot n(r') \cdot n', \end{aligned}$$

and so by uniqueness of decomposition in $\{0, \dots, N_{s\wedge r} - 1\} \times \{0, \dots, N_{s'r'} - 1\}$ imply that

$$\begin{aligned} \mathfrak{m}_{s',r'}(k(s'), k') &= \mathfrak{m}_{r',s'}(m(r'), m') \text{ and} \\ \mathfrak{m}_{s',r'}(j(s'), j') &= \mathfrak{m}_{r',s'}(n(r'), n'). \end{aligned}$$

For the same reason, $k \cdot l = m \cdot g$ and $j \cdot h = n \cdot i$ imply that

$$\begin{aligned} \mathfrak{m}_{s',r'}(k(s'), l) &= \mathfrak{m}_{r',s'}(m(r'), g) \text{ and} \\ \mathfrak{m}_{s',r'}(j(s'), h) &= \mathfrak{m}_{r',s'}(n(r'), i) \end{aligned}$$

Since $s' \wedge r' = e$, the assumption that $\mathfrak{m}_{s',r'}$ respects co-prime pairs implies that

$$k' = l,\ m' = g,\ j' = h, \text{ and } n' = i. \tag{4.13}$$

Hence, by (4.9) and (4.12), the expressions for $\phi(y_1y_2)$ and $\phi(y_2y_1)$ become

$$\begin{aligned} \phi(y_1y_2) &= N(s \vee r)^{-\beta}\phi\big(i_e(\langle \mathbb{1}_{j'}^{r'}, \varphi_{r'}(ab^*)\mathbb{1}_{k'}^{r'}\rangle_{r'}\langle \mathbb{1}_{m'}^{s'}, \varphi_{s'}(cd^*)\mathbb{1}_{n'}^{s'}\rangle_{s'})\big), \\ \phi(y_2y_1)) &= N(s \vee r)^{-\beta}\phi\big(i_e(\langle \mathbb{1}_{j'}^{r'}, \varphi_{r'}(ab^*)\mathbb{1}_{k'}^{r'}\rangle_{r'}\langle \mathbb{1}_{m'}^{s'}, \varphi_{s'}(cd^*)\mathbb{1}_{n'}^{s'}\rangle_{s'})\big), \end{aligned}$$

and so establish that $\phi(y_1y_2) = \phi(y_2y_1)$ in this case.

**Case 2.** $m(s\wedge r) \neq k(s\wedge r)$. In this case we already saw that $\phi(y_2y_1) = 0$. However,

$$\begin{aligned} i_s(\mathbb{1}_k^s)^* i_r(\mathbb{1}_m^r) &= i_{s'}(\mathbb{1}_{k(s')}^{s'})^* i_e(\langle \mathbb{1}_{k(s\wedge r)}^{s\wedge r}, \mathbb{1}_{m(s\wedge r)}^{s\wedge r}\rangle) i_{r'}(\mathbb{1}_{m(r')}^{r'}) \\ &= \delta_{k(s\wedge r),m(s\wedge r)} i_{s'}(\mathbb{1}_{k(s')}^{s'})^* i_{r'}(\mathbb{1}_{m(r')}^{r'}) = 0, \end{aligned}$$

by Definition 3.5(3). Hence $y_1y_2 = 0$ and $\phi$ has the trace property.

**Case 3.** $j(s \wedge r) \neq n(s \wedge r)$. Similarly to case 2, we have $\phi(y_1y_2) = 0$ by previous consideration, and $y_2y_1 = 0$ by Definition 3.5(3), so again $\phi(y_1y_2) = \phi(y_2y_1)$.

**Case 3.** Finally, if $m(s\wedge r) \neq k(s\wedge r)$ and $j(s\wedge r) \neq n(s\wedge r)$ then $y_1y_2 = 0 = y_2y_1$. □

Note that if $\phi \circ i_e$ is injective, then (4.9) and (4.11) show that the product system $X$ must satisfy the condition

$$\langle F^{p,q}(\mathbb{1}_j^p \otimes_A \varphi_q(a)\mathbb{1}_m^q), F^{q,p}(\mathbb{1}_n^q \otimes_A \varphi_p(b)\mathbb{1}_k^p)\rangle_{pq} = \langle \varphi_q(a)\mathbb{1}_m^q, \mathbb{1}_n^q\rangle_q \langle \mathbb{1}_j^p, \varphi_p(b)\mathbb{1}_k^p\rangle_p \tag{4.14}$$

for all $a, b \in A$, all $p, q \in P$ such that $p \wedge q = e$ and all $j, k = 0, \dots, N_p - 1$, $m, n = 0, \dots, N_q - 1$.

4.2. **Ground states and $\mathbf{KMS}_\beta$ states of $\mathcal{NT}(X)$ induced from states of $A$.** We begin this section by recalling the construction of the induced representation via a right Hilbert module. We refer to [20] for details. Let $Y$ be a right Hilbert $A$-module and assume that $\varphi : B \to \mathcal{L}(Y)$ is a $*$-homomorphism. Suppose that $\pi : A \to B(H_\pi)$ is a representation. The balanced tensor product space $Y \otimes_A H_\pi$ is a Hilbert space where the inner-product is characterised by

$$\langle \xi \otimes_A h, \eta \otimes_A k\rangle = (\pi(\langle \eta, \xi\rangle)h \mid k)$$

for $\xi, \eta \in Y$ and $h, k \in H_\pi$. The induced representation $\operatorname{Ind}\pi$ of $B$ on $Y \otimes_A H_\pi$ acts by

$$\operatorname{Ind}\pi(b)(\xi \otimes_A h) = (\varphi(b)\xi) \otimes_A h. \tag{4.15}$$

We apply this construction to the Fock module $F(X)$ associated to a product system $X$ over $P$ of right Hilbert $A$–$A$-bimodules, see section 2.3. For compactly aligned $X$, the Fock representation $l$ of $X$ in $\mathcal{L}(F(X))$ gives rise to a $*$-homomorphism $l_* : \mathcal{NT}(X) \to \mathcal{L}(F(X))$.

**Remark 4.8.** Since the left action has image in $\mathcal{K}(X_s)$ for every $s \in P$, [8, Theorem 6.3] says that $\mathcal{NT}(X)$ is isomorphic to a certain crossed-product $B_P \rtimes_{\tau,X} P$ (the proof there uses that every $X_s$ is essential, but applies in our setting due to Definition 3.5(1)). Then the remark following [8, Definition 7.1] indicates that $B_P \rtimes_{\tau,X} P$, which is a universal crossed product for an action of $P$ on $B_P$ twisted by $X$, is isomorphic to the associated reduced crossed product. We infer from this that $l_*$ is faithful.

Given a state $\tau$ on $A$, let $(\pi_\tau, h_\tau, H_\tau)$ be the corresponding GNS-representation. Denote $\mathbb{1} = \mathbb{1}_e \oplus \bigoplus_{s \neq e} 0_s$ in $F(X)$. Consider the representation

$$\operatorname{Ind}\pi_\tau : \mathcal{NT}(X) \to B(F(X) \otimes_A H_\tau)),$$

and let $\tilde{\omega}_\tau(y) = (\operatorname{Ind}\pi_\tau(y)(\mathbb{1} \otimes h_\tau), \mathbb{1} \otimes h_\tau)$ be the state of $\mathcal{NT}(X)$ arising from this representation. We claim that

$$\tilde{\omega}_\tau(y) = \begin{cases} 0 & \text{unless } s = r = e \\ \tau(ab^*) & \text{if } s = r = e \end{cases} \tag{4.16}$$

for $y = i_e(a)i_s(\mathbb{1}_k^s)i_r(\mathbb{1}_m^r)^* i_e(b)^* \in \mathcal{NT}(X)$. By (4.15) we have $\tilde{\omega}_\tau(y) = \langle (l_*(y)\mathbb{1}) \otimes h_\tau, \mathbb{1} \otimes h_\tau\rangle$. The characterization of $l_*$ shows that $l_r(\varphi_r(b)\mathbb{1}_m^r)^*\mathbb{1} = 0_r$ unless $r = e$, which in turn implies $l_*(y)\mathbb{1} = 0$ when $r \neq e$. Assuming $r = e$, we see next that $l_*(y)\mathbb{1}$, as an element in $F(X)$, has a non-zero coordinate only at $s$, where it equals $\rho_s(b^*)(\varphi_s(a)\mathbb{1}_k^s)$. To compute further in $\tilde{\omega}_\tau(y)$, note that the characterization of the inner-product on $F(X) \otimes_A H_\tau$ involves computing the inner-product in $F(X)$ given by $\langle \mathbb{1}, l_*(y)\mathbb{1}\rangle$. By the definition of the inner-product on $F(X)$ it follows that a non-zero contribution in $\langle \mathbb{1}, l_*(y)\mathbb{1}\rangle$ is only possible at $s = e$, where it equals $\langle \mathbb{1}_e, i_e(ab^*)\mathbb{1}_e\rangle_e$. In other words, we have established that $r \neq e$ or $s \neq e$ imply $\tilde{\omega}_\tau(y) = 0$, while for $s = r = e$ we obtain

$\tilde{\omega}_\tau(y) = \big(\pi_\tau(\mathbb{1}_e, i_e(ab^*)\mathbb{1}_e)h_\tau, h_\tau\big)$, which means $\tilde{\omega}_\tau(y) = \tau(ab^*)$, as required in (4.16). Thus, in connection with Theorem 3.4, we have the following result.

**Proposition 4.9.** *For each state $\tau$ of $A$, the induced state $\tilde{\omega}_\tau$ is a ground state of $\mathcal{NT}(X)$. Moreover, the assignment $\tau \mapsto \tilde{\omega}_\tau$ is an affine isomorphism.*

Next we investigate if a similar construction can induce $\mathrm{KMS}_\beta$ states of $\mathcal{NT}(X)$. For each $s \in P$, let $\bar{\mathbb{1}}^s_j$ be the vector in $F(X)$ with component equal to $\mathbb{1}^s_j$ at $s$ and $0_r$ for $r \neq s$. Note that $l_*(i_s(\mathbb{1}^s_j))\mathbb{1} = l_s(\mathbb{1}^s_j)\mathbb{1} = \bar{\mathbb{1}}^s_j$ for every $s \in P$. We introduce next a condition which is modeled on the reconstruction formula from [14, §10]. Suppose $\beta > 0$ is such that the series

$$\zeta_N(\beta) := \sum_{s\in P} N(s)^{-\beta} N_s \tag{4.17}$$

is convergent and suppose $\phi$ is a $\mathrm{KMS}_\beta$ state of $\mathcal{NT}(X)$. We say $\phi$ satisfies *the reconstruction formula* provided that

$$\phi(y) = \frac{1}{\zeta_N(\beta)} \sum_{s\in P} N(s)^{-\beta} \sum_{j=0}^{N_s-1} \phi(i_s(\mathbb{1}^s_j)^* y i_s(\mathbb{1}^s_j)) \tag{4.18}$$

for all $y \in \mathcal{F}$.

**Theorem 4.10.** *Let $(G, P)$ be lattice ordered and let $X$ be an associative product system of finite type over $P$ such that $\mathfrak{m}_{s,r}$ are bijective and preserve co-prime pairs, for all $s, r \in P$. Assume that the series in* (4.17) *is convergent in an interval $(\beta_c, \infty)$ for some $\beta_c > 0$.*

*Let $\beta > \beta_c$. Then for $\tau$ a trace of $A$ there is a state $\omega_\tau : \mathcal{F} \to \mathbb{C}$ given by*

$$\omega_\tau(y) = \sum_{\{s\in P: r\leq s\}} \frac{N(s)^{-\beta}}{\zeta_N(\beta)} \sum_{j=0}^{N_s-1} \tau(\langle \varphi_{r^{-1}s}(\langle \xi, \mathbb{1}^r_{j(r)}\rangle)\mathbb{1}^{r^{-1}s}_{j'}, \varphi_{r^{-1}s}(\langle \eta, \mathbb{1}^r_{j(r)}\rangle)\mathbb{1}^{r^{-1}s}_{j'}\rangle), \tag{4.19}$$

*for $y = i^{(r)}(\theta_{\xi,\eta}) \in B_r$, where for each $j$ in the summation we denote $j' = j(r^{-1}s)$. Further, the assignment $\tau \mapsto \omega_\tau \circ \Phi^\delta$ is an affine homeomorphism of the tracial states of $A$ onto a subset of $KMS_\beta$ states of $\mathcal{NT}(X)$.*

*If every $KMS_\beta$ state satisfies the reconstruction formula* (4.18)*, then the assignment $\tau \mapsto \omega_\tau \circ \Phi^\delta$ is surjective.*

To argue that (4.19) defines a state of $\mathcal{F}$ for every trace $\tau$ of $A$, we define alternatively a map $\omega_\tau$ on $\mathcal{F}$ by

$$\omega_\tau(y) = \frac{1}{\zeta_N(\beta)} \sum_{s\in P} N(s)^{-\beta} \sum_{j=0}^{N_s-1} \langle \mathrm{Ind}\, \pi_\tau(y)(l_s(\mathbb{1}^s_j)\mathbb{1} \otimes h_\tau),\ (l_s(\mathbb{1}^s_j)\mathbb{1} \otimes h_\tau)\rangle \tag{4.20}$$

for $y \in B_r$, $r \in P$. As in the proof of [11, Theorem 20], $\omega_\tau$ is an absolutely convergent infinite linear combination of vector states in $\mathrm{Ind}\, \pi_\tau$, so it is absolutely continuous with respect to $\tilde{\omega}_\tau$. Since the vectors $\{l_*(i_s(\mathbb{1}^s_j))\mathbb{1} \otimes h_\tau\}_{s\in P}$ form a generating set for $\mathrm{Ind}\, \pi_\tau$, also $\tilde{\omega}_\tau$ is absolutely continuous with respect to $\omega_\tau$.

By definition of the Fock representation, $l_*(y)\bar{\mathbb{1}}^s_j = 0$ when $r \not\leq s$. Assume therefore $s \in rP$. Since $l_s(\mathbb{1}^s_j)\mathbb{1} = \bar{\mathbb{1}}^s_j$, the summand in $s$ and $j$ in the right-hand side of (4.20) is

$$\begin{aligned}\langle \operatorname{Ind}\pi_\tau(y)(l_s(\mathbb{1}^s_j)\mathbb{1}\otimes h_\tau),\ (l_s(\mathbb{1}^s_j)\mathbb{1})\otimes h_\tau\rangle &= \langle \operatorname{Ind}\pi_\tau(y)(\bar{\mathbb{1}}^s_j\otimes h_\tau),\ \bar{\mathbb{1}}^s_j\otimes h_\tau\rangle\\ &= \langle (l_*(y)\bar{\mathbb{1}}^s_j)\otimes h_\tau,\ \bar{\mathbb{1}}^s_j\otimes h_\tau\rangle\\ &= \big(\pi_\tau(\langle \bar{\mathbb{1}}^s_j, l_*(y)\bar{\mathbb{1}}^s_j\rangle_{F(X)})h_\tau \mid h_\tau\big)\\ &= \tau(\langle \mathbb{1}^s_j, i^s_r(\theta_{\xi,\eta})\mathbb{1}^s_j\rangle_s).\end{aligned}$$

At this stage we put $j' = j(r^{-1}s)$, so that $j$ is given uniquely by $\mathfrak{m}_{r,r^{-1}s}(j(r),j') = j$, we decompose $\mathbb{1}^s_j = F^{r,r^{-1}s}(\mathbb{1}^r_{j(r)}\otimes \mathbb{1}^{r^{-1}s}_{j'})$, and we use the properties of the balanced inner-product on $X_r\otimes_A X_{r^{-1}s}$ to write further

$$\begin{aligned}&\langle \operatorname{Ind}\pi_\tau(y)(l(\mathbb{1}^s_j)\mathbb{1}\otimes h_\tau),\ (l(\mathbb{1}^s_j)\mathbb{1})\otimes h_\tau\rangle\\ &\qquad= \tau(\langle \mathbb{1}^{r^{-1}s}_{j'}, \varphi_{r^{-1}s}(\langle \mathbb{1}^r_{j(r)},\xi\rangle\langle \eta,\mathbb{1}^r_{j(r)}\rangle)\mathbb{1}^{r^{-1}s}_{j'}\rangle)\\ &\qquad= \tau(\langle \varphi_{r^{-1}s}(\langle \xi,\mathbb{1}^r_{j(r)}\rangle)\mathbb{1}^{r^{-1}s}_{j'}, \varphi_{r^{-1}s}(\langle \eta,\mathbb{1}^r_{j(r)}\rangle)\mathbb{1}^{r^{-1}s}_{j'}\rangle).\end{aligned}$$

The last term is the summand under $s$ and $j$ in (4.19). The functional $\omega_\tau$ is a state because

$$\begin{aligned}\omega_\tau(1) &= \sum_{s\in P}\frac{N(s)^{-\beta}}{\zeta_N(\beta)}\sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}^s_{j(s)},\mathbb{1}^s_{j(s)}\rangle)\\ &= \sum_{s\in P}\frac{N(s)^{-\beta}}{\zeta_N(\beta)}\sum_{j=0}^{N_s-1}\tau(1)\\ &= (\zeta_N(\beta))^{-1}\sum_{s\in P}N(s)^{-\beta}N_s = 1.\end{aligned}$$

To prove Theorem 4.10 we will employ Theorem 4.6 to show that $\omega_\tau$ given by (4.19) is a trace of $\mathcal{F}$. In the next lemmas we verify that the assumptions of Theorem 4.6 are fulfilled by $\omega_\tau$.

**Lemma 4.11.** *The map $\omega_\tau$ given by* (4.19) *is a trace on $B_e = i_e(A)$ whenever $\tau$ is a trace on $A$.*

*Proof.* Assume $\tau$ is a trace on $A$. Let $c,d\in A$. Then (4.19) implies

$$\omega_\tau(i_e(a)) = \frac{1}{\zeta_N(\beta)}\sum_{s\in P}N(s)^{-\beta}\sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}^s_j,\varphi_s(a)\mathbb{1}^s_j\rangle) \tag{4.21}$$

for each $a\in A$. Thus to prove that $\omega_\tau(i_e(cd)) = \omega_\tau(i_e(dc))$ it suffices to show that

$$\sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}^s_j,\varphi_s(cd)\mathbb{1}^s_j\rangle) = \sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}^s_j,\varphi_s(dc)\mathbb{1}^s_j\rangle) \tag{4.22}$$

for $c,d\in A$. Using Definition 3.5(2) we can write, for every $j,n = 0,\dots,N_s-1$,

$$\varphi_s(d)\mathbb{1}^s_j = \sum_{n=0}^{N_s-1}\rho_s(d_{n,j})\mathbb{1}^s_n \text{ and } \varphi_s(c)\mathbb{1}^s_n = \sum_{m=0}^{N_s-1}\rho_s(c_{m,n})\mathbb{1}^s_m.$$

Then the left-hand side of (4.22) can be written as follows

$$\begin{aligned}\sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \varphi_s(cd)\mathbb{1}_j^s\rangle) &= \sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \varphi_s(c)(\sum_{n=0}^{N_s-1}\rho_s(d_{n,j})\mathbb{1}_n^s)\rangle)\\ &= \sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \sum_{n=0}^{N_s-1}\sum_{m=0}^{N_s-1}\rho_s(c_{m,n}d_{n,j})\mathbb{1}_m^s\rangle))\\ &= \sum_{j=0}^{N_s-1}\sum_{n=0}^{N_s-1}\tau(c_{j,n}d_{n,j}) \text{ by Definition 3.5(3)}\\ &= \sum_{j=0}^{N_s-1}\sum_{n=0}^{N_s-1}\tau(d_{n,j}c_{j,n}) \text{ since } \tau \text{ is a trace.} \qquad (4.23)\end{aligned}$$

Similarly, the right-hand side of (4.22) is

$$\begin{aligned}\sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \varphi_s(dc)\mathbb{1}_j^s\rangle) &= \sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \varphi_s(d)(\sum_{m=0}^{N_s-1}\rho_s(c_{m,j})\mathbb{1}_m^s)\rangle)\\ &= \sum_{j=0}^{N_s-1}\tau(\langle \mathbb{1}_j^s, \sum_{m=0}^{N_s-1}\sum_{n=0}^{N_s-1}\rho_s(d_{n,m}c_{m,j})\mathbb{1}_n^s\rangle))\\ &= \sum_{j=0}^{N_s-1}\sum_{m=0}^{N_s-1}\tau(d_{j,m}c_{m,j}) \text{ by Definition 3.5(3);} \qquad (4.24)\end{aligned}$$

thus (4.23) and (4.24) show that $\omega_\tau$ is a trace on $i_e(A)$.

□

**Lemma 4.12.** *The state $\omega_\tau$ given by* (4.19) *satisfies* (4.6).

*Proof.* Let $a \in A$ and $n, m \in \{0, \dots, N_r - 1\}$. When we compute $\omega_\tau(i_r(\mathbb{1}_n^r)i_e(a)i_r(\mathbb{1}_m^r)^*)$ using (4.19), we have $\xi = \mathbb{1}_n^r$ and $\eta = \rho_r(a^*)\mathbb{1}_m^r$, so in the summation over $j = 0, \dots, N_s - 1$ for every $s \geq r$ the terms $\langle \mathbb{1}_n^r, \mathbb{1}_{j(r)}^r\rangle$ give zero contribution unless $j(r) = n$, that is unless $j = n \cdot j'$ for $j' = 0, \dots, N_{r^{-1}s} - 1$. Thus the summation over $j$ is simply a summation over $j'$. Moreover, since $\langle \rho_r(a^*)\mathbb{1}_m^r, \mathbb{1}_{l(r)}^r\rangle = a\langle \mathbb{1}_m^r, \mathbb{1}_n^r\rangle$, we also get a zero contribution unless $n = m$. In other words, the left-hand side of (4.6) is

$$\omega_\tau(i_r(\mathbb{1}_n^r)i_e(a)i_r(\mathbb{1}_m^r)^*) = \delta_{m,n}\sum_{\{s\in P: r\leq s\}}\frac{N(s)^{-\beta}}{\zeta_N(\beta)}\sum_{j'=0}^{N_{r^{-1}s-1}}\tau(\langle \mathbb{1}_{j'}^{r^{-1}s}, \varphi_{r^{-1}s}(a)\mathbb{1}_{j'}^{r^{-1}s}\rangle). \qquad (4.25)$$

Now by applying (4.21) we can rewrite the right-hand side of (4.6) as follows

$$\begin{aligned}\delta_{n,m}N(r)^{-\beta}\omega_\tau(i_e(a)) &= \delta_{n,m}\frac{1}{\zeta_N(\beta)}\sum_{q\in P}N(r)^{-\beta}N(q)^{-\beta}\sum_{l=0}^{N_q-1}\tau(\langle \mathbb{1}^q_l, \varphi_q(a)\mathbb{1}^q_l\rangle)\\ &= \delta_{n,m}\frac{1}{\zeta_N(\beta)}\sum_{q\in P}N(rq)^{-\beta}\sum_{l=0}^{N_q-1}\tau(\langle \mathbb{1}^q_l, \varphi_q(a)\mathbb{1}^q_l\rangle)\\ &= \delta_{n,m}\frac{1}{\zeta_N(\beta)}\sum_{\{s\in P: r\le s\}}N(s)^{-\beta}\sum_{l=0}^{N_{r^{-1}s}-1}\tau(\langle \mathbb{1}^{r^{-1}s}_l, \varphi_{r^{-1}s}(a)\mathbb{1}^{r^{-1}s}_l\rangle);\end{aligned}$$

comparing this last term with (4.25) proves the claimed scaling identity. □

*Proof of Theorem 4.10.* We verified most of the claims. For every $\beta > \beta_c$, the assignment $\tau \mapsto \omega_\tau \circ \Phi^\delta$ is continuous between compact Hausdorff spaces and has an injective inverse by Lemma 4.1. It also respects convex linear combinations and weak*-limits. If $\tau$ is a tracial state of $A$, then $\omega_\tau$ is a trace on $\mathcal{F}$ by Theorem 4.6, which applies due to the previous two lemmas. Hence $\omega_\tau \circ \Phi^\delta$ is a $\mathrm{KMS}_\beta$ state.

Conversely, suppose $\phi$ is a $\mathrm{KMS}_\beta$ state. Then the restriction $\phi_0$ to $\mathcal{F}$ is a trace satisfying (4.6). Let $\tau = \phi_0|_{i_e(A)}$. We need to show that $\omega_\tau = \phi$. Let $y = i_r(\xi)i_r(\eta)^* \in B_r$ for some $r \in P$. By (4.7) we have

$$i_s(\mathbb{1}^s_j)^* i_r(\xi) = \sum_{i=0}^{N_{s\vee r}-1} i_e(\langle \mathbb{1}^s_j, \mathbb{1}^s_{i(s)}\rangle_s) i_{r'}(\mathbb{1}^{r'}_{i(r')}) i_{s'}(\mathbb{1}^{s'}_{i(s')})^* i_e(\langle \xi, \mathbb{1}^r_{i(r)}\rangle_r)^*$$

and

$$i_r(\eta)^* i_s(\mathbb{1}^s_j) = \sum_{l=0}^{N_{s\vee r}-1} i_e(\langle \eta, \mathbb{1}^r_{l(r)}\rangle_r) i_{s'}(\mathbb{1}^{s'}_{l(s')}) i_{r'}(\mathbb{1}^{r'}_{l(r')})^* i_e(\langle \mathbb{1}^s_j, \mathbb{1}^s_{l(s)}\rangle_s)^*.$$

Hence, with $T := \langle \xi, \mathbb{1}^r_{i(r)}\rangle^* \langle \eta, \mathbb{1}^r_{l(r)}\rangle$ we have

$$i_s(\mathbb{1}^s_j)^* y i_s(\mathbb{1}^s_j) = \sum_{\substack{i=0\\ i(s)=j}}^{N_{s\vee r}-1}\sum_{\substack{l=0\\ l(s)=j}}^{N_{s\vee r}-1} i_{r'}(\mathbb{1}^{r'}_{i(r')}) i_e\left(\langle \mathbb{1}^{s'}_{i(s')}, \varphi_{s'}(T)\mathbb{1}^{s'}_{l(s')}\rangle\right) i_{r'}(\mathbb{1}^{r'}_{l(r')})^*.$$

Now we apply $\phi$ and use (4.18). It follows that

$$\phi(y) = \frac{1}{\zeta_N(\beta)}\sum_{s\in P}N(s)^{-\beta}\sum_{j=0}^{N_s-1}\sum_{\substack{i,l=0\\ i(s)=j=l(s)}}^{N_{s\vee r}-1}\delta_{i(r'),l(r')}N(r')^{-\beta}\phi(i_e(\langle \mathbb{1}^{s'}_{i(s')}, \varphi_{s'}(T)\mathbb{1}^{s'}_{l(s')}\rangle)).$$

We note that $i(s) = j = l(s)$ and $i(r') = l(r')$ imply that $i = i(s)\cdot i(r') = l(s)\cdot l(r') = l$, so the double sum over $i$ and $l$ collapses to a single sum. In particular we have $i(r) = l(r)$ and $i(s') = l(s')$. But letting $s$ run over elements in $P$ is the same as letting $q := s \vee r$ run over elements in $P$ such that $r \le q$, and in this case $s'$ is replaced by $r^{-1}q$ and $r'$ by $s^{-1}q$. Then by performing the change of summation index $s \vee r \mapsto q$ implies that

$N(s)^{-\beta}N(r')^{-\beta}$ gets replaced by $N(q)^{-\beta}$. This now gives

$$\phi(y) = \frac{1}{\zeta_N(\beta)} \sum_{q\in P, r\le q} N(q)^{-\beta} \sum_{j=0}^{N_q-1} \tau\left(\langle \varphi_{r^{-1}q}(\langle \xi, \mathbb{1}^r_{h(r)}\rangle)\mathbb{1}^{r^{-1}q}_{h'}, \varphi_{r^{-1}q}(\langle \eta, \mathbb{1}^r_{h(r)}\rangle)\mathbb{1}^{r^{-1}q}_{h'}\rangle\right)$$

for $h = h(r)\cdot h'$, which is the same as $\omega_\tau(y)$ given by (4.19).

This finishes the proof of surjectivity of the map $\tau \mapsto \omega_\tau$. □

**Proposition 4.13.** *Suppose that $A$ contains a proper isometry $a$. Let $X$ be a compactly aligned product system of finite type over $P$ of right-Hilbert $A$–$A$-bimodules such that $\mathfrak{m}_{s,r}$ is bijective for all $s, r \in P$. Let $N$ be a homomorphism $G \to (0,\infty)$ such that $N(s) = N_s$ for all $s \in P$. Then there are no KMS$_\beta$ states of $(\mathcal{NT}(X), \sigma_N)$ for $\beta < 1$.*

*Proof.* Suppose $\phi$ is a state of $\mathcal{NT}(X)$ which satisfies the KMS$_\beta$ condition for some $\beta > 0$. Then

$$\phi(i_s(\varphi_s(a)\mathbb{1}^s_j)i_s(\varphi_s(a)\mathbb{1}^s_k)^*) = N(s)^{-\beta}\phi(i_s(\mathbb{1}^s_k)^* i_e(a^*a) i_s(\mathbb{1}^s_j)) = \delta_{j,k}N(s)^{-\beta}.$$

Since $\sum_{j=0}^{N_s-1} i_s(\varphi_s(a)\mathbb{1}^s_j)i_s(\varphi_s(a)\mathbb{1}^s_j)^*$ is a projection, we have

$$1 = \phi(1) \ge \phi(\sum_{j=0}^{N_s-1} i_s(\varphi_s(a)\mathbb{1}^s_j)i_s(\varphi_s(a)\mathbb{1}^s_j)^*) = \sum_{j=0}^{N_s-1} N(s)^{-\beta},$$

which is $N_s^{1-\beta}$. Then necessarily $\beta \ge 1$. □

**Remark 4.14.** We note that for a fixed tracial state $\tau$ of $A$, there is a KMS$_\infty$ state $\omega_\infty$ of $\mathcal{NT}(X)$ obtained, for $\beta \to \infty$ in (4.19), as the weak*-limit of the KMS$_\beta$ states $\omega_\tau$.

## 5. Structure of the core $\mathcal{F}$

In this section we identify an action of $P$ by endomorphisms of the core $\mathcal{F}$ and a commutative $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{F}$.

**Proposition 5.1.** *Let $(G, P)$ be lattice ordered and $X$ a product system over $P$ of finite type such that the maps $\mathfrak{m}_{s,r}$ are bijective for all $s, r \in P$. The assignment*

$$\alpha_s(y) = \sum_{j=0}^{N_s-1} i_s(\mathbb{1}^s_j) y i_s(\mathbb{1}^s_j)^* \tag{5.1}$$

*for $s \in P$ and $y \in \mathcal{F}$ defines an action $\alpha$ of $P$ by injective endomorphisms of $\mathcal{F}$.*

*Proof.* Clearly $\alpha_s : \mathcal{F} \to \mathcal{F}$ is well-defined, and $\alpha_s(y_1)\alpha_s(y_2) = \alpha_s(y_1y_2)$ for all $s \in P$ and $y_1, y_2 \in \mathcal{F}$ follows by Definition 3.5(3). If $\alpha_s(y) = 0$ then $0 = i_s(\mathbb{1}^s_0)^*\alpha_s(y)i_s(\mathbb{1}^s_0) = y$, so

each $\alpha_s$ is injective. Let $s, q \in P$. Then

$$\begin{aligned}\alpha_s\alpha_q(y) &= \alpha_s\big(\sum_{j=0}^{N_s-1} i_q(\mathbb{1}_l^q) y i_q(\mathbb{1}_l^q)^*\big)\\ &= \sum_{j=0}^{N_s-1}\sum_{l=0}^{N_q-1} i_{sq}(\mathbb{1}_{j\cdot l}^{sq}) y (i_{sq}(\mathbb{1}_{j\cdot l}^{sq})^*)\\ &= \sum_{k=0}^{N_{sq}-1} i_{sq}(\mathbb{1}_k^{sq}) y (i_{sq}(\mathbb{1}_k^{sq})^* = \alpha_{sq}(y),\end{aligned}$$

which proves that $\alpha$ is an action of $P$ by endomorphisms of $\mathcal{F}$. □

We note that in [10] similar constructions in the case of a single Hilbert bimodule and at the level of relative Cuntz-Pimsner algebras (modeling Exel crossed products) are obtained.

**Corollary 5.2.** *The maps $\alpha_r^q : B_r \to B_q$ given by $\alpha_r^q := \alpha_{r^{-1}q}$ for $r \le q$ give rise to a direct limit $\varinjlim_{r\in P}(B_r, \alpha_r^q)_{r\le q}$ with injective homomorphisms. The canonical embeddings $\alpha^r$ of $B_r$ into $\varinjlim_{r\in P} B_r$ give rise to an increasing union such that*

$$\mathcal{F} = \overline{\bigcup_{r\in P} \alpha^r(B_r)}.$$

*Proof.* Definition 3.5(2) implies that $\alpha_s(B_r) \subseteq B_{rs}$ for all $r, s \in P$, so the maps $\alpha_r^q$ are well-defined from $B_r$ to $B_q$ for all $r \le q$. The fact that $\alpha$ is an action of $P$ implies that $\alpha_r^s = \alpha_s^q \circ \alpha_r^q$ when $r \le q \le s$, so the maps are compatible and give therefore rise to a direct system, as claimed. The inclusion maps $B_r \hookrightarrow \mathcal{F}$ for $r \in P$ are compatible with the bonding maps $\alpha_r^q$, and combine to give an injective homomorphism from $\varinjlim_{r\in P} B_r$ into $\mathcal{F}$, which is also surjective. □

**Proposition 5.3.** *Let $\alpha_s(1) = \sum_{j=0}^{N_s-1} i_s(\mathbb{1}_j^s) i_s(\mathbb{1}_j^s)^*$ for every $s \in P$. Then $\alpha_s(1)\alpha_r(1) = \alpha_{s\vee r}(1)$ for all $s, r \in P$. In particular, $\mathcal{A} := \overline{\operatorname{span}}\{\alpha_r(1) : r \in P\}$ is a commutative $C^*$-subalgebra of $\mathcal{F}$.*

*Proof.* Let $\alpha_s(1) = \sum_{j=0}^{N_s-1} i_s(\mathbb{1}_j^s) i_s(\mathbb{1}_j^s)^*$ and $\alpha_r(1) = \sum_{l=0}^{N_r-1} i_r(\mathbb{1}_l^r) i_r(\mathbb{1}_l^r)^*$ for $s, r \in P$. By (4.7), $i_s(\mathbb{1}_j^s)^* i_r(\mathbb{1}_l^r)$ is a sum of terms indexed over $i = 0, \dots, N_{s\vee r} - 1$ where non-zero terms occur when $i(s) = j$ and $i(r) = l$ simultaneously. Thus

$$\alpha_s(1)\alpha_r(1) = \sum_{j=0}^{N_s-1}\sum_{l=0}^{N_r-1} i_{s\vee r}(\mathbb{1}_{j\cdot j'}^{s\vee r}) i_{s\vee r}(\mathbb{1}_{l'\cdot l}^{s\vee r})^*,$$

where for every $l$ and $j$ the elements $l' = 0, \dots, N_{r^{-1}(s\vee r)} - 1$ and $j' = 0, \dots, N_{s^{-1}(s\vee r)} - 1$ are such that $l' \cdot l = j \cdot j'$ in $\{0, \dots, N_{s\vee r} - 1\}$. Therefore $\alpha_s(1)\alpha_r(1) = \alpha_{s\vee r}(1)$, as claimed. □

**Remark 5.4.** Note that any $\mathrm{KMS}_\beta$ state $\omega_\tau$ given by (4.19) at $\beta > \beta_c$ satisfies

$$\omega_\tau(i_r(\mathbb{1}_n^r) i_r(\mathbb{1}_m^r)^*) = \begin{cases} 0 & \text{if } m \ne n \\ N(r)^{-\beta} & \text{if } n = m. \end{cases} \tag{5.2}$$

Hence $\omega_\tau(\alpha_r(1)) = N_r N(r)^{-\beta}$ for all $r$. If $N(r) = N_r$ for all $r \in P$, this condition is $\omega_\tau(\alpha_r(1)) = N(r)^{1-\beta}$. Also, the restriction of $\omega_\tau$ to $\mathcal{A}$ is independent of $\tau$. One can therefore ask whether certain $\mathrm{KMS}_\beta$ states can be constructed by other methods, and possibly for a larger range of $\beta$'s, by starting from states of $\mathcal{A}$. It is known that a $\mathrm{KMS}_\beta$ state at every $\beta \geq 1$ exists in the case of the product system from example 3.6, as shown by Laca and Raeburn, see [14, Proposition 9.1]. This state is supported on a commutative $C^*$-subalgebra of $\mathcal{F}$, and we conjecture that similar considerations could work more generally.

One would like to apply [13, Theorem 4.1] to the system $(\mathcal{F} \rtimes_\alpha P, \sigma)$, where the dynamics $\sigma$ is trivial on the image of $\mathcal{F}$ in $\mathcal{F} \rtimes_\alpha P$ and scales the implementing isometries $v_s \in \mathcal{F} \rtimes_\alpha P$ by $N(s)^{it}$ for $s \in P$. Then for every $\beta \in \mathbb{R}$, $\mathrm{KMS}_\beta$ states on $\mathcal{F} \rtimes_\alpha P$ would be determined by tracial states $\tau$ on $\mathcal{F}$ which satisfy the scaling condition $\tau \circ \alpha_s = N(s)^{-\beta}\tau$ for every $s \in P$. Note that for a tracial state $\tau$ on $\mathcal{F}$ to satisfy the scaling condition we must have $\tau(\alpha_s(1)) := N(s)^{-\beta}$ for every $s \in P$. However, this last equality does not match $\omega_\tau(\alpha_s(1)) = N_s N(s)^{-\beta}$, so states above $\beta_c$ and states below $\beta_c$ would live on different subalgebras.

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