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On δ-derivations of n-ary algebras

Ivan Kaygorodov

Abstract. We give a description of δ-derivations of $(n+1)$-dimensional n-ary Filippov algebras and, as a consequence, of simple finite-dimensional Filippov algebras over an algebraically closed field of characteristic zero. We also give new examples of non-trivial δ-derivations of Filippov algebras and show that there are no non-trivial δ-derivations of the simple ternary Malcev algebra M_8.

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1 Introduction

The notion of anti-derivation of an algebra, which is a special case of a δ-derivation, that is, a (-1)-derivation, was considered in [6, 11]. Subsequently the definition of a δ-derivation of an algebra appeared in [7]. Recall that for a fixed δ in the ground field F, a δ-derivation of an algebra A is a linear map φ satisfying the condition

$$\varphi(xy) = \delta\big(\varphi(x)y + x\varphi(y)\big) \tag{1}$$

for arbitrary elements $x, y \in A$.

A description of $\frac{1}{2}$-derivations of an arbitrary prime Lie algebra A $\left(\frac{1}{6} \in F\right)$ with a non-degenerate symmetric invariant bilinear form was given in [7]: it was proved that a linear map $\varphi\colon A \to A$ is a $\frac{1}{2}$-derivation if and only if $\varphi \in \Gamma(A)$, where $\Gamma(A)$ is the centroid of A. Hence, if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ with a non-degenerate symmetric invariant bilinear form, then any $\frac{1}{2}$-derivation φ has the form $\varphi(x) = \alpha x$ for some $\alpha \in F$. Filippov [8] proved that a prime Lie algebra cannot have a non-zero δ-derivation if $\delta \neq -1, 0, \frac{1}{2}, 1$. It was shown in [8] that a prime Lie algebra A $\left(\frac{1}{6} \in \Phi\right)$ with a non-zero anti-derivation is a 3-dimensional central simple algebra over the field of fractions of the centre $Z_R(A)$ of its algebra of right multiplications

$R(A)$. In the same paper, an example was constructed of a non-trivial $\frac{1}{2}$-derivation of the Witt algebra W_1, that is, a $\frac{1}{2}$-derivation that is not an element of the centroid of W_1. In [9], δ-derivations were described of prime alternative and non-Lie Malcev algebras with certain restrictions on the ring of operators Φ. It turned out that algebras in these classes have no non-trivial δ-derivations.

A description of δ-derivations of simple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic zero was given in [12], and δ-derivations of classical Lie superalgebras were described in [13]. The paper [14] was devoted to a description of δ-derivations of semisimple finite-dimensional Jordan superalgebras over an arbitrary field of characteristic different from 2 and of δ-(super)derivations of simple finite-dimensional Lie and Jordan superalgebras over an algebraically closed field of characteristic zero. It was shown that the algebras and superalgebras in [12–14] have no non-trivial δ-(super)derivations. Subsequently the results in [13] were generalized by Zusmanovich [27], who gave a description of δ-(super)derivations of prime Lie superalgebras, namely, he proved that a prime Lie superalgebra has no non-trivial δ-(super)derivations for $\delta \neq -1, 0, \frac{1}{2}, 1$. He also showed that if a Lie superalgebra A is perfect ($[A, A] = A$) and has zero centre and a non-degenerate supersymmetric invariant bilinear form, then the space of $\frac{1}{2}$-(super)derivations coincides with the (super)centroid of A. Furthermore, in the case of positive characteristic he gave an affirmative answer to Filippov's question in [8] on the existence of zero-divisors in the ring of $\frac{1}{2}$-derivations of a prime Lie algebra. The present author and Zhelyabin [25] considered δ-(super)derivations of simple unital superalgebras of Jordan brackets. They showed that there are no non-trivial δ-(super)derivations of simple superalgebras of Jordan brackets that are not superalgebras of vector type and gave a description of the δ-(super)derivations of simple finite-dimensional unital Jordan superalgebras over an algebraically closed field of characteristic different from 2. As a consequence, a connection was discovered between the existence of non-trivial δ-derivations of simple unital superalgebras of Jordan brackets and the speciality of the superalgebra. The cycle of papers describing the δ-(super)derivations of simple finite-dimensional Jordan superalgebras includes the paper [16], which contains a complete description of the δ-(super)derivations of semisimple finite-dimensional Jordan superalgebras over an algebraically closed field of characteristic different from 2. In particular, examples of non-trivial $\frac{1}{2}$-(super)derivations were constructed for simple non-unital finite-dimensional Jordan superalgebras.

Here we consider δ-derivations of n-ary Filippov algebras, as well as of the ternary Malcev algebra M_8. We point out that one of the sources of origin of Filippov algebras was the Nambu mechanics proposed in [20]. Broad connections are currently known between problems in the theory of n-ary algebras and mathematical physics, in particular, between Nambu mechanics and Chern–Simons theory (for more details, see [1–4, 10, 19, 21]).

2 New examples of non-trivial δ-derivations of Filippov algebras

A Filippov algebra is an algebra L with one anticommutative n-ary operation $[x_1, \dots, x_n]$ satisfying the identity

$$\big[[x_1, \dots, x_n], y_2, \dots, y_n\big] = \sum_{i=1}^{n} \big[x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n\big].$$

A derivation of an n-ary algebra L is a linear map D satisfying the condition

$$D[x_1,\dots,x_n]=\sum_{i=1}^{n}[x_1,\dots,D(x_i),\dots,x_n]. \tag{2}$$

By analogy with δ-derivations of binary algebras, as studied in the papers [7–9,12–14,16, 25,27], for a fixed element δ of the ground field we can define a δ-derivation of an n-ary algebra as a linear map φ satisfying the condition

$$\varphi[x_1,\dots,x_n]=\delta\sum_{i=1}^{n}[x_1,\dots,\varphi(x_i),\dots,x_n]. \tag{3}$$

Let

$$\Gamma(L)=\{\psi\in \mathrm{End}(L)\mid \psi([x_1,\dots,x_n])=[x_1,\dots,\psi(x_i),\dots,x_n]\}$$

be the centroid of an algebra L. Clearly, in the case of an n-ary algebra L, every element of $\Gamma(L)$ is a $\frac{1}{n}$-derivation. A non-zero δ-derivation φ is considered to be non-trivial if $\delta\neq 0,1$ and $\varphi\notin\Gamma(L)$. The cases $\delta=0,1$ are trivial and therefore are not considered in what follows.

Filippov [5] gave a description of n-ary algebras, which were subsequently called Filippov algebras, of dimension $n+1$. Let A be an n-ary Filippov algebra, $\dim(A)=n+1$, and let $\{e_1,\dots,e_{n+1}\}$ be a basis of A. We write $\hat{x}_i$ to denote the absence of the component x_i. For example, $[x_1,x_2,\hat{x}_3,x_4]=[x_1,x_2,x_4]$.

In accordance with Filippov's classification, the n-ary Filippov algebras of dimension $n+1$ over an algebraically closed field F of characteristic different from 2 are exhausted by the following $n+4$ families of algebras:

(A_1) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $1\leqslant i\leqslant n+1$;

(B_1) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $1<i\leqslant n+1$, $[e_2,\dots,e_{n+1}]=e_1$;

(B_2) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $1\leqslant i\leqslant n$, $[e_1,\dots,e_n]=e_1$;

(C_1) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $1\leqslant i\leqslant n-1$, $[e_1,\dots,\hat{e}_n,e_{n+1}]=e_n$, $[e_1,\dots,e_n]=\alpha e_{n+1}$, $\alpha\neq 0$;

(C_2) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $1\leqslant i\leqslant n-1$, $[e_1,\dots,\hat{e}_n,e_{n+1}]=e_n+\beta e_{n+1}$, $[e_1,\dots,e_n]=e_{n+1}$, $\beta\neq 0$;

(D_r) $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=e_i$ for $3\leqslant r\leqslant n+1$ and $1\leqslant i\leqslant r$, $[e_1,\dots,\hat{e}_i,\dots,e_{n+1}]=0$ for $r+1\leqslant i\leqslant n+1$.

In what follows we shall consider algebras over fields of characteristic zero, but it is easy to see that some of our results can be extended to fields of positive characteristic.

Let φ be a δ-derivation of an algebra A. Then we denote by $[\varphi]$ the matrix of the linear map φ in the basis $\{e_1,\dots,e_{n+1}\}$. Thus, $[\varphi(x)]=[x][\varphi]$, where $[x]$ is the row vector composed of the coordinates of the vector x in the basis $\{e_1,\dots,e_{n+1}\}$.

It is easy to see that for an algebra of type (A_1) any endomorphism is a δ-derivation for any $\delta\in F$, and any endomorphism lies in $\Gamma(A)$.

Lemma 1. *Let A be an algebra of type (B_1) and φ a δ-derivation of A. Then*

$$\varphi(e_1) = \delta \sum_{k=2}^{n+1} \beta_{kk} e_1, \qquad \varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j, \quad i \geqslant 2.$$

Conversely, every map of this form is a δ-derivation. In particular, the map $\varphi(e_2) = e_3$, $\varphi(e_i) = 0$ for $i \neq 2$, is a non-trivial δ-derivation of A.

Proof. Let $\varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j$ with $\beta_{ij} \in F$. We note that

$$\varphi(e_1) = \varphi[e_2, \ldots, e_{n+1}] = \delta \sum_{k=2}^{n+1} [e_2, \ldots, \varphi(e_k), \ldots, e_{n+1}] = \delta \sum_{k=2}^{n+1} \beta_{kk} e_1.$$

It is easy to see that the map given by the rule

$$\varphi(e_1) = \delta \sum_{k=2}^{n+1} \beta_{kk} e_1, \qquad \varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j, \quad i \geqslant 2,$$

is a δ-derivation. The map $\varphi(e_2) = e_3$, $\varphi(e_i) = 0$ for $i \neq 2$, is of this form, and $\varphi \notin \Gamma(A)$, since $[e_2, e_2, e_4, \ldots, e_{n+1}] = 0$, whereas $[e_2, \varphi(e_2), e_4, \ldots, e_{n+1}] = [e_2, \ldots, e_{n+1}] = e_1$. □

Lemma 2. *Let A be an algebra of type (B_2) and φ a δ-derivation of A. Then*

$$\varphi(e_1) = \frac{\delta}{1-\delta} \sum_{k=2}^{n} \beta_{kk} e_1, \qquad \varphi(e_{n+1}) = \beta_{n+1,n+1} e_{n+1}, \qquad \varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j, \quad 2 \leqslant i \leqslant n.$$

Conversely, every map of this form is a δ-derivation. In particular, the map $\varphi(e_2) = e_1$, $\varphi(e_i) = 0$ for $i \neq 2$, is a non-trivial δ-derivation of A.

Proof. Let $\varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j$ with $\beta_{ij} \in F$. We observe that

$$\varphi(e_1) = \varphi[e_1, \ldots, e_n] = \delta \sum_{k=1}^{n} [e_1, \ldots, \varphi(e_k), \ldots, e_n] = \delta \sum_{k=1}^{n} \beta_{kk} e_1,$$

whence $\beta_{11} = \frac{\delta}{1-\delta} \sum_{k=2}^{n} \beta_{kk}$. Note that

$$0 = \varphi[e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}] = \delta[e_1, \ldots, \hat{e}_i, \ldots, e_n, \beta_{n+1,i} e_i],$$

which implies $\beta_{n+1,i} = 0$ for $i \neq n+1$. It is easy to see that the map given by the rule

$$\varphi(e_1) = \frac{\delta}{1-\delta} \sum_{k=2}^{n} \beta_{kk} e_1, \qquad \varphi(e_{n+1}) = \beta_{n+1,n+1} e_{n+1}, \qquad \varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j, \quad 2 \leqslant i \leqslant n,$$

is a δ-derivation. The map $\varphi(e_2) = e_1$, $\varphi(e_i) = 0$ for $i \neq 2$, is of this form, and $\varphi \notin \Gamma(A)$, since $[e_2, e_2, e_3, \ldots, e_n] = 0$, whereas $[\varphi(e_2), e_2, e_3, \ldots, e_n] = [e_1, \ldots, e_n] = e_1$. □

Lemma 3. *Let A be an algebra of type (C_1) and φ a δ-derivation of A. Then $[\varphi] = \begin{pmatrix} A & C \\ G & B \end{pmatrix}$, where $A \in M_{n-1}$, $B \in M_2$, $C \in M_{n-1,2}$, $G = 0 \in M_{2,n-1}$, and if $\delta \neq -1$, then for $w = \frac{\delta}{1-\delta}\sum_{k=1}^{n-1}\beta_{kk}$ we have*

$$B = \begin{pmatrix} w & \beta_{n,n+1} \\ \frac{\delta}{\alpha}\beta_{n,n+1} & w \end{pmatrix},$$

where $\beta_{n,n+1} = 0$; if $\delta = -1$, then

$$B = \begin{pmatrix} \beta_{nn} & \beta_{n,n+1} \\ -\frac{1}{\alpha}\beta_{n,n+1} & \beta_{n+1,n+1} \end{pmatrix}, \qquad \beta_{nn} + \beta_{n+1,n+1} = -\sum_{k=1}^{n-1}\beta_{kk}.$$

Conversely, every map of this form is a δ-derivation. In particular, the map $\varphi(e_1) = e_n$, $\varphi(e_i) = 0$ for $i \neq 1$, is a non-trivial δ-derivation of A.

Proof. Let $\varphi(e_i) = \sum_{j=1}^{n+1}\beta_{ij}e_j$ with $\beta_{ij} \in F$. For $i \leqslant n-1$ we have

$$0 = \varphi[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = \pm\delta\beta_{ni}e_n \pm \delta\alpha\beta_{n+1,i}e_{n+1},$$

whence $G = 0$. We observe that

$$\alpha\varphi(e_{n+1}) = \delta\sum_{k=1}^{n}[e_1, \dots, \varphi(e_k), \dots, e_n] = \alpha\delta\sum_{k=1}^{n}\beta_{kk}e_{n+1} + \delta\beta_{n,n+1}e_n,$$

whence

$$\varphi(e_{n+1}) = \delta\sum_{k=1}^{n}\beta_{kk}e_{n+1} + \frac{\delta}{\alpha}\beta_{n,n+1}e_n.$$

It is easy to see that

$$\varphi(e_n) = \varphi[e_1, \dots, e_{n-1}, e_{n+1}] = \delta\sum_{k=1}^{n-1}\beta_{kk}e_n + \delta^2\sum_{k=1}^{n}\beta_{kk}e_n + \delta^2\beta_{n,n+1}e_{n+1},$$

whence we obtain that $(1-\delta^2)\beta_{nn} = \delta(1+\delta)\sum_{k=1}^{n-1}\beta_{kk}$. Hence, if $\delta \neq -1$, then

$$\beta_{nn} = \frac{\delta}{1-\delta}\sum_{k=1}^{n-1}\beta_{kk}, \qquad \beta_{n+1,n+1} = \delta\sum_{k=1}^{n}\beta_{kk} = \delta\Big(\sum_{k=1}^{n-1}\beta_{kk} + \frac{\delta}{1-\delta}\sum_{k=1}^{n-1}\beta_{kk}\Big) = \frac{\delta}{1-\delta}\sum_{k=1}^{n-1}\beta_{kk},$$

and if $\delta = -1$, then $\beta_{n+1,n+1} = -\sum_{k=1}^{n}\beta_{kk}$, that is, $\beta_{nn} + \beta_{n+1,n+1} = -\sum_{k=1}^{n-1}\beta_{kk}$. We also see that $\beta_{n,n+1} = \delta^2\beta_{n,n+1}$, that is, $\beta_{n,n+1} \neq 0$ only for $\delta = -1$.

We set $w = \frac{\delta}{1-\delta}\sum_{k=1}^{n-1}\beta_{kk}$. It is easy to see that the map φ defined by the formulae

$$\varphi(e_i) = \sum_{j=1}^{n+1}\beta_{ij}e_j,\ i < n, \qquad \varphi(e_n) = we_n + \beta_{n,n+1}e_{n+1}, \qquad \varphi(e_{n+1}) = \frac{\delta}{\alpha}\beta_{n,n+1}e_n + we_{n+1},$$

where $\beta_{n,n+1} = 0$ and $\delta \neq -1$, is a δ-derivation, and for $\delta = -1$ the map defined by the formulae

$$\varphi(e_i) = \textstyle\sum_{j=1}^{n+1}\beta_{ij}e_j,\ i < n,\ \varphi(e_n) = \beta_{nn}e_n + \beta_{n,n+1}e_{n+1},$$
$$\varphi(e_{n+1}) = -\tfrac{1}{\alpha}\beta_{n,n+1}e_n + \beta_{n+1,n+1}e_{n+1},$$

where $\beta_{nn} + \beta_{n+1,n+1} = -\sum_{k=1}^{n-1} \beta_{kk}$, is a δ-derivation. The map $\varphi(e_1) = e_n$, $\varphi(e_i) = 0$ for $i \neq 1$, is of this form, and $\varphi \notin \Gamma(A)$, since $[e_1, \dots, e_{n-1}, e_1] = 0$, whereas $[e_1, \dots, e_{n-1}, \varphi(e_1)] = [e_1, \dots, e_n] = \alpha e_{n+1} \neq 0$. □

Lemma 4. *Let A be an algebra of type (C_2) and φ a δ-derivation of A. Then $[\varphi] = \begin{pmatrix} A & C \\ G & B \end{pmatrix}$, where $A \in M_{n-1}$, $B \in M_2$, $C \in M_{n-1,2}$, $G = 0 \in M_{2,n-1}$, and*

$$B = \begin{pmatrix} \frac{\delta \operatorname{tr}(A)}{1-\delta} - \frac{\beta\delta}{1+\delta}\gamma & \gamma \\ \delta\gamma & \frac{\delta \operatorname{tr}(A)}{1-\delta} + \frac{\beta\delta}{1+\delta}\gamma \end{pmatrix},$$

where $\gamma = \beta_{n,n+1}$ and $\gamma = 0$ if $\delta \neq \frac{-\beta^2 - 2 \pm \sqrt{\beta^4 + 4\beta^2}}{2}$ or $\delta = -1$; if $\delta = -1$, then $\gamma = 0$ and $\beta_{nn} = \beta_{n+1,n+1} = -\frac{\operatorname{tr}(A)}{2}$. Conversely, every map of this form is a δ-derivation. In particular, the map $\varphi(e_1) = e_n$, $\varphi(e_i) = 0$ for $i \neq 1$, is a non-trivial δ-derivation of A.

Proof. Let $\varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij} e_j$ with $\beta_{ij} \in F$. For $i \leqslant n-1$ we have

$$0 = \varphi[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = \pm\delta\beta_{ni}(e_n + \beta e_{n+1}) \pm \delta\beta_{n+1,i} e_{n+1},$$

whence $G = 0$. We observe that

$$\varphi(e_{n+1}) = \varphi[e_1, \dots, e_n] = \delta\Big(\sum_{i=1}^{n} \beta_{ii} e_{n+1} + \beta_{n,n+1}(e_n + \beta e_{n+1})\Big) = \beta_{n+1,n} e_n + \beta_{n+1,n+1} e_{n+1}.$$

Therefore,

$$\begin{aligned} \varphi(e_n) + \beta\beta_{n+1,n} e_n + \beta\beta_{n+1,n+1} e_{n+1} &= \varphi(e_n + \beta e_{n+1}) = \varphi[e_1, \dots, \hat{e}_n, e_{n+1}] \\ &= \delta\Big(\sum_{i=1}^{n-1} \beta_{ii}(e_n + \beta e_{n+1}) + \beta_{n+1,n} e_{n+1} + \beta_{n+1,n+1}(e_n + \beta e_{n+1})\Big). \end{aligned}$$

Writing $\theta = \sum_{i=1}^{n-1} \beta_{ii}$, we find that

$$\beta_{nn} = \delta\theta + \delta\beta_{n+1,n+1} - \beta\beta_{n+1,n}, \tag{4}$$
$$\beta_{n,n+1} = \delta\theta\beta + \delta\beta_{n+1,n} + \delta\beta\beta_{n+1,n+1} - \beta\beta_{n+1,n+1}, \tag{5}$$
$$\beta_{n+1,n} = \delta\beta_{n,n+1}, \tag{6}$$
$$\beta_{n+1,n+1} = \delta\theta + \delta\beta_{nn} + \delta\beta\beta_{n,n+1}. \tag{7}$$

By adding and subtracting (4) and (7) we obtain that

$$(1-\delta)(\beta_{nn} + \beta_{n+1,n+1}) = 2\delta\theta, \tag{8}$$
$$(1+\delta)(\beta_{nn} - \beta_{n+1,n+1}) = -2\delta\beta\beta_{n,n+1}. \tag{9}$$

It follows easily from (8) and (9) that for $\delta \neq -1$ we have

$$\beta_{nn} = \frac{\delta\theta}{1-\delta} - \frac{\beta\delta}{1+\delta}\beta_{n,n+1}, \tag{10}$$
$$\beta_{n+1,n+1} = \frac{\delta\theta}{1-\delta} + \frac{\beta\delta}{1+\delta}\beta_{n,n+1}, \tag{11}$$

and for $\delta = -1$ we have $\beta_{n+1,n} = \beta_{n,n+1} = 0$ and, by (8) and (5), $\beta_{nn} = \beta_{n+1,n+1} = -\frac{\theta}{2}$.

Substituting the expressions for $\beta_{n+1,n+1}$ and $\beta_{n+1,n}$ from equations (11) and (6) into (5), we obtain that

$$(1-\delta^2)\beta_{n,n+1} = \frac{\beta^2\delta(\delta-1)}{1+\delta}\beta_{n,n+1},$$

that is, $\beta_{n,n+1} \neq 0$ only for $\delta = \frac{-\beta^2-2\pm\sqrt{\beta^4+4\beta^2}}{2}$. Thus, we have shown that the map φ has the form described in the statement of the lemma. Clearly, the map thus defined is a δ-derivation. The map $\varphi(e_1) = e_n$, $\varphi(e_i) = 0$ for $i \neq 1$, is of this form, and $\varphi \notin \Gamma(A)$, since $[e_1, \dots, e_{n-1}, e_1] = 0$, whereas $[e_1, \dots, e_{n-1}, \varphi(e_1)] = [e_1, \dots, e_n] = e_{n+1}$. □

Lemma 5. *Let A be an algebra of type (D_r) and φ a δ-derivation of A. Then $[\varphi] = \begin{pmatrix} A & C \\ G & B \end{pmatrix}$, where $A \in M_r$, $B \in M_{n+1-r}$, $C \in M_{r,n+1-r}$, $C = 0$, and furthermore,*

1) *if $\delta = -1$, then $A = (a_{ij})$, where $a_{ij} = (-1)^{i-j}a_{ji}$ for $i \neq j$, and $\operatorname{tr}(A) = -\operatorname{tr}(B)$;*
2) *if $\delta = \frac{1}{r-1}$, then $\operatorname{tr}(B) = 0$ and $A = \beta E$ for $\beta \in F$;*
3) *if $\delta \neq \frac{1}{r-1}, -1$, then $A = \frac{\delta}{1+\delta-r\delta}\operatorname{tr}(B)E$;*
4) *conversely, every map of this form is a δ-derivation; in particular, if $r \leqslant n$, then the map $\varphi(e_{n+1}) = e_2$, $\varphi(e_i) = 0$ for $i \leqslant n$, is a non-trivial δ-derivation of A.*

Proof. Let $\varphi(e_i) = \sum_{j=1}^{n+1} \beta_{ij}e_j$ with $\beta_{ij} \in F$. We observe that for $i \leqslant r$ we have

$$\varphi(e_i) = \varphi[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = \delta \sum_{j\neq i} [e_1, \dots, \varphi(e_j), \dots, \hat{e}_i, \dots, e_{n+1}]$$
$$= \delta \sum_{j\neq i} \beta_{jj}e_i + \delta \sum_{j\leqslant r,\, j\neq i} (-1)^{i-j+1}\beta_{ji}e_j.$$

Hence for $j, i \leqslant r$ we have

$$\beta_{ii} = \delta \sum_{k\neq i} \beta_{kk}, \qquad \beta_{ij} = (-1)^{i-j+1}\delta\beta_{ji} = \delta^2\beta_{ij},$$

that is, for $i, j \leqslant r$, $i \neq j$, we have either $\beta_{ij} = (-1)^{i-j}\beta_{ji}$ and $\delta = -1$, or $\beta_{ij} = 0$ and $\delta \neq -1$. We also obtain that for $i \leqslant r$ we have

$$(1+\delta)\beta_{ii} = \delta \sum_{j=1}^{n+1} \beta_{jj},$$

that is, either $\beta_{ii} = \beta$ and $\delta \neq -1$, or $\sum_{j=1}^{n+1} \beta_{jj} = 0$ and $\delta = -1$. Hence we see that assertion 1) holds.

If $\delta \neq -1$ and $i \leqslant r$, then

$$\beta_{ii} = \frac{\delta}{1+\delta}\operatorname{tr}[\varphi] = \frac{r\delta}{1+\delta}\beta_{ii} + \frac{\delta}{1+\delta}\operatorname{tr}(B),$$

that is, either $\beta_{ii} = \frac{\delta}{1+\delta-r\delta}\operatorname{tr}(B)$ and $\delta \neq \frac{1}{r-1}$, or $\operatorname{tr}(B) = 0$ and $\delta = \frac{1}{r-1}$.

When $i > r$, standard operations yield that

$$0 = \varphi[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = \delta \sum_{j \leqslant r} \beta_{ji} e_j,$$

that is, $\beta_{ji} = 0$ if $i > r$ and $j \leqslant r$, and so $C = 0$.

It is easy to see that a map φ defined as in the hypothesis of the lemma is a δ-derivation.

If $r \leqslant n$, then the map $\varphi(e_{n+1}) = e_2$, $\varphi(e_i) = 0$ for $i \leqslant n$, is of this form, and $\varphi \notin \Gamma(A)$, since $[e_{n+1}, e_3, \dots, e_{n+1}] = 0$, whereas $[\varphi(e_{n+1}), e_3, \dots, e_{n+1}] = [e_2, \dots, e_{n+1}] = e_1$. Hence assertion 4) holds. □

Theorem 6. *Every non-simple $(n+1)$-dimensional n-ary Filippov algebra over an algebraically closed field of characteristic zero, except the algebra of type (A_1), has a non-trivial δ-derivation for an arbitrary $\delta \neq 0, 1$. The algebra of type (A_1) has no non-trivial δ-derivations.*

Proof. In accordance with the classification of $(n+1)$-dimensional n-ary Filippov algebras [5], all such algebras that are not simple are exhausted by the following types: (A_1), (B_1), (B_2), (C_1), (C_2), (D_r) for $r \neq n+1$. For the algebra of type (A_1) every endomorphism lies in $\Gamma(A)$. Thus, the proof of the theorem follows from the classification of algebras given above along with Lemmas 1–5. □

Theorem 7. *Every simple finite-dimensional Filippov algebra over an algebraically closed field of characteristic zero has non-trivial anti-derivations and does not have non-trivial δ-derivations other than anti-derivations.*

Proof. According to [18] every simple finite-dimensional n-ary Filippov algebra over an algebraically closed field of characteristic zero is isomorphic to a simple n-ary Filippov algebra of type (D_{n+1}), as first described by Filippov [5]. Thus, the assertion of the theorem follows from Lemma 5: for $r = n+1$ we have $[\varphi] = A$; if $\delta \neq -1$, then by assertions 2) and 3) of Lemma 5 the map φ is a scalar multiple of the identity map, and hence $\varphi \in \Gamma(A)$; if $\delta = -1$, then by assertion 1) of Lemma 5 the map $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$, $\varphi(e_i) = 0$ for $i \geqslant 3$, is an anti-derivation, and $\varphi \notin \Gamma(A)$, since

$$[e_1, e_1, e_4, \dots, e_{n+1}] = 0, \text{ whereas } [e_1, \varphi(e_1), e_4, \dots, e_{n+1}] = [e_1, e_2, e_4, \dots, e_{n+1}] = e_3.$$

□

We point out that n-ary Filippov algebras of type (D_{n+1}) generalize (to the case of n-ary operations) the well-known simple Lie algebra sl_2, and coincide with it when $n = 2$. Hopkins studied anti-derivations of simple finite-dimensional Lie algebras in [11], where she constructed examples of non-trivial anti-derivations for the algebra sl_2. Our results agree with those in [11] and [7, 8] on anti-derivations and δ-derivations of the simple Lie algebra sl_2. Note that the derivations of simple finite-dimensional Filippov algebras over an algebraically closed field of characteristic zero were described in [5].

3 δ-Derivations of the ternary Malcev algebra M_8

The class of n-ary Malcev algebras was defined in [22] as a certain natural class of n-ary algebras containing the class of n-ary algebras of a cross product of vectors. At present

the only known example of a simple n-ary Malcev algebra that is not a Filippov algebra is the simple ternary Malcev algebra M_8 arising on an 8-dimensional composition algebra. The derivations of M_8 were described in [24], and in [23] its root decomposition was constructed and the structure of a $\mathbb{Z}_3$-grading was introduced.

We define the *n-ary Jacobian* to be the following function defined on an n-ary algebra:

$$J(x_1,\dots,x_n;y_2,\dots,y_n) = \big[[x_1,\dots,x_n],y_2,\dots,y_n\big] - \sum_{i=1}^{n}\big[x_1,\dots,[x_i,y_2,\dots,y_n],\dots,x_n\big].$$

It follows from the definition that if A is an n-ary Filippov algebra, then

$$J(x_1,\dots,x_n;y_2,\dots,y_n) = 0$$

for all $x_1,\dots,x_n,y_2,\dots,y_n \in A$.

An *n-ary Malcev algebra* ($n \geqslant 3$) is defined as an algebra L with one anticommutative n-ary operation $[x_1,\dots,x_n]$ satisfying the identity

$$-J(zR_x,x_2,\dots,x_n;y_2,\dots,y_n) = J(z,x_2,\dots,x_n;y_2,\dots,y_n)R_x,$$

where $R_x = R_{x_2,\dots,x_n}$ is the operator of right multiplication: $zR_x = [z,x_2,\dots,x_n]$.

In what follows we assume that F is a field of characteristic different from $2,3$ and denote by A a composition algebra over F with involution $a \to \overline{a}$ and identity 1 (see, for example, [26]). We assume that the bilinear form $(x,y) = \frac{1}{2}(x\overline{y} + y\overline{x})$ defined on A is non-degenerate and denote by $n(a)$ the norm of an element $a \in A$. We define a ternary multiplication operation $[\,\cdot\,,\cdot\,,\cdot\,]$ on A by the rule

$$[x,y,z] = x\overline{y}z - (y,z)x + (x,z)y - (x,y)z.$$

Then A becomes a ternary Malcev algebra [22], which is denoted by $M(A)$, and by M_8 when $\dim(A) = 8$.

Recall that a *derivation of a ternary algebra* is a linear map D satisfying equation (2) with $n = 3$. The derivations of M_8 were described in [24] and it was shown that every derivation is inner, that is,

$$\mathrm{Der}(M_8) = \langle [R_{x,y},R_{x,z}] + R_{x,[y,x,z]} \mid x,y,z \in M_8\rangle.$$

By a δ-derivation of a ternary algebra we mean a linear map φ satisfying equation (3) with $n = 3$. In the case of a ternary algebra L, every element of the centroid $\Gamma(L)$ is clearly a $\frac{1}{3}$-derivation. We regard a non-zero δ-derivation φ as non-trivial if $\delta \neq 0,1$ and $\varphi \notin \Gamma(L)$.

Let U be a subspace of V and let $x \in V$. Let $x\big|_U$ denote the projection of the vector x onto U.

Theorem 8. *The ternary algebra M_8 has no non-trivial δ-derivations.*

Proof. Let $1,a,b,c$ be orthonormal vectors in A. We choose the following basis in A:

$$\{e_1 = 1,\ e_2 = a,\ e_3 = b,\ e_4 = ab,\ e_5 = c,\ e_6 = ac,\ e_7 = bc,\ e_8 = abc\}.$$

The operator $R_{x,y}$ of right multiplication is said to be *regular* if in the Fitting decomposition $M = M_0 \oplus M_1$ with respect to $R_{x,y}$ the dimension of M_0 is minimal [23]. By

Theorem 1 in [23] we have the following root decomposition of M_8: $M = M_0 \oplus M_\alpha \oplus M_{-\alpha}$, where $\alpha \in F$ is such that $vR_{x,y} = \pm\alpha v$ for any $v \in M_{\pm\alpha}$. It is known from Lemma 3 in [23] that there exists a non-trivial grading on M_8. If we denote $M_{\pm\alpha}$ by $M_{\pm 1}$, then

$$[M_i, M_j, M_k] \subseteq M_{i+j+k \,(\mathrm{mod}\, 3)}.$$

Let $v \in M_\alpha$. Then

$$\alpha\varphi(v) = \varphi(vR_{x,y}) = \delta\big([\varphi(v), x, y] + [v, \varphi(x), y] + [v, x, \varphi(y)]\big). \tag{12}$$

We assume that

$$\varphi(v)\big|_{M_\alpha} = w_\alpha, \qquad \varphi(x)\big|_{M_0} = \alpha_x x + \beta_x y, \qquad \varphi(y)\big|_{M_0} = \alpha_y x + \beta_y y.$$

In view of these relations, equation (12) yields that $w_\alpha = \frac{\alpha_x+\beta_y}{1-\delta}\delta v$. We observe that by Lemma 1 in [23] the operators R_{e_i,e_i+e_j} and R_{e_j,e_i+e_j} for $i \neq j$ are regular. Consequently, we can conclude that

$$\frac{\alpha_{e_i} + \beta_{e_i+e_j}}{1-\delta}\delta v = \frac{\alpha_{e_j} + \beta_{e_i+e_j}}{1-\delta}\delta v,$$

whence $\alpha_{e_i} = \alpha_{e_j}$. Since $\frac{2\alpha_{e_i}}{1-\delta}\delta = \alpha_{e_i}$, we obtain that either $\delta = \frac{1}{3}$ or $\alpha_{e_i} = 0$.

For every $i \in \{2, \dots, 8\}$ it is possible to choose j, k, l, m, s, t depending on i such that

$$e_i = e_j e_k = e_l e_m = e_s e_t, \tag{13}$$
$$e_j = e_s e_m = e_k e_i = e_t e_l, \tag{14}$$
$$e_k = e_i e_j = e_m e_t = e_s e_l, \tag{15}$$
$$e_l = e_m e_i = e_k e_s = e_j e_t, \tag{16}$$
$$e_m = e_i e_l = e_t e_k = e_j e_s, \tag{17}$$
$$e_s = e_l e_k = e_t e_i = e_m e_j, \tag{18}$$
$$e_t = e_i e_s = e_k e_m = e_l e_j. \tag{19}$$

We can assume that $\varphi(e_q) = \sum_{p=1}^{8} a_{qp} e_p$. We already know that $a_{qq} = a_{pp}$ and that $a_{pp} = 0$ when $\delta \neq \frac{1}{3}$. Since

$$\varphi(e_s) = \varphi([e_i, e_j, e_l]) = \delta\big([\varphi(e_i), e_j, e_l] + [e_i, \varphi(e_j), e_l] + [e_i, e_j, \varphi(e_l)]\big),$$

the corresponding multiplication operations yield that

$$\sum_{p=1}^{8} a_{sp} e_p = \delta(-a_{it}e_1 + a_{ik}e_m - a_{im}e_k + a_{i1}e_t - a_{is}e_i - a_{jm}e_1 + a_{j1}e_m$$
$$+ a_{jt}e_k - a_{jk}e_t - a_{js}e_j + a_{lk}e_1 + a_{lt}e_m - a_{l1}e_k - a_{lm}e_t - a_{ls}e_l).$$

(For example, $[e_i, e_l, e_k] = (e_i\overline{e_l})e_k = (e_l e_i)e_k = -e_m e_k = e_k e_m = e_t$.) Consequently, $a_{sp} = -\delta a_{ps}$, where $p \in \{i, j, l\}$. Since the index i is arbitrary, the relations (13)–(19) enable us to conclude that $a_{pq} = -\delta a_{qp}$ for all $p, q \in \{1, \dots, 8\}$. Using this relation we can conclude that $a_{pq} = -\delta a_{qp} = \delta^2 a_{pq}$, whence we have $\delta = -1$ and $a_{pq} = a_{qp}$ or φ is trivial. We can now show that M_8 has no non-trivial anti-derivations. In what follows we follow the scheme of argument used above and take into account the fact that $\delta = -1$.

Suppose that $-\varphi(e_t) = \varphi([e_i, e_k, e_l])$. Then

$$-\sum_{p=1}^{8} a_{tp}e_p = -(a_{i1}e_s - a_{ij}e_m + a_{im}e_j - a_{is}e_1 + a_{it}e_i + a_{k1}e_m + a_{kj}e_s \\ - a_{km}e_1 - a_{ks}e_j + a_{kt}e_k + a_{l1}e_j - a_{lj}e_1 - a_{lm}e_s + a_{ls}e_m + a_{lt}e_l),$$

whence we obtain that

$$a_{ts} = a_{i1} + a_{jk} - a_{ml}. \tag{20}$$

Suppose that $\varphi(e_m) = \varphi([e_i, e_j, e_t])$. Then

$$\sum_{p=1}^{8} a_{mp}e_p = -(a_{i1}e_l - a_{ik}e_s - a_{il}e_1 + a_{im}e_i - a_{is}e_k + a_{j1}e_s - a_{jk}e_l \\ + a_{jl}e_k + a_{jm}e_j - a_{js}e_1 + a_{t1}e_k - a_{tk}e_1 - a_{tl}e_s + a_{tm}e_t + a_{ts}e_l),$$

whence we obtain that

$$a_{ml} = a_{i1} - a_{jk} + a_{ts}. \tag{21}$$

Suppose that $-\varphi(e_t) = \varphi([e_i, e_j, e_m])$. Then

$$-\sum_{p=1}^{8} a_{tp}e_p = -(a_{i1}e_s - a_{ik}e_l + a_{il}e_k - a_{is}e_1 + a_{it}e_i - a_{j1}e_l - a_{jk}e_s \\ + a_{jl}e_1 + a_{js}e_k + a_{jt}e_j - a_{m1}e_k + a_{mk}e_1 + a_{ml}e_s - a_{ms}e_l + a_{mt}e_m),$$

whence we obtain that

$$a_{ts} = a_{i1} - a_{jk} + a_{ml}. \tag{22}$$

Suppose that $-\varphi(e_s) = \varphi([e_i, e_k, e_m])$. Then

$$-\sum_{p=1}^{8} a_{sp}e_p = -(-a_{i1}e_t + a_{ij}e_l + a_{is}e_i + a_{it}e_1 - a_{il}e_j - a_{k1}e_l - a_{kj}e_t \\ + a_{ks}e_k + a_{kt}e_j + a_{kl}e_1 + a_{m1}e_j - a_{mj}e_1 + a_{ms}e_m + a_{mt}e_l - a_{ml}e_t),$$

whence we obtain that

$$a_{st} = -a_{i1} - a_{jk} - a_{ml}. \tag{23}$$

We observe that equations (20) and (21) imply that

$$a_{kj} = a_{st}, \qquad a_{ml} = a_{i1}. \tag{24}$$

The formulae (20) and (22) imply that

$$a_{i1} = a_{st}, \qquad a_{ml} = a_{kj}, \tag{25}$$

and the relations (23) and (24) imply that

$$a_{st} = -a_{ml}. \tag{26}$$

By analysing equations (24)–(26) we obtain that

$$a_{i1} = a_{st} = a_{ml} = a_{kj} = 0.$$

Since the index i is arbitrary, we obtain from the relations (13)–(19) that the map φ is trivial. □

Theorem 8 implies the existence of a simple n-ary Malcev algebra over an algebraically closed field of characteristic zero that is not an n-ary Filippov algebra and has no non-trivial δ-derivations, in contrast to the case of n-ary Filippov algebras over an algebraically closed field of characteristic zero. We point out that these results agree with those of Filippov [9] on the absence of non-trivial δ-derivations on prime binary non-Lie Malcev algebras.

We note that a partial generalization of the results of the present paper was announced in [15], and a detailed survey of the main results in the theory of δ-differential algebras and superalgebras was given in [17].

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Appendix A. Differences from the published version

Lemmas 1–5: statements changed.
Theorem 6: statement changed.
Theorem 7: proof completed.