arXiv:1111.4392v1 [math.OA] 18 Nov 2011

ON CERTAIN UNIVERSAL GRAPH C^*-ALGEBRAS

B. BURGSTALLER

Abstract. For special universal C^*-algebras associated to higher rank graphs or semigraphs we state the universal representations, prove a Cuntz–Krieger uniqueness theorem and state the K-theory.

1. Introduction

In this paper we continue our study of higher rank semigraph C^*-algebras from [3]. The class of higher rank semigraph C^*-algebras present a flexible generalisation of Tomforde's ultragraph algebras [8] or C^*-algebras of labelled graphs [1] to higher rank structures.

In this work we compute the K-theory of weakly free higher rank semigraph algebras in Theorem 5.5. Roughly speaking, a weakly free semigraph algebra is something like the Toeplitz graph algebra [7] in the theory of graph algebras. Actually, Theorem 5.5 extends the K-theory computation of Toeplitz graph algebras in [4]. Also the proof is not much work: it is merely an application of a theorem in [4].

The second aim of this paper covering the main work is the introduction of special higher rank semigraph algebras which we call quell semigraph algebras and which are induced by higher rank graphs or semigraphs. A quell semigraph algebra is defined to be universally generated by a k-semigraph as a generator set, thereby satisfying only a minimum quantity of relations such that it still contains naturally the inducing k-semigraph. We find a concrete faithful representation of a semigraph algebra on a Hilbert space as a left regular representation of a semimultiplicative set in Proposition 4.8. A quell semigraph algebra is free, weakly free, cancelling, and satisfies a Cuntz–Krieger theorem, see Theorem 4.13.

Every semigraph C^*-algebra is generated by partial isometries with commuting source and range projections (see Lemma 2.13). There is some recent interest in universal C^*-algebras generated by partial isometries and their K-theory, see for instance Cho and Jorgensen [5] and Brenken and Niu [2]. Although there is no immediate motivation for a quell semigraph algebra, we suspect it is somehow the freest Cuntz–Krieger like algebra which naturally embedds a given k-graph, see also Lemma 4.14. Moreover we think every universal algebra generated by partial isometries for which one knows

1991 *Mathematics Subject Classification.* 46L05, 46L80.

Key words and phrases. higher rank, graph algebra, semigraph, ultragraph, labelled graph, K-theory.

the K-theory is a benefit; even more so, as there is (despite [9]) an ongoing programm initiated by G. Elliott to classify certain subclasses of nuclear C^*-algebras (as the subclass of purely infinite nuclear C^*-algebras) where the K-theory of a C^*-algebra plays a major role.

We give a brief overview of this paper. In Section 2 we recall the theory of semigraph algebras. In Section 3 we consider two conditions for a semigraph algebra which we call weakly free (Definition 3.3) and free (Definition 3.6). Freeness implies weakly freeness (Proposition 3.9), and free semigraph algebras are cancelling (Proposition 3.10), and so satisfy a Cuntz–Krieger uniqueness theorem. In Section 4 we introduce the quell semigraph algebras (Definition 4.3) and show that they are free and cancelling (Theorem 4.13). In Section 5 we explicitely compute the K-theory of weakly free semigraph C^*-algebras in Theorem 5.5. In Section 6 we use Theorem 5.5 for the computation of the K-theory of a quell semigraph C^*-algebra in Theorem 6.2.

2. Semigraph algebras

In this section we recall briefly the definition and some basic facts about semigraph algebras [3] for further reference.

Definition 2.1. A *semimultiplicative set* T is a set equipped with a subset $T^{(2)} \subseteq T$ and a multiplication $T^{(2)} \longrightarrow T : (s,t) \mapsto st$, which is associative, that is, for all $s,t,u \in T$, $(st)u$ is defined if and only if $s(tu)$ is defined, and both expressions are equal if they are defined.

Definition 2.2. Let k be an index set (which may be regarded as a natural number if k is finite). A *k-semigraph* T is a semimultiplicative set T equipped with a degree map $d : T \longrightarrow \mathbb{N}_0^k$ satisfying the *unique factorisation property* which consists of the following two conditions:

(1) For all $x,y \in T$ for which the product xy is defined one has $d(xy) = d(x) + d(y)$.

(2) For all $x \in T$ and all $n_1, n_2 \in \mathbb{N}_0^k$ with $d(x) = n_1 + n_2$ there exist unique $x_1, x_2 \in T$ with $x = x_1x_2$ satisfying $d(x_1) = n_1$ and $d(x_2) = n_2$.

We call k-semigraph also a higher rank semigraph or just a semigraph. We shall also write $|t|$ rather than $d(t)$ for elements t in a k-semigraph. Let T be a k-semigraph. We denote the set of all elements of T with degree n by $T^{(n)}$ ($n \in \mathbb{N}_0^k$). If $x \in T$ and $0 \leq n_1 \leq n_2 \leq d(x)$ then there are unique $x_1, x_2, x_3 \in T$ such that $x = x_1x_2x_3$, $d(x_1) = n_1, d(x_2) = n_2 - n_1$ and $d(x_3) = d(x) - n_2$. x_2 is denoted by $x(n_1, n_2)$.

Definition 2.3. Let T be a k-semigraph. For $s,t \in T$ we write $s \leq t$ if $\alpha s = t$ for some $\alpha \in T$.

Lemma 2.4. *The last relation is an order relation on a semigraph.*

Proof. Transitivity is clear. So assume $s \leq t$ and $t \leq s$. Then there are $\alpha, \beta \in T$ such that $t = \alpha s$ and $s = \beta t$. Then $s = \beta\alpha s$, and so $d(s) = d(s) +$

$d(\alpha) + d(\beta)$, which implies $d(\alpha) = d(\beta) = 0$. Hence one has $\beta\alpha s = \beta\alpha\beta\alpha s$. By the unique factorisation property $s = \alpha s$. So $s = t$. □

Definition 2.5. A k-semigraph T is called *finitely aligned* if for all $x, y \in T$ the *minimal common extension* of x and y, which is the set

$$\begin{aligned} T^{(\min)}(x,y) &= \{(\alpha,\beta) \in T \times T |\, x\alpha \text{ and } y\beta \text{ are defined,} \\ &\qquad x\alpha = y\beta,\, d(x\alpha) = d(x) \vee d(y)\}, \end{aligned}$$

is finite.

It might help to recall the meaning of the relation $(\alpha, \beta) \in T^{(\min)}(x, y)$ by visualising it by the tautologies $\alpha \leq x\alpha$ and $\beta \leq y\beta$.

Definition 2.6. T is called a *non-unital* k-semigraph if there exists a k-semigraph T_1 which has a unit $1 \in T_1$ such that $T = T_1 \backslash \{1\}$.

We shall use the following notions when we speak about algebras. A $*$-*algebra* means an algebra over $\mathbb{C}$ endowed with an involution. An element s in a $*$-algebra is called a *partial isometry* if $ss^*s = s$, and a *projection* p is an element with $p = p^2 = p^*$. We define $P_a := aa^*$ and $Q_a := a^*a$ for elements a of a $*$-algebra. If I is a subset of a $*$-algebra then $\langle I \rangle$ denotes the self-adjoint two-sided ideal generated by I in this $*$-algebra.

Assume that we are given a set $\mathcal{P}$ and a nonunital k-semigraph $\mathcal{T}$ with $\mathcal{P} \cap \mathcal{T} = \emptyset$. Recall that in a non-unital k-semigraph one has $d(t) > 0$ for all its elements t. We denote by $\mathcal{T}_1 = \mathcal{T} \sqcup \{1\}$ the unital k-semigraph of Definition 2.6. Define $\mathbb{F}$ to be the free nonunital $*$-algebra generated by $\mathcal{P} \cup \mathcal{T}$. We call a $*$-monomial in the letters of $\mathcal{P} \cup \mathcal{T}$ a word.

Definition 2.7. The *degree* $d(x)$ of a word $x = x_1 \dots x_n$ in $\mathbb{F}$ ($n \geq 1$, $x_i \in \mathcal{P} \cup \mathcal{T} \cup \mathcal{P}^* \cup \mathcal{T}^*$) is defined to be $d(x) = d(x_1) + \dots + d(x_n)$, where $d(x_i)$ is to be the semigraph-degree $d(x_i)$ when $x_i \in \mathcal{T}$, $d(x_i) = 0$ if $x_i \in \mathcal{P}$, and $d(x_i^*) = -d(x_i)$ for any $x_i \in \mathcal{T} \cup \mathcal{P}$.

Definition 2.8. The *fiber space* of $\mathbb{F}$ is the union of all fibers $\operatorname{span}(W_n)$, where W_n denotes the set of words with degree $n \in \mathbb{Z}^k$.

Definition 2.9. Let $\sigma : \mathbb{T}^k \longrightarrow \operatorname{Aut}(\mathbb{F})$ be the *gauge action* defined by $\sigma_\lambda(p) = p$ and $\sigma_\lambda(t) = \lambda^{d(t)}t$ for all $p \in \mathcal{P}, t \in \mathcal{T}$ and $\lambda \in \mathbb{T}^k$.

Lemma 2.10 ([3], Lemma 4.7)**.** *X is the quotient of $\mathbb{F}$ by a subset of the fiber space if and only if there is a gauge action on X (defined like in Definition 2.9).*

Definition 2.11 (Semigraph algebra)**.** A *k-semigraph algebra X* is a $*$-algebra which is generated by disjoint subsets $\mathcal{P}$ and $\mathcal{T}$ of X, where

(i) $\mathcal{P}$ is a set of commuting projections closed under taking multiplications,
(ii) $\mathcal{T}$ is a set of nonzero partial isometries closed under nonzero products,

(iii) $\mathcal{T}$ is a non-unital finitely aligned k-semigraph,
(iv) for all $x \in \mathcal{T}$ and all $p \in \mathcal{P}$ there is a $q \in \mathcal{P}$ such that $px = xq$,
(v) for all $x, y \in \mathcal{T}$ there exist $q_{x,y,\alpha,\beta} \in \mathcal{P}$ such that

$$x^*y = \sum_{(\alpha,\beta)\in\mathcal{T}_1^{(\min)}(x,y)} \alpha q_{x,y,\alpha,\beta}\beta^*, \text{ and} \tag{1}$$

(vi) X is canonically isomorphic to the quotient of $\mathbb{F}$ by a subset of the fiber space (Definition 2.8).

It is understood in identity (1) that the unit 1 in $\mathcal{T}_1 = \mathcal{T} \sqcup \{1\}$ is also a unit for all elements of X. The universal C^*-algebra $C^*(X)$ generated by X is called the k-semigraph C^*-algebra associated to X. We often write also $\sum_{x\alpha=y\beta}$ rather than $\sum_{(\alpha,\beta)\in\mathcal{T}_1^{(\min)}(x,y)}$ in (1).

Definition 2.12. We call an element of $\{spt^* \in X |\, s,t \in \mathcal{T}_1, p \in \mathcal{P}\}$ a *standard word* (of the semigraph algebra X). We call an element of $\{sp \in X |\, s \in \mathcal{T}_1, p \in \mathcal{P}\}$ a *half-standard word*.

Lemma 2.13 ([3], Lemma 5.8)**.** (a) *The word set of X is an inverse semigroup of partial isometries.*
(b) *For each word w there exist half-standard words a_i, b_i and c_i such that $ww^* = \sum_{i=1}^n a_i a_i^*$ and $w = \sum_{j=1}^m b_j c_j^*$ with $d(w) = d(b_s c_s^*)$ for all $1 \le s \le m$.*

Corollary 2.14. *(a) A semigraph algebra is linearly spanned by its standard words.*
(b) The range projection of a word is a sum of range projections of half-standard words.
(c) The source projection of a half-standard word is in $\mathcal{P}$.

Corollary 2.15 ([3], Corollary 6.4)**.** *The core is the union of a net of finite dimensional C^*-algebras, each one allowing a matrix representation where each projection on the diagonal is a finite sum of mutually orthogonal standard projections. A C^*-representation of X is injective on the core if and only if it is non-vanishing on nonzero standard projections.*

Definition 2.16. A semigraph algebra X is called *cancelling* if for every standard word w with nonzero degree and every nonzero standard projection p there is a nonzero standard projection q such that $q \le p$ and $qwq = 0$.

Theorem 2.17 (Cuntz–Krieger uniqueness theorem, [3], Theorem 7.3)**.** *The universal representation $X \longrightarrow C^*(X)$ is injective on the core, and so non-vanishing on the nonzero standard projections, and up to isomorphism this is the only existing representation of X in a C^*-algebra which is non-vanishing on nonzero standard projections and has dense image.*

3. Free semigraph algebras

In this section we consider a condition on a semigraph algebra called freeness, and a weaker one called weakly freeness. The rough idea of freeness is that range projections of generators should not sum up to a unit. This is similar as $P_{s_1} + P_{s_2} < 1$ in the Toeplitz version of the Cuntz algebra $\mathcal{O}_2$ with generating isometries s_1 and s_2. The reader may be warned that this section tends to be very technical in a combinatorial and tedious sense. In this section X denotes a semigraph algebra.

Definition 3.1. Write $X^{\setminus i}$ for the $*$-subalgebra of X generated by the elements $\mathcal{P}$ and the non-unital finitely aligned semigraph $\mathcal{T}^{\setminus i} = \{\, t \in \mathcal{T} \mid d(t)_i = 0 \,\}$ (the ith coordinate vanishes). We set $\mathcal{T}_1^{\setminus i} = \mathcal{T}^{\setminus i} \sqcup \{1\}$.

Lemma 3.2. *$X^{\setminus i}$ is a semigraph algebra.*

Proof. All points (i)-(v) of Definition 2.11 are almost obvious. Point (vi) can be seen by Lemma 2.10 and the fact that the restriction of the gauge action on X to $X^{\setminus i}$ is a gauge action on $X^{\setminus i}$. □

We shall regard $X^{\setminus i}$ as a sub-semigraph algebra of X. When we work in X and say "w is a word in $X^{\setminus i}$" (and so on) then "word" refers to the semigraph algebra $X^{\setminus i}$; so w does not mean a word in X which accidentally happens to be in $X^{\setminus i}$.

Definition 3.3. A semigraph algebra X is called *weakly free* if for all coordinates $i \in k$, all nonzero standard projections p of $X^{\setminus i}$ and all finite subsets $\mathcal{B} \subseteq \mathcal{T}^{(e_i)}$ $p \leq \sum_{b\in\mathcal{B}} P_b$ does not hold.

The next two lemmas (Lemma 3.4 and Lemma 3.5) will only be used in the proof of Lemma 5.3. The reader only interested in Theorem 5.5 could go directly to Section 5 after these two lemmas.

Lemma 3.4. *If $i \in k, a \in \mathcal{T}^{(e_i)}$ and w is a word in $X^{\setminus i}$ then there exist $b_1, \ldots, b_n \in \mathcal{T}^{(e_i)}$ such that $P_a w = P_a w(P_{b_1} + \ldots + P_{b_n})$.*

Proof. We may write $w = spt^*$ for $s, t \in \mathcal{T}_1^{\setminus i}$ and $p \in \mathcal{P}$. Then (by Definition 2.11 (v))

$$\begin{aligned}(aa^*)(spt^*) &= a \sum_{a\alpha=s\beta} \alpha q_{a,s,\alpha,\beta}\beta^* pt^* \\ &= \sum_{a\alpha=s\beta,\, t\beta\neq 0} a\alpha q_{a,s,\alpha,\beta} p_\beta \beta^* t^* \\ &= \sum_{a\alpha=s\beta,\, t\beta\neq 0} a\alpha q_{a,s,\alpha,\beta} p_\beta t'^* \beta'^* \sum_{a\alpha=s\beta,\, t\beta\neq 0} P_{\beta'}\end{aligned}$$

for $p_\beta \in \mathcal{P}$ with $p\beta = \beta p_\beta$ (by Definition 2.11 (iv)), and where $d(\beta) = d(a) = e_i$, and $t\beta = \beta' t'$ by the unique factorisation property with $d(\beta') = e_i$ (if $t\beta \neq 0$, which, recall, is equivalent to $t\beta$ being a defined product in $\mathcal{T}$). □

Lemma 3.5. *If X is weakly free then for every coordinate $i \in k$, every element x of the fiber space of $X^{\setminus i}$, and every finite subset $\mathcal{B}$ of $\mathcal{T}^{(e_i)}$ we have that $(\sum_{b\in\mathcal{B}} P_b)x = x$ implies $x = 0$.*

Proof. Suppose that x is a nonzero element of the 0-fiber $X_0^{\setminus i}$ (i.e. the core) of $X^{\setminus i}$, $p = \sum_{b\in\mathcal{B}} P_b$, and $px = x$. The element x is in the core of $X^{\setminus i}$ and so in a finite dimensional C^*-algebra A as described in Corollary 2.15. Say, $x = \sum_{ij} \lambda_{ij} e_{ij}$ for matrix units e_{ij}, where each diagonal unit e_{ii} is a sum of mutually orthogonal standard projections of $X^{\setminus i}$ (Corollary 2.15). Since $x \neq 0$ we may suppose that $0 \neq \lambda_{i_0 j_0} = 1$ for some fixed pair (i_0, j_0). By Corollary 2.15, there is a nonzero standard projection q in $X^{\setminus i}$ (so $q \in \mathbb{P}(\mathcal{T}_1^{\setminus i}\mathcal{P})$) such that $q \leq e_{i_0 i_0}$. Note that p commutes with $e_{i_0 i_0}$ since standard projections commute. Then

$$q \leq e_{i_0 i_0} = e_{i_0 i_0} x e_{j_0 i_0} = e_{i_0 i_0} p x e_{j_0 i_0} = p e_{i_0 i_0} x e_{j_0 i_0} = p e_{i_0 i_0} \leq p.$$

However, $q \leq p$ contradicts the weakly free condition in X. Thus $x = 0$.

Now assume that x is in another fiber $X_n^{\setminus i}$ ($n \in \mathbb{Z}^k$) and $px = x$ (where $p = \sum_{b\in\mathcal{B}} P_b$ again). We may write x as $x = \sum_{k=1}^{l} a_k x_k b_k^*$, where x_k is an element of the core $X_0^{\setminus i}$, $a_k \in \mathcal{T}_1$ with $d(a_k) = \max(n, 0)$, $b_k \in \mathcal{T}_1$ with $d(b_k^*) = \min(n, 0)$, and the pairs (a_k, b_k) are mutually distinct for different k's, for all $1 \leq k \leq l$. Fix $1 \leq k_0 \leq l$. Then, since $px = x$, and by several applications of Lemma 3.4, there exists a subset $\mathcal{B}' \subseteq \mathcal{T}^{(e_i)}$ such that for $p' = \sum_{b\in\mathcal{B}'} P_b$ one has

$$r_{k_0} := a_{k_0}^* x b_{k_0} = Q_{a_{k_0}} x_{k_0} Q_{b_{k_0}} = a_{k_0}^* p x b_{k_0} = p' a_{k_0}^* p x b_{k_0} = p' r_{k_0}.$$

Thus $r_{k_0} = p' r_{k_0}$ and $r_{k_0} \in X_0^{\setminus i}$, and so by what we have already proved, $r_{k_0} = 0$. Since k_0 was arbitrary,

$$0 = \sum_{k_0=1}^{l} a_{k_0} r_{k_0} b_{k_0}^* = \sum_{k_0=1}^{l} a_{k_0} x_{k_0} b_{k_0}^* = x.$$

□

Definition 3.6. A semigraph algebra X is called *free* if

(i) P_a is not in $\mathcal{P}$ for every half-standard word $a \notin \mathcal{P}$, and
(ii) if $p \in \mathcal{P}$ and $a_1, \ldots, a_n$ are half-standard words with $P_{a_i} < p$ for all $1 \leq i \leq n$ then $\bigvee_{i=1}^{n} P_{a_i} < p$.

Lemma 3.7. *Let X be free. If $a, b_1, \ldots, b_n$ are half-standard words satisfying $P_a(1 - P_{b_i}) \neq 0$ for all $1 \leq i \leq n$ then $P_a \prod_{i=1}^{n}(1 - P_{b_i}) \neq 0$.*

Proof. We may write $a = xs$ and $b_i = y_i p_i$ for certain $x, y_i \in \mathcal{T}_1$ and $s, p_i \in \mathcal{P}$. Assume that X is Toeplitz and $P_a(1 - P_{b_i}) \neq 0$ for all $1 \leq i \leq n$. We have

(by Definition 2.11 (v))

$$\begin{aligned} P_a(1-P_{b_i}) &= xsx^*(1-y_ip_iy_i^*) \\ &= xsx^* - \sum_{x\alpha=y_i\beta} xs\alpha q_{x,y_i,\alpha,\beta}\beta^* p_i y_i^* \\ &= xsx^*\Big(1 - \sum_{x\alpha=y_i\beta} xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^*\Big) \\ &= xsx^* \prod_{(\alpha,\beta)\in\mathcal{T}_1^{(\min)}(x,y_i)} (1 - xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^*), \end{aligned}$$

where $p_{i,\beta}$ is chosen in $\mathcal{P}$ such that $p_i\beta = \beta p_{i,\beta}$ (by Definition 2.11 (iv)), and where in the last identity we successively used the formula $(1-p)(1-q) = 1-p-q$ for orthogonal projections p and q. Since the above is nonzero by assumption, we have

$$xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^* < xsx^*$$

for all $1 \le i \le n$ and all $(\alpha,\beta) \in \mathcal{T}_1^{(\min)}(x,y_i)$. Multiplying here from the left and right with x^* and x, respectively, we get $x^*xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* < x^*xs$. By freeness (Definition 3.6) we conclude

$$\bigvee_{i=1}^{n} \bigvee_{x\alpha=y_i\beta} x^*xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* < x^*xs.$$

Thus $\bigvee_{i,\alpha,\beta} xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^* < xsx^*$, whence

$$P_a\prod_{i=1}^{n}(1-P_{b_i}) = xsx^*\prod_{i=1}^{n}\prod_{x\alpha=y_i\beta}(1 - xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^*) \tag{2}$$

$$= xsx^*\Big(1 - \bigvee_{i,\alpha,\beta} xs\alpha q_{x,y_i,\alpha,\beta}p_{i,\beta}\alpha^* x^*\Big) \neq 0, \tag{3}$$

where we used de Morgan's law $\bigwedge(1-\gamma) = (1 - \bigvee\gamma)$. □

Similarly as for elements in a semigraph we write $x_1 \le x_2$ for half-standard words x_1 and x_2 if they allow a representation $x = t_1p_1$ and $x_2 = t_2p_2$ ($t_1, t_2 \in \mathcal{T}$ and $p_1, p_2 \in \mathcal{P}$) with $t_1 \le t_2$. Only in the next corollary $\mathcal{H}$ denotes the set of half-standard words.

Corollary 3.8. *If X is free then these two sets are the set of nonzero standard projections:*

$$\Big\{P_a\prod_{i=1}^{n}(1-P_{b_i}) \in X \,\Big|\, n\in\mathbb{N}_0,\ a,b_i\in\mathcal{H},\ P_a(1-P_{b_i})\neq 0 \text{ for all } i\Big\},$$

$$= \Big\{P_a\prod_{i=1}^{n}(1-P_{b_i}) \in X \,\Big|\, n\in\mathbb{N}_0,\ a,b_i\in\mathcal{H},\ P_{b_i} < P_a \text{ and } a\le b_i\,\forall i\Big\}.$$

Proof. That the first set is the set of nonzero standard projections follows from Lemma 3.7. For the second set just recall the identity (2) how we can write down a standard projection. The assertion can be directly read off from this expansion. □

Proposition 3.9. *If X is free then X is weakly free.*

Proof. Fix $i \in k$. To check weakly freeness, consider a finite subset $\mathcal{B}$ of $\mathcal{T}^{(e_i)}$ and a nonzero standard projection $p = xqx^* \prod_{i=k}^{n}(1 - y_k y_k^*)$ in $X^{\setminus i}$, where $x \in \mathcal{T}_1^{\setminus i}, q \in \mathcal{P}$ and the y_k's are half-standard words in $X^{\setminus i}$. If $b \in \mathcal{B}$ then

$$qx^*x \geq qx^*bb^*x = \sum_{(e,f)\in\mathcal{T}_1^{(\min)}(x,b)} q(eq_{x,b,e,f}f^*)(fq_{x,b,e,f}e^*) \tag{4}$$

$$= \sum_{(e,f)\in\mathcal{T}_1^{(\min)}(x,b)} P_{eq_e q_{x,b,e,f} f^* f} < qx^*x, \tag{5}$$

where we used $qe = eq_e$ from Definition 2.11 (iv). The last inequality is here by freeness. Indeed, $P_{eq_e q_{x,b,e,f} f^* f} \notin \mathcal{P}$ by Definition 3.6 (i). On the other hand, $P_{eq_e q_{x,b,e,f} f^* f} \leq qx^*x \in \mathcal{P}$ by (4), so $P_{eq_e q_{x,b,e,f} f^* f} < qx^*x$. Hence, inequality (5) is true by Definition 3.6 (ii).

We conclude from (4) and (5) that $qx^*P_b x \neq x^*xq$. Hence, applying here the operation $x(\cdot)x^*$ we get $xqx^*P_b \neq xqx^*$. Thus, $xqx^*(1 - P_b) \neq 0$ for all $b \in \mathcal{B}$. Hence

$$p\Big(1 - \sum_{b\in\mathcal{B}} P_b\Big) = p\prod_{b\in\mathcal{B}}(1 - P_b) = xqx^* \prod_{i=k}^{n}(1 - y_k y_k^*) \prod_{b\in\mathcal{B}}(1 - P_b) \neq 0$$

by Lemma 3.7. Consequently $p \leq \sum_{b\in\mathcal{B}} P_b$ does not hold. □

Proposition 3.10. *If X is free then X is cancelling.*

Proof. We are going to check the cancelling condition, Definition 2.16. Let $w = \alpha q \beta^*$ be a standard word with $d(w) \neq 0$ ($\alpha, \beta \in \mathcal{T}_1, q \in \mathcal{P}$). Let P be a nonzero standard projection. We must find a nonzero standard projection Q with $Q \leq P$ and $QxQ = 0$.

We may write $P = pxsx^*$, where $p = \prod_i (1 - y_i y_i^*)$, $x \in \mathcal{T}_1, s \in \mathcal{P}$ and the y_i's are half-standard words. If already $PwP = 0$ then the cancelling condition is verified. So assume that

$$0 \neq PwP = pxs(x^*\alpha)q\beta^* pxsx^*.$$

Then there is a pair $(e, f) \in \mathcal{T}_1^{(\min)}(x, \alpha)$ such that

$$v := pxs(eq_{x,\alpha,e,f}f^*)q\beta^* pxsx^* \neq 0$$

by (1). Consequently $pxse \neq 0$, and so

$$0 \neq P' := pxsee^*x^* = pxes'e^*x^* = px's'x'^* \tag{6}$$

is a standard projection, where $x' := xe$ and $s' \in \mathcal{P}$ satisfies $se = es'$. (We intensively use the fact that the word set forms an inverse semigroup, Lemma 2.13.) Note that $P' \leq P$. If already $P'wP' = 0$, then the cancelling condition is verified.

So assume $P'wP' \neq 0$. Note that $|x'| = |xe| \geq |\alpha|$ since $(e, f) \in \mathcal{T}_1^{(\min)}(x, \alpha)$. Note that by (6) $P' = px's'x'^*$ has the same shape as P, but with $|x'| \geq |\alpha|$. As we are going to search $Q \leq P' \leq P$, we may assume without loss of generality that we are given P with $PwP \neq 0$ and $|x| \geq |\alpha|$. Similar computations as above on the β-side show that we may also assume, by choosing a smaller projection than P, that also $|x| \geq |\beta|$.

So assume without loss of generality that $PwP \neq 0$ and $|x| \geq |\alpha|, |\beta|$. We have

$$0 \neq PwP = pxsx^*(\alpha q \beta^*)xsx^*p.$$

Hence, there are decompositions $x = \alpha x_1 = \beta x_2$ $(x_1, x_2 \in \mathcal{T}_1)$, and a $q' \in \mathcal{P}$ chosen to satisfy $Q_\alpha q Q_\beta x_2 = x_2 q'$, such that

$$\begin{aligned} PwP &= pxs(\alpha x_1)^* \alpha q \beta^* \beta x_2 sx^*p = pxs(x_1^* x_2)q'sx^*p \\ &= pxs \sum_{(e,f)\in\mathcal{T}_1^{(\min)}(x_1,x_2)} eq_{x_1,x_2,e,f} f^* q' sx^*p \\ &= pxsKsx^*p, \end{aligned}$$

where $K := \sum_{(e,f)\in\mathcal{T}_1^{(\min)}(x_1,x_2)} eq_{x_1,x_2,e,f} f^* q'$. Set

$$Q' := \prod_{(e,f)\in\mathcal{T}_1^{(\min)}(x_1,x_2)} (1 - ee^*).$$

Successively using the fourmula $xs(1 - ee^*)(1 - e'e'^*)sx^* = xs(1 - ee^*)sx^*xs(1 - e'e'^*)sx^*$, we get

$$Q := pxsQ'sx^* = p \prod_{(e,f)\in\mathcal{T}_1^{(\min)}(x_1,x_2)} (1 - xsee^*x^*)xsx^* \tag{7}$$

$$= xsx^* \prod_i (1 - y_i y_i^*) \prod_{(e,f)\in\mathcal{T}_1^{(\min)}(x_1,x_2)} (1 - xsee^*x^*). \tag{8}$$

In (8) we have substituted $p = \prod_i (1 - y_i y_i^*)$. By (8), Q is a standard projection. Recall that $P = pxsx^*$, so $Q \leq P$. Since $Q'K = 0$, we obtain

$$QwQ = Q(PwP)Q = QpxsKsx^*pQ = QpxsKsx^*ppxsQ'sx^* = 0.$$

So Q is the sought standard projection that cancels w. It remains to show $Q \neq 0$.

Since $d(\alpha q \beta^*) \neq 0$, we may assume without loss of generality that $|x_1| < |x_1| \vee |x_2|$, that is, $|e| \neq 0$ for every e in (7). By freeness, Definition 3.6 (i), $(x^*x)see^* = P_{x^*xse} \notin \mathcal{P}$, and so $x^*xsee^* \neq x^*xs$. Hence $xsee^*x^* \neq xsx^*$. This shows $0 \neq (1 - xsee^*x^*)xsx^*$. Also $0 \neq (1 - y_i y_i^*)xsx^*$ since $0 \neq P = pxsx^*$. Hence, by formula (8) and Lemma 3.7 $Q \neq 0$. □

Lemma 3.11. *Let X be a semigraph algebra and $\pi : X \longrightarrow C^*(X)$ the universal representation. Then $\pi(X)$ is a semigraph algebra (which is free if X is free).*

Proof. By universality of the universal representation $\pi : X \longrightarrow C^*(X)$, the gauge map $X \xrightarrow{\sigma_\lambda} X \xrightarrow{\pi} C^*(X)$ ($\lambda \in \mathbb{T}^k$), Definition 2.9, induces a gauge map $\tilde{\sigma}_\lambda : C^*(X) \longrightarrow C^*(X)$. Hence, by Lemma 2.10, $\pi(X)$ is a quotient of the free algebra $\mathbb{F}$ by a subset of the fiber space. Thus $\pi(X)$ must be a quotient of X by a subsets of its fiber space, and is thus a semigraph algebra by [3, Lemma 8.1]. Now assume that X is free. By the uniqueness theorem (Theorem 2.17) the cores of X and $\pi(X)$ are isomorphic, and so the validity of Definition 3.6 (ii) carries over from X to $\pi(X)$. If a half-standard word $\pi(a)$ in $\pi(X)$ (a being a half-standard word in X) is not in $\pi(\mathcal{P})$ then $d(a) = d(\pi(a)) > 0$, and so $P_a \notin \mathcal{P}$ since X is free, and thus $P_{\pi(a)} = \pi(P_a) \notin \pi(\mathcal{P})$. This verifies Definition 3.6 (i) for $\pi(X)$. □

4. Quell semigraph C^*-algebras

In this section we define quell semigraph algebras. Let us anticipate roughly what it is. A quell semigraph algebra could be most simply explained by considering a higher rank graph T [6]. Then the quell semigraph C^*-algebra $\mathcal{Q}(T)$ for T is a C^*-algebra which is similar to the Toeplitz graph algebra $\mathcal{T}C^*(T)$ [7] but without the relations $Q_t = s(t)$ ($t \in T$). So the Toeplitz graph algebra is a quotient of the quell semigraph algebra.

Suppose that T is a finitely aligned k-semigraph. Define $T^{(0)}$ to be the set of elements of T which have degree zero ($d(t) = 0$). Note that if $e \in T^{(0)}$ and e^2 is defined then e is automatically idempotent. Indeed, by the factorizatrion property we may choose a unique decomposition $e = ab$ for certain $a, b \in T^{(0)}$. Then $e^2 = abab$. By the unique factorisation property $e = a = b$, and so $e^2 = ab = e$, which proves the claim. Moreover, if $e \in T^{(0)}$ is idempotent and $x \in T$ then either ex is undefined or $ex = x$ (since $ex = e^2x$ and so $x = ex$ by the unique factorisation property). In particular, ef must be undefined for distinct $e, f \in T^{(0)}$.

In this section it is assumed that a semigraph T has only idempotent elements in $T^{(0)}$. Let us summarise the consequences in a lemma.

Lemma 4.1. *The elements of $T^{(0)}$ are idempotent elements and mutually incomposeable. If $e \in T^{(0)}, t \in T$ and et is defined then $et = t$. (Similarly, $te = t$ if te is defined.)*

Definition 4.2. For $x \in T$ we define $r(x) = x(0,0)$ (range of x) and $s(x) = x(d(x), d(x))$ (source of x).

Definition 4.3 (Quell semigraph algebra for T)**.** Suppose that T is a finitely aligned k-semigraph. Then one associates the *quell semigraph algebra* X to T. It is the universal $*$-algebra X generated by the set T subject to the following relations.

(i) T consists of partial isometries,
(ii) $T^{(0)}$ consists of projections,
(iii) X respects the multiplication of T (that is, if $xy = z$ holds in T for $x, y, z \in T$ then this identity should also hold in X),
(iv) $xy = 0$ for all $x, y \in T$ whose product xy is undefined,
(v) Q_x and Q_y commute for all $x, y \in T$, and
(vi)

$$x^*y = \sum_{(e,f)\in T^{(\min)}(x,y)} eQ_{yf}f^* \tag{9}$$

for all $x, y \in T$.

"Quelle" is the German word for source. Since the source projections play an extraordinary role in the last definition (as compared to ordinary graph algebras), we decided for the word quell. One may wish however to say "source semigraph algebra" rather than "quell semigraph algebra".

Note that T is faithfully embedded in the free algebra $\mathbb{F}$, but could perhaps degenerate in the quotient X. Soon we will see below (Corollary 4.9) however that T is also faithfully embedded in X. That is why we should not like to distinguish the k-semigraph T and its embedding in X. For the remainder of this section we shall assume that T is a finitely aligned k-semigraph and X its associated free semigraph algebra.

Definition 4.4. We shall denote an element $(y, \alpha_1, \ldots, \alpha_n) \in T^{n+1}$ $(n \geq 1)$ symbolically by $y\mu_{\alpha_1,\ldots,\alpha_n}$, or for brevity, by $y\mu_\alpha$. Define

$$\Delta_\mu \;=\; \{\, y\mu_{\alpha_1,\ldots,\alpha_n} \mid n \in \mathbb{N},\, y, \alpha_i \in T \text{ and } \exists i \in \{1,\ldots,n\} \text{ with } y \leq \alpha_i \,\}.$$

Definition 4.5. Set $\Delta = T \sqcup \Delta_\mu$.

Lemma 4.6. *Δ is a semimultiplicative set.*

Proof. We endow Δ with the multiplication from T for products within T (as far as defined), and define

$$x(y\mu_\alpha) = (xy)\mu_\alpha \tag{10}$$

if $x \in T, y\mu_\alpha \in \Delta_\mu$ and $(xy)\mu_\alpha \in \Delta_\mu$. Other products in Δ are not allowed (for example, products within Δ_μ are undefined, or products where an element of Δ_μ appears as a left factor are invalid).

We are going to check that Δ is a semimultiplicative set. To this end we have to check associativity in the sense of Definition 2.1. Suppose that $a, b, c \in \Delta$. If $a, b, c \in T$ then $(ab)c$ is defined if and only if $a(bc)$ is defined as T is a semimultiplicative set. If a or b is in Δ_μ then both $(ab)c$ and $a(bc)$ are undefined. Suppose $a, b \in T$ and $c \in \Delta_\mu$. Write $c = y\mu_\alpha$. If $(ab)c = ((ab)y)\mu_\alpha$ is defined then $(ab)y \leq \alpha_i$ for some i. Thus $\alpha_i = zaby$ for some $z \in T$ and so also $a(bc) = a((by)\mu_\alpha) = (a(by))\mu_\alpha$ is defined and we have $(ab)c = a(bc)$. Similarly we see the reverse conclusion. $\square$

If a semimultiplicative set G has left cancellation, that means, $st_1 = st_2$ implies $t_1 = t_2$ (for all $s, t_1, t_2 \in G$) then we can associate a left reduced C^*-algebra to G as defined next. (We shall write e_i or δ_i for the delta function $1_{\{i\}}$.)

Definition 4.7. For a semimultiplicative set G with left cancellation define $\lambda : G \longrightarrow B(\ell^2(G))$ by

$$\lambda_s\Big(\sum_{t\in G}\alpha_t\delta_t\Big) = \sum_{t\,\in\, G,\ st \text{ is defined}} \alpha_t\delta_{st}$$

for all $s \in G$ and $\alpha_t \in \mathbb{C}$. The sub-C^*-algebra of $B(\ell^2(G))$ generated by $\lambda(G)$ is called the *left reduced C^*-algebra* of G and denoted by $C_r^*(G)$.

Proposition 4.8. *There is a representation $\varphi : X \longrightarrow C_r^*(\Delta) : \varphi(t) = \lambda_t$.*

Proof. Of course, $\varphi : \mathbb{F} \longrightarrow C_r^*(\Delta)$ would be a well defined representation of the free algebra $\mathbb{F}$ generated by T. We need to show that this φ respects the defining relations of Definition 4.3. Note that the λ_t's are partial isometries with commuting range and source projections (these are canonical projections onto $\ell^2(Z)$ for subsets $Z \subseteq \Delta$). So, by the property of Δ to be a semimultiplicative set and identity (10) the points (i)-(v) of Definition 4.3 (for $\varphi(T) = \lambda_T$ rather than T) are easy to see. (For (ii) recall Lemma 4.1.)

Let us write down the adjoint operators $\varphi(t)^*$. We have

$$\varphi(t)^*\delta_{ts\mu_\alpha} = \delta_{s\mu_\alpha}, \quad \varphi(t)^*\delta_{ts} = \delta_s, \quad \varphi(t)^*\delta_a = 0 \text{ (else)} \tag{11}$$

To check Definition 4.3 (vi), consider $x, y \in T$. Suppose

$$\varphi(xx^*yy^*)\delta_a = \delta_a \tag{12}$$

for an $a \in \Delta$. Then a is a product $a = ya_y$ (since $\varphi(y)^*\delta_a \neq 0$) for some $a_y \in \Delta$, and similarly $a = xa_x$ for some $a_x \in \Delta$. Say that $a = ys_y\mu_\alpha = xs_x\mu_\alpha$. Then $v := ys_y$ has degree $d(v) \geq d(x) \vee d(y)$, and so there must exist a minimal common extension $(e, f) \in T^{(\min)}(x, y)$ such that

$$v(0, |x| \vee |y|) = xe = yf.$$

Consequently we have

$$\varphi\Big(\sum_{(e,f)\in T^{(\min)}(x,y)} (xe)(yf)^*\Big)\delta_a = \delta_a. \tag{13}$$

On the other hand, (12) follows quite immediately from (13). Since, for arbitrary $a \in \Delta$, the right hand sides of (12) and (13) either give δ_a or zero, we conclude from the shown equivalence that

$$\varphi(xx^*yy^*) = \varphi\Big(\sum_{(e,f)\in T^{(\min)}(x,y)} (xe)(yf)^*\Big). \tag{14}$$

The range projection of $\varphi(ef^*y^*)$ is

$$\varphi(e(yf)^*(yf)e^*) = \varphi(e(xe)^*(xe)e^*) = \varphi(ee^*x^*xee^*) = \varphi(x^*xee^*),$$

which is a smaller projection than $\varphi(x^*x)$. Thus,

$$\varphi(x^*xef^*y^*y) = \varphi(ef^*y^*y) = \varphi(ef^*y^*yff^*).$$

Hence, multiplying in (14) from the left and right with x^* and y, respectively, we see that the identity (9) holds in the image of φ. □

Corollary 4.9. *The canonical map $\iota : T \longrightarrow X$ is an injective k-semigraph homomorphism and non-degenerate.*

Proof. We compose ι with φ of Proposition 4.8 to see this. Let $t \in T$. Let $e = s(t)$, so $t = te$. Then $\varphi(\iota(t))\delta_e = \delta_t \neq 0$. So ι is non-degnerate. If $s \in T$ is distinct from t then we easily see with Lemma 4.1 that $\varphi(\iota(t))\delta_e = \delta_t \neq \varphi(\iota(s))\delta_e$. So ι is injective. □

Lemma 4.10. *X is a semigraph algebra with generators*

$$\mathcal{P} = \{\, Q_{t_1} \dots Q_{t_n} \in X \mid n \in \mathbb{N},\, t_i \in T \,\} \cup \{0\}, \tag{15}$$

$$\mathcal{T} = T \backslash T^{(0)} \tag{16}$$

Proof. We need to show Definition 2.11. That $\mathcal{P}$ is a commuting set of projections closed under multiplications (Definition 2.11 (i)) follows from Definition 4.3 (i) and (v). That $\mathcal{T}$ is a set of partial isometries closed under nonzero products (Definition 2.11 (ii)) follows from Definition 4.3 (i), (iii) and (iv). We are going to check that $\mathcal{T}_1$ is a semigraph (Definition 2.11 (iii)). Let $t \in \mathcal{T}_1$, and $t = t_1t_2$ be the unique decomposition in T subject to $m = d(t_1) > 0$ and $n = d(t_2) > 0$. This is the required decomposition in $\mathcal{T}_1$ also. If however $m = 0$, say, then take the factorisation $t = 1t$.

To prove (1) of Definition 2.11 (v), consider $x, y \in \mathcal{T}$. Note that in (9) $xe = yf$ and $Q_{yf} \in \mathcal{P}$, so (9) looks already similar like (1). We only have to take care whether $\mathcal{T}_1^{(\min)}$ and $T^{(\min)}$ make here a difference. If $d(x) > d(y)$, say, then we have $\{(1, f)\} = \mathcal{T}_1^{(\min)}(x, y)$ and $\{(e', f)\} = T^{(\min)}(x, y)$. Thus $xe' = yf$ so that $e' \in T^{(0)}$ is a right unit for $xe' = yf$ by Lemma 4.1, that is, $yfe' = yf$. Hence, by the unique factorisation property in T, even $fe' = e'$. Thus $1f^*y^*yff^* = e'f^*y^*yff^*$, so there is no difference. The cases $d(x) < d(y)$ and $d(x) = d(y)$ are treated similarly. If $d(x)$ and $d(y)$ are incomparable then there is obviously no difference. To check Definition 2.11 (iv), just note that

$$Q_xy = (x^*x)y = x^*(xy) = yQ_{xy}1 = yQ_{xy}$$

by (9) if $x, y \in T$ and $xy \neq 0$. The algebra X is generated by $\mathcal{P}$ and $\mathcal{T}$ since $e = e^*e = Q_e \in \mathcal{P}$ for $e \in T^{(0)}$ by Definition 4.3 (ii). There is a gauge action σ on X given by $\sigma_\lambda(t) = \lambda^{d(t)}t$ for $t \in T, \lambda \in \mathbb{T}^k$. Indeed, it exists on the free algebra $\mathbb{F}$ generated by T, and so also on X, as the relations of Definition 2.11 are invariant under the σ_λ's. Hence, Lemma 2.10 verifies Definition 2.11 (vi). □

Definition 4.11. Let T be a semigraph and X its free semigraph algebra. Then $\mathcal{Q}(T) := C^*(X)$ is called the *quell semigraph C^*-algebra* associated to T.

Let us use the following abbreviation. If $\alpha = (\alpha_1, \dots, \alpha_n) \in T^n$ then μ_α denotes $s(\alpha_1)\mu_\alpha$. The next lemma shows us that the representation φ can distinguish the elements of $\mathcal{P}$. The restriction $a \mapsto \varphi(a)|_{\ell^2(T)}$ is not able to do this, and this is why we considered Δ at all and not just T, which would have been much simpler.

Lemma 4.12. *Let $p_\alpha = Q_{\alpha_1} \dots Q_{\alpha_n}$ be nonzero for $\alpha = (\alpha_1, \dots, \alpha_n)$ in T^n. Then $\varphi(p_\alpha)\delta_{\mu_\alpha} = \delta_{\mu_\alpha}$ and $\varphi(q)\delta_{\mu_\alpha} = 0$ for every $q \in \mathcal{P}$ with $q < p_\alpha$.*

Proof. Since $Q_{\alpha_i} = \alpha_i^*\alpha_i = \alpha_i^*\alpha_i s(\alpha_i)$, and p_α is nonzero, $s(\alpha_1) = s(\alpha_i)$ for all i by the orthonality of the idempotent elements of T (Lemma 4.1 and Definition 4.3 (iv)). Consequently,

$$\varphi(Q_{\alpha_i})\mu_\alpha = \varphi(\alpha_i^*)\varphi(\alpha_i)(s(\alpha_1)\mu_\alpha) = \varphi(\alpha_i^*)(\alpha_i s(\alpha_1)\mu_\alpha) = \mu_\alpha. \tag{17}$$

This proves $\varphi(p_\alpha)\delta_{\mu_\alpha} = \delta_{\mu_\alpha}$. We may write $q = Q_{y_1} \dots Q_{y_l}$ for some $y_i \in T$ (see (15)). Note that either $\varphi(q)\mu_\alpha = \mu_\alpha$ or $\varphi(q)\mu_\alpha = 0$. Assume the first case. Then by a similar computation as in (17) we see that for every i there is a j_i such that $y_i \le \alpha_{i_j}$. Thus, for every i $Q_{y_i} \ge Q_{\alpha_{i_j}} \ge p_\alpha$. Hence $q \ge p_\alpha$, which is a contradiction to the assumption $q < p_\alpha$. □

Theorem 4.13. *The quell semigraph algebra X associated to a k-semigraph T is free, weakly free and cancelling. It thus satisfies the Cuntz–Krieger uniqueness theorem, that is, there is only one C^*-representation of X (up to isomorphism) which is non-vanishing on nonzero standard projections. This universal C^*-representation, which is also injective on the core, is the representation φ from Proposition 4.8. In particular, there is an isomorphism between the quell semigraph C^*-algebra and the left reduced C^*-algebra of the semimultiplicative set Δ, i.e. $\mathcal{Q}(T) \cong C_r^*(\Delta)$.*

Proof. We are going to check that X is free (Definition 3.6). Let $p \in \mathcal{P}$ and $y_1, \dots, y_n$ be half-standard words. Assume that $P_{y_i} < p$ for all $1 \le i \le n$. By (15) and Lemma 4.12 there is an $\alpha \in T^n$ such that $p = p_\alpha$. Let $\varphi : X \to C^*(\Delta)$ be the representation of Proposition 4.8. By (11) we have $\varphi(y_i^*)\delta_{\mu_\alpha} = 0$ if $d(y_i) > 0$. On the other hand, if $d(y_i) = 0$ then $P_{y_i} \in \mathcal{P}$, and so $\varphi(y_i)\delta_{\mu_\alpha} = 0$ by Lemma 4.12 (as $y_i = P_{y_i} < p_\alpha$). Summarising these facts we get

$$\varphi\Big(\bigvee_{i=1}^n P_{y_i}\Big)\delta_{\mu_\alpha} = 0$$

and $\varphi(p_\alpha)\delta_{\mu_\alpha} = \delta_{\mu_\alpha}$ (Lemma 4.12). Consequently $\bigvee_{i=1}^n P_{y_i} < p_\alpha$. This proves Definition 3.6 (ii).

If y is a half-standard word with $d(y) > 0$ then $\varphi(P_y)\delta_{\mu_\beta} = 0$ for any β by (11). Consequently, P_y cannot be in $\mathcal{P}$ by Lemma 4.12. This verifies Definition 3.6 (i).

We are going to check that φ is faithful on standard projections. By Lemma 3.8 a nonzero standard projection p allows a representation $p = P_a \prod_{i=1}^{n}(1 - P_{b_i})$ with $P_{b_i} < P_a$ and $a \leq b_i$. Say that $a = tp_\alpha = tQ_t p_\alpha$ for $t \in \mathcal{T}_1$ and some $\alpha \in T^n$ according to Lemma 4.12. We may incorporate Q_t in p_α and assume that $\alpha_1 = t$. We have

$$\varphi(P_a)\delta_{t\mu_\alpha} = \delta_{t\mu_\alpha} \tag{18}$$

by (11) and Lemma 4.12. On the other hand, if $d(b_i) > d(a)$ then

$$\varphi(P_{b_i})\delta_{t\mu_\alpha} = 0 \tag{19}$$

by (11). If $d(b_i) > d(a)$ is not true then $b_i = tp_\beta$ (for some $p_\beta \in \mathcal{P}$) since $a \leq b_i$, and then, as $P_{b_i} < P_a$, i.e. $tp_\beta t^* < tp_\alpha t^*$, one has

$$t^* tp_\beta t^* t < t^* tp_\alpha t^* t = p_\alpha$$

(the last identity by the fact that $\alpha_1 = t$). Hence, $\varphi(p_\beta Q_t)\delta_{\mu_\alpha} = \varphi(p_\beta)\delta_{\mu_\alpha} = 0$ by Lemma 4.12. So also in this case we have (19). Identities (18) and (19) show that $\varphi(p) \neq 0$. By Corollary 2.15, φ is injective on the core.

Hence also the universal representation of X must be injective on the core. Since we have also checked that X is free, X is weakly free and cancelling by Propositions 3.9 and 3.10. Thus, by the Cuntz–Krieger uniqueness theorem (Theorem 2.17), φ is the universal representation, which implies $\mathcal{Q}(T) \cong \overline{\varphi(X)} \subseteq C_r^*(\Delta)$.

$C_r^*(\Delta)$ is generated by the operators $(\lambda_t)_{t\in T}$, since the operators λ_t are zero for $t \in \Delta_\mu$ (the composition ts is invalid in Δ for any element $t \in \Delta_\mu$). Consequently, the image of φ is dense in $C_r^*(\Delta)$ and so $\mathcal{Q}(T) \cong \overline{\varphi(X)} = C^*((\lambda_t)_{t\in T}) = C_r^*(\Delta)$. □

The next lemma is intended to serve as an example for a particular quell semigraph algebra. Let ζ_n be the graph induced by the skeleton consisting of one vertex ν and n arrows $s_1, \ldots, s_n$ starting and ending in this single vertex ν; n may be any cardinal number.

Lemma 4.14. *The quell semigraph C^*-algebra $\mathcal{Q}(\zeta_n)$ is the universal unital C^*-algebra generated by the free inverse semigroup (of partial isometries) of n generators $t_1, \ldots, t_n$ with the additional relations that the range projections of these generators are mutually orthogonal, i.e. $P_{t_i}P_{t_j} = 0$ for all $i \neq j$.*

Proof. Set $A = C^*(1, t_1, \ldots, t_n)$. In A, 1 is a unit and the words in the letters t_i form an inverse semigroup (where the inverse element should be the adjoint element in A, that is, inverse semigroup elements happen to be partial isometries); moreover $P_{t_i}P_{t_j} = 0$ for $i \neq j$. A is universal with respect to these relations. The quell semigraph C^*-algebra $\mathcal{Q}(\zeta_n)$ is a semigraph C^*-algebra (Lemma 4.10). We propose a homomorphism

$$\alpha : A \longrightarrow \mathcal{Q}(\zeta_n) : \alpha(1) = \nu,\ \alpha(t_i) = s_i.$$

Since the generators s_i form an inverse semigroup by Lemma 2.13, $P_{s_i}P_{s_j} = 0$ by Definition 4.3 (vi), and ν is a unit in $\mathcal{Q}(\zeta_n)$ by Definition 4.3 (iii) and the fact that $vs_i = s_iv = s_i$, the map α is a well defined homomorphism.

We propose an inverse homomorphism

$$\beta : \mathcal{Q}(\zeta_n) \longrightarrow A : \beta(\nu) = 1,\ \beta(s_{i_1} \dots s_{i_m}) = t_{i_1} \dots t_{i_m}.$$

A is generated by $\beta(\{1, s_1, \dots, s_n\})$, so β is surjective. We need to show that the relations of Definition 4.3, for $\beta(\zeta_n)$ rather than ζ_n, hold in A. Definition 4.3 (iii) is satisfied in the image of β as β is multiplicative. Definitions 4.3 (i)-(ii) and (iv)-(v) are obviously also correct in the image of β. Definition 4.3 (vi) is

$$x^*y = \begin{cases} \nu Q_x \nu = Q_x & \text{if } x = y \\ 0 & \text{if } \zeta_n^{(\min)}(x, y) = \emptyset \\ aQ_y\nu = aa^*x^*xa & \text{if } xa = y \text{ for some } a \in \zeta_n \end{cases}$$

The first case is tautological, the third one reduces to a tautology in an inverse semigroup, so holds in the image of β. The second case we demonstrate for $x = s_1s_2$ and $y = s_1s_3s_4$, say. One has $\beta(x)^*\beta(y) = \beta(x)^*\beta(x)\beta(x)^*\beta(y)\beta(y)^*\beta(y) = 0$ since

$$\begin{aligned} \beta(x)\beta(x)^*\beta(y)\beta(y)^* &= t_1t_2t_2^*(t_1^*t_1)t_3t_4t_4^*t_3^*t_1^* \\ &= t_1t_2t_2^*t_3t_4t_4^*t_3^*(t_1^*t_1)t_1^* = 0, \end{aligned}$$

where we have used inverse semigroup rules (commutativity of projections) and the fact that $P_{s_2}P_{s_3} = 0$. Since all relations of Definition 4.3 evidently hold in the image of β, β must be a well defined homomorphism. This proves the lemma as α and β are inverses to each other. □

5. K-theory of weakly free semigraph C^*-algebras

In this section we are going to compute the K-theory of a weakly free semigraph C^*-algebra by an application of [4, Theorem 2.2]. Since the setting of [4] is somewhat lenghty, we do not recall it here but directly apply it to semigraph algebras.

Definition 5.1. For a semigraph algebra X we say the *source projections cover the generators* if for every $p \in \mathcal{P}$ there is a $t \in \mathcal{T}$ such that $p \leq Q_t$.

Let X be a k-semigraph algebra. Assume that k is finite, the universal representation $X \longrightarrow C^*(X)$ is injective (we shall regard X as a subset of $C^*(X)$), and that the source projections cover the generators (Definition 5.1). Define $\mathcal{A}_i = \{\, tp \mid t \in \mathcal{T}^{(e_i)}, p \in \mathcal{P} \,\}\backslash\{0\}$ for $1 \leq i \leq k$.

Lemma 5.2. *X is generated by $\mathcal{A} := \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$.*

Proof. Let $t \in \mathcal{T}$. Then $t = t_1 \dots t_n = (t_1Q_{t_1}) \dots (t_nQ_{t_n})$ for certain $t_i \in \mathcal{T}$ with $d(t_i) = e_{i_j}$. So t is a product of elements of $\mathcal{A}$ as $Q_s \in \mathcal{P}$ for any $s \in \mathcal{T}$. Let $p \in \mathcal{P}$. Since the source projections cover the generators by assumption, there is a $t \in \mathcal{T}$ such that $p = pQ_t = pt^*tp = (tp)^*(tp)$. This is a product

of elements of $\mathcal{A}$ again as we may write tp as $tp = t_1(t_2p)$ where $t_1, t_2 \in \mathcal{T}$, $d(t_2) = e_i$, so $t_2p \in \mathcal{A}_i$, and t_1 may be further expanded as above. □

Note that by the unique factorisation property in $\mathcal{T}$ every standard word w may be written as

$$w = a_1 \dots a_n p b_m^* \dots b_1^* \tag{20}$$

for suitable letters a_i, b_j in $\bigcup_{i=1}^k \mathcal{T}^{(e_i)}$ and some $p \in \mathcal{P}$.

Define $\mathcal{X}$ to be $C^*(X)$. By Lemma 5.2 $\mathcal{X}$ is generated by a finitely partitioned alphabet $\mathcal{A}$. We have a gauge action (as defined in Definition 2.9) with respect to this alphabet on $\mathcal{X}$ and consequently a degree map determined by $d(a_i) = e_i$ for $a_i \in \mathcal{A}_i$. We define $\mathcal{S}$ to be the set of half-standard words. Their range projections commute by Corollary 2.14. The core is locally matrical by Corollary 2.15. We resolve the core by finite dimensional C^*-algebras as they are described in Corollary 2.15, and in particular choose D to be the set of all their diagonal entries. Since the diagonal elements of these matrices of the core are expressable as direct sums of standard projections (Corollary 2.15), and every standard projection can be written as a sum of range projections of half-standard words (Lemma 2.13), we have $D \subseteq P$, where we set

$$\begin{aligned} P &:= \operatorname{span}_{\mathbb{Z}}\{\, P_x \in \mathcal{X} \mid x \in \mathcal{S} \,\}, \\ Q &:= \operatorname{Alg}^*\{\, Q_x \in \mathcal{X} \mid x \in \mathcal{S} \,\}. \end{aligned}$$

We define W' to be the set of standard words. They linearly span X by Lemma 2.14. We have all requirements for [4, Theorem 2.2], except the technical conditions (a) and (b) from [4].

Lemma 5.3. *If X is weakly free then the technical conditions (a) and (b) from* [4] *hold.*

Proof. The proof is very similar (an almost word by word translation) of [4, Lemma 2.4]. So we ask the reader to prove it along the lines of [4, Lemma 2.4] by using representation (20) and Lemma 3.4, Lemma 3.5 and Lemma 2.14 where necessary. □

We have now all requirements for [4, Theorem 2.2] which states the following.

Theorem 5.4 ([4], Theorem 2.2)**.** *The identical embedding $\theta : C^*(Q) \longrightarrow \mathcal{X}$ induces an isomorphism $K_0(\theta)$, and $K_1(\mathcal{X}) = 0$.*

By (15) we have $Q = \operatorname{span}(\mathcal{P})$. Since the projections $\mathcal{P}$ commute, we have $K_0(C^*(Q)) = \operatorname{Ring}(\mathcal{P})$, where $\operatorname{Ring}(\mathcal{P})$ denotes the subring of $C^*(X)$ generated by $\mathcal{P}$, regarded then as an abelian group under addition. Since Q is a subset of the core, which is locally matrical, $C^*(Q)$ is an AF-algebra and thus $K_1(C^*(Q)) = 0$. Theorem 5.4 states that the K-theory of $C^*(X)$ is the K-theory of $C^*(Q)$, which we have now. Theorem 5.4, Lemma 5.3 and the above discussion now yield the following theorem.

Theorem 5.5. *Let X be a weakly free k-semigraph algebra (with $k < \infty$) whose universal representation $X \longrightarrow C^*(X)$ is injective. Suppose that the source projections cover the generators. Then the semigraph C^*-algebra has the following K-theory:*

$$K_1(C^*(X)) = 0, \qquad K_0(C^*(X)) \cong \mathrm{Ring}(\mathcal{P})$$

via $[p] \longleftrightarrow p$ for $p \in \mathcal{P}$. ($\mathrm{Ring}(\mathcal{P})$ denotes the subring of X generated by $\mathcal{P}$, regarded then as an abelian group under addition.)

We are going to write $\mathrm{Ring}(\mathcal{P})$ as a direct limit of subrings by using common refinements of projections in $\mathcal{P}$. For each finite subset $\mathcal{Q}$ of $\mathcal{P}$ we consider the subring $\mathrm{Ring}(\mathcal{Q})$ generated by $\mathcal{Q}$. This ring is generated by the base elements

$$p_{\mathcal{Q},A} = \prod_{p\in A} p \prod_{q\in\mathcal{Q}\backslash A} (1-q), \tag{21}$$

where A is a nonzero subset of $\mathcal{Q}$. Note that these base elements are mutually orthogonal for different A's. The projection $q_{\mathcal{Q},A}$ is nonzero if and only if

$$\Big(\prod_{p\in A} p\Big)(1-q) \neq 0 \tag{22}$$

for all $q \in \mathcal{Q}\backslash A$. This follows from Lemma 3.7. Let $\hat{\mathcal{Q}}$ denote the family of all subsets A of $\mathcal{Q}$ for which $p_{\mathcal{Q},A}$ is nonzero. Since the the $p_{\mathcal{Q},A}$'s are mutually orthogonal, we have $\mathrm{Ring}(\mathcal{Q}) = \mathbb{Z}^{|\hat{\mathcal{Q}}|}$. We write $\mathrm{Ring}(\mathcal{P})$ as a direct limit

$$\mathrm{Ring}(\mathcal{P}) = \varinjlim_{\mathcal{Q}\subseteq\mathcal{P}} \mathrm{Ring}(\mathcal{Q}) = \varinjlim_{\mathcal{Q}\subseteq\mathcal{P}} \mathbb{Z}^{|\hat{\mathcal{Q}}|}. \tag{23}$$

6. K-theory of quell semigraph C^*-algebras

We aim now to apply Theorem 5.5 to the quell semigraph C^*-algebra. To this end consider the quell semigraph algebra X associated to a finitely aligned k-semigraph T. We may go over to its image X' in $\mathcal{Q}(T)$, and write again X rather than X' for simplicity. This semigraph algebra is weakly free by Theorem 4.13 and Lemma 3.11. By (15) it is clear that the source projections cover the generators. We can thus apply Theorem 5.5 if k is finite. If k is infinite then we write $\mathcal{Q}(T)$ as the direct limit $\mathcal{Q}(T) \cong \varinjlim_{k_0} C^*(X^{(k_0)})$, where k_0 runs over the finite subsets of k, and $X^{(k_0)}$ denotes the sub-semigraph algebra of X which is generated by all elements of T which have degree zero at any coordinate outside of k_0 (same proof as Lemma 3.2). Again, in $X^{(k_0)}$ the source projections cover the generators by (15). Since X is weakly free, $X^{(k_0)}$ is also weakly free. So we can apply Theorem 5.5 to each $C^*(X^{(k_0)})$ and get

$$K_0(\mathcal{Q}(T)) = \varinjlim_{k_0} K_0(C^*(X^{(k_0)})) = \varinjlim_{k_0} \mathrm{Ring}(\mathcal{P}^{(k_0)}) = \mathrm{Ring}(\mathcal{P}). \tag{24}$$

Lemma 6.1. *If $s, t \in T$ then in $\mathcal{Q}(T)$ we have*

$$Q_t = Q_s \iff s = t, \tag{25}$$

$$Q_t < Q_s \iff s < t, \tag{26}$$

$$Q_s Q_t < Q_t \quad \text{if} \quad s \leq t \text{ is wrong} \tag{27}$$

Proof. If $Q_s = Q_t$ then by Lemma 4.12 $\varphi(Q_s)\delta_{\mu_s} = \delta_{\mu_s} = \varphi(Q_t)\delta_{\mu_s}$ and thus $t \leq s$. Similarly, $s \leq t$. We conclude that $s = t$ by Lemma 2.4.

If $Q_t < Q_s$ then by Lemma 4.12 $\varphi(Q_t)\delta_{\mu_t} = \delta_{\mu_t} = \varphi(Q_s)\delta_{\mu_t}$ and thus $s \leq t$. Since $s \neq t$, we have $s < t$. For the reverse implication, if $s < t$ then $t = \alpha s$ for some $\alpha \in T$. So $Q_t \leq Q_s$. Since $s \neq t$, we have $Q_t < Q_s$.

Suppose $Q_t Q_s = Q_t$. Then by Lemma 4.12 $\varphi(Q_t Q_s)\delta_{\mu_t} = \varphi(Q_t)\delta_{\mu_t} = \delta_{\mu_t}$. Hence $s \leq t$. □

Theorem 6.2. *Let T be a k-semigraph. Then the quell semigraph C^*-algebra $\mathcal{Q}(T)$ has the following K-theory. If T is finite then $C^*(T)$ is a finite dimensional C^*-algebra and*

$$K_0(\mathcal{Q}(T)) \cong \mathbb{Z}^{|\hat{\mathcal{P}}|}, \qquad K_1(\mathcal{Q}(T)) = 0. \tag{28}$$

If T is countably infinite then

$$K_0(\mathcal{Q}(T)) \cong \bigoplus_{\mathbb{N}} \mathbb{Z}, \qquad K_1(\mathcal{Q}(T)) = 0. \tag{29}$$

Proof. We have already obtained (24). We aim to analyse $\mathrm{Ring}(\mathcal{P})$ further. By (23) we need to estimate the size of the set $\hat{\mathcal{Q}}$ for a finite subset $\mathcal{Q}$ of $\mathcal{P}$.

Suppose T is finite. Then the set of standard words is a finite set. Since they span $\mathcal{Q}(T)$, $\mathcal{Q}(T)$ is finite dimensional. (28) follows from (23).

Now suppose that T is countably infinite. Suppose $t \in T$ with $d(t) = n\delta_l$ for an $n \in \mathbb{N}$ and some $l \in k$. Then set $\mathcal{Q}_m = \{Q_{t(0,i\delta_l)} \,|\, 1 \leq i \leq m\}$ for $1 \leq m \leq n$. Note that $\mathcal{Q}_m \subseteq \mathcal{Q}_n$. Then $p_{\mathcal{Q}_n,\mathcal{Q}_m}$ of (21) is nonzero. Indeed, for every $m < j \leq n$ (22) is

$$\Big(\prod_{i=1}^{m} Q_{t(0,i\delta_l)}\Big)(1 - Q_{t(0,j\delta_l)}) = Q_{t(0,m\delta_l)}(1 - Q_{t(0,j\delta_l)}), \tag{30}$$

which is nonzero by (27). Thus every $\mathcal{Q}_m$ is an element of $\hat{\mathcal{Q}}_n$. Consequently, $|\hat{\mathcal{Q}}_n| \geq n$ (or even exactly n as one may check). So, if k is finite then we will find a sequence of t's with $d(t)_l \to \infty$ for some coordinate $l \in k$ and so (23) shows (29).

So suppose finally the case that $k = \mathbb{N}$ and there is an infinite family of t_i's such that $d(t_i) = \delta_i$ for all $i \in \mathbb{N}$. Set $\mathcal{Q}_n = \{Q_{t_1}, \ldots, Q_{t_n}\}$ for $n \in \mathbb{N}$. Set $A_m = \{Q_{t_m}\}$ for $1 \leq m \leq n$. Then $A_m \in \hat{\mathcal{Q}}_n$ since $p_{\mathcal{Q}_n,A_m}$ is nonzero. Indeed, (22) is $Q_{t_m}(1 - Q_{t_j})$, which is nonzero by (27). So we have $|\hat{\mathcal{Q}}_n| \geq n$. Again, (23) shows (29). □

References

[1] T. Bates and D. Pask. C^*-algebras of labelled graphs. *J. Oper. Theory*, 57(1):207–226, 2007.
[2] B. Brenken and Z. Niu. The C^*-algebra of a partial isometry. *Proc. Am. Math. Soc.*, pages 1–8, 2011.
[3] B. Burgstaller. A Cuntz–Krieger uniqueness theorem for semigraph C^*-algebras. preprint. arXiv:1111.4166.
[4] B. Burgstaller. The K-theory of certain C^*-algebras endowed with gauge actions. *J. Funct. Anal.*, 256(6):1693–1707, 2009.
[5] I. Cho and P. Jorgensen. C^*-subalgebras generated by partial isometries. *J. Math. Phys.*, 50(2), 2009.
[6] A. Kumjian and D. Pask. Higher rank graph C^*-algebras. *New York J. Math.*, 6:1–20, 2000.
[7] I. Raeburn and A. Sims. Product systems of graphs and the Toeplitz algebras of higher-rank graphs. *J. Oper. Theory*, 53(2):399–429, 2005.
[8] M. Tomforde. A unified approach to Exel-Laca algebras and C^*-algebras associated to graphs. *J. Oper. Theory*, 50(2):345–368, 2003.
[9] A. S. Toms. On the classification problem for nuclear C^*-algebras. *Ann. Math.*, 167(3):1029–1044, 2008.

Doppler Institute for mathematical physics, Trojanova 13, 12000 Prague, Czech Republic

E-mail address: bernhardburgstaller@yahoo.de