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COORDINATE RINGS FOR THE MODULI OF $SL_2(\mathbb{C})$ QUASI-PARABOLIC PRINCIPAL BUNDLES ON A CURVE AND TORIC FIBER PRODUCTS

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Abstract. We continue the program started in [M1] to understand the commutative algebra of the projective coordinate rings of line bundles on the moduli $\mathcal{M}_{C,\vec{p}}(SL_2(\mathbb{C}))$ of quasi-parabolic principal bundles on a marked projective curve. We prove a general theorem about presentations of these rings, which implies that for generic marked curves $(C,\vec{p})$ the square of any effective line bundle has projective coordinate ring generated in degree 1 with a presenting ideal generated in degree 3. When the genus of the curve C is less than or equal to 2, we find that the square of any such line bundle gives a Koszul projective coordinate ring. Both theorems are obtained by studying toric degenerations of the projective coordinate ring. This leads us to formalize the properties of the polytopes used in proving our results by constructing a category of polytopes with term-orders, and studying its closure properties under fiber products.

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1. Introduction

Let G be a simple algebraic group over $\mathbb{C}$. For a complex projective curve C with n points $\{p_1,\dots,p_n\}\subset C$ we consider the moduli $\mathcal{M}_{C,\vec{p}}(G)$ of principal G-bundles on C with quasi-parabolic structure given by the full Borel subgroup B at the points $p_1,\dots,p_n$. This moduli can be expressed as the quotient stack of the product of the affine Grassmannian variety $\mathcal{Q}$ and the projective variety $[G/B]^n$ by an action of an ind-group $L_{C,\vec{p}}$ determined by the points $p_1,\dots,p_n$.

$$\mathcal{M}_{C,\vec{p}}(G) = L_{C,\vec{p}}(G)\backslash\backslash[\mathcal{Q}\times G/B^n] \tag{1}$$

The Picard group of $\mathcal{M}_{C,\vec{p}}(G)$, calculated in [LS], is a product of n copies of the character group of B times a copy of $\mathbb{Z}$.

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$$Pic(\mathcal{M}_{C,\vec{p}}(G)) = \mathcal{X}(B)^n \times \mathbb{Z} \tag{2}$$

The cone of line bundles with non-zero global sections is a subcone of $\Delta^n \times \mathbb{Z}_{\geq 0} \subset \mathcal{X}(B)^n \times \mathbb{Z}$, where Δ is the Weyl chamber of G. The space of global sections $H^0(\mathcal{M}_{C,\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L))$ for a vector of dominant weights $\vec{\lambda}$ and a non-negative integer L appears in mathematical physics as the space of conformal blocks (also know as the space of generalized theta-functions) with weights $\vec{\lambda}$ and level L for the conformal field theory defined by the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to the Lie algebra of G. We are interested in studying spaces of conformal blocks using techniques from combinatorial commutative algebra. Let $R_{C,\vec{p}}(\vec{\lambda}, L)$ be the projective coordinate ring of $\mathcal{M}_{C,\vec{p}}(G)$ defined by $\mathcal{L}(\vec{\lambda}, L)$.

$$R_{C,\vec{p}}(\vec{\lambda}, L) = \bigoplus_{n=0}^{\infty} H^0(\mathcal{M}_{C,\vec{p}}(G), \mathcal{L}(\vec{\lambda}, L)^{\otimes n}) \tag{3}$$

The Hilbert function of this ring outputs the sequence of dimensions of the spaces of conformal blocks associated to $(n\vec{\lambda}, nL)$, which can calculated by the celebrated Verlinde formula from conformal field theory, see [B]. In [M2], we showed that varying $(C, \vec{p})$, gives a flat sheaf of algebras on the moduli of stable $n-$punctured curves of genus g, $\bar{M}_{g,n}$. For $G = SL_2(\mathbb{C})$ this allows us to bring in the toric algebra of the following rational polytope.

Definition 1.1. *For Γ a trivalent graph with n leaves labeled $1, \ldots, n$, and $\beta_1(\Gamma) = g$, we define $P_\Gamma(\vec{r}, L) \subset \mathbb{R}^{|Edge(\Gamma)|}$ to be the set of weightings of the edges of Γ which satisfy the following conditions.*

1. *For any trinode $v \in \Gamma$ with the three incident edge weights w_1, w_2, w_3 must satisfy the triangle inequalities.*
2. *For any trinode as above, $w_1 + w_2 + w_3 \leq 2L$.*
3. *The weight on the edge attached to the $i-$th leaf equals r_i.*

We take the lattice of integer points in $\mathbb{R}^{|Edge(\Gamma)|}$ defined by the condition that the sum $w_1 + w_2 + w_3 \in 2\mathbb{Z}$ for any trinode $v \in \Gamma$. The following was proved in [StX] for C genus 0, and for general genus in [M2] by proposition 1.5.

Theorem 1.2. *Let $\vec{r}$ be a vector of dominant $SL_2(\mathbb{C})$ weights. The algebra $R_{C,\vec{p}}(\vec{r}, L)$ can be flatly deformed to the graded semigroup algebra associated to $P_\Gamma(\vec{r}, L)$ when the genus of C equals the first Betti number of Γ.*

The lattice points of $P_\Gamma(\vec{r}, L)$ are labelings of Γ by dominant weights of $SL_2(\mathbb{C})$ which satisfy the quantum Clebsch-Gordon conditions at each vertex. These combinatorial objects are known by various names in mathematical physics, such as Feynman diagrams or spin diagrams, see [Ko]. Analysis of these polytopes and other convex sets of spin diagrams has also appeared in the computational biology literature, specifically in the work of Buczynska and Wiesniewski on phylogenetic algebraic varieties, [BW], [Bu]. The polytope $P_\Gamma(\vec{r}, L)$ is a cross section of the cone $\tau(\Gamma)$ corresponding to Buczynska's graphical phylogenetic toric variety introduced in [Bu] to study the Jukes-Cantor statistical model on graphs.

A good deal of combinatorial information is preserved by flat deformation, such as the degree and the Hilbert function of $R_{C,\vec{p}}(\vec{r}, L)$. In theory, all these details can be computed from the polytopes $P_\Gamma(\vec{r}, L)$. A non-trivial consequence of this observation is that the equivalent polyhedral information for $P_\Gamma(\vec{r}, L)$, the volume and the number of lattice points, etc, is independent of the graph Γ. The weightings of Γ defined by the lattice points of $P_\Gamma(\vec{r}, L)$ are themselves interesting objects. Considering just the weightings (w_1, w_2, w_3) that satisfy the conditions of definition 1.1, we can always find three numbers x_{12}, x_{13}, x_{23} such that $w_1 = x_{12} + x_{13}, w_2 = x_{12} + x_{23}, w_3 = x_{13} + x_{23}$. We visualize this by making a graph on three vertices with the number of edges between the i and $j-$th vertex equal to x_{ij}. When this graph is placed in a tripod as in figure 1 below, the resulting weights on the edges of the tripod are w_1, w_2, w_3. We call (x_{12}, x_{13}, x_{23}) the "pipes" of the weighting (w_1, w_2, w_3).

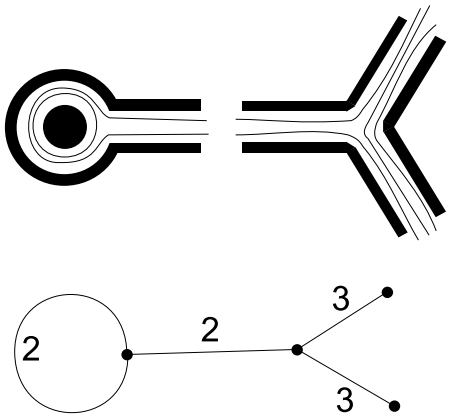

FIGURE 1.

For an arbitrary connected trivalent graph Γ we can form a new disconnected graph $\hat{\Gamma}$ by splitting each non-leaf edge of Γ. A weighting w of Γ then induces a weighting on the forest $\hat{\Gamma}$, and we can form the pipes for each of these weightings, and glue the pipes from two edges of $\hat{\Gamma}$ which come from the same edge of Γ together to form a "pipe" model of w. This is a disjoint union of cycles and a graph on the leaves of Γ. A consequence of this observation is that the number of leaves in Γ which recieve an odd weighting from w is even.

Definition 1.3. *For the polytope $P_\Gamma(\vec{r}, L)$ we say a vector $\vec{r}$ is compatible with a graph Γ if any odd r_i shares a vertex with another odd r_j.*

From now on we will focus on the following graphs.

Definition 1.4. *A trivalent graph Γ of genus g with n leaves is tree-like if it is obtained from a trivalent tree with $g + n$ leaves by adding loops on g leaves.*

When $\vec{r}$ and Γ are compatible, and Γ is tree-like any lattice weighting $w \in P_\Gamma(\vec{r}, L)$ assigns the non-loop edges of Γ only even numbers. This property was critical in the proof of the following theorem, from [M1].

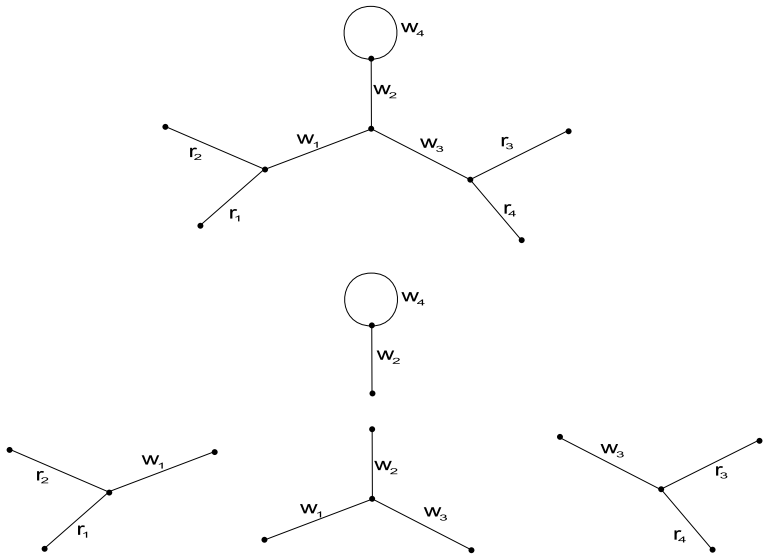

FIGURE 2. The graphs Γ and $\hat{\Gamma}$.

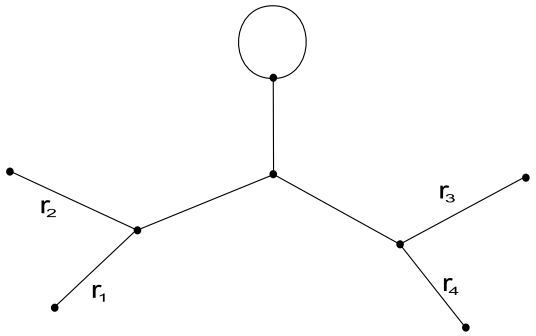

FIGURE 3. (r_1, r_2, r_3, r_4) is adapted when $r_1 + r_2, r_3 + r_4 \in 2\mathbb{Z}$

Theorem 1.5. *Let $\mathcal{T}$ be a trivalent tree, let L be even with $L > 2$, and let $\vec{r}$ be compatible with $\mathcal{T}$, then $\mathbb{C}[P_{\mathcal{T}}(\vec{r}, L)]$ is generated in degree* 1, *and the binomial ideal of relations $I_{P_{\mathcal{T}}(\vec{r},L)}$ is generated in degree* 3.

When $\mathcal{T}$ is a caterpillar tree, see the top of figure 2 below, then the relations are generated in degree 2. The main result presented here on the polytopes $P_\Gamma(\vec{r}, L)$ is a generalization of theorem 1.5 to tree-like graphs.

Theorem 1.6. *Let Γ be tree-like, L even with $L > 2$, and $\vec{r}$ compatible with Γ, then $\mathbb{C}[P_\Gamma(\vec{r}, L)]$ is generated in degree* 1, *and the binomial ideal of relations $I_{P_\Gamma(\vec{r},L)}$ is generated in degree* 3.

In [M1] there are examples which show that certain degree 3 polynomials are necessary to generate $I_{P_\Gamma(\vec{r},L)}$. However, when we restrict the topology of Γ we get a better result. We say Γ is a caterpillar graph if it is obtainable from a caterpillar

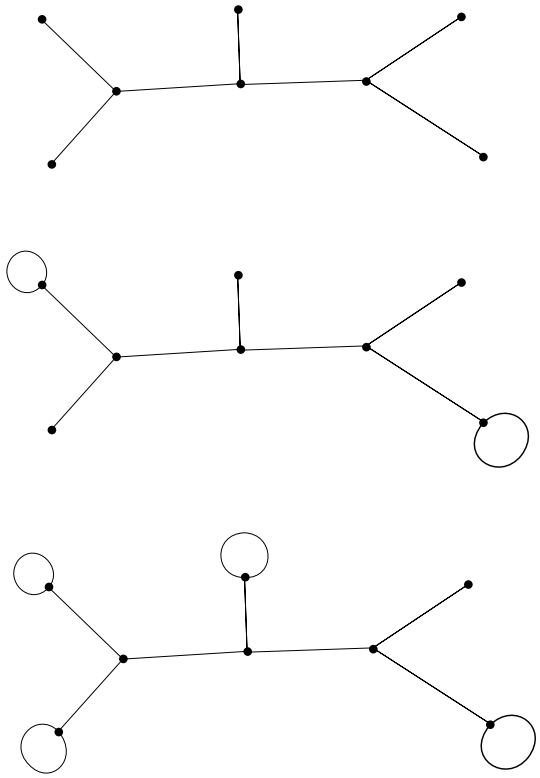

FIGURE 4. From the top, a caterpillar tree, a caterpillar graph, and a tree-like graph.

tree by adding a loop at the head or the tail or both, see the second image in figure 2.

Theorem 1.7. *Let Γ be a caterpillar graph, $\vec{r}, L$ be even with $L > 2$, then the presenting ideal $I_{P_\Gamma(\vec{r},L)}$ of $\mathbb{C}[P_\Gamma(\vec{r},L)]$ has a quadratic, square-free Gröbner basis.*

This improves on the result for trees, which only implies quadtratic generation of the ideal. These theorems can then be lifted back to the algebras $R_{\mathbb{C},\vec{p}}(\vec{r},L)$, the following is a consequence of theorems 1.6 and 1.7.

Theorem 1.8. *For any $\vec{r}$, $L > 2$ an even number, and C a generic curve, $R_{C,\vec{p}}(\vec{r},L)$ is generated in degree 1 with ideal of relations generated in degree 3. If C is genus 2 or less, and $\vec{r}$ is even, then $R_{C,\vec{p}}(\vec{r},L)$ is Koszul.*

Proof. We give a sketch of the argument. Recall that the number of odd r_i in $\vec{r}$ must be even because of the piping construction, so we can always find a tree-like graph Γ which is compatible with $\vec{r}$. We assume that both properties have been proved for the appropriate toric algebra $\mathbb{C}[P_\Gamma(\vec{r},L)]$, and can therefore be shown to hold for the fiber $R_{C,\vec{p}}(\vec{r},L)$ of some closed point in $\bar{\mathcal{M}}_{g,n}$. The moduli $\bar{\mathcal{M}}_{g,n}$ is a connected, Noetherian Deligne-Mumford stack over $\mathbb{C}$, therefore for any closed point $q : Spec(\mathbb{C}) \to \bar{\mathcal{M}}_{g,n}$ there is a dense open substack U with an étale Noetherian affine cover $Q : Spec(A) \to U$. The fiber of our sheaf of algebras is then equal to the fiber over $q' : Spec(\mathbb{C}) \to Spec(A)$ for some closed point of $Spec(A)$. Over a

Noetherian affine base, we obtain the statement on generators and relations from an application of Nakayama's lemma to the graded components of the algebra and its presenting ideal. The statement on the Koszul property results from the fact that Betti numbers of the corresponding generalized Koszul complex (see [E], Ex 17.22) do not increase under specialization because their graded components are locally free modules. We may reduce to this case because the generic point of $\overline{\mathcal{M}}_{g,n}$ is étale-covered by the generic point of $Spec(A)$, and the fiber over this point is obtained by base-change.

□

Remark 1.9. *When* Γ *is a tree and* L *is very large, the algebra* $\mathbb{C}[P_\Gamma(\vec{r}, L)]$ *is also realized as a toric deformation of a projective coordinate ring of the moduli of* $\vec{r}$*-weighted points on the projective line,* $\mathcal{M}_{\vec{r}} = SL_2(\mathbb{C})_{\vec{r}}\backslash\backslash[\mathbb{P}^1]^n$*. Theorem 1.7 then shows that this projective coordinate ring has a quadratic, square-free Gröbner basis when* $\vec{r}$ *is even, a result also obtained by Herring and Howard,* [HH].

Theorems 1.5 ,1.6, and 1.7 are proved by understanding the structure of $P_\Gamma(\vec{r}, L)$ as a fiber product of rational polytopes. In [M1] the polytopes $P_\mathcal{T}(\vec{r}, L)$ were shown to be "balanced" (see section 3). In dimension 2, where most of the polytopes we need to consider live, balanced polytopes are all normal and the presenting ideal of their toric algebra always comes with a quadratic, square-free Gröbner basis. The main technical tool that made theorem 1.5 work is the fact that products and certain fiber products of balanced polytopes remain balanced, and take on critical algebraic features of their components.

Definition 1.10. *Let* $P_1 \subset \mathbb{R}^n$*,* $P_2 \subset \mathbb{R}^m$ *and* $Q \subset \mathbb{R}^k$ *be lattice polytopes, and let* $\pi_1 : \mathbb{R}^n \to \mathbb{R}^k$ *and* $\pi_2 : \mathbb{R}^m \to \mathbb{R}^k$ *be lattice maps which take* P_i *into* Q*. The fiber product* $P_1 \times_Q P_2 \subset \mathbb{R}^{n+m-k}$ *is defined to be the subset of points* $(x, y) \in P \times Q \subset \mathbb{R}^{n+m}$ *such that* $\pi_1(x) = \pi_2(y)$*.*

We will see that the polytope $P_\Gamma(\vec{r}, L)$ is balanced when Γ is tree-like, L is even and greater than 2 and $\vec{r}$ is compatible with Γ. The conditions which define $P_\Gamma(\vec{r}, L)$ are all localized around individual trinodes of Γ, which implies we may represent $P_\Gamma(\vec{r}, L)$ as a fiber product of polytopes defined on the components of $\hat{\Gamma}$. When Γ is tree-like, the polytopes given by weightings of the components of $\hat{\Gamma}$ are single loops with an edge, or a trinode with 0, 1, or 2 fixed edges, see figure 4. We call the polytope corresponding to a loop with an edge $B(L)$, weightings of a trinode with one or two fixed edges $P_r(L)$ or $P_{r,s}(L)$ respectively, and general weightings of a trinode $P_3(L)$. Because Γ is chosen to be tree-like, with $\vec{r}$ compatible, all internal edges of Γ are weighted with even numbers, so for $B(L)$ we consider the lattice which weights the non-loop edge with even numbers. For $P_r(L)$, $P_{r,s}(L)$ and $P_3(L)$ we need only consider even weightings. This restriction forces these polytopes to all be balanced.

When Γ tree-like, it can be obtained from $\hat{\Gamma}$ by an ordered sequence of identifications of pairs of edges in $\hat{\Gamma}$, each identification corresponds to a fiber product of a pair of the above polytopes over the interval $[0, L]$. Note that if Γ is a caterpillar graph, then all the polytopes assigned to components of $\hat{\Gamma}$ are in dimension 1 or 2.

The maps $P_3(L) \to [0, L]$, $B(L) \to [0, L]$, etc, are all coordinate projections given by taking the weight on a particular edge of the component of $\hat{\Gamma}$, these turn out to be the appropriate maps to use with balanced polytopes, and we will find that

a fiber product of normal balanced polytopes is balanced and normal. The main technical part of this paper is to develope and clarify the polytope properties that make this construction work. As fiber products are categorical operations, we view the natural habitat for the above theorems as closure properties of a category of polytopes under certain fiber products. We define this category next.

1.1. **the category $\mathcal{P}$.** To define the environment where our fiber product machinery works, we will need some background on term orders of binomial ideals. For the following definitions see [St]. For a lattice polytope $P \subset \mathbb{R}^n$ let $\Sigma : X_P \to \mathbb{R}$ be an assignment of real numbers to the lattice points of P. The function Σ defines a term order on the polynomial ring $\mathbb{C}[X_P]$, where a monomial $x^{\mathbf{m}} \leq x^{\mathbf{n}}$ when $deg(\mathbf{m}) < deg(\mathbf{n})$ or $deg(\mathbf{m}) = deg(\mathbf{n})$ and $\Sigma(\mathbf{m}) \leq \Sigma(\mathbf{n})$. The polytope P defines a graded algebra $\mathbb{C}[P]$, where the $m-$th graded component has a basis given by the lattice points of the m-th minkowski sum mP, and the product operation is lattice point addition. Let $I_P \subset \mathbb{C}[X_P]$ be the toric ideal which is the kernel of the map $\mathbb{C}[X_P] \to \mathbb{C}[P]$. For a polynomal $f \in I_P$, the initial form $in_\Sigma(f)$ with respect to Σ is defined to be the sum of the monomials in f which have the highest Σ weight.

Definition 1.11. *A monomial $x^{\mathbf{m}}$ is said to be standard with respect to a term order Σ if it is not an initial form of I_P.*

For a lattice point $b \in mP$, we let Δ_b be the set of monomials in $\mathbb{C}[X_P]$ which map to b. The difference of any two elements in Δ_b is a member of I_P. The function Σ defines a partial ordering on Δ_b, and the minimal elements of this partial ordering are the standard monomials.

Definition 1.12. *We define the category $\mathcal{P}$ to have pairs (P, Σ) as objects, where P is a lattice polytope in some $\mathbb{R}^n$, and Σ is a term order function on P. A morphism $\pi : (P, \Sigma) \to (Q, \Gamma)$ is a linear map on $\pi : P \to Q$, induced by a map on lattices, such that the standard monomials of Σ map to standard monomials of Γ.*

The following is the most general result we will prove about the category $\mathcal{P}$. We say (Q, Σ) has unique standard monomials if the sets Δ_b always have a unique minimal element when they are non-empty.

Proposition 1.13. *When (Q, Σ) has unique standard monomials, the fiber product object $(P_1, \Sigma_1) \times_{(Q,\Sigma)} (P_2, \Sigma_2)$ exists in $\mathcal{P}$.*

The category of lattice polytopes comes with a fiber product, so the content of this proposition is in showing that there is a term order on $P_1 \times_Q P_2$ with maps to (P_1, Σ_1), (P_2, Σ_2) satisfying the universal property of a fiber product. Next we formalize properties of balanced polytopes by showing that the category $\mathcal{P}$ has a distinguished subcategory of polytopes with term orders that is closed under fiber product.

Definition 1.14. *A term order Σ on a lattice polytope P is said to flag when a monomial for P is standard if and only if every degree 2 divisor is standard, and x^n is standard for any $x \in X_P$.*

Proposition 1.15. *If $\pi_1 : (P_1, \Sigma_1) \to (Q, \Sigma)$ and $\pi_2 : (P_2, \Sigma_2) \to (Q, \Sigma)$ are maps in $\mathcal{P}$, and if (Q, Σ) has unique standard monomials, then (P_1, Σ_1) and (P_2, Σ_2) flag implies $(P_1, \Sigma_1) \times_{(Q,\Sigma)} (P_2, \Sigma_2)$ flag.*

The connection with the polytopes $P_\Gamma(\vec{r}, L)$ above is that balanced polytopes can be viewed as examples of polytopes with flag term orders. Theorem 1.6 is then a consequence of the following propositions.

Proposition 1.16. *Let $(P_1, \Sigma_1) \times_{(Q,\Sigma)} (P_2, \Sigma_2)$ be a fiber product of flag elements over (Q, Σ) with unique standard monomials. If each (P_i, Σ_i) is normal, then $(P_1, \Sigma_1) \times_{(Q,\Sigma)} (P_2, \Sigma_2)$ is normal.*

For (P, Σ) we call a binomial $x^{\mathbf{n}} - x^{\mathbf{m}} \in I_P$ a standard relation if both monomials are standard.

Proposition 1.17. *Let $(P_1, \Sigma_1) \times_{(Q,\Sigma)} (P_2, \Sigma_2)$ be a fiber product of flag objects over (Q, Σ) with unique standard monomials. If the standard relations of (P_i, Σ_i) are generated by standard relations of degree k_i, then $I_{(P_1,\Sigma_1)\times_{(Q,\Sigma)}(P_2,\Sigma_2)}$ is generated in degree $max\{k_1, k_2\}$.*

Properties of fiber products over a toric algebra have been studied before by Sullivant in [Su]. These theorems can be viewed as refinements of Theorem 2.8 in [Su]. In our results, the assumption that the base of the fiber product have strict unique factorization has been removed by restricting our study to fiber products of binomial ideals.

1.2. **the concatenation product on term orders.** The second technical result, used to prove theorem 1.7, is a version of Sullivant's Theorem 2.9 in [Su]. Once again, we remove the restriction that the base of the fiber product have unique factorization by only considering binomial ideals. We define another operation in $\mathcal{P}$.

Definition 1.18. *Define $(P_1, \Sigma_1) \boxtimes_{(Q,\Sigma)} (P_2, \Sigma_2)$ in $\mathcal{P}$ as the object with the fiber product polytope $P_1 \times_Q P_2$ and term order given by the following rule on monomials. For lattice points $(V_1, W_1), (V_2, W_2) \in (P_1, \Sigma_1) \boxtimes_{(Q,\Sigma)} (P_2, \Sigma_2)$ we say $(V_1, W_1) < (V_2, W_2)$ if*

1. *$\Sigma_1(V_1) < \Sigma_1(V_2)$ or*
2. *the Σ_1 values are equal, and $\Sigma_2(W_1) < \Sigma_2(W_2)$.*

For monomials $x^{(V_1,W_1)\dots(V_k,V_k)}, x^{(U_1,Y_1)\dots(U_\ell,Y_\ell)}$ we first order by degree, then by $\Sigma_1 \oplus \Sigma_2$, we then break ties by writing both sets of exponents in decreasing order with the above ordering and we declare $(V_1, W_1) \dots (V_k, W_k) > (U_1, Y_1) \dots (U_\ell, Y_\ell)$, if $(V_i, W_i) > (U_i, Y_i)$ at the first place they differ.

Proposition 1.19. *If the presenting ideals of (P_1, Σ_1), (P_2, Σ_2), and (Q, Σ) all have quadratic, square-free Gröbner bases, then so does the presenting ideal of $(P_1, \Sigma_1) \boxtimes_{(Q,\Sigma)} (P_2, \Sigma_2)$.*

Theorem 1.7 above then follows from the fact that the presenting ideals of intervals $[0, L]$ and balanced polytopes in dimension 2 have quadratic, square-free Gröbner bases. Although we do not assume that the base of our fiber products has unique factorization, we do assume this "locally" by requiring the polytope to have a quadratic square-free Gröbner basis. This supports the sentiment in [Su] that the toric fiber product can be controlled when the base has unique factorization properties. It would be interesting to see this theorem generalized to arbitrary ideals in an appropriate way.

2. Term orders on toric ideals, the category $\mathcal{P}$ and flag term orders

In this section we review some basic elements of term orders on toric ideals. We discuss decompositions induced by a term order, and the corresponding algebra of the initial ideal. We then go over some properties of the category $\mathcal{P}$, flag term orders, and balanced polytopes, focusing on what a term order says about a polytope geometrically. First, we recall the initial complex $\Delta_\Sigma(P)$ of a term order, see chapter 8 of [St].

Definition 2.1. *Let P be a lattice polytope, and Σ a term order on P. Define the simplicial complex $\Delta_\Sigma(P)$ on the vertex set X_P as follows. A set $S \subset X_P$ defines a face of $\Delta_\Sigma(P)$ if every monomial with support S is standard.*

If $S \in \Delta_\Sigma(P)$ and $F \subset S$ has a monomial $x^{\mathbf{m}}$ supported on its entries which is not standard then one can make a non-standard monomial supported on S by multiplying by the remaining generators in $S \setminus F$, this establishes that $\Delta_\Sigma(P)$ is indeed a simplicial complex. Morphisms in the category $\mathcal{P}$ preserve the information in these simplicial complexes.

Proposition 2.2. *Let $\pi : (P, \Sigma) \to (Q, \Gamma)$ be a morphism in $\mathcal{P}$ then π induces a simplicial map $\pi_* : \Delta_\Sigma(P) \to \Delta_\Gamma(Q)$.*

Proof. Any monomial $x^{\mathbf{m}}$ with support $S \in \Delta_\Sigma(P)$ is standard, implying its image $\pi(x)^{\mathbf{m}}$ is standard with respect to Γ in Q. Since this holds for any monomial, all monomials with support $\pi_*(S) = \{\pi(x_i) | x_i \in S\}$ must be standard, which means $\pi_*(S) \in \Delta_\Gamma(Q)$. □

For flag term orders Σ we will be concerned with the convex hulls $|S| \subset P$ of the faces of $\Delta_\Sigma(P)$. Fix Σ to be a flag term order. If $b \in |S|$ is any rational point, then we have

$$b = \sum_{x_i \in S} s_i x_i \tag{4}$$

for rational s_i. This implies for some $N >> 0$ that the monomial defined by $\sum_{x_i \in S}(Ns_i)x_i = Nb$ is standard. If $b \in |S|$ is a lattice point, this implies $S \cup \{b\}$ is also in $\Delta_\Sigma(P)$ as Nb is defines a standard monomial, implying $(N+1)b = \sum_{x_i \in S} Ns_i x_i + b$ defines a standard monomial as well. The same argument shows that any rational point $b \in P$ is in the convex hull of some face of $\Delta_\Sigma(P)$. The following proposition shows that the interiors of maximal faces of $\Delta_\Sigma(P)$ do not intersect.

Proposition 2.3. *Let (P, Σ) be flag, and let $S, T \subset \Delta_\Sigma(P)$, and suppose the intersection $int|S| \cap int|T|$ is non-empty. Then $S \cup T$ is also in $\Delta_\Sigma(P)$.*

Proof. Let $b \in int|S| \cap int|T|$, be a rational point, this implies that b can be written as convex sums in both sets S and T.

$$b = \sum_{x_i \in S} r_i x_i = \sum_{y_j \in T} t_i y_i \tag{5}$$

In particular the numbers t_j and s_i can be taken non-zero for all i, j. Choose $N >> 0$ so that Ns_i and Nt_j are integers for all i, j, then we have found monomial

expressions for $Nb \in NP$ with support in S and T respectively. This implies that these monomials are both standard, and further implies that the following expressions correspond to standard monomials.

$$2Nb = \sum_{x_i \in S} (2Nr_i)x_i = \sum_{x_i \in S} (Nr_i)x_i + \sum_{y_j \in T} (Nt_i)y_i \tag{6}$$

So the set $S \cup T$ is also in $\Delta_\Sigma(P)$. □

Taken together, these observations imply the following.

Proposition 2.4. *Let (P, Σ) be flag. The polytope P is geometrically subdivided into convex regions $S_i \subset P$, the convex hulls of the maximal facets of $\Delta_\Sigma(P)$. The lattice points in a region S_i are precisely the members of the corresponding maximal facet.*

For any rational point $b \in S$ a maximal face, we can find some set $\{x_0, \ldots, x_n\} \subset S$ such that b is in the convex hull of $x_0, \ldots, x_n$. If this set is not equal to S, this can only be because two monomials in the same Δ_{Nb}, $N >> 0$ get the same minimial Σ weight, so Σ cannot be a total term order. Conversely, when Σ defines a total term order on monomials, all maximal faces $S \in \Delta_\Sigma(P)$ are simplices.

We will now show that that much of the algebra of $\mathbb{C}[P]$ is captured by the algebras $\mathbb{C}[S_i]$, $S_i \in \Delta_\Sigma(P)$. The following lemma allows us to reduce generation and relation degrees for $\mathbb{C}[P]$ to those of the $\mathbb{C}[S_i]$.

Lemma 2.5. *For any monomial $a_1 \ldots a_N \in P$ where (P, Σ) is flag, $a_1 \ldots a_N$ is related to a standard monomial by degree 2 relations.*

Proof. Let $a_1 + \ldots a_N = b \in NP$, and recall Δ_b, the set of monomials which map to b. If $x^{a_1 \cdots a_N}$ is standard, there is nothing to do, so suppose this is not the case, then by definition of flag term orders, it has a non-standard degree 2 divisor, say $x^{a_1 a_2}$. We replace $x^{a_1 a_2}$ with $x^{A_1 A_2}$ standard, then

$$\Sigma(a_1 a_2 \ldots a_N) > \Sigma(A_1 A_2 \ldots a_N) \tag{7}$$

We repeat this procedure on $A_1 A_2 \ldots a_N$. This algorithm must terminate by the finiteness of Δ_b. □

Corollary 2.6. *For (P, Σ) flag with maximal facets $S_1, \ldots S_k$ of K_Σ, any relation $x^{a_1 \ldots a_n} = x^{b_1 \ldots b_N}$ can be taken to a relation in some S_i by degree 2 relations.*

Proof. We must have $a_1 + \ldots + a_N = b_1 + \ldots + b_M = b$ for some $b \in NP$, therefore any standard factorization of b must be made from the lattice points of an S_i which contains $\frac{1}{N}b$. The previous proposition shows that both sides of this relation can be converted to such a standard relation by degree 2 moves. □

The content of this last corollary is that generating sets of relations for the S_i, along with the degree 2 relations of the form $x^{a_1 a_2} = x^{A_1 A_2}$, with one side standard, suffice to generate the ideal I_P.

2.1. **Balanced Polytopes.** We now fix a special term order function Σ^2. We will define balanced polytopes and show that they always have Σ^2 as a flag term order.

Definition 2.7. *Define $\Sigma^2 : \mathbb{Z}^n \to \mathbb{Z}$ to be the function which takes a vector $(a_1, \ldots, a_n)$ to the sum of the squares of its entries $\sum_{i=1}^n a_i^2$.*

Example 2.8. *To motivate the use of this function, consider a number $b \in \mathbb{Z}_{\geq 0}$ and the set $\Delta_N(b)$ of N tuples $(x_1, \ldots, x_N) \in \mathbb{Z}^N_{\geq 0}$ such that $\sum x_i = b$. We can partially order these tuples by weighting them with Σ^2. For $b = 2$, the set $\Delta_3(3) = \{[003], [012], [111]\}$, and these are ordered,*

$$[111] < [012] < [003]. \tag{8}$$

We call the operation with takes a pair of numbers (x, y) to $(\lfloor \frac{x+y}{2} \rfloor, \lceil \frac{x+y}{2} \rceil)$ balancing. For any pair of numbers x, y we have that $x^2 + y^2 \geq \lfloor \frac{x+y}{2} \rfloor^2 + \lceil \frac{x+y}{2} \rceil^2$, while $x+y = \lfloor \frac{x+y}{2} \rfloor + \lceil \frac{x+y}{2} \rceil$. For any pair x_i, x_j from the a tuple as above, balancing yields a tuple which still sums to b, with a lower Σ^2 value, and the value is strictly lower if and only if x_i and x_j differ by more than 1. This implies that the minimal tuple with respect to Σ^2 is of the form $(x+1, x+1, \ldots, x, \ldots, x)$ where $b = Nx + M$ with M the number of entries of the form $x + 1$.

We can play this same game with a tuple of vectors $(\vec{x}_1, \ldots, \vec{x}_N) \subset \mathbb{Z}^m_{\geq 0}$ which sum to a fixed vector $\vec{b} \in \mathbb{Z}^n$. We call the operation that takes two vectors $\vec{x}, \vec{y}$ to the vectors with entries the balancing of the pairs of entries of $\vec{x}, \vec{y}$ a balancing of m-vectors. Once again, this operation lowers the Σ^2 value of $(\vec{x}, \vec{y})$, unless these vectors were already balanced with respect to each other.

Proposition 2.9. *For any tuple of vectors $\{\vec{x}_1, \ldots, \vec{x}_N\} \subset \mathbb{Z}^m_{\geq 0}$ There is a tuple $\{\vec{b}_1, \ldots, \vec{b}_N\} \subset \mathbb{Z}^m_{\geq 0}$ with the following properties.*

1. $\sum \vec{b}_i = \sum \vec{x}_i$
2. *Each tuple of numbers $\{\vec{b}^i_1, \ldots, \vec{b}^i_M\}$ is balanced.*
3. $\Sigma^2(\vec{b}_1, \ldots, \vec{b}_N) \leq \Sigma^2(\vec{x}_1, \ldots, \vec{x}_N)$, *with equality only when each tuple of numbers $\{\vec{x}^i_1, \ldots, \vec{x}^i_M\}$ is balanced.*
4. *The $\vec{b}_i$ all lie in a common translate of the unit cube in $\mathbb{Z}^m_{\geq 0}$.*

Proof. We construct a $\{\vec{b}_1, \ldots, \vec{b}_N\}$ by doing pairwise balancings, beginning with the set $\{\vec{x}_1, \ldots, \vec{x}_N\}$. Since each operation lowers the Σ^2 value, this process terminates. Number 1 above follows because this algorithm does not change the total sum of the vectors, numbers 2 and 3 are consequences of the algorithm and imply number 4. □

Definition 2.10. *We say a lattice polytope $P \subset \mathbb{R}^m$ is balanced if each tuple of lattice points $b_1, \ldots, b_N \in P$ has a balancing in X_P, the lattice points of P.*

Proposition 2.11. *A polytope $P \subset \mathbb{R}^n$ is balanced if and only if (P, Σ^2) is flag, and the convex hulls of the maximal elements in $\Delta_{\Sigma^2}(P)$ are subpolytopes of translates of the unit $n-$cube.*

Proof. If P is balanced, then the balanced monomials of P are clearly the standard monomials with respect to Σ^2, so we verify that the flag properties are satisfied.

First note that for any $x \in X_P$, x^n is a balanced monomial. Furthermore, a monomial $x^{\mathbf{n}}$ is balanced if and only if the entries in any two generators dividing $x^{\mathbf{n}}$ differ by 1 or 0, which is the case if and only if any degree 2 divisor of $x^{\mathbf{n}}$ is balanced. This is exactly the flag condition.

If (P, Σ^2) is flag, and the fundamental regions S_i are all subsets of translates of the unit cube then the algorithm in lemma 2.5 above gives a balancing of any monomial. □

Corollary 2.12. *Let P be a balanced polytope, then P is normal if and only if each maximal cubical region C_i is normal. Furthermore, the ideal I_P is generated in degree bounded by the degrees required to generate the ideals I_{C_i}.*

3. Examples

In this section we'll review examples of flag polytopes that we'll use to analyze $P_\Gamma(\vec{r}, L)$. Serving in the role of Q in this construction are intervals of finite length, $[0, L]$. As discussed in the previous section, these are balanced polytopes, so they are flag with respect to the Σ^2 term order and the maximal facets of $\Delta_{\Sigma^2}([0, L])$ are the unit length intervals, $[k, k+1]$. If P is balanced and dimension 2, then every intersection of a translate of the unit square with P must be a lattice polytope, so P is a convex union of squares and triangles. In particular, all the facets of P are parallel to the lines $y = 0, x = 0, y = x$ or $y = -x$, and conversely any lattice polytope in $\mathbb{R}^2$ with this property is balanced. Since the standard regions of P with respect Σ^2 are unit squares and unit triangles, P is normal, and the ideal I_P is well-behaved.

Proposition 3.1. *Let P be a balanced polytope in $\mathbb{R}^2$, then I_P has a quadratic, square-free Gröbner basis.*

Proof. We define a term order on $\mathbb{C}[X_P]$ as follows, and leave it to the reader to verify that it is quadratic and square-free. First order the elements by degree, then by Σ^2, then by the Lexicographic ordering on the square, where $[1, 1] > [1, 0] > [0, 1] > [0, 0]$. □

The dimension 2 balanced polytopes we will need are pictured below. From now on we use the assumption that the parameters $\vec{r}$ are adapted to the graph in question. In this case, the condition that every trinode $v \in \Gamma$ must have an even sum implies that every edge, except a loop edge, has an even weight. To see this, first consider weightings of the loop, $B(L)$. The trinode in this graph always has two weights which are the same, which forces the third weight to be even. Furthermore, if two weights in a trinode are even, this forces the third weight to be even, an induction argument does the rest for a general tree-like graph. We obtain the following polytopes.

In dimension 3 we will need the balanced polytope $P_3(2L)$ corresponding to the weightings of a trinode. This is the convex hull of $(0, 0, 0), (L, L, 0), (L, 0, L)$, and $(0, L, L)$. For $L > 1$, the standard regions of this polytope with respect to Σ^2 are appeared in [M1], figure 3.

For $L = 1$, this polytope is a non-normal sub polytope of the unit square, this is the reason for the $L > 1$ condition in theorem 1.6. It was shown in [M1] that each of the standard regions of $P_3(2L)$ are normal, with quadratic generating relations,

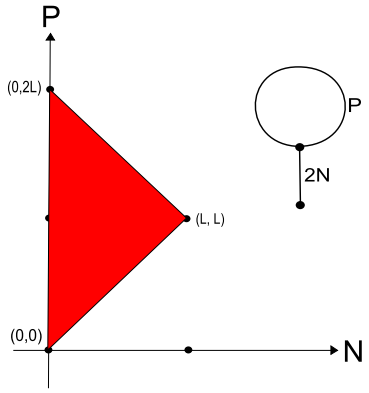

FIGURE 5. The polytope $B(2L)$.

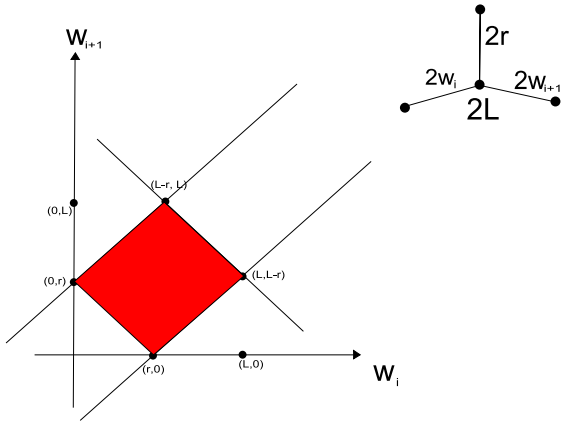

FIGURE 6. The polytope $P_r(2L)$

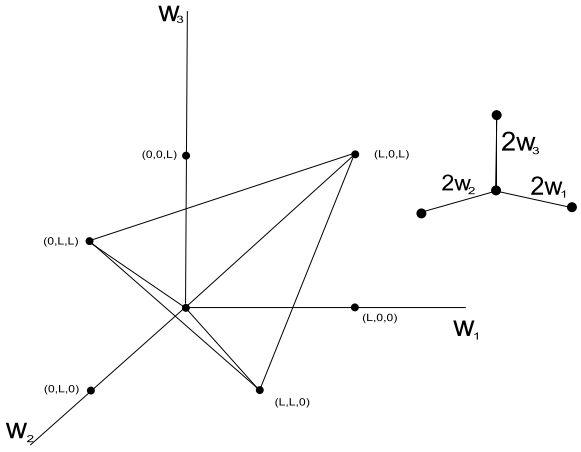

FIGURE 7. The polytope $P_3(2L)$.

except for the region at the origin, the convex hull of $[0,0,0]$, $[1,1,0]$, $[1,0,1]$, $[0,1,1]$, and $[1,1,1]$. This polytope has one cubic relation,

$$[0,1,1][1,0,1][1,1,0] = [0,0,0][1,1,1][1,1,1], \tag{9}$$

which is why theorem 1.6 stipulates that relations are generating by quadrics and cubics instead of just quadrics.

Each of the above polytopes, $P_3(2L)$ $B(2L)$ and $P_r(2L)$ come with distinguished maps to the real line given by projecting onto the weight on a non-loop edge and dividing by 2. The three dimensional polytope $P_3(2L)$ has three such projections, $P_r(2L)$ has two, and $B(2L)$ has one. Each of these maps is given by forgetting coordinates off the edge, so they correspond to a coordinate projection $\mathbb{R}^3 \to \mathbb{R}$, $(a,b,c) \to a$. If a monomial is balanced then it is easy to verify that any map which forgets coordinates gives another balanced monomial. This implies that each of the edge projections above are morphisms in $\mathcal{P}$ to $([0,L],\Sigma^2)$.

In general balanced polytopes $\mathbb{R}^2$ are easy to describe. They are the points (x,y) which satisfy the following inequalities for some choice of 8 numbers, $n_1,\ldots,n_8$.

$$\begin{aligned} x+y &\leq n_1, & x &\leq n_5,\\ x-y &\leq n_2, & y &\leq n_6,\\ -x+y &\leq n_3, & -x &\leq n_7,\\ -x-y &\leq n_4, & -y &\leq n_8, \end{aligned}$$

Generically a balanced polytope in $\mathbb{R}^2$ is an octahedron. Each of these polytopes is normal, and each has a presenting ideal with a quadratic square-free Gröbner basis. They coincide with divisors on the toric variety associated to the fan $\mathcal{F}$ pictured in figure 8.

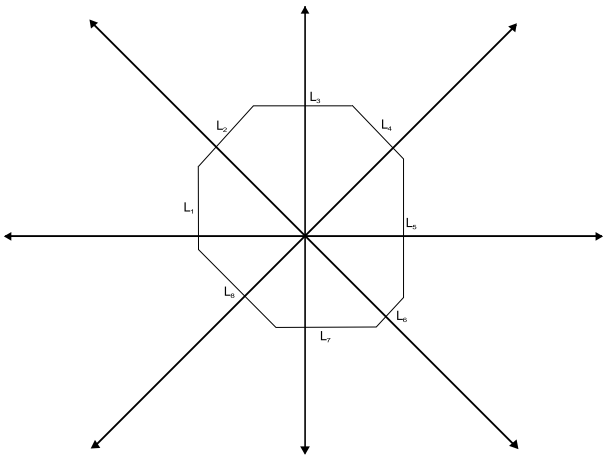

FIGURE 8. Balanced polytope on the fan $\mathcal{F}$

4. Fiber products of polytopes

The previous section establishes that balanced polytopes make up a special subclass of polytopes with a flag term order, which make up a special class of polytopes with term orders. In this section we show that each of these classes is closed under special types of fiber products. The guiding principle here is that fiber products will exist and be well-behaved when the base has unique factorization properties. First we present a useful motivating proposition.

Proposition 4.1. *Let P_1 and P_2 be normal polytopes with maps π_1, π_2 to Q a normal polytope with unique factorization, then $P_1 \times_Q P_2$ is normal.*

Proof. Let $(b_1, b_2) \in N[P_1 \times_Q P_2]$, with $b = \pi_1(b_1) = \pi_2(b_2) \in NQ$. By the normality of P_1 and P_2, we have $b_1 = x^{\mathbf{n}}$ and $b_2 = y^{\mathbf{m}}$ for monomials in the lattice points of P_1 and P_2 respectively. We must also have $\pi_1(x^{\mathbf{n}}) = \pi_2(y^{\mathbf{m}})$, so the the components of these monomials must be the same up to reordering, this means for every component x_i we can find a y_i a component of $y^{\mathbf{m}}$ such that $\pi_1(x_i) = \pi_2(y_i)$. This allows us to express (b_1, b_2) as a product of lattice points of $P_1 \times_Q P_2$. □

For two elements $(P_1, \Sigma_1), (P_2, \Sigma_2) \in \mathcal{P}$ the product polytope $P_1 \times P_2$ has a natural term order given by $\Sigma_1 \oplus \Sigma_2(u, v) = \Sigma_1(u) + \Sigma_2(v)$. This order also makes sense on a fiber product $P_1 \times_Q P_2 \subset P_1 \times P_2$. We now prove that the fiber product of $(P_1, \Sigma_1), (P_2, \Sigma_2) \in \mathcal{P}$ over an element (Q, Σ) with unique standard monomials is a fiber product object in $\mathcal{P}$.

Proof of proposition 1.13. We show that a monomial $x^{[a_1,b_1]\dots[a_k,b_k]}$ is standard with respect to $\Sigma_1 \oplus \Sigma_2$ if and only if $x^{a_1,\dots,a_k}$ and $y^{b_1,\dots,b_k}$ are standard with respect to Σ_1 and Σ_2 respectively. This establishes that the projection maps to P_1 and P_2 are morphisms in $\mathcal{P}$, and that any polytope $(D, \Gamma) \in \mathcal{P}$ with a map to (Q, Σ) which factors by π_1 and π_2 must have a map to the fiber product.

If both $x^{a_1,\dots,a_k}$ and $y^{b_1,\dots,b_k}$ are standard with respect to Σ_1 and Σ_2, then $x^{[a_1,b_1],\dots,[a_k,b_k]}$ must be minimal with respect to $\Sigma_1 \oplus \Sigma_2$, as any alternative factorization induces alternative factorizations of $a_1 + \ldots + a_k$ and $b_1 + \ldots + b_k$. So we must show that $x^{[a_1b_1]\dots[a_kb_k]}$ standard implies that $x^{a_1,\dots a_k}$ and $y^{b_1\dots b_k}$ are standard.

Suppose $x^{[a_1,b_1],\dots,[a_k,b_k]}$ is a monomial in $P_1 \times_Q P_2$ with $x^{a_1\dots a_k}$ not standard, we will show that this implies that $x^{[a_1,b_1],\dots,[a_k,b_k]}$ is not standard. Let $x^{A_1\dots A_k}$ be a standard monomial for $a_1 + \ldots + a_k$, and $y^{B_1\dots B_k}$ the same for $b_1 + \ldots + b_k$. Then $\pi_1(A_1 \dots A_k) = \pi_2(B_1 \dots B_k) = \pi_1(a_1 \dots a_k) = \pi_2(b_1 \dots b_k)$ and it can be arranged that $\pi_1(A_i) = \pi_2(B_i)$ by the unique standard monomial property of Q. This allows us to form the fiber product monomial $x^{[A_1B_1]\dots[A_kB_k]}$, which must be a member of $\Delta_{[a_1b_1]+\dots+[a_kb_k]}$. However, $\Sigma_1 \oplus \Sigma_2([A_1B_1] \dots [A_kB_k] = \Sigma_1(A_1 \dots A_k) + \Sigma_2(B_1 \dots B_k) < \Sigma_1(a_1 \dots a_k) + \Sigma_2(b_1 \dots b_k) = \Sigma_1 \oplus \Sigma_2([a_1b_1] \dots [a_kb_k])$. □

As a corollary we obtain a proof of proposition 1.15.

Proof of proposition 1.15. Let $x^{[a_1,b_1]\dots[a_k,b_k]}$ be a monomial for $P_1 \times_Q P_2$ such that each pair $x^{[a_i,b_i][a_j,b_j]}$ is standard. By the previous proposition this implies that both $a_1, \dots, a_k$ and $b_1, \dots, b_k$ define standard monomials, which implies their image is standard as a Q monomial, and therefore $x^{[a_1,b_1]\dots[a_k,b_k]}$ is standard. □

We also get an expression for the simplicial complex $\Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_Q P_2)$.

Proposition 4.2. *Let (P_1, Σ_1) and (P_2, Σ_2) be flag, then a maximal facet $S \in \Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_Q P_2)$. is of the form $S_1 \times_Q S_2$ for $S_1 \in \Delta_{\Sigma_1}(P_1)$ and $S_2 \in \Delta_{\Sigma_2}(P_2)$.*

Proof. Since any standard support is a subset $S \subset S_1 \times_Q S_2$, it suffices to show that $S_1 \times_Q S_2$ is a facet of $\Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_Q P_2)$. Let $[a_1, b_1] \dots [a_k, b_k]$ be the list of all points in $S_1 \times_Q S_2$. Any monomial $x^{\mathbf{m}}$ with support in this set maps to monomials $\pi_1(\mathbf{m}), \pi_2(\mathbf{m})$ with support in S_1 and S_2 respectively, this means that $\pi_1(\mathbf{m})$ and $\pi_2(\mathbf{m})$ define standard monomials, so $x^{\mathbf{m}}$ must be standard as well. □

When (P_1, Σ_1), (P_2, Σ_2), and (Q, Σ) are flag, a facet $S \in \Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_Q P_2)$. corresponds to a convex region in $P_1 \times_Q P_2$. By proposition 4.2 above, S is the fiber product polytope of its images in P_1 and P_2 over a maximal facet of $\Delta_\Sigma(Q)$, a sub-polytope of Q with unique factorization. When we restrict our attention to balanced polytopes, a fiber product $(P_1, \Sigma^2) \times_{(Q, \Sigma^2)} (P_2, \Sigma^2)$ where the maps are coordinate projections, gives a flag pair $(P_1 \times_Q P_2, \Sigma^2 \oplus \Sigma^2)$. By our discussion, the standard regions of this term order are all fiber products of cubical subpolytopes of P_1 and P_2 over cubical subpolytopes of Q, so they are all cubical. However, the term order $\Sigma^2 \oplus \Sigma^2$ is double Σ^2 over the coordinates from Q.

$$\Sigma^2 \oplus \Sigma^2(V, x, W) = \Sigma^2(V) + 2\Sigma^2(x) + \Sigma^2(W) \tag{10}$$

So this is not quite the Σ^2 term order. An equivalence in the category $\mathcal{P}$ occurs when a polytope P has two term orders with the same standard monomials, we will show that this occurs when we take the fiber product of balanced polytopes.

Proposition 4.3. *Let $P_1 \times_Q P_2$ be a fiber product of balanced polytopes with $\pi_1 : P_1 \to Q$ and $\pi_2 : P_2 \to Q$ induced by coordinate projections. Then the product term order $\Sigma^2 \oplus \Sigma^2$ has the same standard monomials as the term order given by Σ^2 on the fiber product.*

Proof. A monomial $x^{\mathbf{n}}$ is standard with respect to $\Sigma^2 \oplus \Sigma^2$ if and only if it is of the form $x^{(V_1, x_1, W_1) \dots (V_k, x_k, W_k)}$ where $x^{(V_1, x_1) \dots (V_k, x_k)}$ and $y^{(x_1, W_1) \dots (x_k, W_k)}$ are standard for (P_1, Σ^2) and (P_2, Σ^2), and are therefore balanced. This is the case if and only if the tuple $(V_1, x_1, W_1) \dots (V_k, x_k, W_k)$ is balanced. □

In particular, a fiber product of balanced polytopes over maps which are coordinate projections is a balanced polytope. We are interested in fiber products primarily because we can control their generators and relations. We establish this by proving propositions 1.16 and 1.17. These will be used to lift the commutative algebra properties from the examples to $P_\Gamma(\vec{r}, L)$.

Proof of proposition 1.16. The polytope $(P_1 \times_Q P_2, \Sigma_1 \oplus \Sigma_2)$ is flag by proposition 1.15, so it is tiled by the convex hulls of the maximal facets of $\Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_1 P_2)$. By proposition 4.2 these are all fiber products of normal polytopes over polytopes with unique factorization, and are therefore normal. □

Proof of proposition 1.17. If $x^{(a_1, b_1) \dots (a_k, b_k)} - x^{(A_1, B_1) \dots (A_k, B_k)}$ is in the ideal for the fiber product then it can be converted to a relation in some maximal facet of $\Delta_{\Sigma_1 \oplus \Sigma_2}(P_1 \times_Q P_2)$ by degree 2 relations, so we may assume without loss of generality that $x^{a_1 \dots a_k} = x^{A_1 \dots A_k}$ and $y^{b_1 \dots b_k} = y^{B_1 \dots B_k}$ are relations among standard monomials in P_1 and P_2 respectively. Any relation among standard monomials $x^{a_1 \dots a_k} - x^{a'_1 \dots a'_k}$ can be lifted in some way to a relation $x^{(a_1, b_1) \dots (a_k, b_k)} - x^{(a'_1, b_{i_1}) \dots (a'_k, b_{i_k})}$

in $P_1 \times_Q P_2$ because Q has unique factorization. This way, $(a_1, b_1) \dots (a_k, b_k)$ can be converted to $(A_1, B_1) \dots (A_k, B_k)$ by relations among standard monomials of P_1 and P_2. □

This proves theorem 1.6.

5. Fiber products of quadratic square-free Gröbner bases

In this section we show that the concatenation product of term orders which have quadtratic square-free Gröbner bases has a quadratic square-free Gröbner basis. This should be compared with the toric fiber product results of Sullivant [S]. The construction here is essentially the same, except we focus on toric ideals, which allows us to take more general polytopes as the base of the product.

The class of polytopes with flag term orders has a distinguished subclass given by those total term orders with a quadratic square-free Gröbner basis. The "quadratic" part of this distinction is equivalent the condition that any monomial with standard degree 2 divisors must itself be standard, and "square-free" implies that the powers of any lattice point must be standard. The standard regions S_i of these polytopes are unit simplices. We now know that fiber products in this subcategory yield normal polytopes with a flag term order, however there is not enough information in the fiber product order $\Sigma_1 \oplus \Sigma_2$ to yield a total term order, and a Gröbner basis. The issue is that for standard monomials defined by $\vec{v}, \vec{x}, \vec{w}$ in P_1, Q and P_2 respectively, there could be many ways to form a fiber product monomial,

$$(v_1, x_1, w_1) \dots (v_k, x_k, w_k). \tag{11}$$

If two elements w_i and $w_j \in P_1$ map to the same $x_j \in Q$, then we could plausibly permute these entries to obtain distinct, new standard monomials with respect to $\Sigma_1 \oplus \Sigma_2$. The standard monomials in the fiber product are exactly those obtained by fixing $v_1, \dots, v_n$ in some order and permuting the $w_1, \dots, w_k$ while respecting the shared $x_1, \dots, x_k$, so we must establish an order on these sets. Each standard region in $P_1 \times_P P_2$ is a fiber product of simplices with well-ordered lattice points over a simplex with well-ordered lattice points, so to prove proposition 1.19, we can reduce to the following proposition.

Proposition 5.1. *Let P_1, P and P_2 be unit simplices each with a fixed well-ordering $\Sigma_1, \Sigma, \Sigma_2$ on their lattice points. Then $P_1 \times_P P_2$ has a quadratic square-free Gröbner basis defined by the orderings Σ_1, Σ, and Σ_2*

Proof. This follows from a modification of Corollary 2.11 in [Su]. To define the new term-order, we use the $\boxtimes$ term order. We have $(V_1, W_1) < (V_2, W_2)$ if $V_1 < V_2$ or $V_1 = V_2$ and $W_1 < W_2$, and

$$[(V_1^1, W_1^1) \cdots (V_k^1, W_k^1)] > [(V_1^2, W_1^2) \cdots (V_k^2, W_k^2)] \tag{12}$$

if $(V_i^1, W_i^1) = (V_i^2, W_i^2)$ for all $i < \ell$ for some ℓ and $V_\ell^1 > V_\ell^2$ or $V_\ell^1 = V_\ell^2$ with $W_\ell^1 > W_\ell^2$, where we've listed the terms of the monomials in decreasing order. Assume now that both monomials map to the same lattice point, then the sets $\{V_i^1\}$ and $\{V_i^2\}$ are the same, as are $\{W_i^1\}$ and $\{W_i^2\}$, so we may assume without loss of generality that $V_i^1 = V_i^2$. In this case, if $W_\ell^1 > W_\ell^2$ then $W_m^1 = W_\ell^2$ for some $m > \ell$, so for the first monomial we can form the relation

$$(V_\ell, W_\ell^1)(V_m, W_m^1) = (V_\ell, W_m^1)(V_m, W_\ell^1) \tag{13}$$

Performing this exchange must yield a lower monomial. This implies that an arbitrary monomial can be taken to the associated standard monomial with degree 2 weight-lowering relations. Now consider a monomial of the form $x^{(V,W),\dots,(V,W)}$, and let $x^{(V_1,W_1),\dots,(V_k,W_k)}$ be another standard monomial with the same $\Sigma_1 \oplus \Sigma_2$ weight. This implies that $(V_1, \dots, V_k) = (V, \dots, V)$ and $(W_1, \dots, W_k) = (W, \dots, W)$, since both Σ_1 and Σ_2 define total term orders. □

Note that this product is not symmetric, however it is associative. The previous proposition shows that the subcategory of (P, Σ) with quadratic, square-free Gröbner bases is closed under $\boxtimes$. From this result we can deduce a corollary for dimension 2 balanced polytopes.

Theorem 5.2. *Let P be a fiber product of a finite number of dimension 2 balanced polytopes over balanced polytopes of dimension 2 or 1. Then I_P has a Quadratic Square-free Gröbner Basis.*

This proves theorem 1.7. We also remark that the argument above can be adapted to show that if (P_1, Σ_1) and (P_2, Σ_2) are flag, then the so is $P_1 \times_Q P_2$ with respect to the $\boxtimes$ term order.

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