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MAXIMUM OCCUPATION TIME OF A TRANSIENT EXCITED RANDOM WALK ON $\mathbb{Z}$

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Abstract. We consider a transient excited random walk on $\mathbb{Z}$ and study the asymptotic behavior of the occupation time of a currently most visited site. In particular, our results imply that, in contrast to the random walks in random environment, a transient excited random walk does not spend an asymptotically positive fraction of time at its favorite (most visited up to a date) sites.

1. Introduction and statement of results

Excited random walks, also called random walks in cookie environments, are modifications of nearest-neighbor simple random walk in which, during the first several visits to each site of the integer lattice, the jump kernel may prefer one direction and assigns equal probabilities to the remaining $(2d-1)$ directions. If the current location of the random walk has been already visited more than a certain number of times, then the walk moves to one of its nearest neighbors with equal probabilities. The model was introduced by Benjamini and Wilson in [3] and extended by Zerner in [16].

In this paper we focus on the excited random walks in dimension one. To define the transition mechanism of the random walk, fix an integer $M \in \mathbb{N}$ and let

$$\begin{aligned}\Omega_M \;=\; \Big\{&\omega = (\omega(z,i))_{z\in\mathbb{Z},\, i\in\mathbb{N}} : \\ &\omega(z,i) \in [0,1] \text{ for } z \in \mathbb{Z},\ 1 \le i \le M, \\ &\text{and } \omega(z,i) = 1/2 \text{ for } z \in \mathbb{Z},\ i > M\Big\}.\end{aligned}$$

The value of $\omega(z,i)$ determines the probability of the jump from z to $z+1$ upon the i-th visit of the random walk to the site $z \in \mathbb{Z}$. The random walk is assumed to be nearest neighbor, and hence the probability of the jump from z to $z-1$ upon its i-th visit to z is given by the complementary probability $1-\omega(z,i)$. The elements of the set Ω_M are called *cookie environments*.

For a fixed $z \in \mathbb{Z}$, $\big(\omega(z,i)\big)_{1\le i\le M}$ can be thought of as a sequence of numerical characteristics called "strengths", associated with a pile of M "cookies"

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placed at z. Correspondingly, $\omega(z,i)$ is referred to as the *strength of the i-th cookie* at the pile. With a slight abuse of language, using the above introduced jargon we will often identify the strength $\omega(z,i)$ of a cookie with the cookie itself. The transition kernel of the random walk can be informally described as follows: while the supply of the cookies at a given site lasts, the walker eats a cookie upon each visit there and then makes one step in a random direction, such that the probability of moving to the right is equal to the "strength" of the cookie just eaten.

More precisely, the random walk in a cookie environment $\omega \in \Omega_M$ is defined as follows. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let $\Sigma = \mathbb{Z}^{\mathbb{N}_0}$ be the state space of the infinite paths of a discrete-time random walk on $\mathbb{Z}$ ($\mathbb{Z}$ for the location and $\mathbb{N}_0$ for the time), and let $\mathcal{F}$ be its Borel σ-algebra (i. e., the σ-algebra generated by the cylinder sets of the infinite product space Σ). For any $x \in \mathbb{Z}$ and $\omega \in \Omega_M$, an excited random walk (abbreviated in what follows as ERW) starting at $x \in \mathbb{Z}$ in the cookie environment ω is a sequence of random variables $X = \big(X_n\big)_{n\in\mathbb{N}_0}$ defined on $(\Sigma, \mathcal{F}, P_{x,\omega})$ such that $P_{x,\omega}(X_0 = x) = 1$ and

$$P_{x,\omega}(X_{n+1} = X_n + 1 | \mathcal{F}_n) = \omega(X_n, \ell_n),$$

where $\mathcal{F}_n := \sigma(X_i,\, 0 \le i \le n)$ and

$$\ell_n := \#\big\{0 \le i \le n : X_i = X_n\big\} \tag{1}$$

is the number of visits to the current site by time n. The measure $P_{x,\omega}$ is usually referred to as the *quenched law* of the excited random walk in the cookie environment ω. Let $\mathbb{P}$ be a probability measure on Ω_M that makes the collection of "piles" $\omega_z := (\omega(z,i))_{i\in\mathbb{N}}$ indexed by $z \in \mathbb{Z}$ into an i.i.d. sequence. Notice that we do not insist on the independence of the cookies within a given pile, that is the random variables $\omega(z,i)$ for a fixed $z \in \mathbb{Z}$ can be dependent under $\mathbb{P}$. The (associated with $\mathbb{P}$) *annealed* (average) law P_x of the ERW on $(\Sigma, \mathcal{F})$ is defined by setting $P_x(\,\cdot\,) = \mathbb{E}[P_{x,\omega}(\,\cdot\,)]$, where $\mathbb{E}$ is the expectation induced by the probability law $\mathbb{P}$.

Many important aspects of the asymptotic behavior of excited random walks on $\mathbb{Z}$ are by now well-understood. In particular, Zerner in [16], Basdevant and Singh in [1, 2], Kosygina and Zerner in [12], and Kosygina and Mountford in [11] characterized the recurrence-transience behavior and possible speed regimes of the ERW, and proved limit theorems for the fluctuations of the current location of the ERW. The goal of this paper is to study the asymptotic dynamics of the occupation time of a currently most visited site for a transient ERW.

The consumption of a cookie $\omega(z,i)$ creates the local drift (i.e., the bias in the conditional on the history $\mathcal{F}_n$ expectation of the subsequent displacement) equal to $E_{x,\omega}\big[X_{n+1} - X_n \;\big|\; X_n = z, \ell_n = i\big] = 2\omega(z,i) - 1$. Let

$$\delta = \mathbb{E}\Big[\sum_{i=1}^{M}\big(2\omega(z,i) - 1\big)\Big]$$

be the annealed expectation of the total drift (i.e., the "boost" in the positive direction) available to the random walk at a site $z \in \mathbb{Z}$. Since the piles ω_z are assumed i.i.d., δ is independent of z. Following [12] and [11] (and in contrast to the original version presented in [3, 16]) we do not impose the condition that $\mathbb{P}\big(\omega(z,i) \geq 1/2\big) = 1$, thereby allowing both "positive" and "negative" cookies. It turns out (see [11, 12]) that the asymptotic behavior of a one-dimensional excited random walk is largely determined by the value of the parameter δ. In particular, under a mild non-degeneracy assumption on the cookie environment, the random walk is transient to the right if and only if $\delta > 1$.

Throughout this paper we will impose the following conditions on the cookie environment.

Assumption 1.1. The following assumptions hold:

(a) Independence: the sequence of "piles" $\omega_z = (\omega(z,i))_{i\in\mathbb{N}}$ indexed by sites $z \in \mathbb{Z}$ is an i.i.d. sequence under $\mathbb{P}$.
(b) Non-degeneracy: $\mathbb{E}\big[\prod_{i=1}^{M} \omega(0,i)\big] > 0$ and $\mathbb{E}\big[\prod_{i=1}^{M}(1-\omega(0,i))\big] > 0$.
(c) Transience: $\delta > 1$.

The non-degeneracy condition rules out the two one-sided degenerate cases at the level of cookie piles and is the standard ellipticity-type assumption used in the one-dimensional ERW transience theory. It is known [12, 16] that under Assumption 1.1 the ERW is transient to the right (that is, $P_0\big(\lim_{n\to\infty} X_n = \infty\big) = 1$) and, furthermore, has the asymptotic speed

$$\mathrm{v} := \lim_{n\to\infty} \frac{X_n}{n} \in [0,1), \tag{2}$$

which is strictly positive if and only if $\delta > 2$.

Define the occupation time of the ERW at site $x \in \mathbb{Z}$ as

$$\xi_n(x) := \#\big\{0 \leq i \leq n \,:\, X_i = x\big\}. \tag{3}$$

Thus $\xi_n(x)$ is the number of times that the ERW visits $x \in \mathbb{Z}$ during the first n steps. Let

$$\xi_n^* := \max_{x\in\mathbb{Z}} \xi_n(x) \tag{4}$$

be the largest number of visits to a single site during the first n steps. For the sake of notational convenience we will occasionally write $\xi^*(n)$ for ξ_n^*. The asymptotic properties of the process $\xi^* := (\xi_n^*)_{n\in\mathbb{N}}$ can be compared to those of the simple random walk as well as of a random walk in random environment (abbreviated in what follows as RWRE) in dimension one. For a comprehensive up-to-date review of the former topic, see a monograph of Révész [14]. For more recent developments on RWRE, we refer to [6, 8, 9] and references therein.

This paper is devoted to the study of the limit points of the sequence ξ^* for a transient ERW. The approach adopted here is based on a reduction of the study of the asymptotic behavior of the ERW to that of a branching process

and the subsequent reformulation of the problem in terms of the asymptotic dynamics of the most populated generation of the branching process. Similar questions for transient RWRE have been addressed in the interesting paper by Gantert and Shi [8]. The essential branching processes machinery which enables the implementation of our approach to excited random walks was introduced in [1, 2] and further developed in [11, 12].

Our first result concerns non-ballistic ERW.

Theorem 1.2. *Suppose that Assumption 1.1 holds with $\delta \in (1,2]$. Then,*

(i) *The following holds:*

$$\limsup_{n\to\infty} \frac{\xi^*_n}{n^{1/2}} > 0, \qquad P_0 - \text{a. s.} \tag{5}$$

(ii) *Furthermore, for any $\alpha > \frac{1}{\delta}$:*

$$\lim_{n\to\infty} \frac{\xi^*_n}{n^{1/2}(\log n)^\alpha} = 0 \qquad \text{while} \qquad \lim_{n\to\infty} \frac{(\log n)^\alpha \xi^*_n}{n^{1/2}} = \infty, \qquad P_0 - \text{a. s.}$$

The above theorem implies in particular that unlike RWRE (see [8]), a non-ballistic transient ERW does not spend a positive fraction of time at its favorite sites. While the asymptotic behavior of ξ^*_n for transient RWRE seems to be determined by the so-called "traps" created by a random potential (cf. [8] and [6]), and is radically different from that of the simple unbiased random walk, the limsup asymptotic of ξ^*_n for a non-ballistic transient ERW turns out to be rather similar to its counterpart for a simple unbiased random walk (cf. Theorem 11.3 in [14]). We also remark that, based on a comparison with the latter, we believe that in fact $\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/2}} = \infty$ under the conditions of Theorem 1.2, but we are unable to prove this conjecture.

The next theorem deals with the asymptotic behavior of $\xi^*(n)$ for ballistic ERW. Recall that, for $\rho \in \mathbb{R}$, a sequence $\phi : \mathbb{N} \to \mathbb{R}$ is *regularly varying (at infinity) of index* ρ if $\phi(n) = n^\rho L(n)$ for some sequence $L : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} L(\lfloor \lambda n \rfloor)/L(n) = 1$ for all $\lambda > 0$ (that is, L is *slowly varying*). Here and henceforth, $\lfloor x \rfloor$ denotes the integer part of a real number x, that is $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}$.

Theorem 1.3. *Suppose that Assumption 1.1 holds with $\delta > 2$.*

(i) *Let $\phi : \mathbb{N} \to \mathbb{R}$ be a regularly varying sequence of a positive index $\rho > 0$. Then,*

$$\limsup_{n\to\infty} \frac{\xi^*_n}{\phi(n)} = \begin{cases} 0 \\ \infty \end{cases} \quad P_0 - \text{a. s.} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{[\phi(n)]^\delta} \begin{cases} < \infty \\ = \infty. \end{cases}$$

In particular, for any $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{\xi^*_n}{n^{1/\delta}(\log n)^\alpha} = 0, \qquad P_0 - \text{a. s.},$$

and hence $\lim_{n\to\infty} \frac{\xi^*_n}{n} = 0$, $P_0 - \text{a. s.}$

(ii) *For* $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{(\log n)^\alpha \xi^*_n}{n^{1/\delta}} = \infty, \qquad P_0 - \text{a. s.}$$

The rest of the paper is organized as follows. In Section 2, we consider an auxiliary branching process formed by successive level crossings along the random walk path. The proofs of Theorems 1.2 and 1.3 are contained in Section 3.

2. Reduction to a branching process

The proof of our main results, given in this section, relies on the use of a mapping of the paths of ERW into realizations of a suitable branching process with migration. In this section we discuss the branching process framework and recall some auxiliary results related to it; see [11, 12] or expository surveys [7, 13] for more details.

For $m \in \mathbb{N}$, let T_m be the first hitting time of site m, that is

$$T_m = \inf\{n \in \mathbb{N} : X_n = m\}.$$

Since the ERW is transient to the right under Assumption 1.1, the random variables T_m are almost surely finite for all $m \in \mathbb{N}$ under the law P_0. Moreover, it follows from (2) (by passing to the random subsequence of indices $n = T_m$ in (2)) that

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{m\to\infty} \frac{T_m}{X_{T_m}} = \mathrm{v}^{-1} \in [1, \infty], \qquad P_0 - \text{a. s.} \tag{6}$$

Set now $D^m_m := 0$ and for $k \le m-1$ let

$$D^m_k = \sum_{i=0}^{T_m - 1} \mathbf{1}_{\{X_{i+1}=k-1,\, X_i = k\}}$$

be the number of down-crossing steps of the ERW from site k to $k-1$ before time T_m. Then (see, for instance, [8, 12]),

$$T_m = m + 2\sum_{k\le m} D^m_k = m + 2\sum_{0\le k\le m} D^m_k + 2\sum_{k<0} D^m_k. \tag{7}$$

It follows from (3) that

$$\xi_{T_m}(k) = \begin{cases} 0 & \text{for} \quad k > m, \\ 1 & \text{for} \quad k = m, \\ D^m_k + D^m_{k+1} + \mathbf{1}_{\{k\ge 0\}} & \text{for} \quad k < m, \end{cases}$$

and hence

$$\begin{aligned} \max_{0\le k\le m} D^m_k &\le \xi^*_{T_m} = \max\Big\{1, \max_{k<m}\big(D^m_k + D^m_{k+1} + \mathbf{1}_{\{k\ge 0\}}\big)\Big\} \\ &\le 1 + 2\max_{0\le k\le m} D^m_k + 2\max_{k<0} D^m_k. \end{aligned} \tag{8}$$

Notice that $\max_{k<0} D^m_k$ is bounded above by the total time spent by the random walk on the negative half-line. Since the random walk is transient to

the right, the latter quantity is P_0-a. s. finite. Therefore, for any eventually increasing non-negative sequence $\phi : \mathbb{N} \to \mathbb{R}_+$, we have

$$\limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k^m}{\phi(m)} \quad \le \quad \limsup_{m\to\infty} \frac{\xi^*_{T_m}}{\phi(m)} \le 2 \limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k^m}{\phi(m)}. \tag{9}$$

Similar inequalities hold with the lim sup replaced by the lim inf .

To elucidate the probabilistic structure of the sequence D_k^m, it is convenient to exploit the following alternative definition of the random walk $(X_n)_{n\ge 0}$. Assume that the underlying probability space is enlarged to include a sequence of, conditionally on ω, independent Bernoulli random variables ("coins") $\big(B(z,i)\big)_{i\in\mathbb{N},z\in\mathbb{Z}}$ such that

$$P_{0,\omega}\big(B(z,i)=1\big)=\omega(z,i) \qquad \text{and} \qquad P_{0,\omega}\big(B(z,i)=-1\big)=1-\omega(z,i). \tag{10}$$

Then the ERW X can be alternatively defined by specifying the jump sequence recursively, as follows:

$$X_{n+1} = X_n + B\big(X_n, \ell_n\big), \tag{11}$$

where ℓ_n is introduced in (1). We adopt the terminology of [11, 12] and refer to the event $\{B(z,i)=1\}$ as a "success" and to the event $\{B(z,i)=-1\}$ as a "failure". For $z\ge 0$, denote by $F_m^{(z)}$ the number of failures before the m-th success in the sequence $B^{(z)} := \big(B(z,i)\big)_{i\in\mathbb{N}}$. Let $V := (V_k)_{k\ge 0}$ be a Markov chain on $\mathbb{N}_0$ with transition kernel defined (under the law P_0) by means of the following recursion:

$$V_{k+1} = F^{(k)}_{V_k+1}, \qquad k\ge 0.$$

The process V can be thought of as a branching process with the following properties:

(1) There is exactly one immigrant in each generation and the immigration happens before the reproduction.
(2) The number of offspring of m-th individual in generation $k\in\mathbb{N}$, which we denote by $W_m^{(k)}$, is equal to $F_m^{(k)} - F_{m-1}^{(k)}$. For future reference, we write

$$W_m^{(k)} = F_m^{(k)} - F_{m-1}^{(k)}. \tag{12}$$

For non-negative reals $x\ge 0$, denote by P_x^V the law of the process V that starts with $\lfloor x\rfloor$ individuals in the generation zero. It turns out that for every $n\ge 0$, the distribution of $(V_0, V_1, ..., V_n)$ under P_0^V coincides with the distribution of the array $(D_n^n, D_{n-1}^n, ..., D_0^n)$ associated with a transient to the right ERW (see, for instance, Section 2 in [11]):

$$(V_0, V_1, ..., V_n) \stackrel{d}{=} \big(D_n^n, D_{n-1}^n, ..., D_0^n\big) \qquad \forall\ n\ge 0. \tag{13}$$

Under Assumption 1.1, X is transient to the right, and hence there exists an infinite sequence of times when the random walk moves forward to a "fresh point", i. e. to a site which has been never visited before and to which it will never return afterwards [12, 16]. It turns out [1, 12] that for the

branching process V this implies that the following random times are finite with probability one under the law P_0^V :

$$\sigma_{-1} := 0 \quad \text{and} \quad \sigma_k := \inf\big\{i > \sigma_{k-1} : V_i = 0\big\}, \quad k \geq 0. \tag{14}$$

Thus $(\sigma_k)_{k\geq 0}$ is the sequence of renewal times in which the extinction occurs and the branching process starts afresh due to the immigration. Notice that while the immigrants serve as "founders of dynasties" of descendants, they themselves are not counted in the population of the branching process. In what follows we refer to the part of the branching process evolving between two successive extinction times as a *life cycle* of the process. The difference $\sigma_k - \sigma_{k-1}$, $k \geq 0$, represents therefore the duration of the $(k+1)$-th life cycle. We remark that, although under our assumptions the event $\sigma_k - \sigma_{k-1} = 1$ can happen with a positive probability for any $k \geq 0$, the branching process does not get absorbed at zero, and is eventually revived in a future generation with a strictly positive number of immigrants. Let

$$\varrho_m := \min\big\{k \geq 0 : \sigma_k \geq m\big\} \tag{15}$$

denote the index of the first renewal epoch at or after generation m.

Further, let

$$S_k := \sum_{i=\sigma_{k-1}}^{\sigma_k - 1} V_i, \qquad k \geq 0,$$

be the total population present during the $(k+1)$-th cycle, and let

$$M_k := \max_{\sigma_{k-1} \leq i < \sigma_k} V_i, \qquad k \geq 0,$$

be the size of the most populated generation in the $(k+1)$-th cycle. Notice that the sequence $(\sigma_k - \sigma_{k-1})_{k\geq 0}$ as well as the sequence of pairs $(S_k, M_k)_{k\geq 0}$ are i.i.d. under P_0^V. The following asymptotic results hold under Assumption 1.1 (see, for instance, Theorem 2.1, Theorem 2.2, and Lemma 8.1 in [11], respectively):

$$\lim_{n\to\infty} n^{\delta/2} P_0^V(S_0 > n) = K_0 \in (0, \infty), \tag{16}$$

$$\lim_{n\to\infty} n^{\delta} P_0^V(\sigma_0 > n) = K_1 \in (0, \infty), \tag{17}$$

$$\lim_{n\to\infty} n^{\delta} P_0^V(M_0 > n) = K_2 \in (0, \infty). \tag{18}$$

In a few instances, it will be convenient to consider the following stationary version of the branching process. Since the random walk X_n is transient to the right, the Markov chain V_n converges in law to a stationary distribution V_∞ which is the total number of steps left from the site 0 to site -1 for the whole infinite duration of the random walk (see, for instance, [1, p. 631]). Notice that V_∞ is finite with probability one because the walk is transient to the right. Let $Y_{k,n}$ denote the number of descendants in generation n of the immigrant that arrived at time k. To be consistent with the description of the random walk, we enumerate the population in generation n so that,

whenever $k < \ell \le n$, every descendant of the generation-ℓ immigrant has a smaller label than every descendant of the generation-k immigrant. Then,

$$V_n = \sum_{k=0}^{n-1} Y_{k,n}. \tag{19}$$

This labeling convention corresponds to the assumption that at each site $z \ge 0$, the coins $B(z,i)$ are used by the random walk in the order determined by the second index i. In particular, (cf. (13)) (12) ensures that

$$\left(D_k^{(k+1)}, D_k^{(k+2)} - D_k^{(k+1)}, \dots, D_k^{(n+1)} - D_k^{(n)}\right) \stackrel{d}{=} (Y_{n-k,n-k}, Y_{n-k-1,n-k}, \dots, Y_{0,n-k})$$

whenever $1 \le k < n$.

One can introduce "dummy" immigrants arriving at negative times (as it is done, for instance, in [10, 15]) and consider the stationary process

$$\widetilde{V}_n = \sum_{k=-\infty}^{n} Y_{k,n}, \qquad n \in \mathbb{Z}, \tag{21}$$

where $\widetilde{V}_n$ is distributed as V_∞ for each $n \in \mathbb{Z}$. The stationary version of the branching process is useful because

$$\left(\widetilde{V}_{-n}\right)_{n\ge 1} \stackrel{d}{=} (D_n)_{n\ge 1}, \tag{22}$$

where $X \stackrel{d}{=} Y$ indicates that random variables (sequences in this specific case) X and Y have the same distribution, and for $k \ge 0$,

$$D_k := \lim_{m\to\infty} D_k^m = \sum_{i=0}^{\infty} \mathbf{1}_{\{X_{i+1}=k-1,\, X_i=k\}} \tag{23}$$

is the total number of down-crossing steps of the ERW from site k to $k-1$.

For $n \le 0$ and integer $k \in [n, 0]$, let

$$Z_k^n = \sum_{m=n}^{k} Y_{m,k}. \tag{24}$$

Then, for any fixed $n \ge 0$, (cf. (13) and (20)):

$$\left(D_n^n, D_{n-1}^n, ..., D_0^n\right) \stackrel{d}{=} \left(Z_{-n}^{-n}, Z_{-n+1}^{-n}, ..., Z_0^{-n}\right) \stackrel{d}{=} (V_0, \dots, V_n). \tag{25}$$

Most of the time, the identity in (25) will be enough for our purposes. However, occasionally we will use the following stronger version:

$$\left(D_k^n\right)_{n\ge 0,\, 0\le k\le n} \stackrel{d}{=} \left(Z_{-n+k}^{-n}\right)_{n\ge 0,\, 0\le k\le n}. \tag{26}$$

The last identity is the point at which one has to distinguish carefully between a fixed-level branching-process representation and a process-level representation. We record the precise form of this passage, together with the renewal convention used later in the proof.

Lemma 2.1 (Triangular representation and renewal transfer)**.** *Under Assumption 1.1, the equality in distribution in (26) holds as an identity of the whole triangular array, not only row by row. Equivalently, for every $N \in \mathbb{N}$,*

$$\left(D_k^n\right)_{0\le n\le N,\,0\le k\le n} \overset{d}{=} \left(Z_{-n+k}^{-n}\right)_{0\le n\le N,\,0\le k\le n}.$$

Furthermore, let $\tau = \inf\{j \ge 0 : \widetilde{V}_j = 0\}$ for the stationary process $\left(\widetilde{V}_j\right)_{j\in\mathbb{Z}}$ and let

$$\mathcal{C}_i = (\sigma_i - \sigma_{i-1}, S_i, M_i)$$

be the successive cycle triples after a zero of the branching process. The sequence of cycle triples strictly after the finite residual piece ending at τ has the same law as $\left(\mathcal{C}_i\right)_{i\ge 0}$ under P_0^V. Consequently, if an event depending on a one-sided sequence of cycles is unchanged by deleting finitely many initial cycles, then its probability is the same whether the cycles are read from the stationary construction after the residual cycle or from the process V under P_0^V.

Proof. The first assertion follows from the coin construction. For a fixed terminal level n, the down-crossing vector is obtained by running the recursion

$$D_n^n = 0, \qquad D_k^n = F^{(k)}_{D_{k+1}^n+1}, \qquad 0 \le k < n,$$

with the failure-before-success variables determined by the stacks of coins at the sites $0, \dots, n-1$. If the same construction is written with the space-time labels shifted so that the terminal level n is placed at time 0, the descendants of immigrants arriving at times $-n, -n+1, \dots, 0$ give exactly the variables Z_{-n+k}^{-n}. For finitely many terminal levels $0 \le n \le N$, both sides are measurable functions of finitely many coin stacks and the same deterministic failure-before-success maps. The i.i.d. assumption on the cookie piles then permits the common relabeling of the stacks, and hence gives the displayed finite-dimensional equality. Since N is arbitrary, (26) holds as an equality in distribution of triangular arrays.

For the second assertion, observe that zeros of $\widetilde{V}$ are regeneration times. The time τ is finite almost surely because $\widetilde{P}^V(\widetilde{V}_0 = 0) > 0$ and the stationary chain visits zero with positive frequency. By the strong Markov property at τ and by the independence of the cookie piles to the right of that regeneration point, the cycles following τ are independent and have the same common law as the cycles of the branching process started from zero. The part of the stationary process before this first regeneration time is a single finite residual cycle. Therefore deleting it, or deleting any other finite number of initial cycles, cannot change limsup, liminf, or infinitely-often events whose normalizing sequences tend to infinity and whose cycle windows tend to infinity. This proves the transfer statement. □

We will denote by $\widetilde{P}_\omega^V$ and $\widetilde{P}^V$, respectively, the quenched (in a fixed environment ω) and annealed (in a random environment) law of the stationary

branching process. Notice that

$$P_0^V(\,\cdot\,) = \widetilde{P}^V\big(\,\cdot\,\big|\widetilde{V}_0 = 0\big). \tag{27}$$

It follows, for instance from (22), that $\widetilde{P}^V\big(\widetilde{V}_0 = 0\big) > 0$, and in fact $\widetilde{P}^V_\omega\big(\widetilde{V}_0 = 0\big) > 0$ with probability one.

We shall use the notation σ_k, ϱ_k, S_k, and M_k also for cycles read from the stationary process, but with the convention made explicit in Lemma 2.1. The cycle intersecting the time origin in the stationary process is a finite residual cycle and need not have the same distribution as a fresh cycle under P_0^V. After the first zero at or to the right of the origin, however, the future cycle triples are i. i. d. and have the same law as (σ_0, S_0, M_0) under P_0^V. Thus the almost-sure limsup, liminf, and infinitely-often events used below may be evaluated under P_0^V after applying the triangular representation (26), because adding or deleting this finite residual piece does not affect those events.

3. Proof of the main results

We begin with a brief outline of the proof. First, using properties of the regular variation and the tail asymptotic of the renewal times which is given by (17), we reduce the study of ξ_n^* to that of $\xi_{T_n}^*$. The bounds for the latter sequence, stated in (8), enable us then to exploit the connection between the random walk and the branching process V introduced in Section 2. We remark that a similar strategy has been used for RWRE in [8]. The implementation of this approach (which at most stages is technically fairly different in this paper from the one presented in [8]) is based on the existence of the renewal structure (life cycles) for the branching process and the asymptotic results for the distributional tails of the key random variables stated in (16)-(18) above.

A technical point used several times below is that fixed-level identities such as (25) are not, by themselves, enough for limsup or Borel-Cantelli arguments over infinitely many hitting levels. Whenever such an event is transferred from the ERW to a branching process, the transfer is through the triangular-array identity and renewal convention of Lemma 2.1.

We start the proof with the following $0-1$ law for the maximal occupation time of the random walk. A similar statement for one-dimensional random walks in random environment is given in [8, Proposition 3.1].

Lemma 3.1. *Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be an unbounded eventually increasing function such that, for every fixed integer r,*

$$\lim_{n\to\infty} \frac{\phi(n+r)}{\phi(n)} = 1.$$

Then

$$\limsup_{n\to\infty} \frac{\xi_n^*}{\phi(n)} = c_\phi \in [0,\infty] \qquad P_0 - \text{a. s.}$$

Proof. Set

$$L(X) := \limsup_{n\to\infty} \frac{\xi^*_n}{\phi(n)},$$

where the local times are computed along the infinite path X. We first show that, for every fixed $k \geq 0$, the value of $L(X)$ is measurable with respect to the future half-line coin stacks $\sigma(B^{(k)}, B^{(k+1)}, \ldots)$. Let N_k be the total number of times spent by the walk in $(-\infty, k-1]$. Under Assumption 1.1, $N_k < \infty$, P_0-almost surely. If $X^{(k)}$ denotes the path obtained by deleting from X all visits to $(-\infty, k-1]$, then the time parameter of $X^{(k)}$ differs from the original time parameter by the fixed finite amount N_k once the last visit to $(-\infty, k-1]$ has occurred. Moreover, the local times at sites below k are bounded by N_k, while the local times at sites at least k are the local times read from $X^{(k)}$ after adding only finitely many initial visits. Since $\phi(n) \to \infty$ and (28) holds, these finite changes in time and in local times do not change the value of the limsup. Consequently $L(X)$ is determined by $X^{(k)}$.

Following [16] (see the paragraph right before Lemma 3 there), define recursively a sequence of random times $(\eta_{k,m})_{m\in\mathbb{N},k\in\mathbb{Z}}$ by setting

$$\eta_{k,0} := -1 \qquad \text{and} \qquad \eta_{k,m+1} := \inf\{n > \eta_{k,m} : X_n \geq k\}.$$

For a fixed $k \in \mathbb{N}$, the sequence $X^{(k)} = (X_{\eta_{k,m}})_{m\geq 1}$ is the path obtained by retaining only the visits to $\{x \in \mathbb{N} : x \geq k\}$. Zerner's argument [16, Lemma 3] applies here because Assumption 1.1 implies transience to the right, and gives that the quenched law of $X^{(k)}$ in a fixed environment ω is independent of $(\omega_i)_{i\leq k-1}$. In the coin construction this retained path is measurable with respect to

$$\sigma\big(B^{(k)}, B^{(k+1)}, B^{(k+2)}, \ldots\big), \qquad k \geq 0. \tag{28}$$

It follows that, for $\mathbb{P}$-almost every environment ω, the random variable $L(X)$ is a tail random variable of the independent sequence of coin stacks under $P_{0,\omega}$. Kolmogorov's 0-1 law therefore implies that $L(X)$ is $P_{0,\omega}$-almost surely equal to a deterministic value depending only on the environment; denote this value by $L(\omega)$.

The preceding paragraph holds for every k. Hence, for every rational $q \geq 0$, the event $\{L(\omega) \leq q\}$ belongs to $\sigma(\omega_k, \omega_{k+1}, \ldots)$ for every k. Equivalently, it belongs to the tail σ-field of the i.i.d. sequence of cookie piles. Kolmogorov's 0-1 law for the environment implies that $\mathbb{P}(L \leq q) \in \{0, 1\}$ for every rational q. The distribution function of L is therefore a step function with a single jump, possibly at ∞, and L is P_0-almost surely constant. This proves the lemma. □

3.1. **Non-ballistic regime: Proof of Theorem 1.2.**

Part (i). We first consider the case when $\delta \in (1, 2)$. The remaining case $\delta = 2$ is exceptional and is treated separately in Lemma 3.3 below.

Lemma 3.2. *Assumption 1.1 implies (5) when $\delta \in (1,2)$.*

Proof of Lemma 3.2. In order to prove the claim of the lemma, it suffices to show that for all $\delta \in (1,2)$ and any constant $c > 0$ which is small enough we have

$$\liminf_{n\to\infty} P_0\big(\xi^*_{T_n} \geq c\sqrt{T_n}\big) > 0. \tag{29}$$

Indeed, (29) together with the reverse Fatou's lemma imply that

$$P_0\Big(\limsup_{n\to\infty} \frac{\xi^*_n}{\sqrt{n}} > c\Big) \geq P_0\Big(\limsup_{m\to\infty} \frac{\xi^*_{T_m}}{T_m^{1/2}} > c\Big) = E_0\Big(\limsup_{m\to\infty} \mathbf{1}_{\{\xi^*_{T_m} > cT_m^{1/2}\}}\Big)$$
$$\geq \limsup_{m\to\infty} E_0\big(\mathbf{1}_{\{\xi^*_{T_m} > cT_m^{1/2}\}}\big) = \limsup_{m\to\infty} P_0\big(\xi^*_{T_m} > cT_m^{1/2}\big) > 0.$$

By virtue of Lemma 3.1, this yields the claim.

We now turn to the proof that (29) holds for any sufficiently small $c > 0$. Observe that according to (7) and (8), we have

$$\begin{aligned} P_0\big((\xi^*_{T_m})^2 \geq c^2 T_m\big) &\geq P_0\Big(\big(\max_{0\leq k\leq m} D^m_k\big)^2 \geq c^2\Big(m + 2\sum_{k=0}^{m} D^m_k\Big)\Big) \\ &\geq P_0^V\Big(\max_{0\leq i<\varrho_m} M_i^2 \geq c^2\Big(m + 2\sum_{i=0}^{\varrho_m} S_i\Big)\Big). \end{aligned}$$

Denote by $\mu := E_0^V[\sigma_0]$ the annealed (i. e., taken under the law P_0^V) expectation of σ_0, choose an arbitrary $\delta' \in (1,\delta)$, and define

$$a^{\pm}_m = \big\lfloor \mu^{-1}(m \pm m^{1/\delta'})\big\rfloor, \tag{30}$$

where we use the notation $\lfloor x\rfloor$ to denote the integer part of a real number x. We shall also write

$$a_m = \lfloor \mu^{-1} m\rfloor$$

when no upper or lower deviation buffer is needed. It follows from the last inequality stated above that

$$P_0\big((\xi^*_{T_m})^2 \geq c^2 T_m\big)$$
$$\geq P_0^V\Big(\max_{1\leq i\leq a^-_m} M_i^2 \geq c^2\Big(m + 2\sum_{i=1}^{a^+_m+1} S_i\Big)\Big) - P_0^V(\varrho_m > a^+_m) - P_0^V(\varrho_m < a^-_m).$$

Since $\{\varrho_m > a^+_m\}$ implies $\{\sigma_{a^+_m} < m\}$, and since $\sigma_{a^+_m} = \sum_{i=0}^{a^+_m}(\sigma_i - \sigma_{i-1})$, we have

$$P_0^V(\varrho_m > a^+_m) \leq P_0^V\Big(\frac{\sum_{i=0}^{a^+_m}(\sigma_i - \sigma_{i-1} - \mu)}{(a^+_m)^{1/\delta}} \leq -\frac{m^{1/\delta'}}{(a^+_m)^{1/\delta}}\Big). \tag{32}$$

Hence (17) and a stable limit theorem for i.i.d. random variables $\sigma_i - \sigma_{i-1}$ (see, for instance, Theorem 1.5.1 in [5]) imply that $\lim_{m\to\infty} P_0^V(\varrho_m > a^+_m) = 0$. Similarly, one can show that $\lim_{m\to\infty} P_0^V(\varrho_m < a^-_m) = 0$. Therefore, in

order to prove (29), it suffices to show that the following strict lower bound holds:

$$\liminf_{m\to\infty} P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m+2\sum_{i=1}^{a_m^++1} S_i\Big)\Big) > 0. \tag{33}$$

Toward this end, recall K_0 and K_2 from (16) and (18) respectively, fix any positive constants $\beta,\gamma>0$ such that

$$\frac{1}{2}\cdot\mu^{-1}\big(K_2\cdot\beta^{-\delta/2}-K_0\cdot\gamma^{-\delta/2}\big)-K_2^2\cdot\beta^{-\delta}>0, \tag{34}$$

and observe that

$$\begin{aligned} &P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m+2\sum_{i=1}^{a_m^++1} S_i\Big)\Big)\ge\\ &\qquad\ge P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge \beta m^{2/\delta},\, m+2\sum_{i=1}^{a_m^++1} S_i \le \frac{\beta m^{2/\delta}}{c^2}\Big)\ge P_0^V\Big(\bigcup_{i=1}^{a_m^-} A_{i,m}\Big), \end{aligned}$$

where

$$A_{i,m}\overset{(35)}{:=}\Big\{M_i^2\ge\beta m^{2/\delta},\ S_i\le\gamma m^{2/\delta},\ m+2\sum_{\substack{1\le j\le a_m^++1,\\ j\ne i}} S_j<(\beta/c^2-\gamma)m^{2/\delta}\Big\}.$$

Therefore, the inclusion-exclusion formula yields

$$\begin{aligned} P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m+2\sum_{i=1}^{a_m^++1} S_i\Big)\Big) &\ge P_0^V\Big(\bigcup_{i=1}^{a_m^-} A_{i,m}\Big)\\ &\ge \sum_{i=1}^{a_m^-} P_0^V(A_{i,m}) - \sum_{i=1}^{a_m^-}\sum_{j=i+1}^{a_m^-} P_0^V\Big(A_{i,m}\bigcap A_{j,m}\Big)\\ &\ge a_m^- P_0^V(A_{1,m}) - (a_m^-)^2 P_0^V\big(M_1^2\ge\beta m^{2/\delta},\, M_2^2\ge\beta m^{2/\delta}\big). \end{aligned} \tag{36}$$

Using the independence of the life-cycles of the underlying branching process, we obtain that

$$P_0^V(A_{1,m}) = P_0^V\big(M_1^2\ge\beta m^{2/\delta},\, S_1\le\gamma m^{2/\delta}\big)\cdot P_0^V\Big(m+2\sum_{j=2}^{a_m^++1} S_j<(\beta/c^2-\gamma)m^{2/\delta}\Big).$$

Taking into account (16), (18), and (30), one can deduce from the following inequality:

$$P_0^V\big(M_1^2\ge\beta m^{2/\delta},\, S_1\le\gamma m^{2/\delta}\big)\ge P_0^V\big(M_1^2\ge\beta m^{2/\delta}\big)-P_0^V\big(S_1>\gamma m^{2/\delta}\big),$$

that

$$\liminf_{m\to\infty} a_m^-\cdot P_0^V\big(M_1^2\ge\beta m^{2/\delta},\, S_1\le\gamma m^{2/\delta}\big)\ge\mu^{-1}\big(K_2\cdot\beta^{-\delta/2}-K_0\cdot\gamma^{-\delta/2}\big).$$

Furthermore, it follows from (16) and a stable limit theorem for i.i.d. variables S_i (see, for instance, Theorem 1.5.1 in [5]) that the following limit exists and is strictly positive:

$$\lambda(c,\beta,\gamma) := \lim_{m\to\infty} P_0^V\Big(m + 2\sum_{j=2}^{a_m^+ +1} S_j < (\beta/c^2 - \gamma)m^{2/\delta}\Big) > 0.$$

Moreover, given $\beta,\gamma > 0$, we can choose $c > 0$ so small that $\lambda(c,\beta,\gamma) > 1/2$. For such a constant $c > 0$, (36) along with (18) yield

$$\begin{aligned}\liminf_{m\to\infty} P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m + 2\sum_{i=1}^{a_m^+ +1} S_i\Big)\Big)\\ \ge \lambda(c,\beta,\gamma)\cdot\mu^{-1}\big(K_2\cdot\beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}\big) - K_2^2\cdot\beta^{-\delta}\\ \ge \frac{1}{2}\cdot\mu^{-1}\big(K_2\cdot\beta^{-\delta/2} - K_0\cdot\gamma^{-\delta/2}\big) - K_2^2\cdot\beta^{-\delta}.\end{aligned}$$

In view of (34), this implies (33), and hence (29) for the indicated above values of the parameter $c > 0$. □

Lemma 3.2 yields part (i) of Theorem 1.2 for $\delta \in (1,2)$. The remaining case $\delta = 2$ is considered in the next lemma, by using a slightly different approach which is based on a modification of an argument in [8], cf. [8, p. 172].

Lemma 3.3. *Let Assumption 1.1 hold and suppose that $\delta = 2$. Then (5) holds along random subsequence $(T_m)_{m\ge1}$ of integers n. That is,*

$$\limsup_{m\to\infty} \frac{\xi^*_{T_m}}{T_m^{1/2}} > 0, \qquad P_0 - \text{a. s.}$$

Proof of Lemma 3.3. Recall (15) and (30). The proof of the lemma relies on the Borel-Cantelli argument which is applied to the following version of (31):

$$\begin{aligned}P_0\big((\xi^*_{T_m})^2 \ge c^2 T_m \quad \text{i. o.}\big) \;\ge\; P_0^V\Big(\max_{1\le i\le a_m^-} M_i^2 \ge c^2\Big(m + 2\sum_{i=1}^{a_m^+ +1} S_i\Big) \quad \text{i. o.}\Big)\\ -P_0^V(\varrho_m > a_m^+ \quad \text{i. o.}) - P_0^V(\varrho_m < a_m^- \quad \text{i. o.})\end{aligned} \tag{37}$$

with a suitably chosen subsequence of the integer indices m. The inequality in (37) is a process-level statement. We first use the triangular representation of Lemma 2.1, rather than only the fixed-row identity (25), and then use the renewal-transfer part of the same lemma to evaluate the resulting finite-shift invariant i.o. event under P_0^V.

First, we will establish certain large deviation type estimates for the distributional tails of the random variables ϱ_m. To this end, observe that the inequality stated in (32) remains true for any $\delta > 1$, in particular for $\delta = 2$. Furthermore, (17) and the large deviation estimate stated, for instance, in [5,

Theorem 3.4.1] imply that the following holds under Assumption 1.1 (with arbitrary $\delta > 1$) for a suitable constant $c_1 = c_1(\delta) > 0$:

$$P_0^V(\varrho_m > a_m^+) \leq \frac{c_1 a_m^+}{m^{\delta/\delta'}}. \tag{38}$$

Similarly, using the following inequality instead of (32):

$$P_0^V(\varrho_m < a_m^-) \leq P_0^V(m \leq \sigma_{a_m^-}) \leq P_0^V\Big(\frac{\sum_{i=0}^{a_m^-}(\sigma_i - \sigma_{i-1} - \mu)}{(a_m^-)^{1/\delta}} \geq \frac{m^{1/\delta'}}{(a_m^-)^{1/\delta}}\Big), \tag{39}$$

one can deduce from (17) and [5, Theorem 3.4.1] that the following holds under Assumption 1.1 (with arbitrary $\delta > 1$) for some constant $c_2 = c_2(\delta) > 0$:

$$P_0^V(\varrho_m < a_m^-) \leq \frac{c_2 a_m^-}{m^{\delta/\delta'}}. \tag{40}$$

Remark 3.4. We remark that, in the course of proving (38), in order to formally meet the lower tail conditions of Theorem 3.4.1 in [5] one can, for instance, use in (32) the following "unpolarized" version of $\sigma_i - \sigma_{i-1} - \mu$ which has the same structure of upper and lower distribution tails:

$$(\sigma_i - \sigma_{i-1} - \mu)' := U_i \cdot (\sigma_i - \sigma_{i-1} - \mu),$$

where $U = (U_i)_{i \geq 0}$ is a sequence of i.i.d. Bernoulli random variables, independent of "anything else" (i. e., such that the probability law P^U of U is independent of the measure P^V in the enlarged probability space), and such that

$$P^U(U_i = 1) = P^U(U_i = -1) = \frac{1}{2}.$$

Note that (cf. (32)) $P_0^V\Big(\sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1} - \mu) \leq -m^{1/\delta'}\Big) \leq P_0^V\Big(\sum_{i=0}^{a_m^+}(\sigma_i - \sigma_{i-1} - \mu)' \leq -m^{1/\delta'}\Big)$.

Fix now any $c > 0$ and recall (31). For $k \in \mathbb{N}$, let

$$m_k = k^k,$$

and define the following events:

$$B_k = \Big\{\max_{a_{2m_{k-1}}^- < i \leq a_{m_k}^-} M_i^2 \geq c^2\Big(m_k + 2\sum_{i=1}^{a_{m_k}^+ + 1} S_i\Big)\Big\},$$

$$D_k = \Big\{2c^2 \cdot \sum_{i=1}^{a_{m_{k-1}}^+ + 1} S_i > m_k \log m_k\Big\},$$

$$E_k = \Big\{\max_{a_{2m_{k-1}}^- < i \leq a_{m_k}^-} M_i^2 \geq 2m_k \log m_k,\ c^2\Big(m_k + 2\sum_{j=a_{m_{k-1}}^+ + 2}^{a_{m_k}^+ + 1} S_j\Big) \leq m_k \log m_k\Big\}.$$

By virtue of (16), Theorem 3.8.1 along with Theorem 1.5.1-(ii) in [5] (notice that the random centering required in the former is given by the latter), there exists a positive constant $c_3 = c_3(c) > 0$ such that

$$P_0^V(D_k) \le c_3 \frac{m_{k-1}}{m_k \log m_k} \le c_3 \frac{(k-1)^{k-1}}{k^{k+1} \log k} \le c_3 k^{-2}.$$

Thus by the first Borel-Cantelli lemma, $P_0^V(D_k \quad \text{i. o.}) = 0$. On the other hand, similarly to (35) and (36), setting

$$\widetilde{A}_{i,k} := \Big\{ M_i^2 \ge 2m_k \log m_k,\ c^2 \Big(m_k + 2 \sum_{j=a^+_{m_{k-1}}+2}^{a^+_{m_k}+1} S_j \Big) \le m_k \log m_k \Big\},$$

Choose any i_k in the deterministic block $\{i : a_{2m_{k-1}} < i \le a^-_{m_k}\}$ and use the exchangeability of the i.i.d. cycle triples inside this block. We obtain

$$P_0^V(E_k) \ge P_0^V\Big(\bigcup_{i=1+a_{2m_{k-1}}}^{a^-_{m_k}} \widetilde{A}_{i,k} \Big) \ge \sum_{i=1+a_{2m_{k-1}}}^{a^-_{m_k}} P_0^V\big(\widetilde{A}_{i_k,k}\big) - \sum_{i=1}^{a^-_{m_k}} \sum_{j=i+1}^{a^-_{m_k}} P_0^V\Big(\widetilde{A}_{i,k} \bigcap \widetilde{A}_{j,k} \Big)$$

$$\ge (a^-_{m_k} - a_{2m_{k-1}}) P_0^V\big(\widetilde{A}_{i_k,k}\big) - (a^-_{m_k})^2 P_0^V\big(M_1^2 \ge m_k \log m_k,\ M_2^2 \ge m_k \log m_k \big).$$

Thus by virtue of (18) and Theorem 1.5.1-(ii) in [5], if $c > 0$ is sufficiently small, then

$$P_0^V(E_k) \ge \frac{c_4}{\log m_k} = \frac{c_4}{k \log k},$$

for a suitable constant $c_4 > 0$. Since the events E_k are independent of each other, the second Borel-Cantelli lemma yields $P_0^V(E_k \quad \text{i. o.}) = 1$.

For an event A in the underlying probability space, let A^c denote its complement. Observe now that

$$E_k \bigcap D_k^c \subset B_k, \qquad k \ge 0,$$

and hence $P_0^V(B_k \quad \text{i. o.}) = 1$ for any constant $c > 0$ small enough. Finally, (38) and (40) imply that

$$P_0^V(\varrho_{m_k} < a^-_{m_k} \quad \text{i. o.}) = P_0^V(\varrho_{m_k} > a^+_{m_k} \quad \text{i. o.}) = 0.$$

It thus follows from (37) that

$$P_0\big(\xi_n^* \ge c\sqrt{n} \quad \text{i. o.}\big) \ge P_0\big((\xi^*_{T_m})^2 \ge c^2 T_m \quad \text{i. o.}\big) = 1$$

for sufficiently small values of the constant $c > 0$. This completes the proof of the lemma. □

In view of Lemmas 3.2 and 3.3, the proof of part (i) of Theorem 1.2 is completed. □

Part (ii). First, we will show that for any constant $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{\xi^*_n}{n^{1/2}(\log n)^\alpha} = 0, \qquad P_0 - \text{a. s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^\alpha$. Recall the alternative notation $\xi^*(n)$ for ξ^*_n. Observe that in order to prove the above claim, it suffices to show that

$$\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \qquad P_0 - \text{a. s.} \tag{41}$$

Indeed, let k_m, $m \in \mathbb{N}$, be the (uniquely defined) non-negative random integers such that

$$T_{k_m} < m \le T_{k_m+1}, \qquad m \in \mathbb{N}. \tag{42}$$

Then, since $\phi(n)$ is an eventually increasing sequence,

$$\begin{aligned} \limsup_{m\to\infty} \frac{\xi^*(m)}{\phi(m)} &\le \limsup_{m\to\infty} \frac{\xi^*(T_{k_m+1})}{\phi(m)} \le \limsup_{m\to\infty} \frac{\xi^*(T_{k_m+1})}{\phi(T_{k_m})} \\ &\le \limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} = 0, \qquad P_0 - \text{a. s.} \end{aligned} \tag{43}$$

Recall now (38) and (40). Let $\psi_\varepsilon(m) = m^{2/\delta}(\log m)^{-\varepsilon}$ with $\varepsilon \in (0, 2\alpha)$ and let $m_i = 3^i$ for $i \in \mathbb{N}$. Then, for any constant $c > 0$, we have

$$\begin{aligned} \sum_{i=1}^\infty P_0\big(T_{m_i} < c\psi_\varepsilon(m_{i+1})\big) &\le \sum_{i=1}^\infty \Big[P_0^V\Big(2\sum_{k=0}^{a^-_{m_i}-1} S_k < c\psi_\varepsilon(m_{i+1})\Big) + P_0^V\big(\varrho_{m_i} < a^-_{m_i}\big)\Big] \\ &\le \sum_{i=1}^\infty \Big[P_0^V\Big(2\max_{0\le k<a^-_{m_i}} S_k < c\psi_\varepsilon(m_{i+1})\Big) + P_0^V\big(\varrho_{m_i} < a^-_{m_i}\big)\Big] \\ &\le \sum_{i=1}^\infty \Big[\Big(1 - \frac{c_5}{(c\psi_\varepsilon(m_{i+1}))^{\delta/2}}\Big)^{a^-_{m_i}} + \frac{c_2 a^-_{m_i}}{(m_i)^{\delta/\delta'}}\Big] \le \sum_{i=1}^\infty c_6 \cdot \Big[e^{-c_7(i+1)^{\varepsilon\delta/2}} + e^{-c_8 i}\Big] < \infty, \end{aligned}$$

where $c_5, c_6, c_7, c_8 > 0$ are suitable positive constants and c_2 is the constant which appears at (40). Since the preceding summability holds for every fixed $c > 0$, the Borel-Cantelli lemma, applied for instance to $c = 1, 2, \ldots$, gives

$$\liminf_{i\to\infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \qquad P_0 - \text{a. s.}$$

Since for each $n \in \mathbb{N}$,

$$m_i < n \le m_{i+1} \quad \text{for some } i \in \mathbb{N} \text{ which is uniquely determined by } n, \tag{45}$$

then

$$\liminf_{n\to\infty} \frac{T_n}{\psi_\varepsilon(n)} \ge \liminf_{i\to\infty} \frac{T_{m_i}}{\psi_\varepsilon(m_{i+1})} = \infty, \qquad P_0 - \text{a. s.}, \tag{46}$$

and hence $P_0\big(T_m < \psi_\varepsilon(m) \quad \text{i. o.}\big) = 0$.

Recall D_k from (22) and (23). The pathwise bound (9), the identity (22), and the inequality $D_k^m \le D_k$ imply that for any constant $b > 0$,

$$\begin{aligned}
&P_0\Big(\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\Big) \\
&\quad\le P_0\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k^m}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\Big) \le P_0\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} D_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\Big) \\
&\quad\le P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le \varrho_m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\Big) \le P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\Big) \\
&\quad= P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} M_k}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big), \qquad (47)
\end{aligned}$$

where in the last but one step we used the inequality $\varrho_m \le m$. The passage from the ERW down-crossing field to the cycle maxima in (47) uses Lemma 2.1. The fixed-m identity (25) would not be sufficient here, because the event contains a limsup over m.

Let $r_m \in [0, m]$ be the largest integer such that $\max_{0\le k\le m} M_k = M_{r_m}$. That is

$$r_m = \max\big\{n \in [0,m] : M_n = \max_{0\le k\le m} M_k\big\}, \qquad m \in \mathbb{N}. \qquad (48)$$

Note that $r_m \le m$ for any $m \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} M_k}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big) = P_0^V\Big(\limsup_{m\to\infty} \frac{M_{r_m}}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big) \\
&\quad\le P_0^V\Big(\limsup_{m\to\infty} \frac{M_{r_m}}{r_m^{1/\delta}(\log r_m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big) \\
&\quad\le P_0^V\Big(\limsup_{m\to\infty} \frac{M_m}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big). \qquad (49)
\end{aligned}$$

It follows from (18) that if $\varepsilon \in (0, 2\alpha)$ is chosen in such a way that in fact $\alpha - \varepsilon/2 > 1/\delta$, then

$$\sum_{m=1}^{\infty} P_0^V\Big(\frac{M_m}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}}\Big) < \infty.$$

Thus the Borel-Cantelli lemma combined with (45) implies that for $\varepsilon \in (0, 2\alpha - 2/\delta)$,

$$P_0^V\Big(\frac{M_m}{m^{1/\delta}(\log m)^{\alpha-\varepsilon/2}} > \frac{b\delta^\alpha}{2^{1+\alpha}} \quad \text{i. o.}\Big) = 0.$$

It then follows from (47) and (49) that

$$P_0\Big(\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\Big) \le P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\le k\le m} M_k}{\phi(\psi_\varepsilon(m-1))} > \frac{b}{2}\Big) = 0.$$

This completes the proof of (41).

We now turn to the proof that for any constant $\alpha > 1/\delta$,

$$\lim_{n\to\infty} \frac{(\log n)^\alpha \xi^*_n}{n^{1/2}} = \infty, \qquad P_0 - \text{a. s.}$$

Fix any $\alpha > 1/\delta$ and let $\phi(n) = n^{1/2}(\log n)^{-\alpha}$. In order to prove the claim, it suffices to show that

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} = \infty, \qquad P_0 - \text{a. s.} \tag{50}$$

Indeed, in view of (42), it follows from (50) that

$$\infty = \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})} \;\le\; \liminf_{m\to\infty} \frac{\xi^*(T_{k_m})}{\phi(T_{k_m+1})} \le \liminf_{m\to\infty} \frac{\xi^*(m)}{\phi(m)}, \qquad P_0 - \text{a. s.}$$

We will next show that (50) indeed holds true. Fix any $\varepsilon \in (0, \alpha\delta - 1)$ and define

$$\psi_\varepsilon(m) := m^{2/\delta}(\log m)^{2/\delta+\varepsilon}, \qquad m \in \mathbb{N}.$$

By a counterpart of the law of the iterated logarithm for i.i.d. random variables in the domain of attraction of a stable law and with infinite variance, we have (see, for instance, Theorem 1.6.6. in [5]):

$$\limsup_{m\to\infty} \frac{\sum_{k=1}^m S_k}{\psi_\varepsilon(m)} = 0, \qquad P_0^V - \text{a. s.}$$

Combining this result with the fact that $P_0^V\big(\lim_{m\to\infty} \frac{\varrho_m}{m} = \mu^{-1}\big) = 1$ (which is an implication of the renewal theorem applied to the renewal sequence σ_k), and using the triangular representation of Lemma 2.1 for this process-level bound, we obtain that

$$P_0\big(T_m > \psi_\varepsilon(m) \quad \text{i. o.}\big) = 0.$$

Therefore,

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(\psi_\varepsilon(m+1))} \le \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \qquad P_0 - \text{a. s.}$$

Thus it suffices to prove that the left-hand side of the above inequality is infinity. Recall (30) and (40). Then, for $m_i = 3^i$ and any constant $c > 0$, similarly to (44), we have:

$$\begin{aligned}
\sum_{i=1}^\infty P_0\Big(\xi^*(T_{m_i}) < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) &\le \sum_{i=1}^\infty P_0^V\Big(\max_{0\le k<\varrho_{m_i}} M_k < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) \\
&\le \sum_{i=1}^\infty \Big[P_0^V\Big(\max_{0\le k<a^-_{m_i}} M_k < c\phi\big(\psi_\varepsilon(m_{i+1}+1)\big)\Big) + P_0^V(a^-_{m_i} > \varrho_{m_i})\Big] \\
&\le (51) \sum_{i=1}^\infty c_9 \cdot \Big[e^{-c_{10}(i+1)^{\alpha\delta-1-\varepsilon\delta/2}} + \frac{a^-_{m_i}}{(m_i)^{\delta/\delta'}}\Big] < \infty,
\end{aligned}$$

where $c_9 > 0$ and $c_{10} > 0$ are some appropriate positive constants. Therefore, the Borel-Cantelli lemma yields (recall that $\delta \le 2$ and the value of the

parameter ε is chosen from the interval $(0, \alpha\delta - 1)$, so that $\alpha\delta - 1 - \varepsilon\delta/2 \geq \alpha\delta - 1 - \varepsilon > 0$):

$$\liminf_{i\to\infty} \frac{\xi^*(T_{m_i})}{\phi(\psi_\varepsilon(m_{i+1}+1))} = \infty, \qquad P_0 - \text{a. s.}$$

This completes the proof of (50) by using a suitable variation of (46). □

3.2. **Ballistic regime: Proof of Theorem 1.3.** By [4, Theorem 1.5.3] combined with [4, Theorem 1.9.5], the regularly varying sequence $\phi(n)$ is asymptotically equivalent to a non-decreasing sequence $\bar{\phi}(n)$ which is also regularly varying with index ρ. By the asymptotic equivalence we mean that $\lim_{n\to\infty} \bar{\phi}(n)/\phi(n) = 1$. Since

$$\limsup_{n\to\infty} \frac{\xi^*_n}{\phi(n)} = \limsup_{n\to\infty} \frac{\xi^*_n}{\bar{\phi}(n)} \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{\xi^*_n}{\phi(n)} = \liminf_{n\to\infty} \frac{\xi^*_n}{\bar{\phi}(n)},$$

we can assume without loss of generality that $\phi(n)$ is a non-decreasing sequence. In the rest of the paper we will make this additional assumption without further notice.

Part (i). Suppose first that

$$\sum_{m=1}^{\infty} \frac{1}{[\phi(m)]^\delta} < \infty. \tag{52}$$

Recall D_k from (22) and (23). Using (9) and the inequalities $\varrho_m \leq m$, $m \leq T_m$, and $D_k^m \leq D_k$, we obtain that for any constant $b > 0$,

$$\begin{aligned} &P_0\Big(\limsup_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m-1})} > b\Big) \\ &\quad \leq P_0\Big(\limsup_{m\to\infty} \frac{\max_{0\leq k\leq m} D_k^m}{\phi(m-1)} > \frac{b}{2}\Big) \leq P_0\Big(\limsup_{m\to\infty} \frac{\max_{0\leq k\leq m} D_k}{\phi(m-1)} > \frac{b}{2}\Big) \\ &\quad \leq P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\leq k\leq \varrho_m} M_k}{\phi(m-1)} > \frac{b}{2}\Big) \leq P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\leq k\leq m} M_k}{\phi(m-1)} > \frac{b}{2}\Big). \end{aligned} \tag{53}$$

The second inequality in (53) is another use of Lemma 2.1. The bound involves the limiting down-crossing field and a limsup over m, so the row-wise identity (25) alone is not enough.

Recall r_m from (48). Then (recall that the sequence $\phi(m)$ is regularly varying, and hence $\lim_{m\to\infty} \frac{\phi(m)}{\phi(m-1)} = 1$),

$$\begin{aligned} P_0^V\Big(\limsup_{m\to\infty} \frac{\max_{0\leq k\leq m} M_k}{\phi(m-1)} > \frac{b}{2}\Big) &= P_0^V\Big(\limsup_{m\to\infty} \frac{M_{r_m}}{\phi(m)} > \frac{b}{2}\Big) \\ &\leq P_0^V\Big(\limsup_{m\to\infty} \frac{M_{r_m}}{\phi(r_m)} > \frac{b}{2}\Big) \leq P_0^V\Big(\limsup_{n\to\infty} \frac{M_n}{\phi(n)} > \frac{b}{2}\Big). \end{aligned}$$

It follows from (18) and (52) that

$$\sum_{n=1}^{\infty} P_0^V\Big(\frac{M_n}{\phi(n)} > \frac{b}{2}\Big) < \infty.$$

Thus the Borel-Cantelli lemma and (53) imply

$$P_0\Big(\limsup_{m\to\infty}\frac{\xi^*(T_m)}{\phi(T_{m-1})}>b\Big)\le P_0^V\Big(\limsup_{m\to\infty}\frac{\max_{0\le k\le m}M_k}{\phi(m-1)}>\frac{b}{2}\Big)=0.$$

By virtue of (43), this completes the proof of the first half of part (i) of Theorem 1.3.

Suppose now that

$$\sum_{m=1}^{\infty}\frac{1}{[\phi(m)]^{\delta}}=\infty. \tag{54}$$

Recall (2) and the fact that $\mathrm{v}>0$ (the asymptotic speed is positive) when $\delta>2$. Observe that in order to prove that

$$\limsup_{n\to\infty}\frac{\xi^*_n}{\phi(n)}=\infty,\qquad P_0-\text{a. s.},$$

it suffices to show that for any constant $b>0$,

$$\limsup_{m\to\infty}\frac{\max_{0\le i<\varrho_m}M_i}{\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)}>b,\qquad P_0^V-\text{a. s.} \tag{55}$$

Indeed, $P_0\big(T_m>2\mathrm{v}^{-1}m\quad\text{i. o.}\big)=0$ by virtue of (6). Hence (55) implies that for every $b>0$,

$$\begin{aligned}P_0\Big(\limsup_{m\to\infty}\frac{\xi^*(m)}{\phi(m)}>b\Big)&\ge P_0\Big(\limsup_{m\to\infty}\frac{\xi^*(T_m)}{\phi(T_m)}>b\Big)\\&\ge P_0\Big(\limsup_{m\to\infty}\frac{\xi^*(T_m)}{\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)}>b\Big)\ge P_0^V\Big(\limsup_{m\to\infty}\frac{\max_{0\le i<\varrho_m}M_i}{\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)}>b\Big)=1,\end{aligned}$$

where the last inequality is justified by Lemma 2.1; the finite residual cycle in the stationary representation is immaterial for the displayed limsup event. Thus the proof of part (i) of Theorem 1.3 will be completed once we prove (55).

We turn now to the proof of (55). Recall the notation $\mu=E_0^V[\sigma_0]$ that we have used before. By the renewal theorem $P_0^V\big(\lim_{m\to\infty}\frac{\varrho_m}{m}=\frac{1}{\mu}\big)=1$, and hence $P_0^V\big(\varrho_m<\frac{m}{2\mu}\quad\text{i. o.}\big)=0$. It thus suffices to show that

$$\limsup_{m\to\infty}\frac{\max_{0\le i\le\lfloor m\mu^{-1}/2\rfloor}M_i}{\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)}\ge\limsup_{m\to\infty}\frac{M_{\lfloor m\mu^{-1}/2\rfloor}}{\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)}>b,\qquad P_0^V-\text{a. s.} \tag{56}$$

To this end, observe that by virtue of (18) and (54),

$$\sum_{m=1}^{\infty}P_0^V\Big(M_{\lfloor m\mu^{-1}/2\rfloor}>b\phi(\lfloor 2\mathrm{v}^{-1}m\rfloor)\Big)=\infty.$$

The map $m\mapsto\lfloor m\mu^{-1}/2\rfloor$ has uniformly bounded multiplicities. Hence the divergent series above contains a divergent subseries indexed by distinct values of $n=\lfloor m\mu^{-1}/2\rfloor$. Along this subseries the corresponding events depend

on distinct members of the i.i.d. sequence (M_n) and are therefore independent. Regular variation of ϕ ensures that replacing $\lfloor 2\mathrm{v}^{-1}m \rfloor$ by a constant multiple of n changes the tail estimates only by multiplicative constants. The second Borel-Cantelli lemma therefore gives (56), and hence (55). The proof of part (i) of Theorem 1.3 is completed. □

Part (ii). Let $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ for a fixed constant $\alpha > 1/\delta$. In order to prove the claim, it suffices to verify (50) (see the next two lines below (50)). Toward this end, observe that according to the law of large numbers for T_n stated in (6) we have $P_0(T_m > 2\mathrm{v}^{-1}m \quad \text{i. o.}) = 0$, and hence

$$\liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(2\mathrm{v}^{-1}(m+1))} \le \liminf_{m\to\infty} \frac{\xi^*(T_m)}{\phi(T_{m+1})}, \qquad P_0 - \text{a. s.}$$

Thus it suffices to show that the left-hand side of the above inequality is infinity. Let a_m^- be as defined in (30) with the only exception that this time we will use an arbitrary constant $\delta' \in (1,2)$. In view of (17) and (39), Chebyshev's inequality implies

$$P_0^V(a_m^- > \varrho_m) \le \frac{a_m^- + 1}{m^{2/\delta'}} \cdot E_0^V\left[(\sigma_0 - \mu)^2\right]. \tag{57}$$

Therefore, a slight modification of (51) (namely, formally replacing there the composition of two functions $\phi \circ \psi_\varepsilon$ by the "new" $\phi(n) = n^{1/\delta}(\log n)^{-\alpha}$ and also using (57) instead of (40)) along with the Borel-Cantelli lemma imply that $\liminf_{i\to\infty} \frac{\xi^*(T_{m_i})}{\phi(m_{i+1}+1)} = \infty$, $P_0 - \text{a. s.}$. This completes the proof of (50) by using an appropriate variation of (46). □

References

[1] A.-L. Basdevant and A. Singh, *On the speed of a cookie random walk*, Probab. Theory Related Fields **141** (2008), 625–645.

[2] A.-L. Basdevant and A. Singh, *Rate of growth of a transient cookie random walk*, Electron. J. Probab. **13** (2008), 811–851.

[3] I. Benjamini and D. B. Wilson, *Excited random walk*, Electron. Comm. Probab. **8** (2003), 86–92.

[4] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*. Encyclopedia of Mathematics and its Applications **27**, Cambridge University Press, 1989 (paperback edition).

[5] A. A. Borovkov and K. A. Borovkov, *Asymptotic Analysis of Random Walks: Heavy-tailed Distributions*, Encyclopedia of Mathematics and its Applications **118**, Cambridge University Press, 2008.

[6] A. Dembo, N. Gantert, Y. Peres, and Z. Shi, *Valleys and the maximum local time for random walk in random environment*, Probab. Theory and Related Fields **137** (2007), 443–473.

[7] D. Dolgopyat, *Random walks in one dimensional environment*, lecture notes. Available electronically at http://www2.math.umd.edu/~dolgop/RW1d.pdf.

[8] N. Gantert and Z. Shi, *Many visits to a single site by a transient random walk in random environment*, Stoch. Proc. Appl. **99** (2002), 159–176.

[9] N. Gantert, Y. Peres, and Z. Shi, *The infinite valley for a recurrent random walk in random environment*, Ann. Inst. H. Poincaré Probab. Statist. **46** (2010), 525–536.
[10] E. S. Key, *Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment*, Ann. Probab. **15** (1987), 344–353.
[11] E. Kosygina and T. Mountford, *Limit laws of transient excited random walks on integers*, Ann. Inst. H. Poincaré Probab. Statist. **47** (2011), 575–600.
[12] E. Kosygina and M. Zerner, *Positively and negatively excited random walks on integers, with branching processes*, Electron. J. Prob. **13** (2008), 1952–1979.
[13] E. Kosygina and M. P. W. Zerner, *Excited random walks: results, methods, open problems*, Bulletin of the Institute of Mathematics. Academia Sinica (New Series) 8 (2013), 105–157.
[14] P. Revesz, *Random Walk in Random and Non-Random Environments*, World Scientific Publishing Company, 2005.
[15] A. Roitershtein, *A note on multitype branching processes with immigration in a random environment*, Ann. Probab. **35** (2007), 1573–1592.
[16] M. Zerner, *Multi-excited random walks on integers*, Probab. Theory Related Fields **133** (2005), 98–122.

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