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An extension of Stone Duality to fuzzy topologies and MV-algebras

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Abstract

In this paper we introduce the concept of *MV-topology*, a special class of fuzzy topological spaces, and prove a proper extension of Stone Duality to the categories of *limit cut complete MV-algebras* and *Stone MV-spaces*, the latter having MV-topologies whose definition is a natural adaptation of the one of Stone spaces to our framework as objects. Then we describe the object class of limit cut complete MV-algebras, and show that any semisimple MV-algebra has a limit cut completion, namely, a minimum limit cut complete extension.

1 Introduction

In 1965, Zadeh [26] introduced the concept of *fuzzy subset* of a given set X by means of *membership* (or *characteristic*) *functions* defined on X and taking values in the real unit interval $[0, 1]$, the value of such functions at a given point x representing the *degree of membership* of x to the fuzzy subset.

Such a definition was focused only on the objects and left unanswered many questions about the proper logical, algebraic, and categorical settings in which a theory of such objects could be developed. Indeed, while classical set theory finds its natural algebraic framework in the theory of Boolean algebras and its proper logical setting in classical logic, the family of fuzzy subsets of a given set may have many different algebraic structures which, on their turn, correspond to as many logics.

After Zadeh's paper, fuzzy sets and fuzzy logic have been deeply studied both from a strictly mathematical and foundational viewpoint (mainly in connection with many-valued logics, whose introduction came far before the one of fuzzy sets) and as a tool for applications to many areas, especially of Computer Science.

As a matter of fact, all the (propositional) many-valued logics which are standard complete with respect to some algebraic structure defined on $[0, 1]$ are worthy candidates for being a proper logical setting for fuzzy sets. Nonetheless, if we look at the

crisp and fuzzy powersets of a given set X as, respectively, $\{0,1\}^X$ and $[0,1]^X$, it is undoubtable that, among the various fuzzy logics and corresponding algebraic semantics, Łukasiewicz logic and MV-algebras are the ones that best succeed in both having a rich expressive power and preserving many properties of symmetry that are inborn qualities of Boolean algebras. This assertion can be better explained by the following observations.

In any Boolean algebra $\langle B, \vee, \wedge, ^*, 0, 1\rangle$, the conjunction $\wedge$ is a residuated operation, thus it defines a residuum $\rightarrow$ satisfying, among others, the equations $x \rightarrow 0 = x^*$ and $x \rightarrow y = x^* \vee y$. In MV-algebras this property is mirrored by strong conjunction $\odot$, which is residuated and whose residuum $\rightarrow$ verifies similar equations: $x \rightarrow 0 = x^*$ and $x \rightarrow y = x^* \oplus y$. In fact, the equation $x \rightarrow 0 = x^*$, which is meant to express the logical equivalence between "NOT φ" and "φ IMPLIES FALSE", which holds in classical logic, is still verified in Łukasiewicz logic.

In [12, page 28], Hájek states:

> Note that the dual notion of a t-conorm [...] will not play any important role in this book. This is because conjunction and disjunction do not have any dual relation to the implication.

As a matter of fact, this is true for most of the best-known fuzzy logics but not for Łukasiewicz one. Indeed, in Łukasiewicz logic, (strong) conjunction and disjunction have, as standard algebraic semantics, a t-norm and its dual t-conorm — such a duality involving negation and implication too.

Last, De Morgan's laws for Boolean algebras do hold in MV-algebras, for weak and strong conjunction and disjunction:

$$(x \vee y)^* = x^* \wedge y^* \qquad (x \oplus y)^* = x^* \odot y^*$$
$$(x \wedge y)^* = x^* \vee y* \qquad (x \odot y)^* = x^* \oplus y^*.$$

The introduction of several concepts of "fuzzy topology" came a few years after Zadeh's paper, and their study has been pursued for many years (see, for instance, [7, 14–18, 21, 22, 24, 25]).

In this paper we propose a concept of fuzzy topological space which, although rather "naïve" from a foundational perspective, is a natural generalization of the classical (general) topology. Moreover, even if the present paper is more of an algebraic and categorical nature, a strong motivation for the introduction of such fuzzy topologies comes from the area of Mathematical Morphology [13, 19] and its connection to quantales and idempotent semirings discussed in [23]. Indeed, as binary digital images are examples of crisp subsets of a given set, greyscale images are a prototypical example of fuzzy subsets. So, since mathematical morphological operators, for binary images, are designed with the aim of individualizing relevant topological properties of the images, it is clear that classical topology can hardly give satisfactory information in the case of greyscale images.

Our aim is to use MV-algebras as a framework for fuzzy topology which, on the one hand, is sufficiently rich and complex and, on the other hand, reflects (up to a suitable reformulation) as many properties of classical topology as possible. For this reason we introduce the concept of *MV-topology*, a generalization of general topology whose main features can be summarized as follows.

- The Boolean algebra of subsets of the universe is replaced by the MV-algebra of ($[0,1]$-valued) fuzzy subsets.
- Classical topological spaces are examples of MV-topological spaces.
- The algebraic structure of the family of open (fuzzy) subsets has a quantale reduct $\langle \Omega, \bigvee, \oplus \rangle$, which replaces the classical sup-lattice $\langle \Omega, \bigvee \rangle$, and an idempotent semiring one $\langle \Omega, \wedge, \odot, \mathbf{1} \rangle$ in place of the meet-semilattice $\langle \Omega, \wedge, \mathbf{1} \rangle$. Moreover, the lattice reduct $\langle \Omega, \bigvee, \wedge \rangle$ maintains the property of being a frame.
- The MV-algebraic negation * is, in the aforementioned classes of algebras, an isomorphism between the various structures of open subsets and the corresponding ones of closed subsets.
- A classical topology is canonically associated to each MV-topology. It is called the *skeleton topology* and is obtained simply by restricting the family of open subsets to the crisp ones.

The main results of the paper are proved in Sections 5 and 6. In particular, we show an extension of Stone Duality between Boolean algebras and Stone spaces to, respectively, the category of *limit cut complete MV-algebras*, namely, the full subcategory of $\mathcal{MV}$ whose objects are algebras which contain the suprema of certain cuts, and a suitable category of MV-topologies, whose objects are the natural MV-version of Stone (or Boolean) spaces. Such an extension is "proper" in the sense that its restriction to, respectively, Boolean algebras and Stone spaces — which are full subcategories of the ones involved in the duality — yields the classical well-known duality, up to a trivial reformulation in terms of maximal ideals instead of ultrafilters.

2 MV-algebras

In the present section we recall the main definitions and properties on MV-algebras that shall be used in the paper. A reader who is familiar with the basic theory of MV-algebras may skip the present section without compromising the comprehension of the subsequent ones.

Definition 2.1. An *MV-algebra* is an algebra $\langle A, \oplus, ^*, 0 \rangle$ of type $(2,1,0)$ that satisfies the following equations

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus y = y \oplus x$;

(A3) $x \oplus 0 = x$;

(A4) $(x^*)^* = x$;

(A5) $x \oplus 0^* = 0^*$;

(A6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

On every MV-algebra it is possible to define another constant 1 and two further operation as follows:

$1 = 0^*$,

$x \odot y = (x^* \oplus y^*)^*$,

$x \ominus y = x \odot y^*$.

The following properties follow immediately from the definitions

(A7) $1^* = 0$,

(A8) $x \oplus y = (x^* \odot y^*)^*$,

(A5') $x \oplus 1 = 1$ (reformulation of (*A*5)),

(A6') $(x \ominus y) \oplus y = (y \ominus x) \oplus x$ (reformulation of (*A*6)),

(A9) $x \oplus x^* = 1$.

It is well-known that MV-algebras are naturally equipped with an order relation defined as follows

$$x \leq y \quad \text{if and only if} \quad x^* \oplus y = 1. \tag{1}$$

Moreover it is easy to verify that $x^* \oplus y = 1$ is indeed equivalent to each of the following conditions

$x \odot y^* = 0$;

$y = x \oplus (y \ominus x)$;

there exists an element $z \in A$ such that $x \oplus z = y$.

The order relation also determines a structure of bounded distributive lattice on A, with 0 and 1 respectively bottom and top element, and $\vee$ and $\wedge$ defined as follows

$$x \vee y = (x \odot y^*) \oplus y = (x \ominus y) \oplus y,$$
$$x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y).$$

It is worth noticing that $\oplus$, $\odot$, and $\wedge$ distribute over any existing join and, analogously, $\oplus$, $\odot$ and $\vee$ distribute over any existing meet, in any MV-algebra A. In other words, for any $Y \subseteq A$ for which there exists $\bigvee Y$, for any $Z \subseteq A$ for which there exists $\bigwedge Z$, and for any $x \in A$,

$x \bullet \bigvee Y = \bigvee_{y \in Y} (a \bullet y)$, for $\bullet \in \{\oplus, \odot, \wedge\}$,

$x \bullet \bigwedge Z = \bigwedge_{z \in Z} (x \bullet z)$, for $\bullet \in \{\oplus, \odot, \vee\}$.

In the rest of the paper, for any MV-algebra A, for all $a \in A$ and for all $n < \omega$, we shall use the following abbreviations: $a^n := \underbrace{a \odot \cdots \odot a}_{n \text{ times}}$ and $na := \underbrace{a \oplus \cdots \oplus a}_{n \text{ times}}$.

The best-known example of MV-algebra is the real unit interval $[0, 1]$, with the sum $x \oplus y := \min\{x + y, 1\}$ and the involution $x^* := 1 - x$. The product is, then, defined by $x \odot y := \max\{x + y - 1, 0\}$, and the lattice structure resulting is the natural totally ordered one. It is well-known that the MV-algebra $[0, 1]$ generates, both as a variety and as a quasi-variety, the whole variety of MV-algebras and that Łukasiewicz propositional calculus is standard complete w.r.t. to such a structure. We also recall that, for any non-empty set X, $[0, 1]^X$ with pointwise defined operations is an MV-algebra as well, and that Boolean algebras are precisely the MV-algebras satisfying the equation $x \oplus x = x$, hence they form a subvariety of the variety of MV-algebras.

By an *idempotent semiring* we mean a structure $\langle S, \vee, \cdot, 0\rangle$ such that $\langle S, \vee, 0\rangle$ is a semilattice with zero, $\langle S, \cdot\rangle$ is a semigroup and $\cdot$ distributes over $\vee$ from either side; S is called *commutative* if so is $\cdot$.

A *quantale* is a structure $\langle Q, \bigvee, \cdot\rangle$ with a sup-lattice reduct $\langle Q, \bigvee\rangle$ and a semigroup reduct $\langle Q, \cdot\rangle$ such that $\cdot$ distributes over $\bigvee$ or, that is the same, is residuated in both arguments.

For any MV-algebra A, the structures $\langle A, \vee, \oplus, 0\rangle$ and $\langle A, \wedge, \odot, 1\rangle$ are commutative idempotent semirings, and * is an isomorphism between them. If A is complete, then $\langle A, \bigvee, \oplus\rangle$ and $\langle A, \bigwedge, \odot\rangle$ are isomorphic quantales.

We need also to recall some facts about ideals of MV-algebras. A subset I of an MV-algebra A is called an *ideal* if it is a downward closed submonoid of $\langle A, \oplus, 0\rangle$, i.e. if it satisfies the following properties:

- $0 \in I$;
- I is downward closed, that is, for all $a \in I$ and $b \in A$, $b \leq a$ implies $b \in I$;
- $a \oplus b \in I$ for all $a, b \in I$.

It is self-evident that $\{0\}$ and A are ideals; an ideal I is called *proper* if $I \neq A$ or, that is the same, if $1 \notin I$. The set $\mathrm{Id}(A)$ of all ideals of an MV-algebra A is obviously partially ordered by set-inclusion and closed under arbitrary intersections. For any subset S of A, the ideal generated by S, denoted by $(S]$, is characterized by the following well-known result.

Proposition 2.2. *[8] For any non-empty subset S of A,*

$$(S] = \{a \in A \mid a \leq x_1 \oplus \cdots \oplus x_n, \text{ for some } n < \omega \text{ and } x_1, \ldots, x_n \in S\}.$$

An immediate consequence of Proposition 2.2 is the following.

Corollary 2.3. *Let A be an MV-algebra and S a subset of A. Then the ideal $(S]$ generated by S is proper if and only if, for any $n < \omega$ and for any $a_1, \ldots, a_n \in S$, $a_1 \oplus \cdots \oplus a_n < 1$.*

It is also known that any non-trivial MV-algebra has maximal ideals; more precisely, any proper ideal of an MV-algebra is contained in a maximal one. The set of

all maximal ideals of A is called the *maximal spectrum* of A and denoted by $\mathrm{Max}\, A$, the ideal obtained as the intersection of all maximal ideals of A is called the *radical* of A: $\mathrm{Rad}\, A := \bigcap \mathrm{Max}\, A$. *Semisimple* algebras, as usual, are defined as the subdirect products of simple algebras. However, in the theory of MV-algebras, they can be characterized as (non-trivial) algebras whose radical is $\{0\}$, and such a characterization is most often used as a definition. It is worth noticing that Boolean algebras are all semisimple MV-algebras. The following well-known result will be useful in next sections.

Proposition 2.4. *[8] If M is a proper ideal of A then the following are equivalent:*

(a) M is maximal;

(b) for any $a \in A$, if $a \notin M$ then there exists $n < \omega$ such that $(a^*)^n \in M$.

Ideals and congruences of an MV-algebra A are in one-one correspondence. Indeed, for any congruence θ, $0/\theta$ is an ideal and, conversely, for any ideal I, the relation θ_I defined by "$a\theta_I b$ iff $d(a,b) := (a \odot b^*) \oplus (b \odot a^*) \in I$" is a congruence on A — it is, in fact, the only one for which the class of zero is equal to I. Therefore, in MV-algebras, the congruence whose corresponding ideal is I is often denoted by I itself, and the congruence classes and the quotient algebra are denoted, respectively, by a/I (for all $a \in A$) and A/I. The following result is a well-known characterization of congruence classes for MV-algebras.

Lemma 2.5. *Let A be an MV-algebra and $I \in \mathrm{Id}(A)$. For all $a \in A$, $a/I = \{(a \oplus b) \odot c^* \mid b, c \in I\}$.*

Such a correspondence implies also that *simple* MV-algebras, i.e. those algebras whose congruence lattice is the two-element chain, have no non-trivial ideals, that is $\mathrm{Id}(A) = \{\{0\}, A\}$. It may be worth recalling that ideals and filters are, in MV-algebras as well as in Boolean algebras, in one-one correspondence to each other. More precisely, a subset I of an MV-algebra (or a Boolean algebra) A is an ideal if and only if $I^* = \{a^* \mid a \in I\}$ is a filter. Such a correspondece is an order isomorphism between the sets of ideals and filters, ordered by set inclusion, and therefore I is a maximal ideal if and only if I^* is an ultrafilter. In this paper, following the tradition of MV-algebras, we shall deal mainly with (maximal) ideals, but all the results and constructions presented can be suitably reformulated in terms of (ultra)filters.

Besides the fact that Boolean algebras are MV-algebras, it must be mentioned that, in the theory of MV-algebras, the so-called Boolean elements of each algebra play an important role; such a role is even more important to the extent of this work, as we shall see in Section 5.

An element a of an MV-algebra A is called *idempotent* or *Boolean* if $a \oplus a = a$. Boolean elements of MV-algebras enjoy the following properties.

- For any $a \in A$, $a \oplus a = a$ iff $a \odot a = a$.
- An element a is Boolean iff a^* is Boolean.
- If a and b are idempotent, then $a \oplus b$ and $a \odot b$ are idempotent as well; moreover we have $a \oplus b = a \vee b$, $a \odot b = a \wedge b$, $a \vee a^* = 1$ and $a \wedge a^* = 0$.

- The set $\mathrm{B}(A) = \{a \in A \mid a \oplus a = a\}$ is a Boolean algebra, usually called the *Boolean center* of A. In fact, $\mathrm{B}(A)$ is the largest MV-subalgebra of A which is Boolean.
- For any $a \in A$ and $b \in \mathrm{B}(A)$, $a \oplus b = a \vee b$ and $a \odot b = a \wedge b$.

We conclude this section by recalling the definition of hyperarchimedean MV-algebras; the reader may refer to [4, 8] for further information.

Definition 2.6. Let A be an MV-algebra. An element $a \in A$ is *archimedean* if it satisfies the following equivalent conditions:

(a) there exists a positive integer n such that $na \in \mathrm{B}(A)$;

(b) there exists a positive integer n such that $a^* \vee na = 1$;

(c) there exists a positive integer n such that $na = (n+1)a$.

A is called *hyperarchimedean* if all of its elements are archimedean.

It is well-known that any hyperarchimedean MV-algebra is semisimple while the converse is not true.

3 MV-topologies

Throughout the paper, both crisp and fuzzy subsets of a given set will be identified with their membership functions and usually denoted by lower case latin or greek letters. In particular, for any set X, we shall use also $\mathbf{1}$ and $\mathbf{0}$ for denoting, respectively, X and $\emptyset$. In some cases, we shall use capital letters in order to emphasize that the subset we are dealing with is crisp.

We remark that an MV-topological space is basically a special fuzzy topological space in the sense of C. L. Chang [7]. Moreover, most of the definitions and results of the present section and of Section 4 are simple adaptations of the corresponding ones of the aforementioned work to the present context or directly derivable from the same work or from the results presented in the papers [14–18, 21, 22, 24, 25] that we already cited in the introduction.

Definition 3.1. Let X be a set, A the MV-algebra $[0,1]^X$ and $\Omega \subseteq A$. We say that $\langle X, \Omega\rangle$ is an *MV-topological space* if Ω is a subuniverse both of the quantale $\langle [0,1]^X, \bigvee, \oplus\rangle$ and of the semiring $\langle [0,1]^X, \wedge, \odot, \mathbf{1}\rangle$. More explicitly, $\langle X, \Omega\rangle$ is an MV-topological space if

(i) $\mathbf{0}, \mathbf{1} \in \Omega$,

(ii) for any family $\{o_i\}_{i\in I}$ of elements of Ω, $\bigvee_{i\in I} o_i \in \Omega$,

and, for all $o_1, o_2 \in \Omega$,

(iii) $o_1 \odot o_2 \in \Omega$,

(iv) $o_1 \oplus o_2 \in \Omega$,

(v) $o_1 \wedge o_2 \in \Omega$.

Ω is also called an *MV-topology* on X and the elements of Ω are the *open MV-subsets* of X. The set $\Xi = \{o^* \mid o \in \Omega\}$ is easily seen to be a subquantale of $\langle [0,1]^X, \bigwedge, \odot\rangle$ and a subsemiring of $\langle [0,1]^X, \vee, \oplus, \mathbf{0}\rangle$, i.e., it verifies the following properties:

- $\mathbf{0}, \mathbf{1} \in \Xi$,
- for any family $\{c_i\}_{i\in I}$ of elements of Ξ, $\bigwedge_{i\in I} c_i \in \Xi$,
- for all $c_1, c_2 \in \Xi$, $c_1 \odot c_2, c_1 \oplus c_2, c_1 \vee c_2 \in \Xi$.

The elements of Ξ are called the *closed MV-subsets* of X.

Proposition 3.2. *Let $\langle X, \Omega\rangle$ be an MV-topological space. For any subset Y of X, the pair $\langle Y, \Omega_Y\rangle$, where $\Omega_Y := \{o_{\restriction Y} \mid o \in \Omega\}$, is an MV-topology on Y.*

Proof. Trivial. □

Definition 3.3. For any subset Y of X, the pair $\langle Y, \Omega_Y\rangle$ is called an *MV-subspace* of $\langle X, \Omega\rangle$.

Example 3.4. (a) $\langle X, \{\mathbf{0}, \mathbf{1}\}\rangle$ and $\langle X, [0,1]^X\rangle$ are MV-topological spaces.

(b) Any topology is an MV-topology.

(c) Let $d : X \longrightarrow [0, +\infty[$ be a distance function on X. For any fuzzy point α of X, with support x, and any positive real number r, we define the *open ball* of center α and radius r as the fuzzy set $\beta_r(\alpha)$ identified by the membership function $\beta_r(\alpha)(y) = \begin{cases} \alpha(x) & \text{if } d(x,y) < r \\ 0 & \text{if } d(x,y) \geq r \end{cases}$. Analogously, the *closed ball* $\beta_r[\alpha]$ of center α and radius r has membership function $\beta_r[\alpha](y) = \begin{cases} \alpha(x) & \text{if } d(x,y) \leq r \\ 0 & \text{if } d(x,y) > r \end{cases}$. It is immediate to verify that the fuzzy subsets of X that are join of a family of open balls is an MV-topology on X that is said to be *induced* by d. This example can be found also in [17].

Definition 3.5. If $\langle X, \Omega\rangle$ is an MV-topology, then $\langle X, \mathrm{B}(\Omega)\rangle$ — where $\mathrm{B}(\Omega) := \Omega \cap \{0,1\}^X = \Omega \cap \mathrm{B}([0,1]^X)$ — is both an MV-topology and a topology in the classical sense. The topological space $\langle X, \mathrm{B}(\Omega)\rangle$ will be called the *skeleton space* of $\langle X, \Omega\rangle$.

Let X and Y be sets. Any function $f : X \longrightarrow Y$ naturally defines a map

$$\begin{array}{rccc} f^{\leftsquigarrow} : & [0,1]^Y & \longrightarrow & [0,1]^X \\ & \alpha & \longmapsto & \alpha \circ f. \end{array} \tag{2}$$

Obviously $f^{\leftsquigarrow}(\mathbf{0}) = \mathbf{0}$; moreover, if $\alpha, \beta \in [0,1]^Y$, for all $x \in X$ we have $f^{\leftsquigarrow}(\alpha \oplus \beta)(x) = (\alpha \oplus \beta)(f(x)) = \alpha(f(x)) \oplus \beta(f(x)) = f^{\leftsquigarrow}(\alpha)(x) \oplus f^{\leftsquigarrow}(\beta)(x)$ and, analogously, $f^{\leftsquigarrow}(\alpha^*) = f^{\leftsquigarrow}(\alpha)^*$. Then $f^{\leftsquigarrow}$ is an MV-algebra homomorphism and we shall call it the *MV-preimage* of f. The reason of such a name is essentially the fact that $f^{\leftsquigarrow}$ can be seen as the preimage, via f, of the fuzzy subsets of Y. From a categorical viewpoint,

once denoted by $\mathcal{S}$et, $\mathcal{B}$oole and $\mathcal{MV}$ the categories of sets, Boolean algebras, and MV-algebras respectively (with the obvious morphisms), there exist two contravariant functors $\mathscr{P} : \mathcal{S}\text{et} \longrightarrow \mathcal{B}\text{oole}^{\text{op}}$ and $\mathscr{F} : \mathcal{S}\text{et} \longrightarrow \mathcal{MV}^{\text{op}}$ sending each map $f : X \longrightarrow Y$, respectively, to the Boolean algebra homomorphism $f^{\leftarrow} : \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ and to the MV-homomorphism $f^{\leftsquigarrow} : [0,1]^Y \longrightarrow [0,1]^X$.

Moreover, for any map $f : X \longrightarrow Y$ we define also a map $f^{\rightarrow} : [0,1]^X \longrightarrow [0,1]^Y$ by setting, for all $\alpha \in [0,1]^X$ and for all $y \in Y$,

$$f^{\rightarrow}(\alpha)(y) = \bigvee_{f(x)=y} \alpha(x). \tag{3}$$

Clearly, if $y \notin f[X]$, $f^{\rightarrow}(\alpha)(y) = \bigvee \varnothing = \mathbf{0}$ for any $\alpha \in [0,1]^X$.

We can now extend the following basic definitions to MV-topologies.

Definition 3.6. Let $\langle X, \Omega_X \rangle$ and $\langle Y, \Omega_Y \rangle$ be two MV-topological spaces. A map $f : X \longrightarrow Y$ is said to be

- *continuous* if $f^{\leftsquigarrow}[\Omega_Y] \subseteq \Omega_X$,
- *open* if $f^{\rightarrow}(o) \in \Omega_Y$ for all $o \in \Omega_X$,
- *closed* if $f^{\rightarrow}(c) \in \Xi_Y$ for all $c \in \Xi_X$
- an *MV-homeomorphism* if it is bijective and both f and f^{-1} are continuous.

We can use the same words of the classical case because, as it is trivial to verify, if a map between two classical topological spaces is continuous, open, or closed in the sense of the definition above, then it has the same property in the classical sense.

Continuity, as in Definition 3.6, is equivalent to $f^{\leftsquigarrow}[\Xi_Y] \subseteq \Xi_X$. Indeed, since $f^{\leftsquigarrow} : [0,1]^Y \longrightarrow [0,1]^X$ is an MV-algebra homomorphism, it preserves *; therefore, for any closed set c of Y, c^* is an open set, hence $f^{\leftsquigarrow}(c^*) = f^{\leftsquigarrow}(c)^* \in \Omega_X$ implies $f^{\leftsquigarrow}(c) \in \Xi_X$. In a completely analogous way, it can be proved that $f^{\leftsquigarrow}[\Xi_Y] \subseteq \Xi_X$ implies continuity in the sense of the previous definition.

Moreover, it is absolutely obvious that, if $\langle X, \Omega_X \rangle$ and $\langle Y, \Omega_Y \rangle$ are two MV-spaces, and $f : X \longrightarrow Y$ is a continuous function between them, then f is also a continuous map between the two skeleton spaces $\langle X, \mathrm{B}(\Omega_X) \rangle$ and $\langle Y, \mathrm{B}(\Omega_Y) \rangle$.

4 Bases, compactness and separation axioms

In the present section we give the necessary definitions and show some preliminary results in order to prove the extension of Stone Duality. As the reader will notice, the concepts we are going to introduce are direct and natural (and quite obvious, indeed) generalizations of the corresponding ones in classical topology. In order to build a comprehensive theory of MV-topologies, many further material needs to be defined and investigated; nonetheless, as we already mentioned, here we focus our attention to Stone Duality. Therefore we do not intend to introduce notions that shall not be of any utility in this particular paper, leaving such further insights for future works.

Definition 4.1. As in classical topology, we say that, given an MV-topological space $\mathbf{T} = \langle X, \Omega \rangle$, a subset Δ of $[0,1]^X$ is called a *base* for $\mathbf{T}$ if $\Delta \subseteq \Omega$ and every open set of $\mathbf{T}$ is a join of elements of Δ.

Lemma 4.2. *Let $\mathbf{T} = \langle X, \Omega_X \rangle$ and $\mathbf{T}' = \langle Y, \Omega_Y \rangle$ be two MV-topological spaces and let Δ be a base for $\mathbf{T}'$. A map $f : X \longrightarrow Y$ is continuous if and only if $f^{\leftsquigarrow}[\Delta] \subseteq \Omega_X$.*

Proof. One implication is trivial, since Δ is a family of open sets. Conversely, assuming that $f^{\leftsquigarrow}[\Delta] \subseteq \Omega_X$, let $o = \bigvee \Gamma$, with $\Gamma \subseteq \Delta$, be any open set of $\mathbf{T}'$. As we observed, $f^{\leftsquigarrow}$ is an MV-algebra homomorphism, hence $f^{\leftsquigarrow}(o) = f^{\leftsquigarrow}(\bigvee \Gamma) = \bigvee f^{\leftsquigarrow}[\Gamma]$, i.e. $f^{\leftsquigarrow}(o)$ is the join of open sets of $\mathbf{T}$ and, therefore, open itself. □

A *covering* of X is any subset Γ of $[0,1]^X$ such that $\bigvee \Gamma = \mathbf{1}$, while an *additive covering* ($\oplus$-covering, for short) is a finite family $\{\alpha_i\}_{i=1}^n$ of elements of $[0,1]^X$, $n < \omega$, such that $\alpha_1 \oplus \cdots \oplus \alpha_n = \mathbf{1}$. It is worthwhile remarking that we used the expression "finite family" in order to include the possibility for such a family to have repetitions. In other words, an additive covering is a finite subset $\{\alpha_1, \ldots, \alpha_k\}$ of $[0,1]^X$, along with natural numbers $n_1, \ldots, n_k$, such that $n_1\alpha_1 \oplus \cdots \oplus n_k\alpha_k = \mathbf{1}$.

Proposition 4.3. *For any set X, any covering of fuzzy subsets of X which is closed under $\oplus$, $\odot$, and $\wedge$ is a base for an MV-topology on X.*

Proof. Let $\Gamma \subseteq [0,1]^X$ be a covering closed under $\oplus$, $\odot$, and $\wedge$, and let $\Omega = \{\bigvee G \mid G \subseteq \Gamma\}$. We have $\mathbf{1} \in \Omega$, by definition of covering, and $\mathbf{0} = \bigvee \varnothing \in \Omega$.

On the other hand, Ω is trivially closed under arbitray joins and $\odot$, $\oplus$, and $\wedge$ distribute over any existing join. Then, given $o_1, o_2 \in \Omega$, $o_1 = \bigvee_{i \in I} \alpha_i$ and $o_2 = \bigvee_{j \in J} \beta_j$, with $\{\alpha_i\}_{i \in I}, \{\beta_j\}_{j \in J} \subseteq \Gamma$, whence

$$o_1 \bullet o_2 = \left(\bigvee_{i \in I} \alpha_i \right) \bullet \left(\bigvee_{j \in J} \beta_j \right) = \bigvee_{i \in I} \left(\alpha_i \bullet \bigvee_{j \in J} \beta_j \right) = \bigvee_{i \in I} \bigvee_{j \in J} (\alpha_i \bullet \beta_j),$$

for $\bullet \in \{\oplus, \odot, \wedge\}$. So Ω verifies Definition 3.1, i.e. it is an MV-topology, and Γ is a base for it. □

The presence of strong and weak conjunctions and disjunction, in the structure of open sets of an MV-topology, naturally suggests different fuzzy versions (weaker or stronger) of most of the classical topological concepts (separation axioms, compactness etc.). However, we shall limit our attention to the ones that serve the scope of this paper, namely *compactness* and *Hausdorff (or T_2) separation axiom*.

Definition 4.4. An MV-topological space $\langle X, \Omega \rangle$ is said to be *compact* if any open covering of X contains an additive covering; it is called *strongly compact* if any open covering contains a finite covering.

It is obvious that strong compactness implies compactness and, since the operations $\oplus$ and $\vee$ coincide on Boolean elements of MV-algebras, in the case of topologies of crisp subsets the two notions collapse to the classical one. For the same reason, it is evident as well that the skeleton spaces of both compact and strongly compact MV-spaces are compact. The following example shows that compactness does not imply strong compactness, i.e., they are not equivalent.

Example 4.5. Let X be a non-empty set and Ω the set of all constant fuzzy subsets of X, which is clearly an MV-topology. For each $r \in [0,1]$, let o_r be the fuzzy set constantly equal to r. Then, for any family $\{r_i\}_{i\in I} \subseteq [0,1)$ such that $\bigvee_{i\in I} r_i = 1$, the set $\{o_{r_i} \mid i \in I\}$ is an open covering and all the coverings not containing $\mathbf{1}$ are of this form. On the other hand, all of such coverings do not contain finite coverings but do include additive ones.

Lemma 4.6. *A closed subspace $\langle Y, \Omega_Y\rangle$ of a compact (respectively: strongly compact) space $\langle X, \Omega\rangle$ is compact (resp.: strongly compact).*

Proof. Since Y is a subspace, in particular it is a crisp subset of X and, therefore, all of its open sets are of the form $o_{\restriction Y}$ with $o \in \Omega$. So let $\{o_i\}_{i\in I} \subseteq \Omega$ such that $\bigvee_{i\in I} o_i \geq Y$. Since Y is closed, Y^* is open and $\{o_i\}_{i\in I} \cup \{Y^*\}$ is an open covering of X. By compactness of X, there exists a finite family $\{o_j\}_{j=1}^n$ of elements of $\{o_i\}_{i\in I}$ such that $o_1 \oplus \cdots \oplus o_n \oplus Y^* = X$. Then, since $Y \wedge Y^* = \mathbf{0}$, we have (with a slight abuse of notation) $Y = Y \wedge (o_1 \oplus \cdots \oplus o_n) = (Y \wedge o_1) \oplus \cdots \oplus (Y \wedge o_n)$, the latter equality easily following from the properties of Boolean elements of MV-algebras, whence Y is compact.

The case of strong compactness is completely analogous. □

Definition 4.7. Let $\mathbf{T} = \langle X, \Omega\rangle$ be an MV-topological space and $x \neq y \in X$. $\mathbf{T}$ is called an *Hausdorff* (or *separated*) *space* if there exist $o_x, o_y \in \Omega$ such that

(i) $o_x(x) = o_y(y) = 1$,

(ii) $o_x \wedge o_y = \mathbf{0}$.

Remark 4.8. It is important to observe here that there is no interesting "weak" version of the above definition. Indeed, the most natural one would be obtained by requiring condition (i) above and the following two:

$$o_x(y) = o_y(x) = 0, \tag{4}$$

$$o_x \odot o_y = \mathbf{0}. \tag{5}$$

However, the quasi-equation $x \odot y = 0 \Longrightarrow x^2 \wedge y^2 = 0$, which holds in every MV-algebra, implies immediately that such a definition would be equivalent to Definition 4.7 and, moreover, turns (4) redundant. Then an equivalent formulation of Definition 4.7 can be obtained by substituting (ii) with (5).

As for compactness, Definition 4.7 coincide with the classical T_2 property on crisp topologies and implies that the corresponding skeleton space is Hausdorff in the classical sense.

The following result is obvious.

Lemma 4.9. *If $\langle X, \Omega\rangle$ is an Hausdorff space, then all crisp singletons of X are closed.*

5 The extension of Stone Duality

In this section we shall prove that Stone Duality can be extended to a class of semisimple MV-algebras and compact separated MV-topologies having a base of clopens. Before proving the duality theorem, we recall a well-known representation theorem for semisimple MV-algebras.

Theorem 5.1. *[1,5,6] For any set X, the MV-algebra* $[0,1]^X$ *and all of its subalgebras are semisimple. Moreover, up to isomorphisms, all the semisimple MV-algebras are of this type. More precisely, every semisimple MV-algebra can be embedded in the MV-algebra of fuzzy subsets* $[0,1]^{\operatorname{Max} A}$ *of the maximal spectrum of A.*

The proof of the first part of Theorem 5.1 is rather obvious. Before proving our main theorem, it is useful to briefly sketch the proof of the fact that any semisimple MV-algebra is embeddable in $[0,1]^{\operatorname{Max} A}$.

Proof. (Sketch) For any maximal ideal M the quotient algebra A/M is a simple MV-algebra and, therefore, an archimedean MV-chain. Then A/M is isomorphic to a subalgebra of $[0,1]$ and we have this situation:

- for each $M \in \operatorname{Max} A$, there is the natural projection $\pi_M : A \longrightarrow A/M$;
- for each $M \in \operatorname{Max} A$, there exists a unique embedding $\iota_M : A/M \longrightarrow [0,1]$;
- the embedding $\iota : A \longrightarrow [0,1]^{\operatorname{Max} A}$ associates, to each $a \in A$, the fuzzy subset of $\widehat{a}$ of $\operatorname{Max} A$ defined by $\widehat{a}(M) = \iota_M(\pi_M(a)) = \iota_M(a/M)$ for all $M \in \operatorname{Max} A$.

□

It is important to notice that the above construction is possible for any MV-algebra A with the only difference (important, indeed) that the homomorphism ι is not injective if A is not semisimple for the simple reason that $\ker \iota$ always coincides with $\operatorname{Rad} A$, the latter being $\{0\}$ only if A is semisimple.

In what follows, we shall always denote by $\widehat{a}$ and $\widehat{X}$, respectively, $\iota(a) \in [0,1]^{\operatorname{Max} A}$ and $\iota(X) \subseteq [0,1]^{\operatorname{Max} A}$, for $a \in A$ and $X \subseteq A$.

The class of semisimple MV-algebras form a full subcategory of $\mathcal{MV}$ that we shall denote by $\mathcal{MV}^{\mathrm{ss}}$. As usual, for subsets $Z \subseteq Y$ of an ordered set $\langle X \leq \rangle$ we shall denote by $l_Y Z$ (or simply lZ when $Y = X$) the set of lower bounds of Z in Y and by $u_Y Z$ (respectively: uZ) the set of all upper bounds of Z in Y. We set the following

Definition 5.2. Let A be a semisimple MV-algebra. We say that a subset X of A is a *limit cut* iff

$$d(\widehat{X}, \widehat{uX}) = \bigwedge \{d(\widehat{a}, \widehat{b}) \mid b \in uX, a \in X\} = \bigwedge \{\widehat{b} \ominus \widehat{a} \mid b \in uX, a \in X\} = 0. \qquad (6)$$

We shall say that A is *limit cut complete* (*lcc* for short) if, for any limit cut X of A, there exists in A the supremum of X or, equivalently, the supremum of $\widehat{X}$ in $[0,1]^{\operatorname{Max} A}$ belongs to $\widehat{A}$.

Proposition 5.3. *Let A be a semisimple MV-algebra. Then a cut X of A is a limit cut if and only if there exists a cut Y of A such that, in $[0,1]^{\mathrm{Max}\,A}$, $\bigvee \widehat{X} = \bigwedge \widehat{Y^*}$, where $Y^* = \{y^* \mid y \in Y\}$. Moreover, Y is a limit cut too.*

Proof. Let X be a limit cut of A and set $Y = (uX)^*$. From $x \leq y$ iff $x^* \geq y^*$ readily follows that $a \in uY$ iff $a^* \in luX = X$, whence $uY = X^*$. Analogously $a \in luY$ iff $a^* \in uX$. Therefore $luY = (\underline{uX})^* = Y$, i.e., Y is a cut. Now, since $x \ominus y = 0$ iff $x \leq y$ in any MV-algebra, from $d(\widehat{X}, \widehat{uX}) = 0$, we get $\bigvee \widehat{X} = \bigwedge \widehat{uX} = \bigwedge \widehat{Y^*}$. Moreover, from $y^* \ominus x^* = y^* \odot x = x \ominus y$, we have that $d\left(\widehat{Y}, \widehat{uY}\right) = d\left(\widehat{(uX)}^*, \widehat{X^*}\right) = d(\widehat{X}, \widehat{uX}) = 0$, and therefore Y is a limit cut.

Conversely, let X and Y be cuts such that $\bigvee \widehat{X} = \bigwedge \widehat{Y^*}$, so in particular $d(\widehat{X}, \widehat{Y}) = 0$. Then $Y^* \subseteq uX$, whence $d(\widehat{X}, \widehat{uX}) \leq d(\widehat{X}, \widehat{Y^*}) = 0$, and X is a limit cut. The fact that also Y is a limit cut is an immediate consequence of the mutual roles of X and Y in this part of the proof. □

Corollary 5.4. *A semisimple MV-algebra A is lcc if and only if, for all $X, Y \in A$ and $\alpha \in [0,1]^{\mathrm{Max}\,A}$, $\alpha = \bigvee \widehat{X} = \bigwedge \widehat{Y}$ implies $\alpha \in \widehat{A}$.*

Proof. Follows immediately from Proposition 5.3 by observing that, for any subset X of A, $\bigvee \widehat{X} = \bigvee \widehat{luX}$. Then, if $\alpha = \bigvee \widehat{X} = \bigwedge \widehat{Y}$, luX and $lu(Y^*)$ form a pair of limit cuts as in Proposition 5.3. □

We wish to underline that the distance $d(\widehat{X}, \widehat{uX})$ considered in (6) do not necessarily coincide with $\iota(d(X, uX))$, as the following example shows.

Example 5.5. Let B the finite-cofinite Boolean algebra on the natural numbers. Let $\mathbb{E}$ be the set of even numbers and consider the set X of all finite subsets of $\mathbb{E}$ and the set Y of all cofinite subsets of $\mathbb{N}$ which include $\mathbb{E}$. Then it is self-evident that X and Y^* are cuts in B, $Y = uX$, and $d(X, Y) = 0$ in B. However, by the Boolean Prime Ideal Theorem, we know that there exists a maximal ideal M of B which separates X and Y, i.e. such that $X \subset M$ and $Y \cap M = \varnothing$. It follows that $d(\widehat{X}, \widehat{Y}) \neq 0$.

We shall try to describe as much as possible the class (in fact, the full subcategory of $\mathcal{MV}^{\mathrm{ss}}$) $\mathcal{MV}^{\mathrm{lcc}}$ of limit cut complete MV-algebras in Section 6. The rest of the present section is devoted to the extension of Stone Duality to MV-algebras and MV-topologies.

Let us now consider an MV-algebra A. By Theorem 5.1 and the comments following it, up to an isomorphism, $A' = A/RadA$ is a subalgebra of $[0,1]^{\mathrm{Max}\,A}$. Therefore, A' is a covering of $\mathrm{Max}\,A$ and, since it is an MV-subalgebra of $[0,1]^{\mathrm{Max}\,A}$, it is closed under $\oplus$, $\odot$ and $\wedge$. Then, by Proposition 4.3, it is a base for an MV-topology Ω_A on $\mathrm{Max}\,A$. Conversely, given an MV-topological space $\mathbf{T} = \langle X, \Omega \rangle$, the set $\mathrm{Clop}\,\mathbf{T} = \Omega \cap \Xi$ of the *clopen* subsets of X, i.e. the fuzzy subsets of X that are both open and closed, is a semisimple MV-algebra. Indeed $\mathbf{0}, \mathbf{1} \in \mathrm{Clop}\,\mathbf{T}$ and, obviously, $\mathrm{Clop}\,\mathbf{T}$ is closed under $\oplus$ and *; $\mathrm{Clop}\,\mathbf{T}$ is semisimple as an obvious consequence of being a subalgebra of $[0,1]^X$.

Let ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ be the category whose objects are MV-topological spaces and morphisms are MV-continuous functions between them. Moreover, we shall denote by

$^{\mathrm{MV}}\mathcal{S}$tone the full subcategory of $^{\mathrm{MV}}\mathcal{T}$op whose objects are *Stone MV-spaces*, i.e., compact, separated MV-topological spaces having a base of clopen sets (*zero-dimensional*).

In the proof of the following results we shall often identify any semisimple MV-algebra A with its isomorphic image included in $[0,1]^{\mathrm{Max}\,A}$; so any element a of a semisimple MV-algebra will be identified with the fuzzy set $\widehat{a}$. The reader may refer to [1, 5, 6, 8] for further details.

Let us now consider the following class functions:

$$\begin{array}{lccccccc} \mathrm{Clop}: & \mathbf{T} & \in & {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \mathrm{Clop}\,\mathbf{T} & \in & \mathcal{MV} \\ \mathrm{Max}: & A & \in & \mathcal{MV} & \longmapsto & \langle \mathrm{Max}\,A, \Omega_A\rangle & \in & {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}\,. \end{array} \tag{7}$$

Moreover, we set the following:

- for any two MV-topological spaces $\mathbf{T}$ and $\mathbf{T}'$, and for any continuous function $f : \mathbf{T} \to \mathbf{T}'$,
$$\mathrm{Clop}\, f(\alpha) = f^{\leftsquigarrow}(\alpha), \text{ for all } \alpha \in \mathrm{Clop}\,\mathbf{T}';$$
- for any two MV-algebras A and B, and for any MV-algebra homomorphism $h : A \to B$,
$$\mathrm{Max}\, h(N) = h^{\leftarrow}[N], \text{ for all } N \in \mathrm{Max}\, B.$$

Lemma 5.6. *With the above notations,* Clop *and* Max *are two contravariant functors.*

Proof. Let $\mathbf{T} = \langle X, \Omega_X\rangle$ and $\mathbf{T}' = \langle Y, \Omega_Y\rangle$ be two MV-topologies, and let $f : X \longrightarrow Y$ be a continuous map between them. As we already remarked $f^{\leftsquigarrow} : [0,1]^Y \longrightarrow [0,1]^X$ is a homomorphism of MV-algebras. On the other hand, by Definition 3.6, $f^{\leftsquigarrow}[\Omega_Y] \subseteq \Omega_X$ and, as we observed right after the same definition, $f^{\leftsquigarrow}[\Xi_Y] \subseteq \Xi_X$; therefore $f^{\leftsquigarrow}[\mathrm{Clop}\,\mathbf{T}'] \subseteq \mathrm{Clop}\,\mathbf{T}$. Hence, for all $f \in \hom_{{}^{\mathrm{MV}}\mathcal{T}\mathrm{op}}(\mathbf{T}, \mathbf{T}')$, Clop f is an MV-algebra homomorphism from $\mathrm{Clop}\,\mathbf{T}'$ to $\mathrm{Clop}\,\mathbf{T}$, i.e., a morphism from $\mathrm{Clop}\,\mathbf{T}$ to $\mathrm{Clop}\,\mathbf{T}'$ in $(\mathcal{MV}^{\mathrm{ss}})^{\mathrm{op}}$. The fact that Clop preserves composition and identities is absolutely trivial.

Let now A and B be two MV-algebras and $h : A \longrightarrow B$ an MV-algebra homomorphism. It is known that the preimage of a maximal ideal under an MV-algebra homomorphism is a maximal ideal; then it is well-defined the map $\mathrm{Max}\, h : N \in \mathrm{Max}\, B \longmapsto h^{\leftarrow}[N] \in \mathrm{Max}\, A$. The function Max h, on its turn, defines an MV-algebra homomorphism $(\mathrm{Max}\, h)^{\leftsquigarrow} : \alpha \in [0,1]^{\mathrm{Max}\,A} \longmapsto \alpha \circ \mathrm{Max}\, h \in [0,1]^{\mathrm{Max}\,B}$. Let us prove that $(\mathrm{Max}\, h)^{\leftsquigarrow}[A] \subseteq \Omega_B$.

So let N be an arbitrary maximal ideal of B and $M = \mathrm{Max}\, h(N)$. We have

$$(\mathrm{Max}\, h)^{\leftsquigarrow}(\widehat{a})(N) = (\widehat{a} \circ \mathrm{Max}\, h)(N) = \widehat{a}(M), \text{ for all } a \in A.$$

The map $h' : a/M \in A/M \longrightarrow h(a)/N \in B/N$ is well-defined since

$$\begin{array}{l} a/M = a'/M \iff (a \odot a'^*) \oplus (a' \odot a^*) \in M \iff \\ (h(a) \odot h(a')^*) \oplus (h(a') \odot h(a)^*) \in N \iff h(a)/N = h(a')/N; \end{array}$$

moreover it can be proved in a similar way that $h(a)/N = h(a')/N$ implies $a/M = a'/M$, that is, h' is injective. Now, if we look at A/M and B/N as subalgebras of $[0,1]$, we get that the fuzzy set $\widehat{h(a)}$ takes, in any given $N \in \mathrm{Max}\, B$, precisely the same value taken by

the fuzzy set $\widehat{a}$ in $M = \mathrm{Max}\, h(N)$. In other words, the fuzzy set $(\mathrm{Max}\, h)^{\leftsquigarrow}(a)$ is in B, for all $a \in A$. It follows that $(\mathrm{Max}\, h)^{\leftsquigarrow}[A] \subseteq \Omega_B$ and therefore, by Lemma 4.2, $\mathrm{Max}\, h$ is a MV-continuous function from $\langle \mathrm{Max}\, B, \Omega_B\rangle$ to $\langle \mathrm{Max}\, A, \Omega_A\rangle$, i.e., it is a morphism from $\langle \mathrm{Max}\, A, \Omega_A\rangle$ to $\langle \mathrm{Max}\, B, \Omega_B\rangle$ in ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}^{\mathrm{op}}$. Again, it is immediate to see that Max is well-behaved w.r.t. composition and identity morphisms. □

Theorem 5.7 (Duality theorem)**.** Clop *and* Max *form a duality between* $\mathcal{MV}^{\mathrm{lcc}}$ *and* ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$.

Proof. It is immediate to verify that both the functors, restricted to ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ and $\mathcal{MV}^{\mathrm{lcc}}$ respectively, are faithful. Then we shall prove that $\mathrm{Max}\,\mathrm{Clop}\,\mathbf{T} \cong_{\mathrm{MV}\mathcal{T}\mathrm{op}} \mathbf{T}$ and $\mathrm{Clop}\,\mathrm{Max}\, A \cong_{\mathcal{MV}} A$, for all $\mathbf{T} \in {}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ and for all $A \in \mathcal{MV}^{\mathrm{lcc}}$. The assertion will therefore follow from the fact that such isomorphisms, together with faithfulness, yield two natural isomorphisms between the two compositions Max Clop and Clop Max and, respectively, $\mathrm{id}_{\mathcal{S}\,MV}$ and $\mathrm{id}_{\mathcal{MV}^{\mathrm{lcc}}}$.

First, let us prove that $\mathrm{Max}\, A \in {}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ for any semisimple MV-algebra A and that $\mathrm{Clop}\,\mathrm{Max}\, A \cong A$ if $A \in \mathcal{MV}^{\mathrm{lcc}}$.

$\langle \mathrm{Max}\, A, \Omega_A\rangle$ is zero-dimensional by definition. $\mathrm{Clop}\,\mathrm{Max}\, A$ is obviously semisimple, and every element of $\mathrm{Clop}\,\mathrm{Max}\, A$ can be obtained as both a join and a meet of elements of $\widehat{A}$. Therefore, if $A \in \mathcal{MV}^{\mathrm{lcc}}$, by Proposition 5.3 and Corollary 5.4, $A \cong \mathrm{Clop}\,\mathrm{Max}\, A$. Now we need to prove only that $\mathrm{Max}\, A$ is compact and Hausdorff. Let Γ be an open covering of $\mathrm{Max}\, A$ and assume, by contradiction, that it does not contain any additive covering. By Corollary 2.3, $(\Gamma]$ is a proper ideal of A and, therefore, it is contained in some $M \in \mathrm{Max}\, A$; but this implies that, for any $a \in \Gamma$, $a(M) = 0$, i.e. Γ is not a covering of $\mathrm{Max}\, A$, which is absurd.

In order to prove separation, let us consider $M \neq N \in \mathrm{Max}\, A$ and let $a \in M \setminus N$; we have $\widehat{a}(M) = 0$ and $\widehat{a}(N) \neq 0$. So, since $[0,1]$ is hyperarchimedean, there exists $k < \omega$ such that $k\widehat{a}(N) = 1$. Then we have $k\widehat{a}(N) = 1$ and $\widehat{a^*}(M) = \widehat{a}(M)^* = 1$, which implies $(\widehat{a^*})^k(M) = 1$.Moreover, $(\widehat{a^*})^k \odot k\widehat{a} = (k\widehat{a})^* \odot k\widehat{a} = \mathbf{0}$; then, by Remark 4.8, $\langle \mathrm{Max}\, A, \Omega_A\rangle$ is a Stone MV-space.

Now let us prove that $\mathbf{T}$ and $\mathrm{Max}\,\mathrm{Clop}\,\mathbf{T}$ are homeomorphic for any Stone MV-space $\mathbf{T}$. Let $\mathbf{T} = \langle X, \Omega\rangle$ be a Stone MV-space and, for each $x \in X$, let $f(x) = \{o \in \mathrm{Clop}\,\mathbf{T} \mid o(x) = 0\}$. It is self-evident that $f(x)$ is a proper ideal of the algebra $A = \mathrm{Clop}\,\mathbf{T}$ for all $x \in X$. For any fixed x and for each $o \in A$, $o \notin f(x)$ implies $o(x) > 0$ and, therefore, $o^*(x) < 1$. Then there exists $n < \omega$ such that $(o^*)^n(x) = 0$, i.e. $(o^*)^n \in f(x)$, and Proposition 2.4 ensures us that $f(x)$ is a maximal ideal.

Now we must prove that the map $f : X \longrightarrow \mathrm{Max}\, A$ is a homeomorphism of MV-spaces. First, let $x \neq y \in X$; since $\mathbf{T}$ is Hausdorff, there exist $o_x, o_y \in \Omega$ that satisfy Definition 4.7, and each of these open sets is the join of a set of clopens because $\mathbf{T}$ is zero-dimensional. By Lemma 4.9, $\{x\}$ and $\{y\}$ are closed, whence, by Lemma 4.6, they are compact; then there exist two finite families of such sets — say $\{o_{xi}\}_{i=1}^n$ and $\{o_{yj}\}_{j=1}^m$ — which are additive open coverings of $\{x\}$ and $\{y\}$ respectively, and are such that $(o_{x1} \oplus \cdots \oplus o_{xn})(y) = 0 = (o_{y1} \oplus \cdots \oplus o_{ym})(x)$. Moreover, $o_{x1} \oplus \cdots \oplus o_{xn}$ and $o_{y1} \oplus \cdots \oplus o_{ym}$ are both clopen, hence the former belongs to $f(y)$ and the latter to $f(x)$. It follows $f(x) \neq f(y)$, namely, f is injective.

In order to prove that f is onto, let $M \in \operatorname{Max} A$ and assume, by contradiction, that M is not the image under f of any element of X, that is, for all $x \in X$ there exists $o \in M$ such that $o(x) > 0$. Then, for each $x \in X$, there exist $o \in M$ and $m < \omega$ such that $mo(x) = 1$, and $mo \in M$ because M is an ideal. So let, for each $x \in X$, o_x be an element of M whose value in x is 1; the family $\{o_x\}_{x \in X}$ is an open covering of X whence, by the compactness of $\mathbf{T}$, it contains an additive covering $\{o_i\}_{i=1}^n$. It follows that $\mathbf{1} = o_1 \oplus \cdots \oplus o_n \in M$ which contradicts the hypothesis that M is a proper ideal. Such a contradiction follows from the assumption that for all $x \in X$ there exists $o \in M$ such that $o(x) > 0$; hence there exists $x \in X$ such that $o(x) = 0$ for all $o \in M$, i.e., such that $M = f(x)$, and f is onto.

We need to prove that both f and f^{-1} are continuous. To this purpose, we first observe that, for all $x \in X$ and $o \in \operatorname{Clop} \mathbf{T}$, $o/f(x)$ is a real number in $[0,1]$ and coincide with the membership value $o(x)$ of the point x to the clopen o. Indeed, by Lemma 2.5, $o/f(x) = \{(o \oplus p) \odot q^* \mid p, q \in f(x)\}$ and, on the other hand, $((o \oplus p) \odot q^*)(x) = (o(x) \oplus 0) \odot 1 = o(x)$ for all $p, q \in f(x)$. Therefore, $\operatorname{Clop} \mathbf{T}/f(x) = \{o(x) \mid o \in \operatorname{Clop} \mathbf{T}\}$ and $\pi_{f(x)} : o \in \operatorname{Clop} \mathbf{T} \mapsto o(x) \in \operatorname{Clop} \mathbf{T}/f(x) \subseteq [0,1]$.

Now, any clopen o of $\mathbf{T}$ can be identified (see the proof of Theorem 5.1) with a clopen $\widehat{o}$ of $\operatorname{Max} \operatorname{Clop} \mathbf{T}$ in a unique way: $\widehat{o}(M) = \iota_M(\pi_M(o)) = \iota_{f(x)}(\pi_{f(x)}(o)) = \iota_{f(x)}(o(x))$, for all $M = f(x) \in \operatorname{Max} \operatorname{Clop} \mathbf{T}$, and $\iota_{f(x)}$ is simply the inclusion map of $\operatorname{Clop} \mathbf{T}/f(x)$ in $[0,1]$. Therefore, for any basic clopen $\widehat{o}$ of $\operatorname{Max} \operatorname{Clop} \mathbf{T}$, and for each $x \in X$, $f^{\leftsquigarrow}(\widehat{o})(x) = (\widehat{o} \circ f)(x) = \widehat{o}(f(x)) = o(x)$, with $o \in \operatorname{Clop} \mathbf{T}$. It follows that the fuzzy preimage, under f, of any basic open set of $\operatorname{Max} \operatorname{Clop} \mathbf{T}$ is open in $\mathbf{T}$, that is, f is continuous. Analogously, for each $M = f(x) \in \operatorname{Max} \operatorname{Clop} \mathbf{T}$, $(f^{-1})^{\leftsquigarrow}(o)(M) = (o \circ f^{-1})(f(x)) = o(x) = \widehat{o}(M)$, and f^{-1} is continuous as well. We can conclude that $\mathbf{T}$ and $\operatorname{Max} \operatorname{Clop} \mathbf{T}$ are homeomorphic spaces.

The proof is complete. □

Corollary 5.8. *The restriction of the above duality to Boolean algebras and crisp topologies coincide with the classical Stone Duality.*

Proof. This is a trivial consequence of how the functors are defined. □

Theorem 5.9. *For any Stone MV-space* $\mathbf{T}$*, its skeleton space is a Stone space and its image under* Clop *is precisely the Boolean center of* $\operatorname{Clop} \mathbf{T}$.

Conversely, for any semisimple MV-algebra A, $\operatorname{Max} \mathrm{B}(A)$ *coincide with the skeleton topology of* $\operatorname{Max} A$.

Proof. The first part is trivial. For the second part, once observed that, by Proposition 2.4, $M \cap \mathrm{B}(A)$ is a maximal ideal of the Boolean algebra $\mathrm{B}(A)$, for all $M \in \operatorname{Max} A$, it suffices to apply Theorem 5.7. □

It is immediate to verify that

$$\begin{array}{lccc} \mathrm{B}: & A \in \mathcal{MV} & \longmapsto & \mathrm{B}(A) \in \mathcal{B}\mathrm{oole} \\ \mathrm{Sk}: & \langle X, \Omega \rangle \in {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \langle X, \mathrm{B}(\Omega) \rangle \in \mathcal{T}\mathrm{op} \end{array}$$

define two functors, where the action of B on morphisms is simply the restriction of the MV-algebra homomorphism to the Boolean center of the domain, and $\mathrm{Sk}\, f$ is f

itself, for any MV-continous map f. They are, in fact, the left-inverses of the inclusion functors. Then Theorem 5.9 (together with Corollary 5.8) can be reformulated as follows.

Corollary 5.10. $\mathrm{Clop}_{\uparrow} \circ \mathrm{Sk} = \mathrm{B} \circ \mathrm{Clop}$ *and* $\mathrm{Max}_{\uparrow} \circ \mathrm{B} = \mathrm{Sk} \circ \mathrm{Max}$.

Then we have the following commutative diagram of functors, where horizontal arrows are equivalences and vertical ones are inclusions of full subcategories and their respective left-inverses.

$$\begin{array}{ccc} \mathcal{MV}^{\mathrm{lcc}} & \underset{\mathrm{Clop}}{\overset{\mathrm{Max}}{\rightleftarrows}} & {}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}^{\mathrm{op}} \\ \mathrm{B} \downarrow \quad \cup\mathrm{l} & & \cup\mathrm{l} \quad \mathrm{Sk} \downarrow \\ \mathcal{B}\mathrm{oole} & \underset{\mathrm{Max}_{\uparrow}}{\overset{\mathrm{Clop}_{\uparrow}}{\rightleftarrows}} & \mathcal{S}\mathrm{tone}^{\mathrm{op}} \end{array} \tag{8}$$

6 On limit cut complete MV-algebras

In the present section we shall describe the class of limit cut complete MV-algebras, namely, the category which is dual to the one of Stone MV-spaces. Moreover, we will show that, for any semisimple MV-algebra A, the extension $\mathrm{Clop}\,\mathrm{Max}\,A$ is, in a precise sense, the smallest lcc MV-algebra containing A, and it can be obtained with an alternative construction within the class of MV-algebras itself.

We already presented a characterization of $\mathcal{MV}^{\mathrm{lcc}}$ (Theorem 5.4) which, however, may not be handy enough in many cases. As we shall see, a necessary condition for a semisimple MV-algebra to be lcc is that all of its quotients on the maximal ideals must be complete chains or, equivalently, either finite or isomorphic to $[0,1]$. Whether such a condition is sufficient too, is still an open problem, as the author was able neither to prove nor to disprove it so far. It will be stated as a conjecture at the end of the section.

We shall begin by showing two important properties of lcc MV-algebras.

Theorem 6.1. *If A is an lcc MV-algebra, then the following hold.*

(i) For all $a \leq b \in A$, if $\iota([a,b])$ is dense in $[\widehat{a},\widehat{b}]$, then $\iota([a,b]) = [\widehat{a},\widehat{b}]$.

(ii) A is a subdirect product of complete MV-chains, i.e., for all $M \in \mathrm{Max}\,A$, A/M is isomorphic either to $[0,1]$ or to the finite chain $\mathrm{Ł}_n$ for some $n < \omega$.

Proof. (i) If A is lcc and $\iota([a,b])$ is dense in $[\widehat{a},\widehat{b}]$, then every element of $[\widehat{a},\widehat{b}]$ can be obtained as both a supremum and an infimum of elements of $\iota([a,b])$. Then, by Corollary 5.4, $[\widehat{a},\widehat{b}] = \iota([a,b])$.

(ii) Let us consider the dual space $\langle \mathrm{Max}\,A, \Omega_A \rangle$ of A. Then, for all $M \in \mathrm{Max}\,A$, the subspace $\langle \{M\}, \Omega_{A,M} \rangle$ is clearly a Stone MV-space. Now, since the (continuous) inclusion map $\{M\} \to \mathrm{Max}\,A$ corresponds, by Theorem 5.7, to the canonical

projection $\pi_M : A \to A/M$, the quotient A/M must be isomorphic to Clop$\{M\}$. On the other hand, it is easy to see that the only possible MV-topologies on a singleton are either $[0,1]$ or Ł$_n$ for some $n < \omega$. So, if A/M is infinite, it must be dense in $[0,1]$, by [8, Proposition 3.5.3] and $\{\bigvee X \mid X \subseteq A/M\} = [0,1]$, that is, $A/M \cong [0,1]$.

□

Let now, for any semisimple MV-algebra A, $\mathcal{LCC}(A)$ be the following set

$$\mathcal{LCC}(A) = \{B \leq [0,1]^{\mathrm{Max}\,A} \mid B \in \mathcal{MV}^{\mathrm{lcc}} \text{ and } \widehat{A} \leq B\}.$$

$\mathcal{LCC}(A)$ is certainly not empty since it contains at least the Dedekind-MacNeille completion A^{DM} of A (see [3, Section 5]) and $[0,1]^{\mathrm{Max}\,A}$ (which may possibly coincide). Then we can set the following

Definition 6.2. For any semisimple MV-algebra A, the MV-algebra $A^{\mathrm{lcc}} = \bigcap \mathcal{LCC}(A)$ will be called the *limit cut completion* (*lc-completion*, for short) of A.

Since, for any limit cut X of A, $\widehat{X}$ is a limit cut of $\widehat{A}$, and $\bigvee \widehat{X} \in B$ for all $B \in \mathcal{LCC}(A)$, we get immediately that $A^{\mathrm{lcc}} \in \mathcal{MV}^{\mathrm{lcc}}$ for any $A \in \mathcal{MV}^{\mathrm{ss}}$.

Theorem 6.3. *The mapping*

$$(\ )^{\mathrm{lcc}} : A \in \mathcal{MV}^{\mathrm{ss}} \mapsto A^{\mathrm{lcc}} \in \mathcal{MV}^{\mathrm{lcc}}$$

defines a faithful functor.

Proof. From the results of Section 5 it readily follows that, for all $A \in \mathcal{MV}^{\mathrm{ss}}$, $A^{\mathrm{lcc}} = \mathrm{Clop}\,\mathrm{Max}\,A$. Then, by Theorem 5.7, $(\ )^{\mathrm{lcc}}$ is a functor, whose faithfulness is obvious as well. □

Definition 6.4. We shall say that an MV-algebra A is *subdirect factor complete* iff it is a subdirect product of complete chains. In what follows, we shall denote by $\mathcal{MV}^{\mathrm{sfc}}$ the full subcategory of $\mathcal{MV}^{\mathrm{ss}}$ whose objects are subdirect factor complete algebras.

By Theorem 6.1, the class $\mathcal{MV}^{\mathrm{lcc}}$ is contained in $\mathcal{MV}^{\mathrm{sfc}}$. We set the following

Conjecture 6.5. $\mathcal{MV}^{\mathrm{lcc}} = \mathcal{MV}^{\mathrm{sfc}}$, *namely, every subdirect factor complete MV-algebra is limit cut complete and, therefore, is the dual algebra of a Stone MV-space.*

7 Concluding remarks

Theorems 5.7 and 5.9, and Corollaries 5.8 and 5.10, provide, in our opinion, a strong motivation for the development of a more general and comprehensive theory of MV-topologies. Indeed, besides proving once more that MV-algebras are the most natural generalization of Boolean algebras, these results can be combined with the wide variety of equivalences involving categories of MV-algebras, thus giving new equivalences and, therefore, new tools.

For example, we can apply the well-known and celebrated categorical equivalences [20] between MV-algebras and lattice-ordered Abelian groups with a strong order unit (Abelian $u\ell$-groups, for short), thus obtaining a duality between a suitable category of archimedean Abelian $u\ell$-groups and Stone MV-spaces.

It is worth noticing that, on their turn, archimedean Abelian $u\ell$-groups are, up to isomorphisms, subgroups of the $u\ell$-group of bounded functions from a set X to $\mathbb{R}$, with pointwise operations and the 1-constant map as order unit. Then the restriction of such functors yields a duality between Stone spaces and $u\ell$-groups which are, up to isomorphisms, subgroups of the $u\ell$-group of bounded functions from a set X to $\mathbb{Z}$.

Another example can be given with reference to [10], where the authors proved, for each $n > 1$, a categorical equivalence between the MV-subvariety generated by the $(n+1)$-element chain $Ł_n = \{i/n\}_{i=0}^n$ and the category whose objects are pairs (B, R), where B is a Boolean algebra and R is an n-ary relation on B satisfying certain conditions, and a morphism $f : (B, R) \longrightarrow (B', R')$ is a Boolean algebra homomorphism such that $(a_1, \ldots, a_n) \in R$ implies $(f(a_1), \ldots, f(a_n)) \in R'$. So, since all the MV-algebras in such subvarieties are subdirect factor complete, we expect that the combination of our results with the ones in [10] will give a duality between locally finite (in the sense of Universal Algebra) MV-algebras and some categories of classical Stone spaces with additional conditions.

Last, we also point out that all the evidences indicate that a point-free approach to this topic may lead to a fruitful generalization of topos theory; moreover we expect the basic notions and results achieved so far to be plainly applicable to Mathematical Morphology.

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