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Exclusion type spatially heterogeneous processes in continuum

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Abstract

We study deterministic discrete time exclusion type spatially heterogeneous particle processes in continuum. A typical example of this sort is a traffic flow model with obstacles: traffic lights, speed bumps, spatially varying local velocities etc. Ergodic averages of particle velocities are obtained and their connections to other statistical quantities, in particular to particle and obstacles densities (the so called Fundamental Diagram) is analyzed rigorously. The main technical tool is a "dynamical" coupling construction applied in a nonstandard fashion: instead of proving the existence of the successful coupling (which even might not hold) we use its presence/absence as an important diagnostic tool.

1 Introduction

In 1970 Frank Spitzer introduced the (now classical) simple exclusion process as a Markov chain that describes nearest-neighbor random walks of a collection of particles on the one-dimensional infinite[1] integer lattice. Particles interact through the hard core exclusion rule, which means that at most one particle is allowed at each site. This seemingly very particular process appears naturally in a very broad list of scientific fields starting from various models of traffic flows [14, 9, 7, 2, 3], molecular motors and protein synthesis in biology, surface growth or percolation processes in physics (see [15, 5] for a review), and up to the analysis of Young diagrams in Representation Theory [6].

From the point of view of the order of particle interactions there are two principally different types of exclusion processes: with synchronous and asynchronous updating rules. In the latter case at each moment of time a.s. at most one particle may move and hence only a single interaction may take place. This is the main model considered in the mathematical literature (see e.g. [13] for a general account and [1, 8, 10] for recent results), and indeed, the assumption about the asynchronous updating is quite natural in the continuous time setting. The synchronous updating means that *all* particles are trying to move simultaneously and hence an arbitrary large (and even infinite) number

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[1]or finite with periodic boundary conditions

of interactions may occur at the same time. This makes the analysis of the synchronous updating case much more difficult, but this is what happens in the discrete time case.[2] This case is much less studied, but still there are a few results describing ergodic properties of such processes [2, 3, 7, 9, 14].

Recently in [4] we have introduced and studied the synchronous updating version of the simplest exclusion process (TASEP) in continuum.[3] Now our aim is to extend these results to the case when the media is heterogeneous, i.e. contains obstacles. The idea is as follows. Consider a (countable) collection of particles performing (random) walks on a real line with hard core type interactions (we assume that the particles cannot outran each other). Assume also that on the real line there is a (countable) number of obstacles. To overcome an obstacle a particle needs to spend some additional time on it. Thus we have two types of interactions: between particles and with static obstacles. The principal novelty here is that the presence of obstacles leads to a spontaneous creation of "traffic jams" (particles clusters) near obstacles. Indeed, it is easy to construct initial particle configurations having no clusters of particles but such that these clusters will be created in front of obstacles under dynamics. This feature was completely absent in the case without obstacles: a cluster may only disappear but never be created. We start the formalization with the simplest version of the model and later in Sections 6,7 consider what happens in a more complicated setting.

By a *configuration* $x := \{x_i\}_{i\in\mathbb{Z}}$ we mean an ordered (i.e. $x_i \le x_{i+1}\ \forall i \in \mathbb{Z}$) bi-infinite sequence of real numbers $x_i \in \mathbb{R}$. The union X of all such sequences plays the role of the phase space of the system under consideration. Consider also a special configuration $z \in X$ such that $z_i \le z_{i+1}\ \forall i \in \mathbb{Z}$. We refer to elements of x as to positions of particles and to elements of z as to positions of obstacles.

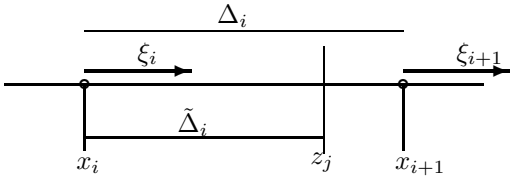


Figure 1: TASEP in heterogeneous continuum.

Let $\Delta_i = \Delta_i(x) := x_{i+1} - x_i$ and let

$$\tilde{\Delta}_i = \tilde{\Delta}_i(x,z) := \min\left(x_{i+1} - x_i,\quad \min_j(z_j :\ z_j > x_i) - x_i\right)$$

stands for the minimum between the distances from the point x_i^t to the point x_{i+1}^t and to the next obstacle (see Fig. 1). We refer to $\Delta_i, \tilde{\Delta}_i$ as *gaps* and *modified gaps* corresponding to the i-th particle in x.

For a given real $v > 0$ (to which we refer as a maximal local velocity) and the configuration z we define a map $T : X \to X$ as follows:

$$(Tx)_i := x_i + \min\left(\tilde{\Delta}_i(x,z), v\right) \qquad \forall i \in \mathbb{Z}. \tag{1.1}$$

[2] if one do not consider some "artificial" updating rules like a sequential or random updating.

[3] Some other continuous space generalizations were considered e.g. in [12, 16].

The dynamical system (T, X) describes the collective motion of particles discussed above. If $\tilde{\Delta}_i(x, z) < v$ we say that the i-th particle is *blocked* (meaning that its motion is blocked by the $i+1$-th particle or an obstacle) at time t and *free* otherwise. By a *cluster* of particles in a configuration $x \in X$ we mean consecutive particles with gaps $\Delta_i(x) < v$ having no obstacles between them.

Associating the number of iterations of the map T with time we often use the notation $T^t x \equiv x^t := \{x_i^t\}$. In this terms the quantity $\xi_i^t := \min\left(\tilde{\Delta}_i(x^t, z), v\right)$ plays the role of the *local velocity* of the i-th particle in the configuration $x \equiv x^0$ at time $t \geq 0$ and thus the dynamics can be rewritten as

$$x_i^{t+1} := x_i^t + \xi_i^t. \tag{1.2}$$

A more general setting including varying/random velocities and waiting times will be considered in Sections 6,7.

It is of interest that even without obstacles (i.e. if $z = \emptyset$) the behavior of the deterministic dynamical system (T, X) is far from being trivial. In [4] it was shown that this system is chaotic in the sense that its topological entropy is positive (and even infinite).

Our main results are concerned with the so called Fundamental Diagram describing the connection between average particle velocities $V(x)$ and particle/obstacle densities $\rho(x), \rho(z)$ (see Section 2 for definitions) and technically the analysis is based on a (somewhat unusual) "dynamical" coupling construction (see Section 3 and also [4]).

For a given $v > 0$ and a configuration of obstacles z denote by $\tilde{z}$ the *extended* configuration of obstacles obtained by inserting between each pair of entries z_i, z_{i+1} new $\lfloor (z_{i+1} - z_i)/v \rfloor$ 'virtual' obstacles at distances v between them starting from the point z_i. Here $\lfloor u \rfloor$ stands for the integer part of the number u.

Theorem 1 *(Fundamental Diagram) Let $\rho(x), \rho(\tilde{z})$ be well defined. Then*

$$V(x) = \min\{1/\rho(\tilde{z}),\ 1/\rho(x)\}. \tag{1.3}$$

Therefore the phase space is divided into two parts: gaseous $\{(x, \tilde{z}) :\ \rho(x) \leq \rho(\tilde{z})\}$ (consisted of configurations of eventually non-interacting particles) and liquid $\{(x, \tilde{z}) :\ \rho(x) > \rho(\tilde{z})\}$ (where clusters of particles are present at any time).

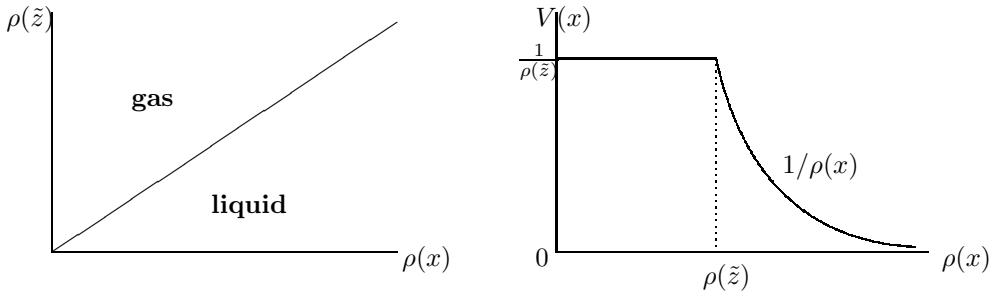


Figure 2: (left) Phase Diagram, (right) Fundamental Diagram.

These results are illustrated in Fig 2. It is worth note that a naive argument tells that the average velocity of a particle in an infinite system is essentially the minimum of the average inter-particle distance and the average inter-obstacle distance. Theorem 1

shows that this is absolutely not the case: instead of the average inter-obstacle distance one needs to take into account the average distance between the elements of the extended configuration $\tilde{z}$. As we shall see the latter is very different from the former, e.g. we always have $\rho(\tilde{z}) \geq 1/v$ independently on $\rho(z)$ (which even might be not well defined). Actually the mere fact that the correspondence between the average velocities and particle/obstacle densities is one-to-one comes as a surprise especially without any regularity assumptions for the positions of obstacles.

The main steps of the proof are as follows. First we show that for given configurations of particles and obstacles upper/lower average particle velocities are the same for all particles. Now to compare average velocities in different particle configurations with the same configuration of obstacles we develop a special dynamical coupling construction. Applying this construction we prove that that upper/lower average particle velocities are the same provided the same particle densities. Thus to calculate the dependence of average velocities on densities it is enough to construct for each density a single configuration having this density for which we are able to calculate its velocity explicitly. To this end we consider an auxiliary zero-range lattice process whose trajectories are invariant under the dynamics of our original system without obstacles. To calculate the average velocity in the zero-range lattice process one uses corresponding results obtained in [4].

Despite that various couplings are widely used in the analysis of Interacting Particle Systems (IPS) (see e.g. [13]), applications of our approach is very different from usual. In particular, we do not prove the existence of the so called successful coupling (which even might not hold) but instead use its presence/absence as an important diagnostic tool. Remark also that typically one uses the coupling argument to prove the uniqueness of the invariant measure and to derive later other results from this fact. In our case there might be a very large number of ergodic invariant measures or no invariant measures at all (e.g. the trivial example of a single particle performing a skewed random walk). This example reminds about another important statistical quantity – average particles velocity. The dynamical coupling will be used directly to find connections between the average particle velocities and other statistical features of the systems under consideration, in particular with the corresponding particle densities.

It is worth note that all approaches used to study discrete time lattice versions of IPS are heavily based on the combinatorial structure of particle configurations. This structure has no counterparts in the continuum setting under consideration. In particular the particle – vacancy symmetry is no longer applicable in our case. This explains the need to develop a fundamentally new techniques for the analysis of IPS in continuum. The presence of obstacles also prevents the direct application of the scheme developed in [4] for the spatially homogeneous case.

2 Basic properties

Here we shall study questions related to particle densities and velocities.

By the *density*

$$\rho(x, I) := \frac{\#\{i \in \mathbb{Z} : \quad x_i \in I\}}{|I|}$$

of a configuration $x \in X$ in a bounded segment $I = [a, b] \in \mathbb{R}$ we mean the number of particles from x whose centers x_i belong to I divided by the Lebesgue measure $|I| > 0$ of

the segment I. If for any sequence of *nested* bounded segments $\{I_n\}$ with $|I_n| \stackrel{n\to\infty}{\longrightarrow} \infty$ the limit

$$\rho(x) := \lim_{n\to\infty} \rho(x, I_n)$$

is well defined we call it the *density* of the configuration $x \in X$. Otherwise one considers upper and lower (with respect to all possible collections of nested intervals I_n) particle densities $\rho_\pm(x)$.

Remark 1 If $\exists \rho(x) < \infty$ then $|x_n - x_m|/|n-m| \stackrel{|n-m|\to\infty}{\longrightarrow} 1/\rho(x)$.

Lemma 2 *The upper/lower densities $\rho_\pm(x^t)$ are preserved by dynamics, i.e. $\rho_\pm(x^t) = \rho_\pm(x^{t+1})$ $\forall t \geq 0$.*

Proof. For a given segment $I \in \mathbb{R}$ the number of particles from the configuration $x^t \in X$ which can leave it during the next time step cannot exceed 1 and the number of particles which can enter this segment also cannot exceed 1. Thus the total change of the number of particles in I cannot exceed 1, because if a particle leaves the segment through one of its ends no other particle can enter through this end. Therefore

$$|\rho(x^t, I) - \rho(x^{t+1}, I)| \cdot |I| \leq 1$$

which implies the claim. □

By the (average) *velocity* of the i-th particle in the configuration $x \in X$ at time $t > 0$ we mean

$$V(x, i, t) := \frac{1}{t} \sum_{s=0}^{t-1} \xi_i^s \equiv (x_i^t - x_i^0)/t.$$

If the limit

$$V(x, i) := \lim_{t\to\infty} V(x, i, t)$$

is well defined we call it the (average) *velocity* of the i-th particle. Otherwise one considers upper and lower particle velocities $V_\pm(x, i)$.

Lemma 3 *Let $x \in X$ and let $\rho(\tilde{z}) < \infty$ be well defined. Then $|V(x, j, t) - V(x, i, t)| \stackrel{t\to\infty}{\longrightarrow} 0$ a.s. $\forall i, j \in \mathbb{Z}$.*

Proof. It is enough to prove this result for $j = i + 1$. Consider the difference between (average) velocities of consecutive particles

$$\begin{aligned} V(x, i+1, t) - V(x, i, t) &= \frac{x_{i+1}^t - x_{i+1}^0}{t} - \frac{x_i^t - x_i^0}{t} \\ &= \frac{x_{i+1}^t - x_i^t}{t} - \frac{x_{i+1}^0 - x_i^0}{t} \\ &= \Delta_i^t/t - \Delta_i^0/t. \end{aligned}$$

The last term vanishes as $t \to \infty$ and it is enough to show that the same happens with Δ_i^t/t. Here and in the sequel we use the notation $\Delta_i^t \equiv \Delta_i(x^t)$.

Normally in the deterministic setting gaps between particles Δ_i^t are uniformly bounded (see e.g. [2, 3, 4]). Surprisingly in the present setting this is not the case and we are only able to show that Δ_i^t may grow not faster than $o(t)$, which fortunately is enough for our aims.

To prove this estimate we introduce a new configuration having only two particles $\acute{x} := \{\acute{x}_1, \acute{x}_2\}$ which have the same initial positions as the i-th and the $(i+1)$-th particles of x, i.e. $\acute{x}_1 = x_i^0, \acute{x}_2 = x_{i+1}^0$. Denoting $\acute{x}^t := T^t\acute{x}$ and $\acute{\Delta}^t := \acute{x}_2^t - \acute{x}_1^t$ we observe that $\acute{\Delta}^t \geq \Delta_i^t \ \ \forall t \geq 0$.

Consider the movement of the leading particle in the process $\tilde{x}$. There is the first moment of time $t_2 := t_2(\tilde{x}_2)$ when this particle encounters an obstacle from z. Denote by $\tilde{z}_k$ the position of this obstacle in the extended configuration $\tilde{z}$. By the definition of the dynamics $T^t\tilde{z}_k = \tilde{z}_{k+t} \ \ \forall t \geq 0$.

The movement of the trailing particle in the process $\tilde{x}$ is a bit more complicated since additionally to obstacles it may be stopped by the leading particle. Anyway there is the first moment of time $t_1 := t_1(\tilde{x}_1, \tilde{x}_2) > t_2$ when this particle encounters the obstacle located at $\tilde{z}_k$. For each $t > t_1$ these two particles move synchronously along the "chain" $\tilde{z}$. Thus the growth of $\acute{\Delta}^t$ is completely determined by the concentration of entries in $\tilde{z}$ (i.e. by $\rho(\tilde{z})$). Due to the assumption about the existence of the density of $\tilde{z}$ these fluctuations cannot exceed $o(t)$ and hence $\Delta_i^t/t \leq \acute{\Delta}^t/t \stackrel{t\to\infty}{\longrightarrow} 0$. □

Corollary 4 *Under the assumptions of Lemma 3 the upper and lower particle velocities $V_\pm(x,i)$ do not depend on i.*

Remark 5 (a) If the density $\rho(\tilde{z})$ is not well defined, the distance Δ_i^t may grow linearly with time and the limit average velocities might differ for different particles. (b) The existence of $\rho(z) < \infty$ does not imply the existence of $\rho(\tilde{z}) < \infty$ but fir "typical" z this implication takes place (see Section 7).

Let us estimate upper/lower densities of the extended configuration $\tilde{z}$ of obstacles for a given configuration z with $\rho(z) > 0$ and a given local velocity $v > 0$.

Lemma 6 $\max\{1/v, \rho(z)\} \leq \rho_-(\tilde{z}) \leq \rho_+(\tilde{z}) \leq \rho(z) + 1/v$.

Proof. Observe that for each $i \in \mathbb{Z}$ the spatial segment $[z_i, z_{i+1}]$ contains exactly $\lfloor (z_{i+1} - z_i)/v \rfloor \in [(z_{i+1} - z_i)/v - 1, (z_{i+1} - z_i)/v]$ elements. Therefore $\forall n \in \mathbb{Z}_+$

$$\begin{aligned}\rho(\tilde{z}, [z_0, z_n]) &= \left(n + \sum_{i=0}^{n-1} \lfloor (z_{i+1} - z_i)/v \rfloor\right) / (z_n - z_0) \\ &= \rho(z, [z_0, z_n]) + \left(\sum_{i=0}^{n-1} \lfloor (z_{i+1} - z_i)/v \rfloor\right) / (z_n - z_0) \\ &\leq \rho(z, [z_0, z_n]) + 1/v.\end{aligned}$$

Similarly

$$\rho(\tilde{z}, [z_0, z_n]) \geq \rho(z, [z_0, z_n]) - n/(z_n - z_0) + 1/v = 1/v.$$

Passing to the limit as $n \to \infty$ and using that $\tilde{z}$ contains all elements of z we get the result. □

3 Dynamical coupling through particle's overtaking

Consider two independent particle processes $x^t, \acute{x}^t$. By $i(x^t)$ we denote the i-th particle of the process x^t (i.e. x_i^t is the location of the particle $i(x^t)$ at time t).

We say that the particle $i(x^t)$ *overtakes* at time $t > 0$ the particle $j(\acute{x}^t)$ if $x_i^{t-1} < \acute{x}_j^{t-1}$ and $x_i^t \geq \acute{x}_j^t$, and denote this event as $i(x^t) \hookrightarrow j(\acute{x}^t)$.

Lemma 7 *If* $i(x^t) \hookrightarrow j(\acute{x}^t)$ *then*
(a) $\nexists n \in \mathbb{Z} :\ n(x^t) \hookrightarrow j(\acute{x}^t)$;
(b) $\nexists m \in \mathbb{Z} :\ j(\acute{x}^t) \hookrightarrow m(x^t)$.
Additionally $i(x^t) \hookrightarrow j(\acute{x}^t)$ *and* $(j-1)(\acute{x}^t) \hookrightarrow i(x^t)$ *might happen only if* $x_i^t = \acute{x}_{j-1}^t$.

Note that there is no contradiction between two parts of this Lemma since the former part concerns a particle being overtaken while the latter concerns a overtaking particle.

Proof. By the construction of the process under consideration and the definition of the overtaking we have:

$$x_{i-1}^t \leq x_i^{t-1} < \acute{x}_j^{t-1} \leq \acute{x}_j^t \leq x_i^t \leq x_{i+1}^{t-1}. \tag{3.1}$$

Now (a) follows from the observation that by (3.1) neither particles preceding $i(x^t)$ nor particles succeeding it may overtake the particle $j(\acute{x}^t)$ at time t.

If (b) would hold then $\acute{x}_j^{t-1} < x_m^{t-1} \leq x_m^t \leq \acute{x}_j^t$. Thus by (3.1) we get $\acute{x}_j^{t-1} \leq x_m^{t-1} \leq \acute{x}_j^t$, which might be possible only if $\acute{x}_j^{t-1} = \acute{x}_j^t$. The latter contradicts to the assumption about the overtaking.

To show that the event discussed in the additional part may take place, consider a pair of configurations satisfying for some $i, j, t \in \mathbb{Z}$ the following inequalities:

$$\acute{x}_{j-1}^{t-1} < x_i^{t-1} < \acute{x}_j^{t-1} = \acute{x}_{j+1}^{t-1} = x_{i+1}^{t-1}, \qquad \max\left(\tilde{\Delta}_i(x^{t-1}, z), \tilde{\Delta}_{j-1}(\acute{x}^{t-1}, z)\right) \leq v.$$

Then $x_i^t = \acute{x}_{j-1}^t = \acute{x}_j^t$ which implies that $i(x^t) \hookrightarrow j(\acute{x}^t)$ and $(j-1)(\acute{x}^t) \hookrightarrow i(x^t)$. Assume now that on the contrary $x_i^t \neq \acute{x}_{j-1}^t$. If $x_i^t > \acute{x}_{j-1}^t$ then $\acute{x}_{j-1}^t \leq \acute{x}_j^t < x_i^t$ which contradicts to $(j-1)(\acute{x}^t) \hookrightarrow i(x^t)$. Similarly $x_i^t < \acute{x}_{j-1}^t$ implies $x_i^t < \acute{x}_{j-1}^t \leq \acute{x}_j^t$ which contradicts to $i(x^t) \hookrightarrow j(\acute{x}^t)$. □

Let us introduce the *dynamical coupling* of the processes $x^t, \acute{x}^t$ which consists in a consequent pairing of particles of opposite processes. The pairing in our deterministic setting is a pure formal action which does not change the dynamics. The idea is that if a particle overtakes some particles from the opposite process it becomes paired with one of them. The construction starts at time $t = 0$ and initially all particles are assumed to be unpaired.

Denote by

$$J_i(x^t) := \{j \in \mathbb{Z} :\ i(x^t) \hookrightarrow j(\acute{x}^t)\}$$

the set of particles overtaken simultaneously by the particle $i(x^t)$ at time t, by

$$i_\bullet(x^t) := \min\left\{\infty,\ \min\{j \in J_i(x^t) :\ j(\acute{x}^{t-1}) \text{ is paired}\}\right\}$$

the paired particle with the minimal index among them, and finally

$$i_\circ(x^t) := \begin{cases} i_\bullet(x^t) - 1 & \text{if } i_\bullet(x^t) - 1 \in J_i(x^t) \\ \infty & \text{otherwise} \end{cases}.$$

In words, $i_\bullet(x^t)$ stands for the paired particle in J_i^t having the minimal index, and $i_\circ(x^t)$ stands for the unpaired particle in $J_i(x^t)$ having the maximal index among those preceding $i_\bullet(x^t)$. A typical connection between those indices and positions of the i-th particle at times $t-1$ and t is shown in Fig 3.

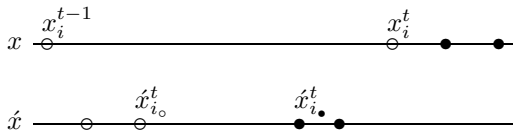


Figure 3: Connections between indices and particle's positions.

To simplify the description of the dynamical coupling we shall use a diagrammatic representation for coupled configurations, where paired particles are denoted by black circles and unpaired ones by open circles, and use the upper line of the diagram for the x-particles (i.e. particles from the x-process) and the lower line for the $\acute{x}$-particles. In this representation a typical pairing event looks as follows

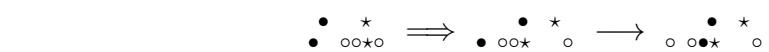

(3.2)

Here $\Longrightarrow$ corresponds to the dynamics and $\longrightarrow$ to the pairing, while $\overset{\bullet}{\bullet}$; $\overset{\star}{\star}$ stand for two different mutual pairs of particles created after the particle overtaking under dynamics.

Now we are ready to define the pairing rigorously. We proceed first with all x-particles, overtaking some $\acute{x}$-particles at time t, and then with all $\acute{x}$-particles, overtaking some x-particles at this time.

Let the overtaking takes place for i-th x-particle at time t, then if

(a) $i(x^{t-1})$ is paired and $i_\circ(x^t) < \infty$ we re-pair these particles:

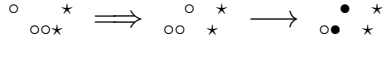


(b) $i(x^{t-1})$ is unpaired and

(b') $i_\bullet(x^t) < \infty$ and $x_i^t > \acute{x}_{i_\bullet(x^t)}$ we re-pair these particles:



(b") else if $i_\circ(x^t) < \infty$ this unpaired particle forms a new pair with $i(x^t)$:



The pairing rules when the overtaking takes place for i-th $\acute{x}$-particle are exactly the same except for the exchange of x by $\acute{x}$ and vice versa.

The complexity of these rules reflects that first a particle may overtake simultaneously several particles from another process, and second an arbitrary number of particles may share the same spatial position. In particular, in the following event we have:



rather than $\longrightarrow$ $\begin{smallmatrix} & \bullet & \circ \\ \circ & \bullet & \star \end{smallmatrix}$. Indeed, the trailing x-particle overtakes both two trailing $\acute{x}$-particles and since the re-pairing rules in (b) demand strict inequalities only the pairing of trailing unpaired particles takes place.

By Lemma 7 each overtaking event may be treated separately. Therefore we apply the pairing rules first for all x-particles, overtaking some $\acute{x}$-particles, and then for all $\acute{x}$-particles, overtaking some x-particles.

According to the definition, particles from the same pair move synchronously until either the movement of one of them is blocked or one of the members of the pair is swapped with an unpaired particle from the same process. It is convenient to think about the coupled process as a "gas" of single (unpaired) particles and "dumbbells" (pairs). A previously paired particle may inherit the role of the unpaired one from one of its neighbors. In order to keep track of positions of unpaired particles we shall refer to them as x- and $\acute{x}$-*defects* depending on the process they belong.

We say that a pair of configurations $(x, \acute{x})$ is *proper* if for each two mutually paired particles the open segment between them cannot exceed v, does not contain neither a defect nor an obstacle (i.e. the situations $\bullet^{\circ}\,\bullet$, $\bullet|^{\bullet}$ cannot happen), and there are no crossing mutually paired pairs $(\begin{smallmatrix} & \bullet & \star \\ \star & & \bullet \end{smallmatrix})$.

Lemma 8 *If the pair of configurations $(x^{t-1}, \acute{x}^{t-1})$ is proper for some $t > 1$, then the pair $(x^t, \acute{x}^t)$ is proper as well.*

Proof. The situation $\bullet^{\circ}\,\bullet$ may happen if either a defect overtakes the trailing particle in a pair, or if a pair is born around this defect. Both such possibilities contradict to the definition of pairing.

Assume that at time t there is a pair of mutually paired particles $i(x^t), j(\acute{x}^t)$ such that the open segment between them contains an obstacle: $x_i^t < z_k < \acute{x}_j^t$. Assume also that this pair was present at time $t-1$ as well. According to the construction of the dynamics this may happen only if $x_i^{t-1} < z_k - v$ and $z_k = \acute{x}_j^{t-1}$, which contradicts to the assumption that the pair $(x^{t-1}, \acute{x}^{t-1})$ is proper: namely the distance between members of the same pair exceeds v. On the other hand if the particles $i(x^t), j(\acute{x}^t)$ were not paired at time $t-1$ and the pair is just created at time t then one of the particles should overtake another at this time. Hence the distance between these particles cannot exceed v.

It remains to show that the last property still holds for "old" pairs. In order to enlarge the distance between the existing mutually paired particles one of them should be blocked by another particle or by an obstacle. On the other hand, the obstacle cannot belong to the open segment between pair members and hence it may block only the leading particle in the pair, which may only decrease the distance.

The blocking particle might be paired or unpaired. The former case implies that the 'companion' of the blocking particle is at distance at most v and hence it will block the enlargement of the distance by more than v. In the letter case the non-blocked paired particle overtakes the unpaired one and hence the pair under consideration will be re-paired.

The observation that in the moment of the creation of a pair the distance between its members cannot exceed v finishes this part of the proof.

The analysis of the absence of crossing mutually paired pairs is completely similar to the absence of defects between elements of a pair and therefore we skip this point. $\square$

Denote by $\rho_u(x, I)$ the density of the x-defects belonging to a finite segment I, and by $\rho_u(x) := \rho_u(x, \mathbb{R})$ the upper limit of $\rho_u(x, I_n)$ taken over *all* possible collections of nested finite segments I_n whose lengths go to infinity.

We say that a coupling of two Markov particle processes $x^t, \acute{x}^t$ is *nearly successful* if the upper density of the x-defects $\rho_u(x)$ vanishes with time a.s. This definition differs significantly from the conventional definition of the successful coupling (see e.g. [13]), which basically means that the coupled processes converge to each other in finite time.

Lemma 9 *Let $x, \acute{x} \in X$ with $\rho(x) = \rho(\acute{x}) > 0$, and let there exist a nearly successful dynamical coupling $(x^t, \acute{x}^t)$. Then*

$$|V(x, 0, t) - V(\acute{x}, 0, t)| \overset{t\to\infty}{\longrightarrow} 0.$$

Proof. Consider an integer valued function n_t which is equal to the index of the $\acute{x}$-particle paired at time $t > 0$ with the 0-th x-particle. If the 0-th x-particle is not paired at time t we set $n_t := \begin{cases} n_{t-1} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$.

To estimate the growth rate of $|n_t|$ at large t observe that n_t changes its value only at those moments of time when the 0-th x-particle meets a $\acute{x}$-defect. By the assumption about the nearly successful coupling at time $t \gg 1$ the average distance between the defects at time t is of order $1/\rho_u(\acute{x}^t)$ while the amount of time needed for two particles separated by the distance L to meet cannot be smaller than $L/(2v)$. Therefore the frequency of interactions of the 0-th x-particle with $\acute{x}$-defects may be estimated from above by the quantity of order $\rho_u(\acute{x}^t) \overset{t\to\infty}{\longrightarrow} 0$, which implies $n_t/t \overset{t\to\infty}{\longrightarrow} 0$.

Now we are ready to prove the main claim.

$$\begin{aligned} |V(x, 0, t) - V(\acute{x}, 0, t)| &= |(x_0^t - x_0^0) - (\acute{x}_0^t - \acute{x}_0^0)|/t \\ &\le |x_0^t - \acute{x}_0^t|/t + |x_0^0 - \acute{x}_0^0|/t \\ &\le |x_0^t - \acute{x}_{n_t}^t|/t + \frac{|n_t|}{t}\, |\acute{x}_{n_t}^t - \acute{x}_0^t|/|n_t| + |x_0^0 - \acute{x}_0^0|/t. \end{aligned}$$

The 2nd addend is a product of two terms $|n_t|/t$ and $|\acute{x}_{n_t}^t - \acute{x}_0^t|/|n_t|$. As we have shown, the 1st of them vanishes with time. If $|n_t|$ is uniformly bounded, then the 2nd term is obviously uniformly bounded on t. Otherwise, for large $|n_t|$ by Remark 1 and the density preservation the 2nd term is of order $1/\rho(\acute{x})$, which proves its uniform boundedness as well. Thus the 2nd addend goes to 0 as $t \to \infty$. Noting finally that the 1st and the last addend also vanishes with time at rate $1/t$ we are getting the result. □

Lemma 10 *Let $\rho(x) = \rho(\acute{x})$ and let in the coupled process $\forall i, j \;\; \exists$ a (random) moment of time $t_{ij} < \infty$ such that $x_i^t > \acute{x}_j^t$ for each $t \ge t_{ij}$. Then the coupling is nearly successful.*

Proof. By the assumption each x-particle will overtake eventually each $\acute{x}$-particle located originally to the right from its own position and thus will form a pair with it or with one of its neighbors (if they are so close that were overtaken simultaneously). Thus the creation of pairs is unavoidable. To show that the upper density of defects cannot remain positive, consider how the defects move under our assumptions. Assume that at time

$t \geq 0$ the i-th x-particle is paired with the j-th $\acute{x}$-particle. Then by Lemma 8 in order to overtake at time $s > t$ the j-th $\acute{x}$-particle significantly (by a distance larger than v) the i-th x-particle necessarily needs to break the pairing with the j-th $\acute{x}$-particle. Thus by the definition of the dynamical coupling either a x-defect overtakes the j-th $\acute{x}$-particle: $\overset{\circ}{\bullet}\ \bullet \longrightarrow \bullet\ \overset{\circ}{\bullet} \longrightarrow \bullet\ \overset{\bullet}{\circ}$, or the i-th x-particle overtakes a $\acute{x}$-defect: $\bullet\ \overset{\bullet}{\circ} \longrightarrow \bullet\ \overset{\bullet}{\circ} \longrightarrow \circ\ \overset{\bullet}{\bullet}$. (Otherwise this pair will not be broken.) Therefore during this process the x-defects move to the right while the $\acute{x}$-defects move to the left. Hence they inevitably meet each other and "annihilate". The assumption about the equality of particle densities implies the result. □

4 Auxiliary lattice zero-range process

Consider now a lattice process which we shall need in the sequel. This process is defined on an integer lattice $\mathbb{Z}$ occupied by a bi-infinite configuration of particles y ordered with respect to their positions y_i, i.e. $y_i \leq y_{i+1} \quad \forall i$. The dynamics is defined as follows. For each $i \in \mathbb{Z}$ consider all particles occupied the site i and choosing the one with the largest index (say k_i) among them we move this particle by one position to the right, i.e. $y_{k_i} := y_{k_i} + 1$. This is a deterministic version of the so called zero-range process on $\mathbb{Z}$ with parallel updating rules illustrated in Fig 4.

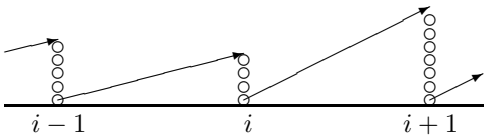


Figure 4: Zero-range process.

Setting $v = 1,\ z = \emptyset$ and assuming that $x_i \in \mathbb{Z} \quad \forall i \in \mathbb{Z}$ we observe that in this case (without obstacles) the trajectory $T^t x$ coincides with the trajectory of the zero-range process. Therefore according to [4] the average particle velocity for the zero-range process is equal to

$$V(y) = \min\{1,\ 1/\rho(y)\}. \tag{4.1}$$

Assume now that the lattice under consideration is not uniform (e.g. $\mathbb{Z}$) but the distance between the i-th and $(i+1)$-th site is equal to a nonnegative number ℓ_i for each $i \in \mathbb{Z}$. Considering the zero-range process on this heterogeneous lattice, but assuming that the site's occupation is the same in both cases we are able to calculate the corresponding statistical quantities.

Denote by $\acute{y}$ a configuration on a heterogeneous lattice $\acute{\mathbb{Z}}$ in which distances between the i-th site and the $(i+1)$-th one are given by the sequence of numbers $\{\ell_i\}$. Let y be a configuration on $\mathbb{Z}$ such that the i-th particle of the configuration $\acute{y}$ is located on the site whose index coincides with the index of the site occupied by the i-th particle of the configuration y.

Lemma 11 *The deterministic zero-range processes on $\mathbb{Z}$ and $\acute{\mathbb{Z}}$ starting with configurations $y, \acute{y}$ described above for each $t \geq 0$ preserves the connection between the particle configurations. Therefore the particle velocities in these processes satisfy the relation*

$$\acute{V}(\acute{y}) = V(y)/\rho(\acute{\mathbb{Z}}), \tag{4.2}$$

where $\rho(\acute{\mathbb{Z}})$ is assumed to be positive and is defined as the density of the particle configuration having exactly one particle at each site of $\acute{\mathbb{Z}}$.

Proof. The first claim follows from the definition of the zero-range process. Denote $L(t) := y_0^t - y_0^0 \ \ \forall t$. Then

$$(\acute{y}_0^t - \acute{y}_0^0)/t = \frac{1}{t} \sum_{j=0}^{y_0^t - y_0^0 - 1} \ell_{j+y_0^0} = \frac{L(t)}{t} \ \frac{1}{L(t)} \sum_{j=0}^{L(t)-1} \ell_{j+y_0^0} \overset{t\to\infty}{\longrightarrow} V(y)/\rho(\acute{z}).$$

□

5 Proof of Theorem 1

Lemma 12 *Let $x, \acute{x} \in X$ $\rho(x) = \rho(\acute{x})$ and let $V(x)$ be well defined for given v, z. Then $V(\acute{x})$ is also well defined and $V(\acute{x}) = V(x)$.*

Proof. Let $x, \acute{x} \in X_\rho := \{z \in X : \ \ \rho(z) = \rho\}$ be two admissible configurations of the same particle density. If one assumes that the dynamical coupling procedure leads to the nearly successful coupling of particles in these configurations then by Lemma 8 the assumptions of Lemma 9 are satisfied and hence $|V(x,0,t) - V(\acute{x},0,t)| \overset{t\to\infty}{\longrightarrow} 0$ which by Lemma 3 implies the claim. In general the assumption about the nearly successful coupling may not hold,[4] however as we demonstrate below the pairing construction is still useful.

Define random variables

$$W_{ij}^t := x_i^t - \acute{x}_j^t, \ \ i, j \in \mathbb{Z}, \ \ t \in \mathbb{Z}_0.$$

Then

$$V(x,i,t) - V(\acute{x},j,t) = W_{ij}^t/t - W_{ij}^0/t.$$

Since by Lemma 3 the differences between average velocities of different particles belonging to the same configuration vanish with time it is enough to consider only the case $i = j = 0$. For W_{00}^t there might be three possibilities which we study separately:

(a) $\lim\limits_{t\to\infty} W_{00}^t/t = 0$. Then $|V(x,0,t) - V(\acute{x},0,t)| \leq |W_{00}^t|/t + |W_{00}^0|/t \overset{t\to\infty}{\longrightarrow} 0$, which by Corollary 4 implies that the sets of limit points of the average velocities coincide.

[4]Consider e.g. the setting with $1/\rho > 5v$ and the configurations $x_i := i/\rho$ and $\acute{x}_i := i/\rho + 2v$, and assume that there are no obstacles. Then $\rho(x) = \rho(\acute{x}) = \rho$, $V(x) = V(\acute{x}) = v$ but no pair will be created.

(b) $\limsup_{t\to\infty} W_{00}^t/t > 0$. Then $\forall i \in \mathbb{Z}$ the i-th particle of the x-process will overtake eventually each particle of the $\acute{x}$-process located at time $t = 0$ to the right from the point x_i^0. This together with the assumption of the equality of particle densities allows to apply Lemma 10 according to which the coupling is nearly successful. On the other hand, by Lemma 8 the distance between mutually paired particles cannot exceed v. Therefore by Lemma 9 we have $|V(x,0,t) - V(\acute{x},0,t)| \stackrel{t\to\infty}{\longrightarrow} 0$, which contradicts to the assumption (b).

(c) $\limsup_{t\to\infty} W_{00}^t/t < 0$. Changing the roles of the processes $x^t, \acute{x}^t$ one reduces this case to the case (b).

Thus only the case (a) may take place. □

To apply this result to prove Theorem 1 one needs to construct for each particle density α a configuration having this density for which we are able to calculate its velocity explicitly. There are two possibilities to realize this idea. In both cases we use the auxiliary lattice zero-range process y^t constructed in Section 4 on the heterogeneous lattice $\acute{\mathbb{Z}}$ whose i-th site coincides with the location of the i-th element in the extended configuration of obstacles $\tilde{z}$.

The first construction is as follows. Choose an arbitrary initial configuration of particles y of a given density for the zero-range process. By Lemma 11 and the relation (4.1) we obtain the relation (1.3). Observing that the trajectory of the zero-range process y^t coincides with the trajectory of our original process $T^t y$ we get the desired result.

An alternative way to derive the relation (1.3) is to construct a specific initial configuration of the zero-range process, for which we can calculate the corresponding average velocity directly. The key observation here is that for $x := \tilde{z}$ obviously we have $V(x) = 1/\rho(\tilde{z})$.

For a given $x \in X$ for which $\exists\ 0 < \rho(x) < \infty$ and $\alpha \in \mathbb{R}_+$ we define the configuration $\alpha x \in X$ in three steps depending on arithmetic properties of α:

(a) kx for $\alpha = k \in \mathbb{Z}_+$;

(b) $\frac{k}{n}x$ for $\alpha = \frac{k}{n}$, $k = mn + k'$ and $m, k' \in \mathbb{Z}_0$, $k' < n \in \mathbb{Z}_+$;

(c) $\frac{k_n}{n}x \stackrel{n\to\infty}{\longrightarrow} \alpha x$ if $\frac{k_n}{n} \stackrel{n\to\infty}{\longrightarrow} \alpha$.

Here by kx we mean a configuration in which each particle in x is replaced by k particles. To get $y := \frac{k}{n}x$ we divide x into blocks having n consecutive particles each and construct $y := \{y_i\}$ as follows. For each i at location x_i we set exactly m particles at positions $y_j = x_i$. Besides at positions of each of the first k' particles in each of the blocks we add an additional particle and enumerate the particles according to their positions.

If $\alpha \neq k/n$ one considers a sequence of its rational approximations: $k_n/n \stackrel{n\to\infty}{\longrightarrow} \alpha$. For each n according to the rule (b) we construct a configuration $x^{(\alpha_n)} := \frac{k}{n}x$. Observe that for $\ell > 1$ the configuration $x^{(\alpha_{n\ell})}$ differs from $x^{(\alpha_n)}$ only by the presence of some additional particles in each of the blocks of length $n\ell$. Using the existence of $\rho(x)$ one proves that the limit configuration exists, and we denote the latter by αx.

Now for any $\alpha \in \mathbb{R}_+$ we construct $x := \alpha\tilde{z}$ and calculate $V(\alpha\tilde{z})$ as follows. We start with the case $\alpha = 1$, i.e. $x = \tilde{z}$. In this case obviously Tx coincides with the left shift of the configuration x. Therefore $(\tilde{z}_t - \tilde{z}_0) \cdot \rho(\tilde{z}, [\tilde{z}_0, \tilde{z}_t]) \equiv t \ \ \forall t \geq 0$ and thus

$$V(x,0,t) = (\tilde{z}_t - \tilde{z}_0)/t = 1/\rho(\tilde{z}, [\tilde{z}_0, \tilde{z}_t]) \stackrel{t\to\infty}{\longrightarrow} 1/\rho(\tilde{z}).$$

If $\alpha = \frac{k}{n}$ the configuration $x := \alpha\tilde{z}$ is "almost" spatially periodic: it consists of blocks of the configuration $\tilde{z}$ having equal number of n particles. The spatial periodicity immediately implies the existence of the average velocity $V(x)$. Observe now that for any rational $\alpha < 1$ the particles in the configuration $\alpha\tilde{z}$ on average move exactly as in the original configuration $\tilde{z}$ (i.e. $V(\alpha\tilde{z}) \equiv V(\tilde{z})$). If $\alpha = \frac{k}{n} > 1$ then the direct calculation gives $V(x) = \frac{n}{k}V(\tilde{z})$. Passing to the limit as $\frac{k_n}{n} \overset{n\to\infty}{\longrightarrow} \alpha$ we get eventually $V(\alpha\tilde{z}) := \begin{cases} \frac{1}{\rho(\tilde{z})} & \text{if} \quad \alpha \le 1 \\ \frac{1}{\alpha\rho(\tilde{z})} & \text{otherwise} \end{cases}$. Here one uses that $V(\frac{k}{n}\tilde{z}) \ge V(\frac{k}{n\ell}\tilde{z}) \ \ \forall \ell \in \mathbb{Z}_+$.

6 Varying waiting times and local velocities

In this Section we consider a more general model which includes both varying waiting times and local velocities. Namely we assume that the waiting time at the j-th obstacle is equal to $\tau_j \in \{1, 2, \dots, \tau\}$, and the local velocity for particles in the spatial segment (z_j, z_{j+1}) is equal to $v_j \in (0, v]$. It is useful to think about this model as a road divided into parts separated by obstacles (e.g. traffic lights) and having different quality of the pavement, which we describe by varying from part to part local velocities.

If $\tau_j > 1$ it might be possible that a new particle is coming to an obstacle when some preceding particles are still waiting there. We assume that for the given particle the waiting time τ_j at the j-th obstacle starts only when the succeeding particle leaves it. The movement of a particle waiting at a certain obstacle in the next moment of time depends on the time which this and preceding particles already spent there. Therefore this model is non Markovian. To cure this pathology using methods developed in [3] (for a very different situation) one adds a new variable *type* (for each particle) whose value is equal to the amount of time the particle will wait at an obstacle if it is located on the obstacle and zero otherwise.

Strictly speaking in order to use the notion of the coupling this is important (since it is defined for a pair of Markov processes). Nevertheless in the deterministic setting which we consider in this paper this issue is not crucial (as we shall see) and we proceed without this Markovian extension.

Denote by $\overline{\tau}, \overline{v}$ the collections of waiting times and local velocities corresponding to corresponding obstacles. If $\tau_j \equiv \tau$ and $v_j \equiv v$ we recover the previous setting. Surprisingly the analysis of this general setting is very similar to the previous one. Therefore we shall discuss only the points when the analysis differs.

First we refine the configuration of obstacles z exchanging the original i-th obstacle by $\tau_i + 1$ consequent obstacles located at the same position z_i but having 0 waiting times, i.e. leaving one these new obstacles the particle immediately moves to the next of them. The resulted ordered collection of obstacles we again call z. To take care about the change of indices we change also the collection of local velocities inserting τ_i unit velocities (before the original element v_i) corresponding to new obstacles and re-enumerating them.

Obviously for any configuration of particles x the movements of its elements in the case of the original configurations of obstacles and velocities and refined ones are exactly the same.

To construct the *extended configuration* of obstacles $\tilde{z}$ we proceed as follows: between the obstacles z_i and z_{i+1} we insert $\lfloor (z_{i+1} - z_i)/v_i \rfloor$ virtual obstacles at distance v_i starting from the point z_i for each $i \in \mathbb{Z}$. If $v_i \equiv v$ this construction boils down to the one

described in Section 1.

It is straightforward to check that all constructions and results obtained in Sections 2-4 remain valid in this more general setting except only one point. In the definition of the proper pair of configurations one changes that the open segment between two mutually paired particles belonging to the interval $[z_i, z_{i+1}]$ for some $i \in \mathbb{Z}$ cannot exceed v_i (instead of v in the original definition).

Therefore using the same arguments as in the proof of Theorem 1 we get its generalization.

Theorem 2 *(Fundamental Diagram) Let* $z,\ \tau_j \in \{1, 2, \dots, \tau\},\ v_j \in (0, v]$ *be given and let* $\rho(x), \rho(\tilde{z})$ *be well defined and positive. Then*

$$V(x) = \min\{1/\rho(\tilde{z}),\ 1/\rho(x)\}. \tag{6.1}$$

7 Discussion and generalizations

1. The density of a configuration in the way how it was defined in Section 2 depends sensitively on the statistics of both left and right tails of the configuration. A close look shows that in fact if all particles move in the same direction, say right, one needs only the information about the corresponding (right) tail, which allows to expand significantly the set of configurations having densities and for which our results can be applied.

For a configuration $x \in X$ by a *one-sided particle density* we mean the limit

$$\hat{\rho}(x) := \lim_{\ell \to \infty} \rho(x, [0, \ell]). \tag{7.1}$$

The upper an lower one-sided densities correspond to the upper and lower limits.

Theorem 3 *All previous results formulated in terms of "two-sided" densities remain valid if one replaces the usual particle density* ρ *to the one-sided density* $\hat{\rho}$.

Proof. The key observation here is that the movement of a given particle in a configuration $x^t \in X$ depends only on particles with larger indices. Therefore if one changes positions of all particles with negative indices the particles with positive indices will still have the same average velocity. On the other hand, by Lemma 3 the average velocity does not depend on the particle index. This allows to apply the following trick.

For each configuration $x \in X$ of density $\rho(x)$ we associate a new configuration $\hat{x} \in X$ defined by the relation:

$$\hat{x}_i := \begin{cases} x_i^t & \text{if } i \geq 0 \\ x_0 + i/\rho(x) & \text{otherwise .} \end{cases}$$

Then obviously $\hat{\rho}(x) = \rho(\hat{x}) = \rho(x)$.

Therefore for all purposes related to the average velocities all results valid for the configuration $\hat{x}$ remain valid for x as well. □

2. A close look to the proof of Theorem 1 may lead to the hypothesis that for a given configuration of obstacles z "typical" trajectories eventually become supported only by the locations belonging to the extended configuration $\tilde{z}$. Indeed, the calculation of the average velocity is given exactly for the configurations of this type. Let us show that this

is not the case even in the simplest setting when the configuration of obstacles has density one and is supported by integer points (i.e. a single obstacle at each point of $\mathbb{Z}$). Let $\rho(x) \geq 2$ and let the initial configuration has at least 2 particles at each half-integer point $\frac{1}{2}\,\mathbb{Z}$. Then for each $v \geq 1/2$ and $t \geq 1$ the configuration x^t is supported by the lattice $\mathbb{Z} \cup \frac{1}{2}\,\mathbb{Z}$.

3. Question about the existence of $\rho(\tilde{z})$. Let $\rho(z) > 0$ be well defined. For each given $v > 0$ we say that the configuration z is *regular* if the density of the corresponding extended configuration $\tilde{z}$ is well defined. In the topology induced by the uniform metric in the space of sequences there exists an open set containing irregular configurations z. Nevertheless assuming a reasonable model for the creation of the configuration of obstacles one can show that a "typical" configuration of obstacles is regular.

Indeed, assume that the sequence z is a realization of an ergodic stochastic process with stationary increments. Then Birkhoff Ergodic Theorem implies that Cesaro means for the sequence of fractional parts $\{(z_{i+1} - z_i)/v\}$ converge to the limit and thus our claim follows.

If $\rho_+(\tilde{z}) \neq \rho_-(\tilde{z})$ then one may consider upper/lower particle velocities and the corresponding statement for the Fundamental Diagram may be rewritten as follows.

Theorem 4 *Let $v, \rho(z) > 0$ be given. Then*

$$V_\pm(x) = \min\{1/\rho_\mp(\tilde{z}),\ 1/\rho_\mp(x)\}.$$

The proof of this result follows basically the same arguments as in the case of Theorem 1 but some additional technical estimates related to partial limits are necessary.

4. Assume now that we consider our original setting (i.e. $\tau_j \equiv 0\ \forall j \in \mathbb{Z}$) and the local velocity of the i-th particle does not depends on space (as in Section 6) but depends on time according to a given collection of local velocities $\{v_i^t\}$, i.e. v_i^t stands for the local velocity of the i-th particle at time t. Thus using the notation introduced in Section 1 we get $\xi_i^t := \min\left(\tilde{\Delta}_i(x,z), v_i^t\right)$ and $x_i^{t+1} := x_i^t + \xi_i^t$.

Theorem 5 *For any given configuration of obstacles z such that $z_k \overset{k\to\infty}{\longrightarrow} \infty$ and two one-particle configurations $x := \{x_0\} \neq \acute{x} := \{\acute{x}_0\}$ there exists a sequence of local velocities $\{v_0^t\}$ such that $|x_0^t - \acute{x}_0^t| \geq Ct$ for all $t \in \mathbb{Z}_+$ and some $C > 0$. Hence average particle velocities cannot coincide.*

The proof is based on the observation that one can choose $\{v_0^t\}$ such that each time t when the x-particle meets an obstacle it makes a "full" step equal to $\{v_0^t\}$ while the $\acute{x}$-particle which meets with a different obstacle makes only a half step.

This result shows that a direct generalization of our pure deterministic setting to the random case is not available.

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