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BEFRIENDING ASKEY–WILSON POLYNOMIALS

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Abstract. Although our main interest is with the Askey–Wilson (AW) polynomials we recall and review four other families of the so-called Askey–Wilson scheme of polynomials. We do this for completeness as well as for better exposition of AW properties. Our main results concentrate on the complex parameters case, revealing new fascinating symmetries between the variables and some of the parameters. In particular we express Askey–Wilson polynomials as linear combinations of Al-Salam–Chihara (ASC) polynomials which together with the obtained earlier expansion of the Askey–Wilson density forms complete generalization of the situation met in the case of Al-Salam–Chihara and $q-$Hermite polynomials and the Poisson–Mehler expansion formula. As a by-product we get useful identities involving ASC polynomials. Finally by certain re-scaling of variables and parameters we arrive to AW polynomials and AW densities that have clear probabilistic interpretation. We recall some known and present some believed to be unknown identities and integrals.

1. Introduction

As announced in the abstract first we recall and review properties of the 5 families of polynomials that belong to the so-called Askey–Wilson scheme of polynomials. Then we present some believed to be new, finite and infinite expansions of one family of the polynomials with respect to the other as well as the densities of measures that make these families orthogonal. The families of polynomials that we are going to present are the families of the so called continuous $q-$Hermite (qh), big continuous $q-$Hermite (bqh), Al-Salam–Chihara (ASC), continuous dual Hahn (C2H) and finally Askey–Wilson polynomials (AW). They are related by the fact that respectively 4, 3, 2, 1 and 0 of parameters that appear in the definition of Askey–Wilson polynomials (apart of the 5th parameter - the base) are set to zero.

As it is well known AW polynomials were introduced in [9] and constitute the biggest (as of now) family of orthogonal polynomials depending on 5 parameters. One of these parameters is special, called base and denoted by q. In most of the applications where AW polynomials appear q is real from $(-1, 1]$. Sometimes one considers also the case $q > 1$. Our presentation of AW polynomials as the last element of the sequence of families of orthogonal polynomials (AW scheme) will reveal some regularities like e.g. (2.14), (2.26), (2.31), (2.45) or (2.12), (2.17), (2.30), (2.29) that if properly extended might lead to a generalization of AW polynomials.

Much is known about these polynomials however far from all. In particular only recently moments of the AW distribution that makes these polynomials orthogonal ware calculated (see e.g. [7] or [11]).

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The paper aims to increase this knowledge. In particular we derive the so-called 'connection coefficients' between considered families of polynomials and try to derive as much as possible information about these polynomials from these coefficients. In particular knowing these coefficients we are able relate densities (if they exist) of measures that make these polynomials orthogonal. The idea how to get it was presented in [10]. We will recall this idea now briefly since below we use it to quickly and intuitively derive the densities that make involved polynomials orthogonal.

More precisely let us assume that we have two positive probability measures $d\alpha$ and $d\beta$ with the densities respectively $A(x)$ and $B(x)$. Moreover let us assume that $\{a_n(x)\}_{n\geq 0}$ and $\{b_n(x)\}_{n\geq 0}$ are sets of monic polynomials orthogonal with respect to $d\alpha$ and $d\beta$ respectively. Besides assume that $\operatorname{supp}\beta = \operatorname{supp}\alpha$.

Further let us also assume that we know the so called 'connection coefficients' between sets $\{b_n\}$ and $\{a_n\}$. More precisely let us assume that we know the numbers $\gamma_{k,n}$ such that for every $n \geq 0$:

$$a_n(x) = \sum_{k=0}^{n} \gamma_{k,n} b_k(x)$$

for every $x \in \mathbb{R}$. Besides assume that $\int_{\operatorname{supp}(B)} (B^2(x)/A^2(x)) d\alpha(x) < \infty$, then we have

$$B(x) = A(x) \sum_{n=0}^{\infty} \frac{\gamma_{0,n}\hat{b}_0}{\hat{a}_n} a_n(x), \tag{1.1}$$

where $\hat{a}_n = \int_{\operatorname{supp}(A)} a_n(x)^2 d\alpha(x)$, similarly for the polynomials b_n. Convergence in (1.1) is L^2 convergence, however if coefficients $\frac{\gamma_{0,n}\hat{b}_0}{\hat{a}_n}$ are such that $\sum_{n\geq 0} \left(\frac{\gamma_{0,n}\hat{b}_0}{\hat{a}_n}\right)^2 \log^2 n < \infty$, then by the Rademacher–Menshov theorem we have almost pointwise, absolute convergence. In most cases interesting in the $q-$series theory this condition is trivially satisfied hence all expansions we are going to consider will be absolutely almost pointwise convergent.

Our main results concentrate on the complex parameters case and are presented in Proposition 6 and Theorem 1. For such parameters the considered families of polynomials are orthogonal with respect to distributions that have densities. Besides for such parameters after additional re-scaling and re-normalization we get clear probability interpretation of the analyzed densities and polynomials. The interpretation is important for both commutative -classical as well as in non-commutative probability case. In particular in the relatively new and intensively developing so-called 'free probability'.

As a by-product we get nontrivial finite identities involving rational functions of 4 variables (see Corollary 4).

It turns out that in this 'complex parameter' case there appear new fascinating symmetries between the variables and some of the parameters. In particular we express Askey–Wilson polynomials as linear combinations of Al-Salam–Chihara polynomials which together with obtained earlier expansion of Askey–Wilson density forms complete generalization of situation met in the case of Al-Salam–Chihara and $q-$Hermite polynomials and Poisson–Mehler expansion formula (see Corollary 3 and Remark 9).

The paper is organized as follows. In the next section we present elements of the useful notation used in the so called $q-$series theory, present polynomials that are the object of this paper and point out relationships between them that will be important in the sequel. In Section 3 we present our main results. Starting with the useful conversion proposition (Proposition 6) that allows to express a certain combination of complex exponentials in the form of a certain combination of $q-$Hermite and Al-Salam–Chihara polynomials, we present our main result (Theorem 1) that is the finite expansion of n-th Askey–Wilson polynomial with complex parameters as a linear combination of ASC polynomials with coefficients that have a form of some polynomial closely related to ASC polynomials and conversely. That is we present $n-$th ASC polynomial as a linear combination of AW polynomials with coefficients being ASC polynomials of certain parameters. In Section 4 we give probabilistic interpretation of the AW polynomials as well as the AW density. We also present the exact form of these polynomials and AW density for the special cases $q = 1$ and $q = 0$. The last case being important for the free probability. Last Section 5 present less interesting or longer proofs.

2. Notation and definitions of families of orthogonal polynomials

2.1. **Notation.** q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. We will use traditional notation of the $q-$series theory i.e. $[0]_q = 0$, $[n]_q = 1 + q + \ldots + q^{n-1}$, $[n]_q! = \prod_{j=1}^{n} [j]_q$, with $[0]_q! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}.$$

It will be useful to use the so called $q-$Pochhammer symbol for $n \geq 1$:

$$\begin{aligned} (a;q)_n &= \prod_{j=0}^{n-1} \left(1 - aq^j\right), \\ (a_1, a_2, \ldots, a_k; q)_n &= \prod_{j=1}^{k} (a_j; q)_n . \end{aligned}$$

with $(a;q)_0 = 1$. Often $(a;q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if the base will be q and if it will not cause misunderstanding. We will need also the following four well known, easy to derive, formulae

$$(a)_n = (-1)^n q^{\binom{n}{2}} a^n \left(a^{-1}; q^{-1}\right)_n, \tag{2.1}$$

$$\left(a; q^{-1}\right)_n = (-1)^n q^{-\binom{n}{2}} a^n (1/a)_n, \tag{2.2}$$

$$(a)_n = \left(aq^{n-1}; q^{-1}\right)_n, \tag{2.3}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}. \tag{2.4}$$

It is easy to notice that $(q)_n = (1-q)^n [n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases}.$$

To support intuition let us notice that $[n]_1 = n, [n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $(a;1)_n = (1-a)^n$ and $[n]_0 = \begin{cases} 1 & if \quad n \geq 1 \\ 0 & if \quad n = 0 \end{cases}$, $[n]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $(a;0)_n = \begin{cases} 1 & if \quad n = 0 \\ 1-a & if \quad n \geq 1 \end{cases}$.

2.2. **Polynomials: $q-$Hermite, big $q-$Hermite, Al-Salam–Chihara, continuous dual Hahn and Askey–Wilson polynomials.** These families of orthogonal polynomials form a part of the so called Askey–Wilson scheme of orthogonal polynomials. It means that they are interrelated and each member of this sequence depends on one more parameter. All the families that will be presented below depend on one common parameter - the base denoted by q such that $q > -1$. For $q \in (-1,1]$ these polynomials are orthogonal with respect to some positive measure on $[-1,1]$ under some conditions imposed on the values of parameters. Hence we will mostly concentrate on the case $q \in (-1,1]$. The conditions imposed on the parameters consist in fact of the requirement that the products of all pairs of these parameters are real and have absolute values not exceeding 1. If one of the parameters, say a is real and greater than 1 then the measure has $\#\left\{k : aq^k > 1\right\}$ jumps located at points $x_k = \left((aq^k + (aq^k)^{-1}\right)/2$. The masses that are assigned to these points depend on the polynomial. Since our main concern is with the regular 'without jumps' case we will give those masses only once for the so called big $q-$Hermite polynomials presented below as an example.

All the properties of the presented below families of polynomials as well as the relationships between these families are either taken directly or slightly modified from [18] or from [17]. Only some, clearly marked, are believed to be new.

2.2.1. *$q-$Hermite.* The simplest are the so called $q-$Hermite polynomials. They are defined by the following 3-term recurrence:

$$h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q), \tag{2.5}$$

with $h_{-1}(x|q) = 0$, $h_0(x|q) = 1$.

Their generating function is given by the following formula

$$\varphi_h(x|\rho,q) = \sum_{n\geq 0} \frac{\rho^n}{(q)_n} h_n(x|q) = \frac{1}{\prod_{i=0}^{\infty} v(x|\rho q^i)}, \tag{2.6}$$

convergent for $|\rho| < 1$, $|x| \leq 1$, where we denoted

$$v(x|t) = 1 - 2xt + t^2. \tag{2.7}$$

Let us observe that $\forall x \in [-1,1]$, $t \in \mathbb{R} : v(x|t) \geq 0$.

Besides we have:

$$\int_{[-1,1]} h_n(x|q)\, h_m(x|q)\, f_h(x|q)\, dx = \delta_{m,n}(q)_n, \tag{2.8}$$

with

$$f_h(x|q) = \frac{2(q)_\infty \sqrt{1-x^2}}{\pi} \prod_{i=1}^{\infty} l\left(x|q^i\right),$$

where

$$l(x|a) = (1+a)^2 - 4ax^2. \tag{2.9}$$

Notice that $\forall\, |x| \leq 1 : l(x|a) \geq 0$.

2.2.2. *Big $q-$Hermite.* The next polynomials i.e. depending on one more parameter) in the AW scheme are the so called big $q-$Hermite polynomials $\{h_n(x|a,q)\}_{n\geq -1}$ (briefly bqh). They are defined by the following 3-term recurrence:

$$h_{n+1}(x|a,q) = (2x - aq^n)h_n(x|a,q) - (1-q^n)h_{n-1}(x|a,q), \tag{2.10}$$

with $h_{-1}(x|a,q) = 0$, $h_0(x|a,q) = 1$. It turns out (see e.g. [13]) that they are connected with $q-$Hermite polynomials by the following two relationship

$$h_n(x|a,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k q^{\binom{k}{2}} h_{n-k}(x|q), \tag{2.11}$$

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k h_{n-k}(x|a,q). \tag{2.12}$$

It is known that that the characteristic function of bqh polynomials is given by the following formula

$$\varphi_{bh}(x|a,\rho,q) = \sum_{n\geq 0} \frac{\rho^n}{(q)_n} h_n(x|a,q) = \frac{(a\rho)_\infty}{\prod_{i=0}^{\infty} v(x|\rho q^i)},$$

convergent for $|\rho|, |a| < 1$, $|x| \leq 1$, where polynomials $v(x|a)$ are defined by (2.7). Notice that

$$\varphi_{bh}(x|a,\rho,q) = (a\rho)_\infty \varphi_h(x|\rho,q). \tag{2.13}$$

It is also known (follows directly Favard's theorem and (2.10)) that bqh polynomials are orthogonal with respect to the positive measure that has density f_{bh} defined below if $|a| < 1$.

If $a > 1$ it turns out that apart from absolutely continuous part with the density f_{bh} this measure has also $\#\left\{k : aq^k > 1\right\}$ jump points concentrated at points with masses $w_k = \frac{(1/a^2)_\infty (1-a^2q^{2k})(a^2)_k}{(q)_\infty (1-a^2)(q)_k (-a^4)^k} q^{-3k^2/2-k/2}$.

So that we have for $|a| < 1$:

$$\int_{-1}^{1} h_n(x|a,q)\, h_m(x|a,q)\, f_{bh}(x|a,q)\, dx = (q)_n \delta_{nm},$$

where

$$f_{bh}(x|a,q) = \frac{f_h(x|q)}{\prod_{i=0}^{\infty} v(x|aq^i)} = f_h(x|q)\, \varphi_h(x|a,q), \tag{2.14}$$

for $|x| \leq 1$, $|a| < 1$.

Notice that using the idea of the density expansion presented in the introduction we could have obtained (2.14) straightforwardly, first getting

$$f_{bh}(x|a,q) = f_h(x|q) \sum_{j\geq 0} \frac{a^j}{(q)_j} h_j(x|q)$$

by (1.1) and (2.12) and then summing the series by (2.6).

2.2.3. *Al-Salam–Chihara.* Next family of polynomials that we are going to consider depends on 2 (apart from q) parameters denoted by a and b. These polynomials will be called Al-Salam–Chihara polynomials $\{Q_n(x|a,b,q)\}_{n\geq -1}$ briefly ASC. They satisfy the following 3-term recurrence:

(2.15)
$$Q_{n+1}(x|a,b,q) = (2x-(a+b)q^n)Q_n(x|a,b,q)-(1-abq^{n-1})(1-q^n)Q_{n-1}(x|y,\rho,q),$$

with $Q_{-1}(x|a,b,q)=0$, $Q_0(x|a,b,q)=1$. It follows from Favard's theorem that the measure that makes these polynomials orthogonal is positive if $\forall n\geq 1:(1-abq^{n-1})(1-q^n)$, which for $|q|\leq 1$ is reduced to $|ab|\leq 1$.

They are related to bqh polynomials by the following 2 relationships (see either [18] (16.4.3) or [13] after obvious modifications):

$$Q_n(x|a,b,q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}}(-a)^{n-k}h_k(x|b,q), \tag{2.16}$$

$$h_n(x|b,q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k}Q_k(x|a,b,q). \tag{2.17}$$

Let us mention also two relationships between ASC and continuous $q-$ Hermite polynomials. We have the following Proposition:

Proposition 1. *For $x,a,b,q\in\mathbb{C}$:*

$$Q_n(x|a,b,q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q q^{\binom{n-j}{2}}(-1)^{n-j}s_{n-j}\left(a,b|q^{-1}\right)h_j(x|q), \tag{2.18}$$

$$h_n(x|q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q s_{n-j}(a,b|q)\,Q_j(x|a,b,q). \tag{2.19}$$

Where we denoted:

$$s_n(a,b|q) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q a^{n-m}b^m. \tag{2.20}$$

Proof. Proof is shifted to Section 5. □

Notice that

$$s_n(a,b|q) = a^n R_n(b/a|q),$$

where $R_n(x|q)$ are the so called Rogers–Szegö polynomials defined by:

$$R_n(x|q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q x^j. \tag{2.21}$$

It is well known that (see [18] formula 13.1.7) that

$$h_n(x|q) = \exp(in\theta)R_n(\exp(-2i\theta)|q), \tag{2.22}$$

where $x=\cos(\theta)$.

One knows also the generating function of ASC polynomials. Namely following [17] we have for $|t|<1$:

$$\varphi_Q(x|a,b,t,q) \overset{df}{=} \sum_{k=0}^{\infty}\frac{t^k}{(q)_k}Q_k(x|a,b,q) = \frac{(at,bt)_\infty}{\prod_{k=0}^{\infty}v(x|tq^k)}. \tag{2.23}$$

Notice that

$$\varphi_Q(x|a,b,t,q) = (at,bt)_\infty \varphi_h(x|t,q). \tag{2.24}$$

As mentioned above polynomials Q_n are orthogonal with respect to the positive measure if $|ab|<1$. Like in the case of polynomials bqh this measure has density if absolute values of both parameters a and b are less than 1. If one of them is greater than 1 then this orthogonalizing measure has jumps.

To get known (of course from i.e. [18]) density of measure that makes polynomials Q orthogonal under condition that $|a|,|b|<1$ one can apply idea of density of expansion presented in the introduction and use formula (2.8), obtaining

$$f_Q(x|a,b,q) = f_h(x|q)\sum_{j\geq 0}\frac{s_j(a,b|q)}{(q)_j}h_j(x|q). \tag{2.25}$$

Now we use relationship $s_n(a,b|q) = a^n W_n(b/a|q)$, the definition (2.21) of polynomials W_n, change the order of summation and finally apply Lemma 1, i) getting

$$f_Q(x|a,b,q) = f_h(x|q)\,\varphi_h(x|a,q)\,\varphi_h(x|b,q)\,(ab)_\infty. \tag{2.26}$$

Remark 1. Notice that comparing (2.25) and (2.26) and recalling (2.6) we obtain nice universal formula concerning $q-$Hermite polynomials. Namely we have:

$$(ab)_\infty\sum_{j,k\geq 0}\frac{a^j b^k}{(q)_j(q)_k}h_j(x|q)\,h_k(x|q) = \sum_{k\geq 0}\frac{s_k(a,b|q)}{(q)_k}h_k(x|q),$$

valid for $|a|,|b|<1$, $|x|\leq 1$.

One also shows (e.g. in [18]) that

$$\int_{[-1,1]} Q_n(x|a,b,q)\,Q_m(x|a,b,q)\,f_Q(x|a,b,q)\,dx = (ab,q)_n\,\delta_{nm}. \tag{2.27}$$

2.2.4. *Continuous dual Hahn.* The continuous dual Hahn (briefly C2H) polynomials $\{\psi\}_{n\geq -1}$ satisfy the following 3-term recurrence:

$$\psi_{n+1}(x|a,b,c,q) = (2x-\delta_n)\psi_n(x|a,b,c,q) - \zeta_{n-1}(1-q^n)\psi_{n-1}(x|a,b,c,q), \tag{2.28}$$

with $\psi_0(x|a,b,c,q)=1$ and $\psi_{-1}(x|a,b,c,q)=0$ and $\delta_n = (a+b+c)q^n + abcq^{n-1}(1-q^n-q^{n+1})$, $\zeta_n = (1-abq^n)(1-acq^n)(1-bcq^n)$.

These polynomials are related to ASC and bqh polynomials through the following relationships recently presented in [5]:

$$\psi_n(x|a,b,c,q) = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q (-a)^{n-i} q^{\binom{n-i}{2}} (bcq^i)_{n-i} Q_i(x|b,c,q), \tag{2.29}$$

$$Q_n(x|b,c,q) = \sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q a^{n-i}\left(bcq^i\right)_{n-i}\psi_i(x|a,b,c,q). \tag{2.30}$$

In fact in [5] only (2.29) was given. (2.30) can be simply derived from it.

The generating function for C2H polynomials does not have a simple form and thus we will not present it. It can be found e.g. in [17]. However under condition that $|a|,|b|,|c|<1$ it is relatively easy to get the density of the measure with

respect to which polynomials ψ are orthogonal. Namely using described in the introduction idea of expansion, (2.30) and (2.27) we get:

$$f_{\psi}\left(x|a,b,c,q\right)=f_{Q}\left(x|b,c,q\right)\sum_{j\geq 0}\frac{a^{j}\left(bc\right)_{j}}{\left(q\right)_{j}\left(bc\right)_{j}}Q_{j}\left(x|b,c,q\right).$$

Now we apply (2.26) and (2.23) getting:

$$f_{\psi}\left(x|a,b,c,q\right)=f_{h}\left(x|q\right)\varphi_{h}\left(x|a,q\right)\varphi_{h}\left(x|b,q\right)\varphi_{h}\left(x|c,q\right)\left(ab,ac,bc\right)_{\infty}. \tag{2.31}$$

Remark 2. Since $\int_{-1}^{1}f_{\psi}\left(x|a,b,c,q\right)dx=1$ as a corollary of the (2.31) we get the following summation formula

$$\sum_{i,j,k\geq 0}\frac{a^{i}b^{j}c^{k}}{\left(q\right)_{i}\left(q\right)_{j}\left(q\right)_{k}}\int_{-1}^{1}h_{i}\left(x|q\right)h_{j}\left(x|q\right)h_{k}\left(x|q\right)f_{h}\left(x|q\right)dx=\frac{1}{\left(ab,ac,bc\right)_{\infty}},$$

which is in fact a generating function of the set of numbers (in fact functions of q) $\left\{\int_{-1}^{1}h_{i}\left(x|q\right)h_{j}\left(x|q\right)h_{k}\left(x|q\right)f_{h}\left(x|q\right)dx\right\}_{i.j.k\geq 0}$. It was obtained for free. Numbers $\int_{-1}^{1}h_{i}\left(x|q\right)h_{j}\left(x|q\right)h_{k}\left(x|q\right)f_{h}\left(x|q\right)dx$ have been calculated and have many forms. One of them can be found in [18].

(2.31) is known in a slightly different form (see e.g. [17]). It is known (again see e.g. [17]) that

$$\int_{[-1,1]}\psi_{n}\left(x|a,b,c,q\right)\psi_{m}\left(x|a,b,c,d,q\right)f_{\psi}\left(x|a,b,c,q\right)dx=\left(ab,ac,bc,q\right)_{n}\delta_{nm}. \tag{2.32}$$

2.2.5. *Askey–Wilson.* Askey–Wilson (AW) polynomials $\{\alpha_n\}$ were introduced in [9] and as of now constitute the largest (in the sense of the number of parameters) family of orthogonal polynomials. Its original definition was given through the basic hypergeometric function

$$p_{n}\left(x|a,b,c,d,q\right)=\frac{\left(ab,ac,ad\right)_{n}}{a^{n}}\ {}_{4}\phi_{3}\left(\begin{array}{cccc} q^{-n} & abcdq^{n-1} & ae^{i\theta} & ae^{-i\theta} \\ & ab & ac & ad \end{array}\Big|q,q\right),$$

with $x=\cos\theta$. In the formula above ${}_4\phi_3$ denotes basic hypergeometric function given for example by formula 1.3 of [9]. Fortunately there exist other characterizations of this family of polynomials. Particularly there exists a characterization through 3-term recurrence. Namely starting from [17], (3.1.5) and defining

$$\alpha_{n}\left(x|a,b,c,d,q\right)=p_{n}\left(x|a,b,c,d,q\right)/\left(abcdq^{n-1}\right)_{n} \tag{2.33}$$

we get the following 3-term recurrence to be satisfied by the polynomials a_n:

$$2x\alpha_{n}\left(x\right)=\alpha_{n+1}\left(x\right)+\left(a+a^{-1}-A_{n}-C_{n}\right)\alpha_{n}(x)-A_{n-1}C_{n}\alpha_{n-1}(x),$$

with $a_{-1}\left(x\right)=0$, $a_{0}\left(x\right)=1$, where for simplicity we denoted $\alpha_{n}(x)=\alpha_{n}\left(x|a,b,c,d,q\right)$,

$$\begin{aligned} A_{n} &= \frac{(1-abq^{n})(1-acq^{n})(1-adq^{n})(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \\ C_{n} &= \frac{a(1-q^{n})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned}$$

Now let us denote $f_{n}\left(a,b,c,d,q\right)=A_{n-1}C_{n}$, $e_{n}\left(a,b,c,d,q\right)=\left(a+a^{-1}-A_{n}-C_{n}\right)$.

Again from Favard's theorem it follows that polynomials $\{\alpha_n\}$ are orthogonal with respect to the positive measure provided $A_{n-1}C_n$ is positive for all $n\geq 1$.

Again examining the form of A_n and C_n we see that this requirement can be reduced to the requirement that $|ab|, |ac|, |ad|, |bc|, |bd|, |cd| \le 1$. As in the case of polynomials bqh, ASC, C2H if $|a|, |b|, |c|, |d| \le 1$ then this measure has density.

We have immediate

Proposition 2. $f_n(a,b,c,d,q) = (1-q^n)\frac{\left(abq^{n-1},acq^{n-1},adq^{n-1},bcq^{n-1},bdq^{n-1},cdq^{n-1},abcdq^{n-2}\right)_1}{\left(abcdq^{2n-3}\right)_3\left(abcdq^{2n-2}\right)_1}$,

$e_n(a,b,c,d,q) = q^{n-2}\frac{((a+b+c+d)(q^2-abcbdq^n(1+q-q^{n+1}))+(abc+abd+acd+bcd)(q-q^{n+2}-q^{n+1})+abcdq^{2n})}{(1-abcdq^{2n-2})(1-abcdq^{2n})}$

Proof. Technical, uninteresting proof is moved to Section 5 □

Lemma 1. *We have*

$$\alpha_n(x|a,b,c,d,q) = \tag{2.34}$$

$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q (-a)^{n-i} q^{\binom{n-i}{2}} \frac{\left(bcq^i, bdq^i, cdq^i\right)_{n-i}}{\left(abcdq^{n+i-1}\right)_{n-i}} \psi_i\left(x|b,c,d,q\right), \tag{2.35}$$

$$\psi_n\left(x|b,c,d,q\right) = \tag{2.36}$$

$$\sum_{i=0}^{n}\begin{bmatrix} n \\ i \end{bmatrix}_q a^{n-i} \frac{\left(bcq^i, bdq^i, cdq^i\right)_{n-i}}{\left(abcdq^{2i}\right)_{n-i}} \alpha_i\left(x|a,b,c,d,q\right). \tag{2.37}$$

Proof. Is shifted to Section 5 □

As a corollary we have

Corollary 1.

$$\alpha_n(x|a,b,c,d,q) = \tag{2.38}$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(cdq^k\right)_{n-k} Q_k\left(x|c,d,q\right) \tag{2.39}$$

$$\times \sum_{m=0}^{n-k}\begin{bmatrix} n-k \\ m \end{bmatrix}_q q^{m(m-n+k)} a^m b^{n-k-m} \frac{\left(bcq^{n-m}, bdq^{n-m}\right)_m}{\left(abcdq^{2n-m-1}\right)_m} \tag{2.40}$$

$$Q_n\left(x|c,d,q\right) = \tag{2.41}$$

$$\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \left(cdq^j\right)_{n-j} \alpha_j\left(x|a,b,c,d,q\right) \times \tag{2.42}$$

$$\sum_{m=0}^{n-j}\begin{bmatrix} n-j \\ m \end{bmatrix}_q b^{n-j-m} a^m \frac{\left(bcq^j, bdq^j\right)_m}{\left(abcdq^{2j}\right)_m} \tag{2.43}$$

Proof. Is shifted to Section 5. □

Remark 3. In the literature (e.g. [18],16.4.3 or [9], 6.4-6.5) there exist formulae for the so-called 'connection-coefficients' i.e. expanding polynomials of one family of polynomials of the AW scheme in terms of the polynomials of the other. However apart from the cases when 3 out of 4 parameters appearing on both sides of these formulae are the same and the 4-th parameter being equal to zero, it is difficult to use these formulae. In fact we utilize [9], 6.5 in this situation to get (2.36) and (2.34). The formulae (2.41) and (2.38) we get by direct calculations without referring to [9], 6.5. We get them by combining (2.36) and (2.34) and (2.30) and (2.29).

Remark 4. Following the idea of expansion, assuming $|a|,\ |b|,\ |c|,\ |d| \leq 1$ (2.36) and (2.32), we can relate density of the Askey–Wilson polynomials and the density of C2H polynomials. Namely we have:

$$f_{AW}(x|a,b,c,d,q) = f_{\psi}(x|b,c,d,q) \sum_j \frac{a^j}{(abcd)_j} \psi_j(x|b,c,d,q). \tag{2.44}$$

Expression $\sum_j \frac{a^j}{(abcd)_j} \psi_j(x|b,c,d,q)$ was summed recently by Atakishiyeva and Atakishiyev in [5] yielding

$$\sum_j \frac{a^j}{(abcd)_j} \psi_j(x|b,c,d,q) = \frac{(ab,ac,ad)_\infty}{(abcd)_\infty} \varphi_h(x|a,q).$$

Hence following (2.31) we have:

$$f_{AW}(x|a,b,c,d,q) = f_h(x|q)\,\varphi_h(x|a,q)\,\varphi_h(x|b,q)\,\varphi_h(x|c,q)\,\varphi_h(x|d,q)\,\frac{(ab,ac,ad,bc,bd,cd)_\infty}{(abcd)_\infty}. \tag{2.45}$$

This formula is known for nearly 30 years of course in a different form (see [9], [18], [17]). The presented above one exposes symmetry of this density with respect to the parameters. However notice that if it was assumed to be known then we would be able by (2.44) to sum $\sum_j \frac{a^j}{(abcd)_j} \psi_j(x|b,c,d,q)$ saving Atakishiyeva and Atakishiyev a lot of work. Whatever point of view we chose, (2.44) gives a new interesting interpretation to the generating function $\sum_j \frac{a^j}{(abcd)_j} \psi_j(x|b,c,d,q)$.

Remark 5. Like in the case of C2H polynomials following the fact that $\int_{-1}^{1} f_{AW}(x|a,b,c,d,q)\,dx = 1$ and of course (2.45) we have:

$$\sum_{j,k,n,m} \frac{a^j b^k c^n d^m}{(q)_j (q)_k (q)_n (q)_m} \int_{-1}^{1} h_j(x|q)\,h_k(x|q)\,h_n(x|q)\,h_m(x|q)\,f_h(x|q)\,dx = \frac{(abcd)_\infty}{(ab,ac,ad,bc,bd,cd)_\infty},$$

i.e. characteristic function of the numbers $\left\{\int_{-1}^{1} h_j(x|q)\,h_k(x|q)\,h_n(x|q)\,h_m(x|q)\,f_h(x|q)\,dx\right\}_{j,k,n,m\geq 0}$ for free.

Remark 6. For completeness let us remark that following [18] or [17] we have;

$$\int_{[-1,1]} \alpha_n(x|a,b,c,d,q)\,\alpha_m(x|a,b,c,d,q)\,f_{AW}(x|a,b,c,d,q)\,dx = \delta_{mn} \frac{\left(abcdq^{n-1}\right)_n (ab,ac,ad,bc,bd,cd)_n}{(abcd)_{2n}} \tag{2.46}$$

3. Complex parameters

As mentioned above all analyzed polynomials are orthogonal with respect to a positive measure if only products of all pairs of their parameters have absolute values less that 1. Hence these parameters can also be complex but forming conjugate pairs.

Specially important will be in this section polynomials that depend on even numbers of parameters i.e. $q-$Hermite, ASC and AW polynomials, since then the measures that make these polynomials orthogonal have densities.

First let us consider ASC polynomials with complex parameters.

Let us put $a = \rho \exp(i\eta)$, $b = \rho \exp(-i\eta)$.

$$Q_n(x|\rho \exp(i\eta), \rho \exp(-i\eta), q) = p_n(x|y,\rho,q),$$

where $\cos\eta = y$.

One can easily notice that polynomials p_n satisfy the following 3-term recurrence:

(3.1)
$$p_{n+1}(x|y,\rho,q) = 2(x-\rho y q^n)p_n(x|y,\rho,q) - (1-q^n)(1-\rho^2 q^{n-1})p_{n-1}(x|y,\rho,q).$$

Following (2.23) we have

$$\varphi_p(x,t|y,\rho,q) = \sum_{j=0}^{\infty}\frac{t^j}{(q)_j}p_j(x|y,\rho,q) = \prod_{j=0}^{\infty}\frac{v\left(y|\rho t q^j\right)}{v\left(x|tq^j\right)}. \tag{3.2}$$

Notice that

$$\varphi_p(x,t|y,\rho,q) = \frac{\varphi_h(x|t,q)}{\varphi_h(y|t\rho,q)}. \tag{3.3}$$

Let us introduce also the following $\{b_n(x|q)\}_{n\geq -1}$ family of polynomials

$$b_n(x|q) = (-1)^n q^{\binom{n}{2}} h_n\left(x|q^{-1}\right). \tag{3.4}$$

Polynomials were studied in [28], [29] and [27] and some of their basic properties can be found there. Its rôle in the complex parameters case was disclosed in [23]. We collect these properties and expose the rôle of these polynomials in the following Lemma,

Lemma 2. *For $\forall n \geq 0$, $x, y, \rho \in \mathbb{R}$,*

i) $b_n(y|q) = (-1)^n q^{\binom{n}{2}} s_n\left(\rho e^{i\eta}, \rho e^{-i\eta}|q^{-1}\right)/\rho^n$, where polynomials are given by (2.20),

ii) $b_n\left(x|q^{-1}\right) = (-1)^n q^{-\binom{n}{2}} h_n(x|q)$,

iii) $\forall n \geq 0 : b_{n+1}(x|q) = -2q^n x b_n(x|q) + q^{n-1}(1-q^n)b_{n-1}(x|q)$,

iv) $\sum_{j\geq 0}\frac{t^n}{(q)_n}b_j(x|q) = \prod_{k=0}^{\infty} v\left(x|tq^k\right) = 1/\varphi_h(x|t,q)$, consequently $\forall n \geq 1$: $\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q h_j(x|q)\, b_{n-j}(x|q) = 0$,

v) $p_n(x|y,\rho,q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \rho^{n-j} h_j(x|q)\, b_{n-j}(y|q)$, $p_n(x|y,0,q) = h_n(x|q)$,

vi) $h_n(x|q) = \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q p_j(x|y,\rho,q)\,\rho^{n-j} h_{n-j}(y|q)$.

Proof. We prove first our identities for $x, y \in [-1,1]$, $|\rho| < 1$. Then since all involved functions are polynomials we extend them to all values of variables. i) and ii) follow directly definition of b_n and s_n. iii) We start with 3−term recurrence satisfied for polynomials h_n i.e. $h_{n+1}\left(x|q^{-1}\right) = 2xh_n\left(x|q^{-1}\right) - (1-q^{-n})h_{n-1}\left(x|q^{-1}\right)$. Then we multiply both sides by $(-1)^{n+1}q^{\binom{n+1}{2}}$ and utilize the fact that $\binom{n}{2} + n = \binom{n+1}{2}$. iv) see [28] and [27]. v) We rewrite formulae (2.18) and (2.19) getting $p_n(x|y,\rho,q) = \sum_{j=0}^{n} q^{\binom{n-j}{2}}\begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{n-j} h_j(x|q)\, s_{n-j}\left(a,b|q^{-1}\right)\Big|_{a=\rho e^{i\theta}, b=\rho e^{-i\theta}}$. Now we apply assertion i). This formula with different scaling and normalization was derived in [23]. vi) was proved in [19] although its direct prove is easy in view of 2.19. □

Let us introduce also the following family of polynomials $\{g_n(x|y,\rho,q)\}_{n\geq -1}$ that are related to polynomials p_n in the following way:

$$g_n(x|y,\rho,q) = \begin{cases} \rho^n p_n\left(y|x,\rho^{-1},q\right) & if \quad \rho \neq 0 \\ b_n(x|q) & if \quad \rho = 0 \end{cases}. \tag{3.5}$$

We have the following easy Proposition:

Proposition 3. *i)* $\forall n \geq 0$:

$$g_n\left(x|y,\rho,q\right)=(-1)^n q^{\binom{n}{2}} p_n\left(y|x,\rho,q^{-1}\right),$$

ii) $\forall n \geq 0$:

$$g_{n+1}\left(x|y,\rho,q\right)=-2(xq^n-\rho y)g_n\left(x|y,\rho,q\right)-(1-q^n)(\rho^2-q^{n-1})g_{n-1}\left(x|y,\rho,q\right),$$

with $g_{-1}\left(x|y,\rho,q\right)=0,\ g_0\left(x|y,\rho,q\right)=1,$

iii)

$$\sum_{j=0}^{\infty}\frac{t^n}{(q)_n}g_n\left(x|y,\rho,q\right)=1/\varphi_p\left(x,t|y,\rho,q\right),$$

iv) $\forall n \geq 1, x, y, \rho \in \mathbb{R}$:

$$\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q p_j\left(x|y,\rho,q\right)g_{n-j}\left(x|y,\rho,q\right)=0.$$

Proof. Is shifted to Section 5. □

Let us introduce the following polynomials of order 2 in x and y and 4 in ρ :

$$\omega\left(x,y|\rho\right)=(1-\rho^2)^2-4xy\rho(1+\rho^2)+4\rho^2\left(x^2+y^2\right). \tag{3.6}$$

We have an obvious observations collected in the following proposition:

Proposition 4. *i)* $\omega\left(x,y|\rho\right)=v\left(x|\rho e^{i\eta}\right)v\left(x|\rho e^{-i\eta}\right)$, *where* $y=\cos\left(\eta\right)$ *and polynomials* v *are defined by (2.7)*

ii) $\left(te^{i\theta},te^{-i\theta},te^{i\eta},te^{-i\eta}\right)_1=\omega\left(z,y|t\right)$, *with* $y=\cos\left(\eta\right),\ z=\cos\left(\theta\right).$

Besides one knows also the density with respect to which these polynomials are orthogonal. Namely in the present setting formula (2.27) has the following form:

$$\int_{-1}^{1}p_n\left(x|y,\rho,q\right)p_m\left(x|y,\rho,q\right)f_p\left(x|y,\rho,q\right)=\delta_{nm}\left(\rho^2,q\right)_n, \tag{3.7}$$

where

$$f_p\left(x|y,\rho,q\right)=f_h\left(x|q\right)\frac{\left(\rho^2\right)_\infty}{\prod_{i=0}^{\infty}\omega\left(x,y|\rho q^i\right)},$$

is obtained from (2.26) after an obvious adjustment to complex parameters and application of Proposition 4, i).

Similarly we will consider the AW polynomials with two pairs of complex conjugate parameters. We will denote new parameters ρ_1, ρ_2, y, z.

They will be related to parameters a, b, c, d by the following relationship:

$$a=\rho_1\exp\left(i\theta\right),\ b=\rho_1\exp\left(-i\theta\right),\ c=\rho_2\exp\left(i\eta\right),\ d=\rho_2\exp\left(i\eta\right), \tag{3.8}$$

with $y=\cos\theta$, $z=\cos\eta$. Notice that :

Proposition 5. *With new parameters defined by (3.8) we have i)* $f_n(a,b,c,d,q)=(1-q^n)\frac{\omega\left(z,y|\rho_1\rho_2q^{n-1}\right)\left(\rho_1^2q^{n-1},\rho_2^2q^{n-1},\rho_1^2\rho_2^2q^{n-2}\right)_1}{\left(\rho_1^2\rho_2^2q^{2n-3}\right)_3\left(\rho_1^2\rho_2^2q^{2n-2}\right)_1}$,

ii) $v_n(a,b,c,d,q)=2q^{n-1}\frac{q(\rho_1y+\rho_2z)\left(1-\rho_1^2\rho_2^2q^{n-2}(1+q-q^{n+1})\right)+\rho_1\rho_2(\rho_1z+\rho_2y)\left(1-q^{n+1}-q^n+\rho_1^2\rho_2^2q^{2n-1}\right)}{\left(\rho_1^2\rho_2^2q^{2n-2}\right)_1\left(\rho_1^2\rho_2^2q^{2n}\right)_1}$,

iii) $(ab,ac,ad,bc,bd,cd)_1=\left(\rho_1^2,\rho_2^2\right)\omega\left(y,z|\rho_1\rho_2,q\right),\ abcd=\rho_1^2\rho_2^2.$

Let us denote also $\forall n \geq 1$:

$$w_n\left(x|y,\rho_1,z,\rho_2,q\right)=\alpha_n\left(x|\rho_1e^{i\theta},\rho_1e^{-i\theta},\rho_2e^{i\eta},\rho_2e^{-i\eta},q\right).$$

Corollary 2.

$$f_{AW}\left(x|y,\rho_1,z,\rho_2,q\right)=f_h\left(x|q\right)\frac{\left(\rho_1^2,\rho_2^2\right)_\infty\prod_{j=0}^{\infty}\omega\left(y,z|\rho_1\rho_2q^j\right)}{\prod_{j=0}^{\infty}\omega\left(x,y|\rho_1q^j\right)\omega\left(x,z|\rho_2q^j\right)}.$$

Proof. Follows (2.45) with (3.8) and the simplified by Proposition 4 i) and ii). □

Remark 7. Notice that: i) $\forall n\geq 0: w_n\left(x|y,0,\rho_2,z,q\right)=p_n\left(x|z,\rho_2,q\right),$
ii) $f_{AW}\left(x|y,0,z,\rho_2,q\right)=f_p\left(x|z,\rho_2,q\right).$

Proof. $\rho_1=0$ implies $a=b=0$ which leads to $f_n\left(0,0,c,d,q\right)=\left(1-q^n\right)\left(1-\rho_2^2q^{n-1}\right),$ and $v_n\left(0,0,c,d,q\right)=2q^n\rho_2z$. Now we confront these values with (3.1) getting first assertion. The second is trivial. □

We will use also the following useful

Proposition 6. *Let i denote imaginary unit, then $\forall n,m\geq 0$:*

$$\begin{aligned}(3.9)\quad &\sum_{k=0}^{n}\sum_{j=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}m\\j\end{bmatrix}_q\frac{\left(te^{i(-\theta+\eta)}\right)_k\left(te^{i(\theta-\eta)}\right)_j\left(te^{i(\theta+\eta)}\right)_{k+l}}{\left(t^2\right)_{k+j}}e^{i(2k-n)\theta}e^{i(2j-m)\eta}\\ &=\sum_{l=0}^{m}\begin{bmatrix}m\\l\end{bmatrix}_q(-1)^lq^{\binom{l}{2}}\frac{h_{m-l}\left(z|q\right)p_{n+l}\left(y|z,t,q\right)}{\left(t^2\right)_{n+l}},\end{aligned}$$

with $y=\cos\theta$ and $z=\cos\eta$.

Proof. Is shifted to Section 5 □

As a corollary we have our main result

Theorem 1. $\forall n\geq 0:$
(3.10)

$$w_n\left(x|\rho_1,y,\rho_2,z,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q p_j\left(x|y,\rho_1,q\right)\frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{n+j-1}\right)_{n-j}}g_{n-j}\left(z|y,\rho_1\rho_2q^{n-1},q\right),$$

(3.11)

$$p_n\left(x|y,\rho_1,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q w_j\left(x|y,\rho_1,z,\rho_2,q\right)\frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{2j}\right)_{n-j}}p_{n-j}\left(z|y,\rho_1\rho_2q^j,q\right).$$

where $g_n\left(z|y,\tau,q\right)=\tau^np_n\left(y|z,\tau^{-1},q\right)$ if $\tau\neq 0$, and $g_n\left(z|y,0,q\right)=b_n\left(z|q\right).$

Proof. Is shifted to Section 5. □

As a corollary we have the following expansion of the Askey–Wilson density with complex parameters.

Corollary 3. *i)*

$$(3.12)\quad f_{AW}\left(x|y,\rho_1,z,\rho_2,q\right)=f_p\left(x|y,\rho_1,q\right)\sum_{j\geq 0}\frac{\rho_2^jp_j\left(z|y,\rho_1\rho_2,q\right)}{\left(q\right)_j\left(\rho_1^2\rho_2^2\right)_j}p_j\left(x|y,\rho_1,q\right),$$

hence for $\forall n\geq 1,|y|,|z|\leq 1,|\rho_1|,|\rho_2|<1:$

$$(3.13)\quad \int_{[-1,1]}p_n\left(x|y,\rho_1,q\right)f_{AW}\left(x|y,\rho_1,z,\rho_2,q\right)dx=\frac{\rho_2^n\left(\rho_1^2\right)_n}{\left(\rho_1^2\rho_2^2\right)_n}p_n(z|y,\rho_1\rho_2,q),$$

ii)

(3.14)

$$f_p\left(x|y,\rho_1,q\right)=f_{AW}\left(x|y,\rho_1,z,\rho_2,q\right)\sum_{j\geq 0}\frac{\left(\rho_1^2\rho_2^2\right)_{2n}\rho_2^j g_j\left(z|y,\rho_1\rho_2q^{j-1},q\right)}{\left(q\right)_j\left(\rho_1^2\rho_2^2q^{j-1}\right)_j^2\left(\rho_2^2\right)_j\prod_{k=1}^j\omega\left(y,z|\rho_1\rho_2q^k\right)}w_j\left(x|y,\rho_1,z,\rho_2,q\right),$$

hence $\forall n\geq 1,\left|y\right|,\left|z\right|\leq 1,\left|\rho_1\right|\left|\rho_2\right|<1:$

(3.15)

$$\int_{[-1,1]}w_n(x|y,\rho_1,z,\rho_2,q)f_p\left(x|y,\rho_1,q\right)dx=\frac{\rho_2^n\left(\rho_1^2\right)_n}{\left(\rho_1^2\rho_2^2q^{n-1}\right)_n}g_n\left(z|y,\rho_1\rho_2q^{n-1},q\right).$$

Proof. We use idea of density expansion presented at the beginning of the paper. To get i) we use (1.1), (3.11) and (2.27). ii) We use (1.1), (3.10) and (2.46) . □

Remark 8. Expansion given in assertion i) was (with slightly different scaling) obtained by the similar method in [13] or as a particular case of an other expansion in [15].

Remark 9. Notice that if consider the following correspondence between two groups of polynomials. Namely to one group let us take polynomials $\{h_n(x|q),\ b_n\left(y|q\right),\ p_n\left(x|y,\rho,q\right)\}$ and to the second $\{p_n\left(x|y,\rho_1,q\right),g_n\left(z|y,\rho_1\rho_2q^{n-1},q\right),w_n\left(x|y,\rho_1,z,\rho_2,q\right)\}$ then we see that formula (3.10) is a generalization of Lemma 2, v) while (3.11) is a generalization of Lemma 2, vi). Besides (3.12) is a generalization of the Poisson–Mehler expansion formula

$$f_p\left(x|y,\rho,q\right)=f_h\left(x|q\right)\sum_{j=0}^{\infty}\frac{\rho^j}{\left(q\right)_j}h_j\left(x|q\right)h_j\left(y|q\right),\tag{3.16}$$

which has many proofs see e.g. [1], [18] or [10] where the presented above idea of the density expansion is applied. Similarly (3.14) is a generalization of [10], (5.3). Notice that following Remark 7 we see that (3.12) as well (3.10) reduce to (3.16) and Lemma 2, v) when it is set $\rho_1=0$. Similarly (3.14) and (3.11) reduce to [10], (5.3) and Lemma 2, vi) when we set $\rho_1=0$.

Corollary 4. *i)* $\forall n\geq 1,0\leq k<n,z,y,t\in\mathbb{R}:$

$$\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_q\frac{p_j\left(z|y,tq^k,q\right)}{\left(t^2q^{2k}\right)_j}\frac{g_{n-k-j}\left(z|y,tq^{n-1},q\right)}{\left(t^2q^{n+j+k-1}\right)_{n-k-j}}=0,$$

ii) $\forall n\geq 1,0\leq k<n,z,y,t\in\mathbb{R}:$

$$\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\ m\end{bmatrix}_q\frac{p_{n-k-m}\left(z|y,tq^{m+k},q\right)g_mz|y,tq^{m+k-1},q)}{\left(t^2q^{2m+2k}\right)_{n-k-m}\left(t^2q^{m+2k-1}\right)_m}=0.$$

Proof. Is shifted to Section 5. □

Remark 10. Notice that apart from the two above mentioned identities involving polynomials $\{p_n\}$ and $\{g_n\}$ we have also one exposed in Proposition 3, iv.

4. Towards probabilistic interpretation

To get nice probabilistic interpretation of Askey–Wilson density and polynomials we have to make certain re-scaling and re-normalization of all polynomials that appeared above. Let us introduce the following sets:

$$S\left(q\right)=\begin{cases} S\left(q\right)=\left[-2/\sqrt{1-q},2/\sqrt{1-q}\right] & if \quad |q|<1 \\ \mathbb{R} & if \quad q=1 \end{cases}.$$

Hence we will consider the following polynomials:

$$\begin{aligned} H_n\left(x|q\right) &= h_n\left(x\sqrt{1-q}/2\right)/\left(1-q\right)^{n/2}, \\ P_n\left(x|y,\rho,q\right) &= p_n\left(x\sqrt{1-q}/2|y\sqrt{1-q}/2,\rho,q\right), \end{aligned}$$

$$\begin{aligned} A_n\left(x|y,\rho_1,z,\rho_2,q\right) &= w_n\left(x\sqrt{1-q}/2|y\sqrt{1-q}/2,\rho_1,z\sqrt{1-q}/2,\rho_2,q\right), \\ B_n\left(x|q\right) &= b_n\left(x\sqrt{1-q}/2|q\right)/\left(1-q\right)^{n/2}, \\ G_n\left(x|y,\rho,q\right) &= g_n\left(x\sqrt{1-q}/2|y\sqrt{1-q}/2,\rho,q\right). \end{aligned}$$

One can easily check that these new polynomials satisfy the following 3-term recurrences:

$$H_{n+1}\left(x|q\right)=xH_n\left(x|q\right)-[n]_qH_{n-1}(x|q), \tag{4.1}$$

$$P_{n+1}\left(x|y,\rho,q\right)=(x-\rho yq^n)P_n\left(x|y,\rho,q\right)-[n]_q(1-\rho^2q^{n-1})P_{n-1}\left(x|y,\rho,q\right), \tag{4.2}$$

$$B_{n+1}\left(x|q\right)=-q^nxB_n\left(x\right)+q^{n-1}[n]_qB_{n-1}\left(x\right), \tag{4.3}$$

$$G_{n+1}\left(x|y,\rho,q\right)=-(xq^n-\rho y)G_n\left(x|y,\rho,q\right)-[n]_q\left(\rho^2-q^{n-1}\right)G_{n-1}\left(x|y,\rho,q\right), \tag{4.4}$$

$$A_{n+1}\left(x|y,\rho_1,z,\rho_2,q\right)=(x-\beta_n)A_n\left(x|y,\rho_1,z,\rho_2,q\right)-\gamma_{n-1}A_{n-1}\left(x|y,\rho_1,z,\rho_2\right), \tag{4.5}$$

where we use Pochhammer symbol and (3.6) for brevity:

$$\beta_n=\frac{q^n\left(y\rho_1+z\rho_2\right)\left(1-q^{n-2}(1+q-q^{n+1})\rho_1^2\rho_2^2\right)+q^{n-1}\rho_1\rho_2\left(z\rho_1+y\rho_2\right)\left(1-q^{n+1}-q^n+\rho_1^2\rho_2^2q^{2n-1}\right)}{(1-\rho_1^2\rho_2^2q^{2n-2})(1-\rho_1^2\rho_2^2q^{2n})}, \tag{4.6}$$

$$c_n=\frac{[n]_q\left(\rho_1^2q^{n-1}\right)_1\left(\rho_2^2q^{n-1}\right)_1\left(\rho_1^2\rho_2^2q^{n-2}\right)_1\omega\left(y\frac{\sqrt{1-q}}{2},z\frac{\sqrt{1-q}}{2}|\rho_1\rho_2q^{n-1}\right)}{\left(\rho_1^2\rho_2^2q^{2n-3}\right)_3\left(\rho_1^2\rho_2^2q^{2n-2}\right)_1}, \tag{4.7}$$

with $H_{-1}\left(x|q\right)=H_{-1}\left(x|a,q\right)=P_{-1}\left(x|y,\rho,q\right)=B_{-1}\left(x|q\right)=G_{-1}\left(x|y,\rho,q\right)=A_{-1}\left(x|y,\rho_1,z,\rho_2,q\right)=0$, $H_0\left(x|q\right)=H_0\left(x|a,q\right)=P_0\left(x|y,\rho,q\right)=B_0\left(x|q\right)=G_0\left(x|y,\rho,q\right)=A_0\left(x|y,\rho_1,z,\rho_2,q\right)=1$.

Adapted to current setting Lemma 2 has the following form

Lemma 3. *For* $\forall n\geq 0$, $x,y,\rho\in\mathbb{R}$,

i)

$$P_n\left(x|y,\rho,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\rho^{n-j}H_j\left(x|q\right)B_{n-j}\left(y|q\right),$$

ii)

$$H_n(x|q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \rho^{n-j} H_{n-j}(y|q) P_j(x|y,\rho,q).$$

Proof. i) was shown in this form in [23]. ii) follows almost Lemma 2, v). □

Let us denote by $H_n(x)$ monic polynomials that are orthogonal with respect to measure with the density $\exp\left(-x^2/2\right)/\sqrt{2\pi}$. By $U_n(x)$ let us denote Chebyshev polynomials of the second kind . Thus these polynomials are defined by the following 3-term recurrences:

$$H_{n+1}(x) = xH_n(x) + nH_{n-1}(x), \tag{4.8}$$

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{4.9}$$

with $H_{-1}(x) = U_{-1}(x) = 0$, $H_0(x) = U_0(x) = 1$.

Polynomials $\{H_n(x|q)\}$ are orthogonal with respect to measures having the following densities

(4.10)

$$f_N(x|q) = f_h\left(\frac{x\sqrt{1-q}}{2}|q\right)\frac{\sqrt{1-q}}{2} = \frac{\sqrt{1-q}\,(q)_\infty}{2\pi\sqrt{4-(1-q)x^2}}\prod_{i=0}^{\infty} l\left(x\frac{\sqrt{1-q}}{2}|q^i\right),$$

$x \in S(q)$.

Polynomials $\{P_n\}$ are orthogonal with respect to the measure with the density

$$f_{CN}(x|y,\rho,q) = f_N(x|q)\frac{\left(\rho^2\right)_\infty}{\prod_{j=0}^{\infty}\omega\left(x\frac{\sqrt{1-q}}{2}, y\frac{\sqrt{1-q}}{2}|\rho q^i\right)}, \tag{4.11}$$

with $x, y \in S(q)$.

Polynomials $\{A_n\}$ are orthogonal with respect to measure with the density:

$$f_{AW}(x|y,\rho_1,z,\rho_2,q) = \frac{f_{CN}(y|x,\rho_1,q)\,f_{CN}(x|z,\rho_2,q)}{f_{CN}(y|z,\rho_1\rho_2,q)}. \tag{4.12}$$

We have several easy observations that are believed to help intuition which we will put two remarks one concerning densities the second polynomials:

Remark 11. i)

$$f_N(x|1) = \exp\left(-x^2/2\right)/\sqrt{2\pi}, x \in \mathbb{R}; f_N(x|0) = \frac{1}{2\pi}\sqrt{4-x^2}, |x| \le 2,$$

ii)

$$f_{CN}(x|y,\rho,1) = \frac{\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right)}{\sqrt{2\pi\left(1-\rho^2\right)}},$$

for $x \in \mathbb{R}$,

$$f_{CN}(x|y,\rho,0) = \frac{\left(1-\rho^2\right)\sqrt{4-x^2}}{2\pi\omega\left(x\frac{\sqrt{1-q}}{2}, y\frac{\sqrt{1-q}}{2}|\rho\right)}$$

for $|x|, |y| \le 1,\ |\rho| < 1$,

iii)

$$f_{AW}\left(x|y,\rho_1,z,\rho_2,1\right)=\frac{\exp\left(-\frac{\left(x-\frac{y\rho_1\left(1-\rho_2^2\right)+z\rho_2\left(1-\rho_1^2\right)}{1-\rho_1^2\rho_2^2}\right)^2}{2\left(\frac{\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)}{1-\rho_1^2\rho_2^2}\right)}\right)}{\sqrt{2\pi\left(\frac{\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)}{1-\rho_1^2\rho_2^2}\right)}},$$

for : $x\in\mathbb{R}$,

$$f_{AW}\left(x|y,\rho_1,z,\rho_2,0\right)=\frac{\sqrt{4-x^2}\omega\left(z\frac{\sqrt{1-q}}{2},y\frac{\sqrt{1-q}}{2}|\rho_1\rho_2\right)}{2\pi\omega\left(x\frac{\sqrt{1-q}}{2},y\frac{\sqrt{1-q}}{2}|\rho_1\right)\omega\left(x\frac{\sqrt{1-q}}{2},z\frac{\sqrt{1-q}}{2}|\rho_2\right)},$$

for $|x|,|y|,|z|\leq 2$, $|\rho_1|,\rho_2|<1$.

Proof. i) For $q=1$ see [18], for $q=0$ one can see directly, ii) For $q=1$ see [18], for $q=0$ we have get it directly from (4.11). iii) For $q=1$ it follows (4.12) and assertion ii), for $q=0$ we apply (4.11) and direct calculation. □

That is why we changed slightly notation and we use f_N (4.10) to denote $q-$Normal density, and f_{CN} (4.11) to denote conditional $q-$Normal density.

Remark 12. $\forall n\geq 0$: i)

$$H_n\left(x|1\right)=H_n\left(x\right),H_n\left(x|0\right)=U_n\left(x/2\right),$$

ii)

$$H_n\left(x|a,1\right)=H_n\left(x-a\right),H_n\left(x|a,0\right)=U_n\left(x/2\right)-aU_{n-1}\left(x/2\right),$$

iii)

$$B_n\left(x|1\right)=i^nH_n\left(ix\right),B_n\left(y|0\right)=\left\{\begin{array}{ccc}-y & if & n=1\\ 1 & if & n=2\vee n=0\\ 0 & if & n\geq 3\end{array}\right.,$$

iv)

$$\begin{aligned}P_n\left(x|y,\rho,1\right) &= H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\left(1-\rho^2\right)^{n/2},\\ P_n\left(x|y,\rho,0\right) &= U_n\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)+\rho^2U_{n-2}\left(x/2\right),\end{aligned}$$

v)

$$\begin{aligned}G_n\left(x|y,\rho,1\right) &= \rho^nP_n\left(y|x,\rho^{-1},1\right)=\left(-i\right)^nH_n\left(i\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\left(1-\rho^2\right)^{n-1},\\ G_n\left(x|y,\rho,0\right) &= \rho^nU_n\left(y/2\right)-\rho^{n-1}xU_{n-1}\left(y/2\right)+\rho^{n-2}U_{n-2}\left(y/2\right),\end{aligned}$$

vi)

$$\begin{aligned}A_n\left(x|y,\rho_1,z,\rho_2,1\right) &= \left(\frac{\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)}{1-\rho_1^2\rho_2^2}\right)^{n/2}H_n\left(\frac{x(1-\rho_1^2\rho_2^2)-\rho_1\left(1-\rho_2^2\right)y-\rho_2(1-\rho_1^2)z}{\sqrt{\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)\left(1-\rho_1^2\rho_2^2\right)}}\right),\\ A_n\left(x|y,\rho_1,z,\rho_2,0\right) &= U_n\left(x/2\right)-\left(\rho_1y+\rho_2z\right)U_{n-1}\left(x/2\right)+(\rho_1^2+\rho_2^2+yz\rho_1\rho_2)U_{n-2}\left(x/2\right)\\ &\quad -\rho_1\rho_2\left(\rho_1z+\rho_2y\right)U_{n-3}\left(x/2\right)+\rho_1^2\rho_2^2U_{n-4}\left(x/2\right).\end{aligned}$$

Proof. i) and ii) follow immediately comparison of (4.1) and respectively (4.8) and (4.9). iii) For $q=1$ follow (4.3) and (4.8) considered for $x->ix$ while for $q=0$ follows directly (4.3). iv) Follow (4.2) and for $q=1$ (4.8) modified in a standard way to fit (4.8). The case $q=0$ follows directly Lemma 3, i) and iii). v) Follow definition of polynomials G_n and assertion iv). vi) Follows the fact that β_n for $q=1$ is equal to $\frac{y\rho_1\left(1-\rho_2^2\right)+z\rho_2\left(1-\rho_1^2\right)}{1-\rho_1^2\rho_2^2}$ -does not depend on n while for $q=0$ is equal to 0 for $n\geq 2$. For $q=1$ c_n is equal to $\frac{n\left(1-\rho_1^2\right)\left(1-\rho_2^2\right)}{1-\rho_1^2\rho_2^2}$ while for $q=0$ to 1 for $n\geq 2$. For $q=1$ it is easy to notice that 3-term recurrence satisfied polynomials A_n is a modified one given by (4.8). For $q=0$ one can reason in two ways. One way of reasoning is to observe that the density given by (4.12) has a form of the ratio of $\sqrt{4-x^2}$ divided by certain polynomial of order 4. Hence following Proposition 1 of [10] we deduce that polynomials that are orthogonal with respect to the density of this form must be of the form of linear combination of successive $4+1=5$ polynomials that are orthogonal with respect to the density $\sqrt{4-x^2}/2\pi$ i.e. Chebyshev polynomials of the second kind. Or we can notice that the 3-term recurrence satisfied by AW polynomials for $q=0$ is the same as that of the Chebyshev polynomials of the second kind with different initial conditions. Now it remains to find coefficients of the linear combination of the Chebyshev polynomials of the second kind. □

Now we are ready to present probabilistic applications of the densities f_N , f_{CN} and f_{AW} as well as polynomials H_n, P_n , A_n.

Let us consider 3 random variable say X, Y, Z their joint density is equal to $f_N(y|q)\, f_{CN}(x|y,\rho_1,q)\, f_{CN}(z|x,\rho_2,q)$. Another words we assume that Y,X,Z form a finite (of length 3) Markov chain.

Lemma 4. *For almost all $x,y,z\in S(q)$ and all $n\geq 1$we have:*
i) $(Y,X)\sim f_N(x|q)\, f_{CN}(y|x,\rho_1,q)$, $(Z,X)\sim f_N(x|q)\, f_{CN}(z|x,\rho_2,q)$,
ii) $Y\sim f_N$, $Z\sim f_N$, $(Z,Y)\sim f_N(z|q)\, f_{CN}(y|z,\rho_1\rho_2,q)$,
iii) $X|(Y=y,Z=z)\sim f_{AW}(x|y,\rho_1,z,\rho_2,q)$,
iv) $\mathbb{E}(H_n(X|q))=0$,
v) $\mathbb{E}(H_n(X|q)|Y=y)=\rho_1^n H_n(y|q)$, $\mathbb{E}(H_n(Z|q)|X=x)=\rho_2^2 H_n(x|q)$, $\mathbb{E}(H_n(Y|q)|Z=z)=(\rho_1\rho_2)^n H_n(z|q)$,
vi) $\mathbb{E}(P_n(X|y,\rho_1,q)|Y=y)=0$, $\mathbb{E}(P_n(X|z,\rho_2,q)|Z=z)=0$, $\mathbb{E}(P_n(Y|z,\rho_1\rho_2,q)|Z=z)=0$
vii) $\mathbb{E}(A_n(X|y,\rho_1,z,\rho_2,q)|Y=y,Z=z)=0$.
viii)

$$\begin{aligned}\mathbb{E}(H_n(X|q)|Y=y,Z=z) &= \frac{1}{\left(\rho_1^2\rho_2^2\right)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_2^{2k}\rho_1^{2k}\left(\rho_1^2,\rho_2^2\right)_k\\ &\times\sum_{j=0}^{n-2k}\begin{bmatrix}n-2k\\j\end{bmatrix}_q\left(\rho_1^2q^k\right)_j\left(\rho_2^2q^k\right)_{n-2k-j}\rho_1^{n-2k-j}\rho_2^j H_j(z|q)\, H_{n-2k-j}(y|q),\end{aligned}$$

ix)

$$\mathbb{E}(P_n(X|y,\rho_1,z,\rho_2,q)|Y=y,Z=z)=\frac{\rho_2^n\left(\rho_1^2\right)_n}{\left(\rho_1^2\rho_2^2\right)_n}P_n(z|y,\rho_1\rho_2,q),$$

x)

$$\mathbb{E}(A_n\left(X|y,\rho_1,z,\rho_2,q\right)|Y=y)=\frac{\rho_2^n\left(\rho_1^2\right)_n}{\left(\rho_1^2\rho_2^2q^{n-1}\right)_n}G_n\left(z|y,\rho_1\rho_2q^{n-1},q\right).$$

Proof. i) The simplest proof of the fact that $\int_{S(q)}f_N\left(x|q\right)f_{CN}\left(y|x,\rho_1,q\right)dx=f_N\left(y|q\right)$ follows (4.11) from which it follows that $f_N\left(y|q\right)f_{CN}\left(x|y,,t,q\right)=f_N\left(x|q\right)f_{CN}\left(y|x,t,q\right)$. ii) First two fact are obvious, the third follows the fact that $\int_{S(q)}f_{CN}\left(x|y,\rho_1,q\right)f_{CN}\left(y|z,\rho_2,q\right)dy=f_{CN}\left(x|z,\rho_1\rho_2,q\right)$ for $|\rho_1|,|\rho_2|<1$ which was proved in [23]. iii) Conditional density $X|(Y=y,Z=z)$ is equal to the ratio of joint density divided by the marginal density of (Y,z) given in assertion ii) .iv) Follows ii) and the fact that polynomials $\{H_n\}$ are orthogonal with respect to measure with the density f_N. v) First we use properties of conditional expectation, the fact that $X|Y=y\sim f_{CN}\left(x|y,\rho_1,q\right)$. Now we use the fact that $\int_{S(q)}H_n\left(x|q\right)f_{CN}\left(x|y,\rho,q\right)=\rho^nH_n\left(y|q\right)$ proved among others in [3] or [2]. vi) The fact that ASC polynomials are orthogonal with respect to the measure with the density f_{CN} was pointed out in [23]. vii) Follows iii) and the fact that polynomials A_n are orthogonal with respect to the measure with the density f_{AW}. viii) Was shown in [11]. ix) and x) are in fact equivalent to assertions of Corollary 3. □

Let us mention that assertions of Lemma 4 does not exist all probabilistic aspects of mentioned in this paper polynomials and densities. First such probabilistic aspects appeared an excellent paper of Bożejko et all [3] where two stochastic processes that constitute generalization of ordinary Wiener and Ornstein–Uhlenbeck processes appeared. This was done in noncommutative probability context however is important also in for classical probability. In [3] there appeared re-scaled densities f_h and f_p that this densities f_N and f_{CN}. Further analysis of the stochastic properties of these processes was done recently in [16].

Generalization of the presented above finite 3-step Markov chain was done in [14].

An attempt to overcome the Markov scheme was done in [15]. Unfortunately it was an unsuccessful attempt in the sense that it turned out that it is impossible to construct trough generalization of the Kible-Slepian formula a 3−dimensional joint density that would have marginals of the form $f_N\left(x|q\right)f_{CN}\left(y|x,\rho,q\right)$ and many other properties described in Lemma 4 (except of course vii) and viii)).

Let us mention also that in 2001 W. Bryc in [2] constructed a stationary Markov chain $\{X_n\}_{n=-\infty}^{\infty}$ satisfying simple intuitive conditions imposed on its first two conditional moments. More precisely he assumed that there exit constants a, A, B, C such that for every $n\in\mathbb{Z}$ (set of integers):

$$\begin{aligned}\mathbb{E}\left(X_n|\mathcal{F}_{\neq n}\right) &= a\left(X_{n-1}+X_{n+1}\right)\\ \mathbb{E}(X_n^2|\mathcal{F}_{\neq n}) &= A\left(X_{n-1}^2+X_{n+1}^2\right)+BX_{n-1}X_{n+1}+C.\end{aligned}$$

It turns out that this Markov chain does not exist for all values of parameters a,A,B,C. However for some it does. Moreover its stationary distribution has density f_N while its transfer function f_{CN} with a proper constant ρ. Note that in original definition of this Markov chain there is no q. This fact surprised Richard Askey (as he discloses in the introduction to [18]). Bryc has defined parameter q as a certain rational function of parameters a,A,B,C. For details refer to original

paper of Bryc [2], but also some recollection of known and presentation of new properties of these Markov processes is also done in [23][16][14].

Finally let us mention also the fact that polynomials and densities analyzed in this paper appear also in the so called 'quadratic harnesses context'. For detail see [26],[25],[24].

5. Proofs

Proof of Proposition 1. Formulae (2.18) and (2.19) follow the following easy calculations:

$$\begin{aligned}
Q_n(x|a,b,q) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-a)^{n-k}q^{\binom{n-k}{2}}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q(-b)^{k-j}q^{\binom{k-j}{2}}h_j(x|q)\\
&= \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q(-1)^{n-j}h_j(x|q)\sum_{k=j}^{n}\begin{bmatrix}n-j\\k-j\end{bmatrix}_q a^{n-k}b^{k-j}q^{\binom{n-k}{2}+\binom{k-j}{2}}\\
&= \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q(-1)^{n-j}h_j(x|q)\sum_{m=0}^{n-j}\begin{bmatrix}n-j\\m\end{bmatrix}_q a^{n-j-m}b^{m}q^{\binom{n-j-m}{2}+\binom{m}{2}}\\
&= \sum_{j=0}^{n}q^{\binom{n-j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_q(-1)^{n-j}h_j(x|q)\sum_{m=0}^{n-j}\begin{bmatrix}n-j\\m\end{bmatrix}_q q^{m(m-n+j)}a^{n-j-m}b^{m}.
\end{aligned}$$

Now we notice that $\begin{bmatrix}n\\k\end{bmatrix}_{q^{-1}}=\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(n-k)}$. Next we have:

$$\begin{aligned}
h_n(x|q) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b^{n-k}h_k(x|b,q)\\
&= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b^{n-k}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q a^{k-j}Q_j(x|a,b,q)\\
&= \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q Q_j(x|a,b,q)\sum_{k=j}^{n-j}\begin{bmatrix}n-j\\k-j\end{bmatrix}_q b^{n-k}a^{k-j}\\
&= \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q Q_j(x|a,b,q)\sum_{m=0}^{n-j}\begin{bmatrix}n-j\\m\end{bmatrix}_q b^{n-j-m}a^{m}
\end{aligned}$$

□

Proof of Proposition 2. As far as f_n is concerned the situation is simple straightforward. So let us concentrate on justifying the form of e_n. We have $a^{-1}-A_n = (1-\frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{(1-abcdq^{2n-1})(1-abcdq^{2n})})/a = (-(abc+abd+acd+bcd)q^{2n}+abcd(a+b+c+d)q^{3n}+abcd(abc+abd+acd+bcd)q^{3n-1}-a^2b^2c^2d^2(a+b+c+d)q^{4n-1}+bq^n+cq^n+dq^n+a^2b^3c^2d^2q^{4n-1}+a^2b^2c^3d^2q^{4n-1}+a^2b^2c^2d^3q^{4n-1}+bcd(q^{n-1}-q^{2n-1})+ab^2cd(-q^{3n}-q^{2n-1})+abc^2d(-q^{3n}-q^{2n-1})+abcd^2(-q^{3n}-q^{2n-1})+ab^2c^2d^2(-q^{3n-1}+q^{4n-1}))/(1-abcdq^{2n-1})(1-abcdq^{2n})$, and $a-C_n = a(1-\frac{(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}) = ((a+b+c+d)q^n+a^2b^2c^2d^2(a+b+c+d)q^{4n-3}+(abc+abd+acd+bcd)(q^{n-1}-q^{2n-1})-abcd(a+b+c+d)(q^{2n-2}+q^{2n-1})-bq^n-cq^n-dq^n-a^2b^3c^2d^2q^{4n-3}-a^2b^2c^3d^2q^{4n-3}-a^2b^2c^2d^3q^{4n-3}+bcd(-q^{n-1}+q^{2n-1})+ab^2cd(q^{2n-1}+q^{3n-2})+abc^2\ d(q^{2n-1}+q^{3n-2})+abcd^2(q^{2n-1}+q^{3n-2})$

$+ ab^2c^2d^2(q^{3n-3} - q^{4n-3}))/((1 - abcdq^{2n-2})(1 - abcdq^{2n-1}))$. Now notice that each of the expressions has both symmetric in parameters a, b, c, d, and non-symmetric parts. To get e_n we have to add $a^{-1} - A_n$ and $a - C_n$ which leads to the fact that $(-bq^n - cq^n - dq^n - a^2b^3c^2d^2q^{4n-3} - a^2b^2c^3d^2q^{4n-3} - a^2b^2c^2d^3q^{4n-3} + bcd(-q^{n-1} + q^{2n-1}) + ab^2cd(q^{2n-1} + q^{3n-2}) + abc^2\ d(q^{2n-1} + q^{3n-2}) + abcd^2(q^{2n-1} + q^{3n-2}) + ab^2c^2d^2(q^{3n-3} - q^{4n-3}))(1 - abcdq^{2n}) + (1 - abcdq^{2n-2})(bq^n + cq^n + dq^n + a^2b^3c^2d^2q^{4n-1} + a^2b^2c^3d^2q^{4n-1} + a^2b^2c^2d^3q^{4n-1} + bcd(q^{n-1} - q^{2n-1}) + ab^2cd(-q^{3n} - q^{2n-1}) + abc^2d(-q^{3n} - q^{2n-1}) + abcd^2(-q^{3n} - q^{2n-1}) + ab^2c^2d^2(-q^{3n-1} + q^{4n-1}))$ $= 0$ that is non-symmetric parts cancel out. Thus it remains to calculate $(1 - abcdq^{2n})((a + b + c + d)q^n + a^2b^2c^2d^2(a + b + c + d)q^{4n-3} + (abc + abd + acd + bcd)(q^{n-1} - q^{2n-1}) - abcd(a+b+c+d)(q^{2n-2} + q^{2n-1})) + (1 - abcdq^{2n-2})(-(abc + abd + acd + bcd)q^{2n} + abcd(a + b + c + d)q^{3n} + abcd(abc + abd + acd + bcd)q^{3n-1} - a^2b^2c^2d^2(a + b + c + d)q^{4n-1})$ which gives e_n. □

Proof of Lemma 1. Let us consider $a = 0$. We get then $w_n(0, b, c, d, q) = (1-q^n)(1-bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})$ and $v_n(0, b, c, d, q) = q^n(b + c + d) + bcdq^{n-1}(1 - q^{n+1} - q^n)$, Thus comparing this with (2.28) we deduce that $a_n(x|0, b, c, d, q) = \psi_n(x|b, c, d, q)$. To get (2.34) we apply formula 6.4 of [9] with $\alpha = a$, $a = 0$. One has to be aware that this formula is valid for polynomials p_n. Thus taking into account (2.33) we get using on the way formula $a^{n-k}(\alpha/a)_{n-k} = (-\alpha)^{n-k} q^{\binom{n-k}{2}}$ when $a = 0$: Hence we are getting:

$$c_{k,n} = (-1)^k q^{nk-\binom{k}{2}} \left(q^{-n}\right)_k q^{\binom{n-k}{2}} \frac{(-a)^{n-k}\left(abcdq^{n-1}\right)_k \left(bcq^k, bdq^k, cdq^k\right)_{n-k}}{(q)_k} \frac{1}{\left(abcdq^{n-1}\right)_n}.$$

Now notice that $\left(q^{-n}\right)_k = (-1)^k q^{-nk+\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (q)_k$. Hence we have:

$$c_{k,n} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} (-a)^{n-k} \frac{\left(bcq^k, bdq^k, cdq^k\right)_{n-k}}{\left(abcdq^{n+k-1}\right)_{n-k}}.$$

To get (2.36) we use the same formula but with $\alpha = 0$. □

Proof of the Proposition 3. i) Let us notice that using (3.4) and Lemma 2, ii) we get

$$\begin{aligned}
p_n\left(y|x, \rho, q^{-1}\right) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{-1}} \rho^{n-j} h_j\left(y|q^{-1}\right) b_{n-j}\left(x|q^{-1}\right) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} \rho^{n-j} h_j\left(y|q^{-1}\right) (-1)^{n-j} q^{-\binom{n-j}{2}} h_{n-j}(x|q) \\
&= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n)} \rho^{n-j} (-1)^j (-1)^{n-j} q^{j(j-n)-\binom{j}{2}-\binom{n-j}{2}} b_j(y|q) h_{n-j}(x|q) \\
&= q^{-\binom{n}{2}} (-1)^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \rho^{n-j} b_j(y|q) h_{n-j}(x|q) = q^{-\binom{n}{2}} (-1)^n \rho^n p_n\left(y|x, \rho^{-1}, q\right).
\end{aligned}$$

ii) We have

$$\begin{aligned}
g_{n+1}(x|y, \rho, q) &= \rho^{n+1}(2(y - \rho^{-1}q^n x) p_n\left(y|x, \rho^{-1}, q\right) - (1 - q^n)(1 - \rho^{-2}q^{n-1}) p_{n-1}\left(y|x, \rho^{-1}q\right)) \\
&= -2(q^n x - \rho y) g_n(x|y, \rho, q) - (1 - q^n)(\rho^2 - q^{n-1}) g_{n-1}(x|y, \rho, q).
\end{aligned}$$

iii) Assume $\rho \neq 0$. We have $\sum_{j=0}^{\infty} \frac{t^n}{(q)_n} g_n\left(x|y,\rho,q\right) = \sum_{j=0}^{\infty} \frac{\rho^n t^n}{(q)_n} p_n\left(y|x,\rho^{-1},q\right)$ $= \varphi_p\left(y,\rho t|x,\rho^{-1},q\right) = \frac{\varphi_h(y,\rho t|q)}{\varphi_h(x,t|q)} = 1/\varphi_p\left(x,t|y,\rho,q\right)$ by (3.3). iv) Follows directly iii). □

Proof of Proposition 6. Proof is based on two independently obtained formulae for the sum $\sum_{k\geq 0} h_{n+k}\left(x|q\right) h_{m+k}\left(y|q\right) \frac{t^k}{(q)_k}$. On one hand Carlitz in [6], formula 1.4 showed that

$$\sum_{k\geq 0} R_{m+k}\left(x|q\right) R_{n+k}\left(y|q\right) \frac{\tau^k}{(q)_k} = \frac{\left(xy\tau^2\right)_\infty}{\left(\tau, x\tau, y\tau, xy\tau\right)_\infty} \times \sum_{s=0}^{m}\sum_{r=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{\left(x\tau\right)_s \left(y\tau\right)_r \left(xy\tau\right)_{s+r}}{\left(xy\tau^2\right)_{s+r}} x^{m-s} y^{n-r}, \tag{5.1}$$

where $R_n\left(x|q\right)$ denotes the so called Rogers-Szegö polynomial defined by (2.21). Notice that setting $m = n = 0$ in (5.1) we get

$$\sum_{k\geq 0} R_k\left(x|q\right) R_k\left(y|q\right) \frac{\tau^k}{(q)_k} = \frac{\left(xy\tau^2\right)_\infty}{\left(\tau, x\tau, y\tau, xy\tau\right)_\infty}. \tag{5.2}$$

Now let us recall formula (2.22) expressing relationship between RS and qh polynomials.

Applying (5.1) with $x = \exp\left(-2i\theta\right)$, $y = \exp\left(-2i\eta\right)$ and $\tau = \exp\left(i(\theta+\eta)\, t\right.$ we get: (5.2)

$$\sum_{k\geq 0} R_{m+k}\left(e^{-2i\theta}|q\right) R_{n+k}\left(e^{-2i\eta}|q\right) \frac{e^{ik(\theta+\eta)}t^k}{(q)_k} = \sum_{k\geq 0} R_k\left(e^{-2i\theta}|q\right) R_k\left(e^{-2i\eta}|q\right) \frac{e^{ik(\theta+\eta)}t^k}{(q)_k} \times \sum_{s=0}^{m}\sum_{r=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{\left(te^{i(-\theta+\eta)}\right)_s \left(te^{i(\theta-\eta)}\right)_r \left(te^{-i(\theta+\eta)}\right)_{s+r}}{\left(t^2\right)_{s+r}} e^{-2i(m-s)\theta} e^{-2i(n-r)\eta},$$

Multiplying both sides by $e^{i(m\theta+n\eta)}$ and passing to polynomials h_n we get

$$\sum_{k\geq 0} h_{m+k}\left(x|q\right) h_{n+k}\left(y|q\right) \frac{t^k}{(q)_k} = \sum_{k\geq 0} h_m\left(x|q\right) h_n\left(y|q\right) \frac{t^k}{(q)_k} \times \sum_{s=0}^{m}\sum_{r=0}^{n} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{\left(te^{i(-\theta+\eta)}\right)_s \left(te^{i(\theta-\eta)}\right)_r \left(te^{-i(\theta+\eta)}\right)_{s+r}}{\left(t^2\right)_{s+r}} e^{i\theta(2s-m)} e^{i\eta(2r-n)}. \tag{5.3}$$

Now recall that in [10], Lemma 3 i) we have proved certain formula for the polynomials H_n related to polynomials h_n by

$$H_n\left(x|q\right) = h_n\left(\frac{x\sqrt{1-q}}{2}|q\right) / (1-q)^{n/2}.$$

After necessary adjustment to polynomials h_n (5.3) reads:

$$\sum_{k\geq 0} h_{m+k}\left(x|q\right) h_{n+k}\left(y|q\right) \frac{t^k}{(q)_k} = \sum_{k\geq 0} h_k\left(x|q\right) h_k\left(y|q\right) \frac{t^k}{(q)_k} \times \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q (-1)^s q^{\binom{s}{2}} h_{k-s}\left(y|q\right) \frac{p_{m+s}\left(x|y,t,q\right)}{\left(t^2\right)_{m+k}}. \tag{5.4}$$

Comparing (5.3) and (5.4) we get our assertion. □

Proof of Corollary 1. We start with (2.34), then we use (2.29)

$$\alpha_n(x|a,b,c,d,q)=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q(-a)^{n-i}q^{\binom{n-i}{2}}\frac{\left(bcq^i,bdq^i,cdq^i\right)_{n-i}}{\left(abcdq^{n+i-1}\right)_{n-i}}\psi_i\left(x|b,c,d,q\right)$$

$$=\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q(-a)^{n-i}q^{\binom{n-i}{2}}\frac{\left(bcq^i,bdq^i,cdq^i\right)_{n-i}}{\left(abcdq^{n+i-1}\right)_{n-i}}\sum_{k=0}^{i}\begin{bmatrix}i\\k\end{bmatrix}_q(-b)^{i-k}q^{\binom{i-k}{2}}\left(cdq^k\right)_{i-k}Q_k\left(x|c,d,q\right)$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}\left(cdq^k\right)_{n-k}Q_k\left(x|c,d,q\right)\sum_{i=k}^{n}\begin{bmatrix}n-k\\i-k\end{bmatrix}_q q^{\binom{n-i}{2}+\binom{i-k}{2}}a^{n-i}b^{i-k}\frac{\left(bcq^i,bdq^i\right)_{n-i}}{\left(abcdq^{n+i-1}\right)_{n-i}}$$

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}\left(cdq^k\right)_{n-k}Q_k\left(x|c,d,q\right)\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\m\end{bmatrix}_q q^{\binom{n-k-m}{2}+\binom{m}{2}}a^{n-k-m}b^{m}\frac{\left(bcq^{k+m},bdq^{k+m}\right)_{n-k-m}}{\left(abcdq^{n+k+m-1}\right)_{n-k-m}}.$$

Now we notice that $(n-m)(n-m-1)/2+m(m-1)/2-n(n-1)/2=m\,(m-n)$

Hence we see that $q^{\binom{n-k-m}{2}+\binom{m}{2}}=q^{\binom{n-k}{2}}q^{m(m-n+k)}$ and consequently we get (2.38). □

Proof of Theorem 1. First we will prove (3.11). We start with (2.41), apply (3.8) getting

$$p_n\left(x|y,\rho_1,q\right)=$$

$$\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\left(\rho_1^2q^j\right)_{n-j}\rho_2^{n-j}w_n\left(x|y,\rho_1,z,\rho_2,q\right)\times$$

$$\sum_{m=0}^{n-j}\begin{bmatrix}n-j\\m\end{bmatrix}_q e^{-i(n-j-m)\eta}e^{im\eta}\frac{\left(\rho_1\rho_2e^{i(\eta-\theta)}q^j,\rho_1\rho_2e^{-i(\eta+\theta)}q^j\right)_m}{\left(\rho_1^2\rho_2^2q^{2j}\right)_m}.$$

Now we apply (3.9) with $m->0$ and $j->0$ and $n->n-j$, $t->\rho_1\rho_2q^j$ and get $\sum_{m=0}^{n-j}\left[\begin{smallmatrix}n-j\\m\end{smallmatrix}\right]_q e^{-i(n-j-m)\eta}e^{im\eta}\frac{\left(e^{i(\theta-\eta)}q^j,e^{-i(\theta+\eta)}q^j\right)_m}{\left(\rho_1^2\rho_2^2q^{2j}\right)_m}=p_{n-j}\left(z|y,\rho_1\rho_2q^j,q\right)/\left(\rho_1^2\rho_2^2q^{2j}\right)_{n-j}$.

Now let us concentrate on proving (3.10). Using (2.38) and (3.8) we get:

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}\left(cdq^k\right)_{n-k}Q_k\left(x|c,d,q\right) \tag{5.5}$$

$$\times\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\m\end{bmatrix}_q q^{m(m-n+k)}a^mb^{n-k-m}\frac{\left(bcq^{n-m},bdq^{n-m}\right)_m}{\left(abcdq^{2n-m-1}\right)_m} \tag{5.6}$$

$$w_n\left(x|\rho_1,y,\rho_2,z,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^{n-k}q^{\binom{n-k}{2}}\left(\rho_1^2q^k\right)_{n-k}p_k\left(x|y,\rho_1,q\right)$$

$$\times\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\m\end{bmatrix}_q q^{m(m-n+k)}\rho_2^me^{im\eta}\rho_2^{n-k-m}e^{-i(n-k-m)\eta}\frac{\left(q^{n-m}\rho_1\rho_2e^{i(-\eta+\theta)},q^{n-m}\rho_1\rho_2e^{i(-\eta-\theta)}\right)_m}{\left(\rho_1^2\rho_2^2q^{2n-m-1}\right)_m}$$

Now we apply formula (3.9) with $m \longrightarrow 0$, $n \longrightarrow n-k$, $t=\rho_1\rho_2 q^{n-1}$. We have applying (2.3)

$$
\begin{aligned}
w_n\left(x|\rho_1,y,\rho_2,z,q\right) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(\rho_1^2 q^k\right)_{n-k} \rho_2^{n-k} p_k\left(x|y,\rho_1,q\right) \\
&\times \sum_{m=0}^{n-k}\begin{bmatrix} n-k \\ m \end{bmatrix}_{q^{-1}} e^{-i(n-k-2m)\eta} \frac{\left(q^{-m+1}te^{i(-\eta+\theta)}, q^{-m+1}te^{i(-\eta-\theta)}\right)_m}{\left(t^2q^{-m+1}\right)_m} \\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(\rho_1^2 q^k\right)_{n-k} \rho_2^{n-k} p_k\left(x|y,\rho_1,q\right) \\
&\times \sum_{m=0}^{n-k}\begin{bmatrix} n-k \\ m \end{bmatrix}_{q^{-1}} e^{-i(n-k-2m)\eta} \frac{\left(e^{i(-\eta+\theta)}, te^{i(-\eta-\theta)}; q^{-1}\right)_m}{\left(t^2;q^{-1}\right)_m} \\
= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} &\left(\rho_1^2 q^k\right)_{n-k} \rho_2^{n-k} p_k\left(x|y,\rho_1,q\right) \frac{p_{n-k}\left(y|x,t,q^{-1}\right)}{\left(t^2;q^{-1}\right)_{n-k}}.
\end{aligned}
$$

Now it remains to apply Proposition 3, i).and once more (2.3). □

Proof of the Corollary 4. i) We combine (3.10) and (3.11) getting:

$$
\begin{aligned}
w_n\left(x|\rho_1,y,\rho_2,z,q\right) &= \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q p_j\left(x|y,\rho_1,q\right) \frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{n+j-1}\right)_{n-j}} g_{n-j}\left(z|y,\rho_1\rho_2q^{n-1},q\right) \\
&= \sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{n+j-1}\right)_{n-j}} g_{n-j}\left(z|y,\rho_1\rho_2q^{n-1},q\right) \times \\
&\quad \sum_{k=0}^{j}\begin{bmatrix} j \\ k \end{bmatrix}_q w_k\left(x|y,\rho_1,z,\rho_2,q\right) \frac{\rho_2^{j-k}\left(\rho_1^2q^k\right)_{j-k}}{\left(\rho_1^2\rho_2^2q^{2k}\right)_{j-k}} p_{j-k}\left(z|y,\rho_1\rho_2q^k,q\right)
\end{aligned}
$$

Changing the order of summation leads:

$$
\begin{aligned}
w_n\left(x|\rho_1,y,\rho_2,z,q\right) &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q w_k\left(x|y,\rho_1,z,\rho_2,q\right) \rho_2^{n-k}\left(\rho_1^2q^k\right)_{n-k} \times \\
&\sum_{j=k}^{n}\begin{bmatrix} n-k \\ j-k \end{bmatrix}_q \frac{g_{n-j}\left(z|y,\rho_1\rho_2q^{n-1},q\right)}{\left(\rho_1^2\rho_2^2q^{n+j-1}\right)_{n-j}} \frac{p_{j-k}\left(z|y,\rho_1\rho_2q^k,q\right)}{\left(\rho_1^2\rho_2^2q^{2k}\right)_{j-k}} \\
= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q w_k\left(x|y,\rho_1,z,\rho_2,q\right) \rho_2^{n-k}\left(\rho_1^2q^k\right)_{n-k} &\sum_{m=0}^{n-k}\begin{bmatrix} n-k \\ m \end{bmatrix}_q \frac{g_{n-k-m}\left(z|y,\rho_1\rho_2q^{n-1},q\right)}{\left(\rho_1^2\rho_2^2q^{n+k+m-1}\right)_{n-k-m}} \frac{p_m\left(z|y,\rho_1\rho_2q^k,q\right)}{\left(\rho_1^2\rho_2^2q^{2k}\right)_m}.
\end{aligned}
$$

Since expansions in AW polynomials is unique we deduce that $\sum_{m=0}^{n-k}\begin{bmatrix} n-k \\ m \end{bmatrix}_q \frac{g_{n-k-m}\left(z|y,\rho_1\rho_2q^{n-1},q\right)}{\left(\rho_1^2\rho_2^2q^{n+k+m-1}\right)_{n-k-m}} \frac{p_m\left(z|y,\rho_1\rho_2q^k,q\right)}{\left(\rho_1^2\rho_2^2q^{2k}\right)_m} = 0$ for $k=0,\ldots,n-1$. Now we replace $\rho_1\rho_2$ by t. Since we deal with rational functions we can extend range of all variables to all reals.

ii) We combine (3.11) and (3.10) getting

$$p_n\left(x|y,\rho_1,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q w_j\left(x|y,\rho_1,z,\rho_2,q\right)\frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{2j}\right)_{n-j}}p_{n-j}\left(z|y,\rho_1\rho_2q^j,q\right)$$

$$=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\frac{\rho_2^{n-j}\left(\rho_1^2q^j\right)_{n-j}}{\left(\rho_1^2\rho_2^2q^{2j}\right)_{n-j}}p_{n-j}\left(z|y,\rho_1\rho_2q^j,q\right)\times$$

$$\sum_{k=0}^{j}\begin{bmatrix}j\\k\end{bmatrix}_q p_k\left(x|y,\rho_1,q\right)\frac{\rho_2^{j-k}\left(\rho_1^2q^k\right)_{j-k}}{\left(\rho_1^2\rho_2^2q^{j+k-1}\right)_{j-k}}g_{j-k}\left(z|y,\rho_1\rho_2q^{j-1},q\right).$$

We change the order of summation getting:

$$\begin{aligned}p_n\left(x|y,\rho_1,q\right) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q p_k\left(x|y,\rho_1,q\right)\rho_2^{n-k}\left(\rho_1^2q^k\right)_{n-k}\times\\ &\sum_{j=k}^{n}\begin{bmatrix}n-k\\j-k\end{bmatrix}_q\frac{p_{n-j}\left(z|y,\rho_1\rho_2q^j,q\right)}{\left(\rho_1^2\rho_2^2q^{2j}\right)_{n-j}}\frac{g_{j-k}\left(z|y,\rho_1\rho_2q^{j-1},q\right)}{\left(\rho_1^2\rho_2^2q^{j+k-1}\right)_{j-k}}\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q p_k\left(x|y,\rho_1,q\right)\rho_2^{n-k}\left(\rho_1^2q^k\right)_{n-k}\times\\ &\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\m\end{bmatrix}_q\frac{p_{n-k-m}\left(z|y,\rho_1\rho_2q^{m+k},q\right)}{\left(\rho_1^2\rho_2^2q^{2k+2m}\right)_{n--k-m}}\frac{g_m\left(z|y,\rho_1\rho_2q^{m+k-1},q\right)}{\left(\rho_1^2\rho_2^2q^{2k+m-1}\right)_m}\end{aligned}$$

Since expansion in ASC polynomials is unique we deduce that $\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\m\end{bmatrix}_q\frac{p_{n-k-m}\left(z|y,\rho_1\rho_2q^{m+k},q\right)}{\left(\rho_1^2\rho_2^2q^{2k+2m}\right)_{n--k-m}}\frac{g_m\left(z|y,\rho_1\rho_2q^{m+k-1},q\right)}{\left(\rho_1^2\rho_2^2q^{2k+m-1}\right)_m}=0$ for $k=0,\ldots,n-1$. Now it remains to replace $\rho_1\rho_2$ by t. □

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