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FROM BROWNIAN-TIME BROWNIAN SHEET TO A FOURTH ORDER AND A KURAMOTO-SIVASHINSKY-VARIANT INTERACTING PDE SYSTEMS

HASSAN ALLOUBA

Abstract. We introduce n-parameter $\mathbb{R}^d$-valued Brownian-time Brownian sheet (BTBS): a Brownian sheet where each "time" parameter is replaced with the modulus of an independent Brownian motion. We then connect BTBS to a new system of n linear, fourth order, and interacting PDEs and to a corresponding fourth order interacting nonlinear PDE. The coupling phenomenon is a result of the interaction between the Brownian sheet, through its variance, and the Brownian motions in the BTBS; and it leads to an intricate, intriguing, and random field generalization of our earlier Brownian-time-processes (BTPs) connection to fourth order linear PDEs. Our BTBS does not belong to the classical theory of random fields; and to prove our new PDEs connections, we generalize our BTP approach in [5, 4] and we mix it with the Brownian sheet connection to a linear PDE system, which we also give along with its corresponding nonlinear second order PDE and a $2n$-th order linear PDE that we also connect to Brownian sheet. In addition, we introduce the n-parameter d-dimensional linear Kuramoto-Sivashinsky sheet (LKSS) kernel (or "transition density"); and we link it to an intimately connected system of new linear Kuramoto-Sivashinsky-variant interacting PDEs, generalizing our earlier one parameter imaginary-Brownian-time-Brownian-angle kernel and its connection to a linear KS PDE. The interacting PDEs connections established here open up another new fundamental front in the rapidly growing field of iterated-type processes and their connections to both new and important higher order PDEs and to some equivalent fractional Cauchy problems. We connect the BTBS fourth order interacting PDEs system given here with an interacting fractional PDE system and further study it in another article. We note that Burdzy posed a question regarding path properties of an iterated Brownian sheet intimately related to our BTBS when $n = 2$.

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1. Introduction and results statements

1.1. **Brownian-time Brownian sheet and its fourth order interacting PDE system.** Let $B^{(1)},\dots,B^{(n)}$ be n independent copies of a standard one-dimensional Brownian motion starting at 0 and independent of an n-parameter ($n \geq 1$) $\mathbb{R}^d$-valued Brownian sheet

$$W^0 = \left\{W^0(\mathbf{t}) = \left(W_1^0(\mathbf{t}),\dots,W_d^0(\mathbf{t})\right); \mathbf{t} = (t_1,\dots,t_n) \in \mathbb{R}_+^n\right\},$$

"starting" at $0 \in \mathbb{R}^d$ under $\mathbb{P}$—$\mathbb{P}\{W^0(\mathbf{t}) = 0\} = 1$ for $\mathbf{t} \in \partial\mathbb{R}_+^n$ (see Notation 1.1)—all defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Of course, the Brownian sheet coordinates $\left\{W_1^0(\mathbf{t}); t \geq 0\right\},\dots,\left\{W_d^0(\mathbf{t}); t \geq 0\right\}$ are assumed independent. For any $x = (x_1,\dots,x_d) \in \mathbb{R}^d$, let

$$W^x := W^0 + x = \left\{W^x(\mathbf{t}) = (W_1^{x_1}(\mathbf{t}),\dots,W_d^{x_d}(\mathbf{t})); \mathbf{t} = (t_1,\dots,t_n) \in \mathbb{R}_+^n\right\}.$$

We define the n-parameter $\mathbb{R}^d$-valued Brownian-time Brownian sheet (BTBS) based on W^x and $B^{(1)},\dots,B^{(n)}$, and starting at $x \in \mathbb{R}^d$, by

$$\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t}) := W^x\left(\left|B^{(1)}(t_1)\right|,\dots,\left|B^{(n)}(t_n)\right|\right);\ \mathbf{t} \in \mathbb{R}_+^n. \tag{1.1}$$

Clearly, $\mathbb{P}\left[\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t}) = x\right] = 1$ for $\mathbf{t} \in \partial\mathbb{R}_+^n$. Our first main result gives a nonlinear fourth order interacting PDE and a corresponding linear system of fourth order interacting PDEs that are solved by running the BTBS in (1.1). Before stating our main result, it is helpful to adopt some simplifying notational conventions.

Notation 1.1. We always use the notation $\mathbb{N}_n := \{1,\dots,n\}$; and we denote by $\overset{\circ}{\mathbb{R}}{}_+^n$ and $\partial\mathbb{R}_+^n$ the interior and boundary of $\mathbb{R}_+^n$, respectively. From this point on, we will alternate freely between $u_{\mathbf{t}}$, $u(\mathbf{t})$, and $u(t_1,\dots,t_n)$ for typsetting convenience and ease of exposition. Moreover, $\mathbf{t}_{\not j} = (t_i)_{i\in\mathbb{N}_n\setminus\{j\}} \in \mathbb{R}_+^{n-1}$ will denote the $(n-1)$-tuple point in $\mathbb{R}_+^{n-1}$ that is obtained from $\mathbf{t} = (t_1,\dots,t_n) \in \mathbb{R}_+^n$ by removing the j-th variable, t_j. The notations $u_{\mathbf{t}}$, $u(\mathbf{t})$, and $u_{\mathbf{t}_{\not j}}(t_j)$ will all mean the n-parameter function $u : \mathbb{R}_+^n \to \mathbb{R}^d$ evaluated at $\mathbf{t} = (t_1,\dots,t_n)$, and we use $u_{\mathbf{t}_{\not j}}(t_j)$ whenever

we need to highlight the variation in the variable t_j for clarity. The same comment applies for the notations $u(\mathbf{t},x)$ and $u_{\mathbf{t}_{\not j}}(t_j,x)$.

Remark 1.1. We note that in [15] Burdzy first discussed a process intimately related to the case $n=2$ of our BTBS in the context of a question he posed regarding the path properties of such a process.

We are now ready to state our first main result.

Theorem 1.1 (Brownian-time Brownian sheet: fourth order interacting PDEs connections)**.** *Let $\{\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t});\mathbf{t}\in\mathbb{R}^n_+\}$ be an n-parameter $\mathbb{R}^d$-valued BTBS based on a Brownian sheet W^x and Brownian motions $\left\{B^{(i)}\right\}_{i=1}^n$ and starting on $\partial\mathbb{R}^n_+$ at $x\in\mathbb{R}^d$ on $(\Omega,\mathscr{F},\mathbb{P})$. Let $f:\mathbb{R}^d\to\mathbb{R}$ be bounded and measurable such that all second order partial derivatives $\partial^2_{kl}f$ are bounded and Hölder continuous, with exponent $0<\alpha\le 1$, for $1\le k,l\le d$. If*

$$
\begin{aligned}
u(\mathbf{t},x)&=\mathbb{E}_{\mathbb{P}}\left[f\left(\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t})\right)\right],\\
\mathscr{U}^{(j)}(\mathbf{t},x)&=\mathbb{E}_{\mathbb{P}}\left[\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}\left|B^{(i)}(t_i)\right|\right)^2 f\left(\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t})\right)\right];\ j\in\mathbb{N}_n,
\end{aligned}
\tag{1.2}
$$

for $(\mathbf{t},x)\in\mathbb{R}^n_+\times\mathbb{R}^d$, then the family $\left\{u(\mathbf{t},x),\mathscr{U}^{(1)}(\mathbf{t},x),\dots,\mathscr{U}^{(n)}(\mathbf{t},x)\right\}$ is a solution to both

(i) *the system of fourth order interacting Linear PDEs:*

$$
\begin{cases}
(a)\ \dfrac{\partial u}{\partial t_j}=\sqrt{\dfrac{\prod_{i\in\mathbb{N}_n\setminus\{j\}}t_i}{2^{4-n}\pi^n t_j}}\Delta_x f(x)+\dfrac{1}{8}\Delta_x^2\mathscr{U}^{(j)}; & \mathbf{t}\in\overset{\circ}{\mathbb{R}}{}^n_+,x\in\mathbb{R}^d,\\
(b)\ u(\mathbf{t},x)=f(x); & \mathbf{t}\in\partial\mathbb{R}^n_+,x\in\mathbb{R}^d,\\
(c)\ \mathscr{U}^{(j)}(\mathbf{t},x)=0; & \mathbf{t}_{\not j}\in\partial\mathbb{R}^{n-1}_+,x\in\mathbb{R}^d,\\
(d)\ \mathscr{U}^{(j)}(\mathbf{t},x)=\left[\prod_{i\in\mathbb{N}_n\setminus\{j\}}t_i\right]f(x); & t_j=0,x\in\mathbb{R}^d,
\end{cases}
\tag{1.3}
$$

for $j\in\mathbb{N}_n$; and

(ii) *the corresponding fourth order nonlinear interacting PDE:*

$$
\begin{aligned}
\prod_{j\in\mathbb{N}_n}\frac{\partial u}{\partial t_j}&=\sqrt{\frac{(t_1\cdots t_n)^{n-2}}{2^{4n-n^2}\pi^{n^2}}}\left(\Delta_x f(x)\right)^n+\frac{1}{8^n}\prod_{j\in\mathbb{N}_n}\Delta_x^2\mathscr{U}^{(j)}\\
&\quad+\sum_{(k_1,\dots,k_n)\in\mathbb{S}_n}\prod_{j\in\mathbb{N}_n}T_{k_j,j}(\mathbf{t},x);\ \mathbf{t}\in\overset{\circ}{\mathbb{R}}{}^n_+,x\in\mathbb{R}^d
\end{aligned}
\tag{1.4}
$$

together with the temporal boundary conditions (b), (c), and (d) in (1.3), where

$$
T_{k_j,j}(\mathbf{t},x)=\begin{cases}\sqrt{\dfrac{\prod_{i\in\mathbb{N}_n\setminus\{j\}}t_i}{2^{4-n}\pi^n t_j}}\Delta_x f(x); & k_j=1,j\in\mathbb{N}_n\\ \dfrac{1}{8}\Delta_x^2\mathscr{U}^{(j)}; & k_j=2,j\in\mathbb{N}_n,\end{cases}
\tag{1.5}
$$

and where

$$
\mathbb{S}_n=\left\{(k_1,\dots,k_n);k_1,\dots,k_n\in\{1,2\},n+1\le\sum_{j=1}^n k_j\le 2n-1\right\},\ n\ge 2.
$$

The intriguing interaction (or coupling) in Theorem 1.1 between u and $\mathscr{U}^{(j)}$, $j = 1, \ldots, n$, shows that the PDEs in (1.3) and (1.4) are nontrivial intricate generalizations of our BTP-PDE connections in [5, 4] (the case $n = 1$). This coupling phenomenon is caused by the interaction between the Brownian-times $\left|B^{(1)}\right|, \ldots, \left|B^{(n)}\right|$ and the variance of the Brownian sheet W^x through $\partial_j \mathrm{Var}\left(W^x(t)\right) = \prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i$. The nonlinear feature of the BTBS-PDE connection and its corresponding linear system connection are inherited from the Brownian sheet PDE together with its corresponding linear system of PDEs, which we give below in Lemma 1.1. However, in addition to the new coupling above, and due to the iterative nature of BTBS, those PDEs in (1.3) and (1.4) are fourth order versus the much simpler second order Brownian sheet nonlinear PDE and linear system of PDEs. On the other hand, when $n = 1$, our BTBS becomes a Brownian-time Brownian motion (see [5, 4]), $u = \mathscr{U}^{(j)}$, and (1.3) and (1.4) reduce to our fourth order BTP-PDE (0.2) in [5]:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{\Delta_x f(x)}{\sqrt{8\pi t}} + \dfrac{1}{8} \Delta_x^2 u(t,x); & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = f(x); & x \in \mathbb{R}^d. \end{cases} \tag{1.6}$$

Note that if $n = 1$ then $\prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i = \prod_{i \in \mathbb{N}_n \setminus \{j\}} \left|B^{(i)}(t_i)\right|^2 = 1$ since $\mathbb{N}_n \setminus \{j\} = \phi$ (the empty set), and $\sum_{(k_1, \ldots, k_n) \in \mathbb{S}_n} \prod_{j \in \mathbb{N}_n} T_{k_j, j}(\mathbf{t}, x) = 0$ since $\mathbb{S}_n = \phi$.

As with its one-parameter BTP counterpart, the BTBS is memory preserving as is indicated by the inclusion of the Laplacian of the initial function $\Delta_x f(x)$ in the BTBS PDEs in (1.3) and (1.4). Also, just as BTPs are not classical (not semimartingales, not Markovian, and not Gaussian), BTBS is not a classical random field. To prove Theorem 1.1 we generalize and mix our Brownian-time approach from [5, 4] with the Brownian-sheet-PDEs connection which we give in Lemma 1.1 below.

Theorem 1.1 gives intriguing connections between the Brownian-time Brownian sheet and a new nonlinear fourth order and interacting PDE (1.4) and its corresponding interacting system (1.3). This connection opens up another new fundamental front in the rapidly growing field of Brownian-time and iterated processes and their connections to both (1) new and important higher order deterministic and stochastic PDEs and (2) some equivalent fractional Cauchy problems (e.g., Allouba and Allouba et al. [5, 4, 3, 1, 2, 6, 9, 10, 8] and followup papers; as well as the recent articles by Nane [25, 26], Baeumer et al. [13], Meerschaert et al. [24], Orsingher et al. [27, 14], and DeBlassie [19]). In this new Brownian-time Brownian sheet PDEs direction, the BTBS-PDEs connections (1.4) and (1.3) introduce a way for a random field to shed light on nonlinear PDEs—and their corresponding system of n linear PDEs—that are fourth order and interacting; and these PDEs, in turn, could be used to gain insight into the underlying Brownian-time random field. Guided by our one-time-parameter ($n = 1$) work in [5, 4, 3, 6, 1], we explore ramifications of such a program in planned upcoming work. In addition, following our work with Zheng in [5], where we initiated the link between Brownian-time processes and fractional time derivatives by way of the BTP half-derivative generator—an implicit connection to temporally-fractional PDEs that was later made explicit and extended in several directions in the nice works of Nane, Meerchart, Baeumer, Vellaisamy, Orsingher, and Beghin [26, 24, 13, 27, 14]—we intend to connect our BTBS with systems of fractional interacting PDEs and their corresponding nonlinear fractional coupled PDE in future work.

1.1.1. *Three Brownian sheet-PDEs links.* Here, we give the connections between the n-parameter $\mathbb{R}^d$-valued Brownian sheet W^x and (1) a nonlinear second order PDE, (2) a corresponding system of n-linear second order PDEs, and (3) a linear $2n$-th order PDE. This Brownian sheet-PDEs connection, which characterizes Brownian sheet similar to the way the heat equation characterizes Brownian motion, is one of the needed ingredients in our proof of Theorem 1.1; and it contains within it the first element causing the coupling between u and $\mathscr{U}^j$ in Theorem 1.1 above. Although some of its aspects are perhaps known as "folklore" among the specialists (we obtained the Brownian sheet-PDEs connection—as part of our unpublished preprint [7]—and we circulated it as early as 1996), this connection does not seem to be recorded explicitly anywhere in the standard literature. We present it here, along with a proof that we give in the Appendix for completeness. From a mathematical finance point of view, the system (1.7) below is at the heart of our new Black-Scholes-Merton system for semi-SPDEs bonds and stocks models that we introduce and study in a separate joint paper. Due to the fundamental nature of Brownian sheet and its connection to space-time white noise, the Brownian sheet-PDE connection in Lemma 1.1 below (as well as Corollary 1.1 and Corollary 1.2)—a connection which is as of yet really untapped—and natural generalization thereof are useful to further study the interactions between random fields and SPDEs solutions on one hand and nonlinear PDEs or higher order linear PDEs and their corresponding system of linear PDEs on the other (more on that in [7]).

The following result gives the Brownian sheet system of PDEs.

Lemma 1.1 (Brownian sheet linear second order PDEs system)**.** *Let $\{W^x(\mathbf{t}); \mathbf{t} \in \mathbb{R}^n_+\}$ be an n-parameter $\mathbb{R}^d$-valued Brownian sheet on $(\Omega, \mathscr{F}, \mathbb{P})$ with $\mathbb{P}\{W^x(\mathbf{t}) = x\} = 1$, for $\mathbf{t} \in \partial\mathbb{R}^n_+$. Let $u(\mathbf{t}, x) = \mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]$, for a continuous bounded real-valued function $f : \mathbb{R}^d \to \mathbb{R}$. Then, $u(\mathbf{t}, x)$ is a bounded solution to the system of heat-type PDEs*

$$\begin{cases} (a)\ \dfrac{\partial u}{\partial t_j} = \left(\dfrac{1}{2}\displaystyle\prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i\right)\Delta_x u; & j \in \mathbb{N}_n, \mathbf{t} \in \overset{\circ}{\mathbb{R}}{}^n_+, x \in \mathbb{R}^d, \\ (b)\ u(\mathbf{t}, x) = f(x); & \mathbf{t} \in \partial\mathbb{R}^n_+, x \in \mathbb{R}^d, \end{cases} \tag{1.7}$$

Moreover, $u(\mathbf{t}, x)$ is the unique bounded solution to

$$\begin{cases} (a)\ \dfrac{\partial u}{\partial t_j} = \left(\dfrac{1}{2}\displaystyle\prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i\right)\Delta_x u; & \mathbf{t} \in \overset{\circ}{\mathbb{R}}{}^n_+, x \in \mathbb{R}^d, \\ (b)\ u_{\mathbf{t}_{\hat{j}}}(0, x) = f(x); & \mathbf{t}_{\hat{j}} \in \mathbb{R}^{n-1}_+, x \in \mathbb{R}^d, \end{cases} \tag{1.8}$$

for each $j \in \mathbb{N}_n$. In particular, $u(\mathbf{t}, x)$ is the unique bounded solution to the system (1.7)*.*

As immediate corollaries to Lemma 1.1 we connect the Brownian sheet W^x to both a nonlinear second order PDE and a higher order linear PDE.

Corollary 1.1 (The Brownian sheet second order nonlinear PDE)**.** *Under the same conditions in Lemma 1.1 $u(\mathbf{t}, x) = \mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]$ is a bounded solution to the*

nonlinear second order PDE

$$\begin{cases} \left(\frac{\partial u}{\partial t_1}\right)\cdots\left(\frac{\partial u}{\partial t_n}\right) = \frac{(t_1\cdots t_n)^{n-1}}{2^n}\left(\Delta_x u\right)^n; & \mathbf{t}\in \overset{\circ}{\mathbb{R}}{}^n_+, x\in\mathbb{R}^d, \\ u(\mathbf{t},x) = f(x); & \mathbf{t}\in\partial\mathbb{R}^n_+, x\in\mathbb{R}^d, \end{cases} \tag{1.9}$$

Corollary 1.2 (The Brownian sheet $2n$-th order linear PDE)**.** *Under the same conditions in Lemma 1.1* $u(\mathbf{t},x) = \mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]$ *is a bounded solution to the* $2n$*-th order linear PDE*

$$\begin{cases} \dfrac{\partial^n u}{\partial t_1\cdots\partial t_n} = \mathscr{L}_n(\mathbf{t},u); & \mathbf{t}\in \overset{\circ}{\mathbb{R}}{}^n_+, x\in\mathbb{R}^d, \\ u(\mathbf{t},x) = f(x); & \mathbf{t}\in\partial\mathbb{R}^n_+, x\in\mathbb{R}^d, \end{cases} \tag{1.10}$$

where, for each $\mathbf{t}$*, the* $2n$*-th order operator* $\mathscr{L}_n(\mathbf{t},\cdot)$*, acting on* u*, is obtained by using the Brownian sheet PDE system* (1.7) *and iteratively applying the* t_j *derivatives for* $j\in\mathbb{N}_n$*. In particular, if* $n=1,\ldots,4$*, then*

$$\begin{aligned} \mathscr{L}_1(\mathbf{t},u) &= \frac{1}{2}\Delta u \\ \mathscr{L}_2(\mathbf{t},u) &= \frac{1}{2}\Delta u + \frac{t_1t_2}{4}\Delta^2 u \\ \mathscr{L}_3(\mathbf{t},u) &= \frac{1}{2}\Delta u + \frac{3}{4}t_1t_2t_3\Delta^2 u + \frac{t_1^2t_2^2t_3^2}{8}\Delta^3 u \\ \mathscr{L}_4(\mathbf{t},u) &= \frac{1}{2}\Delta u + \frac{8}{4}t_1t_2t_3\Delta^2 u + \frac{5}{8}t_1^2t_2^2t_3^2\Delta^3 u + \frac{t_1^3t_2^3t_3^3}{16}\Delta^4 u \end{aligned} \tag{1.11}$$

Remark 1.2.

(a) The uniqueness assertion in Lemma 1.1 is not needed in our proof of Theorem 1.1, but we give a uniqueness proof in Appendix A that judiciously exploits the relationship between Brownian sheets and their corresponding space-time white noises to essentially reduce the proof to stochastic analysis in one time parameter.

(b) The uniqueness for (1.8) for each $j\in\mathbb{N}_n$ together with the fact that $u(\mathbf{t},x)$ solves (1.7) means that, in the Brownian-sheet case, the system (1.7) is overdetermined or reducible. I.e., reducing the system (1.7) to (1.8) (for any j) leaves the bounded-solutions set unchanged.

(c) The Brownian sheet heat-type system (1.7) is symmetric in $\mathbf{t}=(t_1,\ldots,t_n)$: if $\pi(\mathbf{t})$ is any one of the $n!$ permutations of $\mathbf{t}=(t_1,\ldots,t_n)$, then $u(\mathbf{t},x)$ and $u_\pi(\mathbf{t},x) = u(\pi(\mathbf{t}),x)$ satisfy (1.8) for each $j\in\mathbb{N}_n$, without change. This is not surprising since the unique solution $u(\mathbf{t},x)=\mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]$ is symmetric in $\mathbf{t}$. Restricting our attention to $\mathbf{t}$-symmetric bounded solutions $s(\mathbf{t},x)$ to the nonlinear PDE (1.9), we obtain that $s(\mathbf{t},x)$ must satisfy (1.8) for each j, and hence s is the unique solution $s(\mathbf{t},x)=\mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]$. A similar observation holds for the higher order PDE (1.10). We leave the details to the interested reader.

Unlike the Brownian motion, "running" the Brownian sheet W^x solves heat PDEs with variable conductivity. The system of heat-type PDEs (1.7) is one in which the j-th PDE (where time is given by t_j) may be thought of as modeling heat flow in a d-dimensional medium whose *thermal diffusivity* (conductivity) is determined by the product of all other $n-1$ parameters t_i, $i\neq j$, in the driving Brownian sheet. I.e., the conductivity of the d-dimensional medium varies with

$n-1$ outside factors represented by the parameters t_i, $i \neq j$, in the driving Brownian sheet. Probabilistically, this is a manifestation of the fact that $\partial_j \mathrm{Var}(W^x_{\mathbf{t}}) = \prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i$—which interacts with the Brownian times $\left|B^1\right|, \ldots, |B^n|$ in the BTBS $\mathbb{W}^x_{B^{(1)},\ldots,B^{(n)}}(\mathbf{t})$ to result in the coupling between u and $\mathscr{U}^j$ in (1.3) and (1.4). Naturally, when $n = 1$, the higher order PDE (1.10), the nonlinear PDE (1.9), and the corresponding system (1.7) reduce to the standard heat equation associated with Brownian motion.

1.2. **A new intimately connected Kuramoto-Sivashinsky-variant interacting PDE system.** Following our Kuramoto-Sivashinsky (KS) links to variants of BTPs in [3, 6, 2, 1], we can modify the BTBS above to obtain solutions to systems of new interacting variants of the celebrated KS equation—both deterministic and stochastic. For a sample of some deterministic KS results and background, the reader may consult [28] and the references therein. Working with a complex variant of the BTBS density, we now briefly provide a link to such deterministic KS-variant interacting system. In what follows $\mathbf{i} = \sqrt{-1}$ and should not be confused with the index variable i.

The first step is to generalize the propagator (complex Gaussian density on $\mathbb{R}^d$) in [3] to the n-parameter case to obtain

$$p^{(d)}_{\mathbf{is}}(x,y) = \frac{e^{-|x-y|^2/2\mathbf{i}\prod_{i=1}^n s_i}}{\left[2\pi\mathbf{i}\prod_{i=1}^n s_i\right]^{d/2}};\ x,y\in\mathbb{R}^d,\ \mathbf{s}=(s_1,\ldots,s_n)\in\mathbb{R}^n. \tag{1.12}$$

Now, motivated by the BTBS case above and our KS link in [3], we define

$$\begin{aligned}
v(\mathbf{s},x) &:= \exp\left(\mathbf{i}\sum_{i=1}^n s_i\right)\int_{\mathbb{R}^d} f(y)p^{(d)}_{\mathbf{is}}(x,y)dy\\
u(\mathbf{t},x) &= \int_{\mathbb{R}^n} v(\mathbf{s},x)\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\\
\mathbb{U}^{(j)}(\mathbf{t},x) &= \int_{\mathbb{R}^n}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}} s_i\right) v(\mathbf{s},x)\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\\
\mathscr{U}^{(j)}(\mathbf{t},x) &= \int_{\mathbb{R}^n}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}} s_i\right)^2 v(\mathbf{s},x)\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}
\end{aligned} \tag{1.13}$$

for $(\mathbf{t},x)\in \overset{\circ}{\mathbb{R}}{}^n_+\times\mathbb{R}^d$ and $j\in\mathbb{N}_n$. To define u, $\mathbb{U}^{(j)}$, and $\mathscr{U}^{(j)}$ on $\partial\mathbb{R}^n_+\times\mathbb{R}^d$ we adopt the following notational convenience: for $\mathbf{t}\in\mathbb{R}^n_+$ and $I\subset\mathbb{N}_n$ we denote by $\zeta_I(\mathbf{t})$ the point in $\mathbb{R}^n_+$ obtained from $\mathbf{t}=(t_1,\ldots,t_n)$ by setting $t_k = 0$ for every $k\in I$. For any function g on $\mathbb{R}^n_+\times\mathbb{R}^d$, $g_{\zeta_I}(\mathbf{t},x) := g(\zeta_I(\mathbf{t}),x)$. With this in mind,

we give the following temporal boundary definitions:

(1.14)

$$u_{\zeta_I}(\mathbf{t},x)=f(x)\exp\left(-\frac{1}{2}\sum_{k\in I^c}t_k\right);\quad I\subset\mathbb{N}_n,$$

$$\mathbb{U}^{(j)}_{\zeta_I}(\mathbf{t},x)=\begin{cases}f(x)\left[\prod_{k\in\mathbb{N}_n\setminus\{j\}}\mathbf{i}t_k\right]\exp\left(-\frac{1}{2}\sum_{k\in\mathbb{N}_n\setminus\{j\}}t_k\right); & I=\{j\},\\ 0; & I\subset\mathbb{N}_n\setminus\{j\},\end{cases}$$

$$\mathscr{U}^{(j)}_{\zeta_I}(\mathbf{t},x)=\begin{cases}f(x)\left[\prod_{k\in\mathbb{N}_n\setminus\{j\}}\left(t_k-t_k^2\right)\right]\exp\left(-\frac{1}{2}\sum_{k\in\mathbb{N}_n\setminus\{j\}}t_k\right); & I=\{j\},\\ 0; & I\subset\mathbb{N}_n\setminus\{j\},\end{cases}$$

Of course, if $I=\mathbb{N}_n$ in (1.14) then $I^c=\phi$ and $u_{\zeta_I}(\mathbf{t},x)=f(x)$.

We call the complex kernel

$$K^{\mathrm{KSS}(n,d)}_{\mathbf{t};x,y}:=\int_{\mathbb{R}^n}\exp\left(\mathbf{i}\sum_{i=1}^n s_i\right)p^{(d)}_{\mathbf{is}}(x,y)\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\tag{1.15}$$

the "transition density" (or kernel) of the n-parameter d-dimensional linear Kuramoto-Sivashinsky sheet (LKSS). We are now ready to state our second main result.

Theorem 1.2 (Linear KS-variant interacting PDEs system)**.** *Let $f\in C^2_c(\mathbb{R}^d;\mathbb{R})$ such that the partial derivatives $\partial_{ij}f$ are Hölder continuous with exponent $0<\alpha\leq 1$, for all $1\leq i,j\leq d$. If $u(\mathbf{t},x)$, $\mathbb{U}^{(j)}(\mathbf{t},x)$, and $\mathscr{U}^{(j)}(\mathbf{t},x)$ are given by* (1.13) *then the family*

$$\left\{u(\mathbf{t},x),\mathbb{U}^{(j)}(\mathbf{t},x),\mathscr{U}^{(j)}(\mathbf{t},x)\right\}_{j\in\mathbb{N}_n}$$

solves the system of linear Kuramoto-Sivashinsky-variant interacting PDEs

$$\frac{\partial}{\partial t_j}u=-\frac{1}{8}\Delta^2\mathscr{U}^{(j)}-\frac{1}{2}\Delta\mathbb{U}^{(j)}-\frac{1}{2}u;\quad \mathbf{t}\in\overset{\circ}{\mathbb{R}}{}^n_+,\,x\in\mathbb{R}^d,\tag{1.16}$$

for $j\in\mathbb{N}_n$, with temporal boundary conditions given by (1.14).

2. Proof of Theorem 1.1

2.1. **A technical lemma: the BTBS case.** We start with a differentiating-under-the-integral type lemma that is needed in the proof of Theorem 1.1.

Lemma 2.1. *Let W^x be an n-parameter d-dimensional Brownian sheet starting at $x\in\mathbb{R}^d$ under $\mathbb{P}$; and let $f:\mathbb{R}^d\to\mathbb{R}$ be bounded and measurable such that $\partial^2_{kl}f$ is Hölder continuous, with exponent $0<\alpha\leq 1$, for $1\leq k,l\leq d$. Fix $j\in\mathbb{N}_n$ and let*

$$\mathscr{E}^{(j)}(\mathbf{t},x):=\int_{\mathbb{R}^n_+}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i^2\right)\mathbb{E}_{\mathbb{P}}\left[f(W^x(\mathbf{s}))\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\tag{2.1}$$

for $j \in \mathbb{N}_n$, $(\mathbf{t}, x) \in \overset{\circ}{\mathbb{R}}{}^n_+ \times \mathbb{R}^d$, *where* $\mathbf{s} = (s_1, \ldots, s_n)$ *and* $K^{\mathrm{BM}}_{t_i;0,s_i}$ *is the transition density of* $B^{(i)}(t_i)$. *Then* $\Delta^2_x \mathscr{E}^{(j)}(\mathbf{t}, x)$ *is finite and*

$$\Delta^2_x \mathscr{E}^{(j)}(\mathbf{t}, x) = \int_{\mathbb{R}^n_+} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i^2 \right) \Delta^2_x \mathbb{E}_{\mathbb{P}} \left[f(W^x(\mathbf{s})) \right] \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \tag{2.2}$$

If we additionally assume that $\partial^2_{kl} f$ *are bounded (for all* $1 \le k, l \le d$*), then* $\Delta^2_x \mathscr{E}^{(j)}(\mathbf{t}, x)$ *is continuous on* $\overset{\circ}{\mathbb{R}}{}^n_+ \times \mathbb{R}^d$.

Proof. For notational simplicity we show that

$$\frac{\partial^4 \mathscr{E}^{(j)}}{\partial x_k^4} = \int_{\mathbb{R}^n_+} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i^2 \right) \frac{\partial^4}{\partial x_k^4} \mathbb{E}_{\mathbb{P}} \left[f(W^x(\mathbf{s})) \right] \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s}; \ k = 1, \ldots, d, \tag{2.3}$$

the mixed derivatives cases follow the same steps. In the remainder of the proof, C denotes a constant that depends only on n, d, and α and that may change its value from line to line. Fix an arbitrary $k \in \{1, \ldots d\}$, and let

$$K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} := \frac{1}{dy} \mathbb{P}[W^x(\mathbf{s}) \in dy] = \frac{\exp\left(\dfrac{-|x-y|^2}{2 \prod_{i=1}^n s_i} \right)}{\left(2\pi \prod_{i=1}^n s_i\right)^{d/2}} \tag{2.4}$$

be the transition density for the Brownian sheet W^x, starting on $\partial \mathbb{R}^n_+$ from $x \in \mathbb{R}^d$, under $\mathbb{P}$, and moving to $y \in \mathbb{R}^d$ at $\mathbf{s} = (s_1, \ldots, s_n) \in \overset{\circ}{\mathbb{R}}{}^n_+$. Using the boundedness on f and Problem 3.1 p. 254 in [23] (the case $\mathbb{R}^n_+ \times \mathbb{R}^d$ with $n, d > 1$ is a simple extension when f is bounded), the symmetry of $K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y}$ in x and y, and the facts that

$$\lim_{y_k \to \pm\infty} f(y) \frac{\partial^3}{\partial y_k^3} K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} = \lim_{y_k \to \pm\infty} \frac{\partial}{\partial y_k} f(y) \frac{\partial^2}{\partial y_k^2} K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} = 0,$$

(since f is bounded and $\frac{\partial}{\partial y_k} f(y)$ is Lipschitz in y_k), we get

$$\begin{aligned} \frac{\partial^4}{\partial x_k^4} \mathbb{E}_{\mathbb{P}} f(W^x(\mathbf{s})) &= \left(\int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial x_k^4} K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} dy \right) \\ &= \left(\int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial y_k^4} K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} dy \right) \\ &= \left(\int_{\mathbb{R}^d} \frac{\partial^2}{\partial y_k^2} f(y) \frac{\partial^2}{\partial y_k^2} K^{\mathrm{BS}(n,d)}_{\mathbf{s};x,y} dy \right). \end{aligned} \tag{2.5}$$

Rewriting the last term in (2.5), and letting $h_k(y) := \partial^2 f(y)/\partial y_k^2$, we have

$$\begin{aligned} &\left(\int_{\mathbb{R}^d} \left(2\pi \prod_{i=1}^n s_i \right)^{-d/2} \left(\frac{(x_k - y_k)^2 - \prod_{i=1}^n s_i}{\prod_{i=1}^n s_i^2} \right) e^{\frac{-|x-y|^2}{2\prod_{i=1}^n s_i}} h_k(y) dy \right) \\ &= \mathbb{E}_{\mathbb{P}} \left[\left(\frac{(x_k - W^{x_k}_k(\mathbf{s}))^2 - \prod_{i=1}^n s_i}{\prod_{i=1}^n s_i^2} \right) h_k(W^x(\mathbf{s})) \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[\left(\frac{(x_k - W^{x_k}_k(\mathbf{s}))^2 - \prod_{i=1}^n s_i}{\prod_{i=1}^n s_i^2} \right) (h_k(W^x(\mathbf{s})) - h_k(x)) \right], \end{aligned} \tag{2.6}$$

where we used the fact that $\mathbb{E}_{\mathbb{P}}\left((x_k - W_k^{x_k}(\mathbf{s}))^2 - \prod_{i=1}^n s_i\right) = 0$ to obtain the last equality. Now, using the Brownian sheet scaling, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left|(x_k - W_k^{x_k}(\mathbf{s}))^2 - \prod_{i=1}^n s_i\right|^2 &= \prod_{i=1}^n s_i^2 \mathbb{E}_{\mathbb{P}}\left|\left(\frac{W_k^0(\mathbf{s})}{\sqrt{\prod_{i=1}^n s_i}}\right)^2 - 1\right|^2 \\ &= \prod_{i=1}^n s_i^2 \mathbb{E}_{\mathbb{P}}\left|\left(W_k^0(1,\dots,1)\right)^2 - 1\right|^2 = C\prod_{i=1}^n s_i^2,\end{aligned} \tag{2.7}$$

for some constant C. Using (2.6) and Cauchy-Schwarz inequality yield

$$\begin{aligned}&\left|\frac{\partial^4}{\partial x_k^4}\mathbb{E}_{\mathbb{P}} f(W^x(\mathbf{s}))\right| \\ &\le \left(\mathbb{E}_{\mathbb{P}}\left(\frac{(x_k - W_k^{x_k}(\mathbf{s}))^2 - \prod_{i=1}^n s_i}{\prod_{i=1}^n s_i^2}\right)^2 \mathbb{E}_{\mathbb{P}}\left|h_k(W^x(\mathbf{s})) - h_k(x)\right|^2\right)^{1/2} \\ &\le \frac{C}{\prod_{i=1}^n s_i}\left(\mathbb{E}_{\mathbb{P}}\left|W^x(\mathbf{s}) - x\right|^{2\alpha}\right)^{1/2} = \frac{C}{\prod_{i=1}^n s_i^{1-\alpha/2}},\end{aligned} \tag{2.8}$$

where the next to last inequality follows from (2.7) and the Hölder condition on h_k. Fixing an arbitrary $j \in \{1,\dots n\}$, we therefore have

$$\begin{aligned}&\int_{\mathbb{R}_+^n}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}} s_i^2\right)\left|\frac{\partial^4}{\partial x_k^4}\mathbb{E}_{\mathbb{P}}\left[f(W^x(\mathbf{s}))\right]\right|\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \\ &\le \frac{C}{\sqrt{2\pi t_j}}\int_{\mathbb{R}_+^{n-1}}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}} s_i^{1+\frac{\alpha}{2}}\right)\prod_{i\in\mathbb{N}_n\setminus\{j\}} K^{\mathrm{BM}}_{t_i;0,s_i}\int_{\mathbb{R}_+}\frac{e^{-s_j^2/2t}}{s_j^{1-\alpha/2}} ds_j d\mathbf{s}_{\not j} \\ &= \frac{C\prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i^{\frac{2+\alpha}{4}}}{t_j^{\frac{2-\alpha}{4}}} < \infty; \qquad \mathbf{t}\in \overset{\circ}{\mathbb{R}}{}_+^n,\ 0<\alpha\le 1.\end{aligned} \tag{2.9}$$

So, (2.3), as well as the continuity assertion for $\partial^4\mathscr{E}^{(j)}/\partial x_k^4$, follow by a standard classical argument. □

2.2. **Proof of the BTBS-PDEs system link.** Using (1.7) in Lemma 1.1 (the proof of Lemma 1.1 is given in the Appendix A), Lemma 2.1, and generalizing our BTP-PDEs approach in [5, 4], we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. It is enough to prove Theorem 1.1 (i) since a solution of (1.3) is automatically a solution of (1.4). First, fix an arbitrary $j \in \{1,\dots,n\}$ and observe that owing to the independence of W^x and $\left\{B^{(i)}\right\}_{i=1}^n$ as well as the independence of all the $B^{(i)}$'s and the boundedness of f together with Fubini's theorem we have

$$u(\mathbf{t},x) = \mathbb{E}_{\mathbb{P}}\left[f\left(\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t})\right)\right] = 2^n\int_{\mathbb{R}_+^n}\mathbb{E}_{\mathbb{P}}\left[f(W^x(\mathbf{s}))\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \tag{2.10}$$

$$
\begin{aligned}
\mathscr{U}^{(j)}(\mathbf{t},x) &= \mathbb{E}_{\mathbb{P}}\left[\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}\left|B^{(i)}(t_i)\right|\right)^2 f\left(\mathbb{W}^x_{B^{(1)},\dots,B^{(n)}}(\mathbf{t})\right)\right] \\
&= 2^n\left[\int_{\mathbb{R}^n_+}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}} s_i\right)^2 \mathbb{E}_{\mathbb{P}}\left[f\left(W^x_{\mathbf{s}}\right)\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\right]
\end{aligned}
\tag{2.11}
$$

for $(\mathbf{t},x)\in\mathring{\mathbb{R}}^n_+\times\mathbb{R}^d$, where $K^{\mathrm{BM}}_{t_i;0,s_i}$ is the transition density of $B^{(i)}(t)$. Differentiating (2.10) with respect to t_j and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $K^{\mathrm{BM}}_{t_j;0,s_j}$ satisfies the heat equation

$$
\frac{\partial}{\partial t_j}K^{\mathrm{BM}}_{t_j;0,s_j} = \frac{1}{2}\frac{\partial^2}{\partial s_j^2}K^{\mathrm{BM}}_{t_j;0,s_j} \tag{2.12}
$$

we have

$$
\begin{aligned}
\frac{\partial}{\partial t_j}u(\mathbf{t},x) &= 2^n\int_{\mathbb{R}^n_+}\mathbb{E}_{\mathbb{P}_x}\left[f(W_{\mathbf{s}})\right]\frac{\partial}{\partial t_j}K^{\mathrm{BM}}_{t_j;0,s_j}\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s} \\
&= 2^{n-1}\int_{\mathbb{R}^n_+}\mathbb{E}_{\mathbb{P}_x}\left[f(W_{\mathbf{s}})\right]\frac{\partial^2}{\partial s_j^2}K^{\mathrm{BM}}_{t_j;0,s_j}\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}
\end{aligned}
\tag{2.13}
$$

Integrating by parts twice in s_j and observing that the boundary terms always vanish at ∞ (as $s_j\nearrow\infty$) and that $(\partial/\partial s_j)K^{\mathrm{BM}}_{t_j;0,s_j}|_{s=0}=0$ but $K^{\mathrm{BM}}_{t_j;0,0}>0$, and then using the Brownian-sheet PDEs connection in Lemma 1.1 (1.7) successively twice we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t_j}u(\mathbf{t},x) &= 2^{n-1}\int_{\mathbb{R}^n_+}\frac{\partial^2}{\partial s_j^2}\mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{s}}\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s} \\
&\quad + 2^{n-1}\int_{\mathbb{R}^{n-1}_+}K^{\mathrm{BM}}_{t_j;0,0}\left(\frac{\partial}{\partial s_j}\mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{s}})\right]\right)\Big|_{s_j=0}\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}_{\not j} \\
&= 2^{n-1}\int_{\mathbb{R}^n_+}\left(\frac{1}{2}\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)^2\Delta_x^2\mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{s}})\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s} \\
&\quad + \frac{2^{n-1}}{\sqrt{2\pi t_j}}\int_{\mathbb{R}^{n-1}_+}\left(\frac{1}{2}\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)\Delta_x f(x)\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}_{\not j} \\
&= 2^{n-3}\Delta_x^2\left[\int_{\mathbb{R}^n_+}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)^2\mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{s}})\right]\prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\right] \\
&\quad + \frac{2^{n-2}}{\sqrt{2\pi t_j}}\Delta_x f(x)\int_{\mathbb{R}^{n-1}_+}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}_{\not j} \\
&= \sqrt{\frac{\prod_{i\in\mathbb{N}_n\setminus\{j\}}t_i}{2^{4-n}\pi^n t_j}}\Delta_x f(x) + \frac{1}{8}\Delta_x^2\mathscr{U}^{(j)}(\mathbf{t},x),
\end{aligned}
$$

where pulling Δ_x^2 outside the integral in the next to last step is justified by Lemma 2.1 under the Hölder continuity and boundedness assumptions on the second partial derivatives of f. The boundary conditions (b), (c), and (d) follow easily from our assumptions on f and from the independence of the $B^{(i)}$'s and from their independence of the Brownian sheet W^x. $\square$

3. Proof of Theorem 1.2

3.1. **A technical lemma: the Kuramoto-Sivashinsky-variant case.** As in the proof of Theorem 1.1, we need the following differentiating-under-the-integral type lemma for the proof of Theorem 1.2.

Lemma 3.1. *Assume $f : \mathbb{R}^d \to \mathbb{R}$ is bounded and measurable such that $\partial_{kl}^2 f$ is Hölder continuous, with exponent $0 < \alpha \leq 1$, for $1 \leq k, l \leq d$. Let v, $\mathscr{U}^{(j)}$, and $\mathbb{U}^{(j)}$ be defined as in* (1.13):

$$\begin{aligned} \mathscr{U}^{(j)}(\mathbf{t}, x) &:= \int_{\mathbb{R}^n} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i^2 \right) v(\mathbf{s}, x) \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \\ \mathbb{U}^{(j)}(\mathbf{t}, x) &:= \int_{\mathbb{R}^n} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i \right) v(\mathbf{s}, x) \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \end{aligned} \tag{3.1}$$

for $j \in \mathbb{N}_n$, $(\mathbf{t}, x) \in \overset{\circ}{\mathbb{R}}{}_+^n \times \mathbb{R}^d$, and $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$; and where $K^{\mathrm{BM}}_{t_i;0,s_i}$ is the transition density of a one dimensional BM, $B^{(i)}(t_i)$, starting at 0. Then $\Delta_x^2 \mathscr{U}^{(j)}(\mathbf{t}, x)$ and $\Delta_x \mathbb{U}^{(j)}(\mathbf{t}, x)$ are finite and

$$\begin{aligned} \Delta_x^2 \mathscr{U}^{(j)}(\mathbf{t}, x) &:= \int_{\mathbb{R}^n} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i^2 \right) \Delta_x^2 v(\mathbf{s}, x) \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \\ \Delta_x \mathbb{U}^{(j)}(\mathbf{t}, x) &:= \int_{\mathbb{R}^n} \left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} s_i \right) \Delta_x v(\mathbf{s}, x) \prod_{i=1}^n K^{\mathrm{BM}}_{t_i;0,s_i} d\mathbf{s} \end{aligned} \tag{3.2}$$

If we additionally assume that $\partial_{kl}^2 f$ are bounded (for all $1 \leq k, l \leq d$), then $\Delta_x^2 \mathscr{U}^{(j)}(\mathbf{t}, x)$ and $\Delta_x \mathbb{U}^{(j)}(\mathbf{t}, x)$ are continuous on $\overset{\circ}{\mathbb{R}}{}_+^n \times \mathbb{R}^d$.

In light of our adaptation of the proof of Lemma 2.1 in [4] to that of Lemma 2.1 in [3] in the one parameter case ($n = 1$), the proof of Lemma 3.1 is a straightforward and similar adaptation of Lemma 2.1 above with only minor changes, and we omit these details.

3.2. **Proof of the Kuramoto-Sivashinsky-variant interacting system link.** We now have the needed ingredients for the proof of Theorem 1.2, which we present next.

Proof of Theorem 1.2. Let u, v, $\mathbb{U}^{(j)}$, and $\mathscr{U}^{(j)}$ be as given in (1.13). Differentiating $u(\mathbf{t}, x)$ with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using (2.12)

and integrating by parts twice using the fact that the boundary terms vanish at $\pm\infty$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t_j}u(\mathbf{t},x) &= \int_{\mathbb{R}^n} v(\mathbf{s},x)\frac{\partial}{\partial t_j}K^{\mathrm{BM}}_{t_j;0,s_j}\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\\
&= \frac{1}{2}\int_{\mathbb{R}^n} v(\mathbf{s},x)\frac{\partial^2}{\partial s_j^2}K^{\mathrm{BM}}_{t_j;0,s_j}\prod_{i\in\mathbb{N}_n\setminus\{j\}}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\\
&= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\partial^2}{\partial s_j^2}v(\mathbf{s},x)\prod_{i\in\mathbb{N}_n}K^{\mathrm{BM}}_{t_i;0,s_i}d\mathbf{s}\\
&= \frac{1}{2}\int_{\mathbb{R}^n}\prod_{i\in\mathbb{N}_n}K^{\mathrm{BM}}_{t_i;0,s_i}\left[-\frac{1}{4}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)^2\Delta_x^2 v(\mathbf{s},x)\right.\\
&\qquad\qquad\left.-\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)\Delta_x v(\mathbf{s},x)-v(\mathbf{s},x)\right]d\mathbf{s}\\
&= -\frac{1}{8}\Delta_x^2\int_{\mathbb{R}^n}\prod_{i\in\mathbb{N}_n}K^{\mathrm{BM}}_{t_i;0,s_i}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)^2 v(\mathbf{s},x)d\mathbf{s}\\
&\quad-\frac{1}{2}\Delta_x\int_{\mathbb{R}^n}\prod_{i\in\mathbb{N}_n}K^{\mathrm{BM}}_{t_i;0,s_i}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)v(\mathbf{s},x)d\mathbf{s}-\int_{\mathbb{R}^n}\prod_{i\in\mathbb{N}_n}K^{\mathrm{BM}}_{t_i;0,s_i}v(\mathbf{s},x)d\mathbf{s}\\
&= -\frac{1}{8}\Delta_x^2\mathscr{U}^{(j)}(\mathbf{t},x)-\frac{1}{2}\Delta_x\mathbb{U}^{(j)}(\mathbf{t},x)-\frac{1}{2}u(\mathbf{t},x)
\end{aligned}\tag{3.3}$$

where we have used the fact that, for each $j\in\mathbb{N}_n$, v satisfies

$$\begin{aligned}
\frac{\partial v}{\partial s_j} &= \frac{\mathbf{i}}{2}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)\Delta_x v(\mathbf{s},x)+\mathbf{i}v(\mathbf{s},x)\\
\frac{\partial^2 v}{\partial s_j^2} &= -\frac{1}{4}\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)^2\Delta_x^2 v(\mathbf{s},x)-\left(\prod_{i\in\mathbb{N}_n\setminus\{j\}}s_i\right)\Delta_x v(\mathbf{s},x)-v(\mathbf{s},x),
\end{aligned}\tag{3.4}$$

and the conditions on f and Lemma 3.1 to take the applications of the derivatives outside the integrals in (3.3) and (3.4). The temporal boundary conditions are trivially satisfied in light of (1.14). The proof is complete. □

Appendix A. Proof of Lemma 1.1

Existence proof. First, let $K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y}$ be the transition density of an n-parameter $\mathbb{R}^d$-valued Brownian sheet $W^x(\mathbf{t}) = W^x(t_1 \ldots, t_n)$, starting on $\partial\mathbb{R}^n_+$ from $x \in \mathbb{R}^d$, under $\mathbb{P}$, and moving to $y \in \mathbb{R}^d$ at $\mathbf{t} = (t_1, \ldots, t_n) \in \overset{\circ}{\mathbb{R}}{}^n_+$:

$$K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} := \frac{1}{dy}\mathbb{P}[W^x_{\mathbf{t}} \in dy] = \frac{1}{(2\pi t_1 \cdots t_n)^{d/2}} \exp\left(\frac{-|x-y|^2}{2t_1 \cdots t_n}\right).$$

Then, it is easy to see that for any $1 \le j \le n$,

$$\frac{\partial}{\partial t_j} K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} = \frac{1}{2}\Delta_x K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} \prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i. \tag{A.1}$$

Now, by the definition of u, the assumptions on f, the dominated convergence theorem, and (A.1), we readily get:

$$\begin{aligned}
\frac{\partial u}{\partial t_j} &= \frac{\partial}{\partial t_j}\int_{-\infty}^{\infty} f(y) K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} dy = \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial t_j} K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} dy \\
&= \frac{1}{2}\left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i\right) \int_{-\infty}^{\infty} f(y) \Delta_x K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} dy \\
&= \frac{1}{2}\left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i\right) \Delta_x \int_{-\infty}^{\infty} f(y) K^{\mathrm{BS}(n,d)}_{\mathbf{t};x,y} dy = \frac{1}{2}\left(\prod_{i \in \mathbb{N}_n \setminus \{j\}} t_i\right) \Delta_x u;
\end{aligned}$$

for $j = 1, \cdots, n$, $\mathbf{t} \in \overset{\circ}{\mathbb{R}}{}^n_+$. By definition and by the continuity and boundedness of f we easily have $\lim_{\substack{\mathbf{s}\to\mathbf{t} \\ y \to x}} u(\mathbf{s}, y) = u(\mathbf{t}, x) = f(x)$, for $\mathbf{t} \in \partial\mathbb{R}^n_+$ and $x \in \mathbb{R}^d$, proving the existence part of Lemma 1.1. □

Uniqueness proof. Fix an arbitrary $j \in \{1, \ldots, n\}$ and fix arbitrary points $\mathbf{t} = (t_1, \ldots, t_j, \ldots, t_n) \in \mathbb{R}^n_+$ and $x \in \mathbb{R}^d$. Let λ be Lebesgue measure on $\mathscr{B}\left(\mathbb{R}^{n-1}_+\right)$, let $\mathscr{R} = \left\{A \subset \mathbb{R}^{n-1}_+; A \in \mathscr{B}\left(\mathbb{R}^{n-1}_+\right), \lambda(A) < \infty\right\}$. Now, let

$$\mathscr{W}^{j,x} = \left\{\mathscr{W}^{j,x}_t(A) = \left(\mathscr{W}^{j,x_1}_{t,1}(A), \ldots, \mathscr{W}^{j,x_d}_{t,d}(A)\right); 0 \le t < \infty, A \in \mathscr{R}\right\} \tag{A.2}$$

be the d-dimensional space-time white noise associated with an n-parameter $\mathbb{R}^d$-valued Brownian-sheet W^x on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$:

$$\mathscr{W}^{j,x}_{s_j}\left(\otimes_{i \in \mathbb{N}_n \setminus \{j\}}[0, t_i]\right) = W^x_{\mathbf{t}_{\not j}}(s_j);\ 0 \le s_j < \infty,$$

(see Walsh [29] for the case $d = 1$) where $\otimes$ denotes the cartesian product of sets, and where the j-th parameter s_j is treated as time and all the other parameters $\mathbf{t}_{\not j} = (t_i)_{i \in \mathbb{N}_n \setminus \{j\}} \in \mathbb{R}^{n-1}_+$ are called space (not to be confused with the spatial variable $x \in \mathbb{R}^d$ in the PDEs (1.7), which is the starting point in $\mathbb{R}^d$ of W^x). Let $\left\{\mathscr{F}^{(j)}_t\right\}_{t \ge 0}$ be the filtration generated by $\mathscr{W}^{j,x}$; i.e.,

$$\mathscr{F}^{(j)}_t = \sigma\left(\left\{\mathscr{W}^{j,x}_{s_j}(A); 0 \le s_j \le t, A \in \mathscr{R}\right\}\right);\ t \ge 0.$$

As is well known, the space-time white noise $\mathscr{W}^{j,x}$ is a continuous orthogonal martingale measure [29]. In particular, for any arbitrary fixed $\mathbf{t}_{\not j} \in \mathbb{R}_+^{n-1}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have that

$$W^x_{\mathbf{t}_{\not j}}(s_j) = \mathscr{W}^{j,x}_{s_j}\left(\otimes_{i\in\mathbb{N}_n\setminus\{j\}}[0,t_i]\right), \mathscr{F}^{(j)}_{s_j};\ 0 \le s_j < \infty$$

is a d-dimensional martingale, in s_j, with coordinates quadratic covariations given by

$$\left\langle W^{x_k}_{k,\mathbf{t}_{\not j}}(\cdot), W^{x_l}_{l,\mathbf{t}_{\not j}}(\cdot)\right\rangle_{s_j} = \delta_{kl} s_j \prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i;\ s_j \in \mathbb{R}_+,\ k,l \in \{1,\ldots,d\}, \tag{A.3}$$

where δ_{kl} is the Kronecker delta ($\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$).

Now, fix arbitrary points $\mathbf{t}_{\not j} \in \mathbb{R}_+^{n-1}$ and $x \in \mathbb{R}^d$. Our first step is to prove

$$\begin{aligned} &M^x_{\mathbf{t}_{\not j}}(s_j) := u_{\mathbf{t}_{\not j}}(t_j - s_j, W^x_{\mathbf{t}_{\not j}}(s_j)),\ \mathscr{F}^{(j)}_{s_j} \text{ is local martingale in } s_j \text{ on } [0,t_j),\\ &\text{for any fixed but arbitrary } t_j > 0, \text{ whenever } u \text{ solves part (a) in (1.8).}\end{aligned} \tag{A.4}$$

To see (A.4), we suppose that u solves the j-th PDE in the system (1.7) (a) ((a) in (1.8)); and we apply Itô's formula to the function $u_{\mathbf{t}_{\not j}}$, in the s_j variable, to get

$$\begin{aligned} &u_{\mathbf{t}_{\not j}}(t_j - s_j, W^x_{\mathbf{t}_{\not j}}(s_j)) - u_{\mathbf{t}_{\not j}}(t_j, W^x_{\mathbf{t}_{\not j}}(0)) = \int_0^{s_j} -\partial_j u_{\mathbf{t}_{\not j}}(t_j - r_j, W^x_{\mathbf{t}_{\not j}}(r_j))dr_j\\ &+ \int_0^{s_j} \nabla u_{\mathbf{t}_{\not j}}(t_j - r_j, W^x_{\mathbf{t}_{\not j}}(r_j)) \cdot dW^x_{\mathbf{t}_{\not j}}(r_j)\\ &+ \frac{1}{2}\left[\prod_{i\in\mathbb{N}_n\setminus\{j\}} t_i\right]\int_0^{s_j} \Delta u_{\mathbf{t}_{\not j}}(t_j - r_j, W^x_{\mathbf{t}_{\not j}}(r_j))dr_j;\ \text{a.s. } \mathbb{P},\end{aligned} \tag{A.5}$$

where ∂_j is the partial derivative with respect to the j-th variable, where we used the dot product notation for the second stochastic integral, and where the third term on the right of (A.5) is seen by (A.3). This, of course, means that $M^x_{\mathbf{t}_{\not j}}(s_j) := u_{\mathbf{t}_{\not j}}(t_j - s_j, W^x_{\mathbf{t}_{\not j}}(s_j))$, $\mathscr{F}^{(j)}_{s_j}$ is a local martingale in s_j on $[0,t_j)$, since the first and the third terms on the right of (A.5) vanish because u is assumed to solve the j-th PDE in the system (1.7) (a). So, (A.4) is proved.

The last step is to show that

$$\text{If } u \text{ is bounded and satisfies (1.8), then } u(\mathbf{t},x) = \mathbb{E}_{\mathbb{P}}[f(W^x(\mathbf{t}))]. \tag{A.6}$$

If u is bounded and satisfies (1.8) (a); then, using (A.4), $M_{\mathbf{t}_{\not j}}(s_j)$, $0 \le s_j < t_j$, is a bounded martingale. This combined with the assumption that u satisfies (1.8) (b) give us that the limit as $s_j \nearrow t_j$ exists and must be

$$M^x_{\mathbf{t}_{\not j}}(t_j) := \lim_{s_j\nearrow t_j} M^x_{\mathbf{t}_{\not j}}(s_j) = f(W^x_{\mathbf{t}_{\not j}}(t_j)),\ \text{a.s.}$$

Since $M^x_{\mathbf{t}_{\not j}}(s_j)$ is uniformly integrable then

$$M^x_{\mathbf{t}_{\not j}}(s_j) = \mathbb{E}_{\mathbb{P}}\left[M^x_{\mathbf{t}_{\not j}}(t_j)|\mathscr{F}^{(j)}_{s_j}\right] = \mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{t}_{\not j}}(t_j))|\mathscr{F}^{(j)}_{s_j}\right];\quad \text{a.s. } 0 \le s_j \le t_j.$$

Taking $s_j = 0$, taking expectation, and remembering the notational conventions in Notation 1.1 we get that

$$u(\mathbf{t},x) = u_{\mathbf{t}_{\not j}}(t_j,x) = \mathbb{E}_{\mathbb{P}} M^x_{\mathbf{t}_{\not j}}(0) = \mathbb{E}_{\mathbb{P}}\left[f(W^x_{\mathbf{t}_{\not j}}(t_j))\right] = \mathbb{E}_{\mathbb{P}}\left[f(W^x(\mathbf{t}))\right], \tag{A.7}$$

and the proof is complete since $j \in \{1, \ldots, n\}$, $\mathbf{t} = (t_1, \ldots, t_j, \ldots, t_n) \in \mathbb{R}^n_+$, and $x \in \mathbb{R}^d$ are arbitrary. □

References

[1] H. Allouba, *BTP SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: Ultra regular BTRW SIEs limits solutions, the K-martingale approach, and fourth order SPDEs links.* Submitted. arXiv:0708.3419v3

[2] H. Allouba, *A Brownian-time excursion into fourth-order PDEs, linearized Kuramoto-Sivashinsky, and BTP-SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$.* Stoch. Dyn. **6** no. 4, (2006), 521–534. MR2285514

[3] H. Allouba, *A linearized Kuramoto-Sivashinsky PDE via an imaginary-Brownian-time-Brownian-angle process.* C. R. Math. Acad. Sci. Paris **336** no. 4, (2003), 309–314. MR1976309

[4] H. Allouba, *Brownian-time processes: the PDE connection II and the corresponding Feynman-Kac formula.* Trans. Amer. Math. Soc. **354** no. 11, (2002), 4627–4637. MR1926892

[5] H. Allouba, and W. Zheng. *Brownian-time processes: the PDE connection and the half-derivative generator.* Ann. Probab. **29** no. 4, (2001), 1780–1795. MR1880242

[6] H. Allouba, *Kuramoto-Sivashinsky-type SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ via the imaginary-Brownian-time-Brownian-angle representation.* To be submitted soon.

[7] H. Allouba, *Hyperbolic SPDEs: Linear system of PDEs and corresponding nonlinear PDEs connections.* Preprint (v1. (1996)–v4. (2009)).

[8] H. Allouba and J. Duan, *Swift-Hohenberg SPDEs driven on $\mathbb{R}_+ \times \mathbb{R}^d$ and their attractors.* In preparation.

[9] H. Allouba and J. A. Langa, *Nonlinear Kuramoto-Sivashinsky type SPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$ and their attractors.* In preparation.

[10] H. Allouba and Y. Xiao, *BTP SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: modulus of continuity, hitting probabilities, Hausdorff dimensions, and d-dependent variation.* In preparation.

[11] R. Bañuelos and D. DeBlassie *The exit distribution of iterated Brownian motion in cones.* Stochastic Process. Appl. **116** no. 1, (2006), 36–69. MR2186839

[12] R. Bass, *Probabilistic techniques in analysis.* Springer-Verlag, New York, 1995. MR1329542

[13] B. Baeumer, M. Meerschaert, and E. Nane *Brownian subordinators and fractional Cauchy problems.* Trans. Amer. Math. Soc. **361** no. 7, (2009), 3915–3930. MR2491905

[14] L. Beghin and E. Orsingher, *Iterated elastic Brownian motions and fractional diffusion equations.* Stochastic Process. Appl. **119** no. 6, (2009), 1975–2003. MR2519353

[15] K. Burdzy, *Some path properties of iterated Brownian motion.* Seminar on Stochastic Processes 1992, Birkhäuser, (1993), 67–87. MR1278077

[16] K. Burdzy, *Variation of iterated Brownian motion.* Workshop and conf. on measure-valued processes, stochastic PDEs and interacting particle systems. CRM Proceedings and Lecture Notes **5**, (1994), 35–53. MR1278281

[17] R. Dalang and F. Russo, *A prediction problem for the Brownian sheet.* J. Multivariate Anal. **26** no. 1, (1988), 16–47. MR0955202

[18] R. Dalang and T. Mountford *Non-independence of excursions of the Brownian sheet and of additive Brownian motion.* Trans. Amer. Math. Soc. **355** no. 3, (2003), no. 3, 967–985. MR1938741

[19] R. DeBlassie, *Iterated Brownian motion in an open set.* Ann. Appl. Prob. **14** no. 3, (2004), 1529–1558. MR2071433

[20] R. Durrett *Stochastic calculus. a practical introduction.* Probability and Stochastics Series. CRC Press, Boca Raton, FL. 1996. MR1398879

[21] T. Funaki, *Probabilistic construction of the solution of some higher order parabolic differential equation.* Proc. Japan Acad. Ser. A Math. Sci. **55** no. 5, (1979), 176–179.

[22] K. Hochberg and E. Orsingher, *Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations.* J. Theoret. Probab. **9**, no. 2, (1996), 511–532.

[23] I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus.* Springer-Verlag, 1988. MR0917065

[24] M. M. Meerschaert, E. Nane, and P. Vellaisamy, *Fractional Cauchy problems on bounded domains.* Ann. Probab. **37** no. 3, (2009), 979–1007. MR2537547

[25] E. Nane, *Higher order PDE's and iterated processes.* Trans. Amer. Math. Soc. **360** no. 5, (2008), 2681–2692. MR2373329

[26] E. Nane, *Stochastic solutions of a class of Higher order Cauchy problems in* $\mathbb{R}^d$. arXiv:0809.4824

[27] E. Orsingher and L. Beghin *Fractional diffusion equations and processes with randomly varying time.* Ann. Probab. **37** no. 1, (2009), 206–249. MR2489164

[28] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics.* Second edition. Applied Mathematical Sciences, 68. Springer-Verlag, New York, 1997. xxii+648 pp. ISBN: 0-387-94866-X MR1441312

[29] J. B. Walsh, *An introduction to stochastic partial differential equations.* Ecole d'Eté de Probabilités de Saint-Flour XIV. Lecture Notes in Math. **1180**. Springer, New York (1986). MR0876085

[30] Y. Xiao, *Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields.* Probab. Th. Rel. Fields **109**, (1997),129–157. MR1469923

[31] Y. Xiao, *Dimension results for Gaussian vector fields and index-*α *stable fields.* Ann. of Probab. **23**, (1995), 273–291. MR1330771

[32] Y. Xiao *A packing dimension theorem for Gaussian random fields.* Statist. Probab. Lett. **79** no. 1, (2009), 88–97. MR2483400

[33] Y. Xiao *Local times and related properties of multidimensional iterated Brownian motion.* J. Theoret. Probab.**11** no. 2, (1998), no. 2, 383–408. MR1622577

Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242
E-mail address: allouba@math.kent.edu