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Abstract

The main goal of this paper is to define the *Chow weight structure* w_{Chow} for the category $DM_c(S)$ of (constructible) Beilinson motives over any excellent separated finite-dimensional base scheme S (this is the version of Voevodsky's motives over S described by Cisinski and Deglise). We also study the functoriality properties of w_{Chow} (they are very similar to those for weights of mixed complexes of sheaves, as established in §5 of [BBD82]).

As shown in a preceding paper, (the existence of) w_{Chow} automatically yields a certain exact conservative weight complex functor $DM_c(S) \to K^b(Chow(S))$. Here $Chow(S)$ is the *heart* of w_{Chow}; it is 'generated' by motives of regular schemes that are projective over S. We also obtain that $K_0(DM_c(S)) \cong K_0(Chow(S))$ (and define a certain 'motivic Euler characteristic' for S-schemes).

Besides, we obtain (Chow)-weight spectral sequences and filtrations for any (co)homology of motives; we discuss their relation with Beilinson's 'integral part' of motivic cohomology and with weights of mixed complexes of sheaves. For the study of the latter we also introduce a new formalism of *relative weight structures*.

Weights for relative motives; relation with mixed complexes of sheaves

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Contents

1 Preliminaries: relative motives and weight structures 8
1.1 Beilinson motives (after Cisinski and Deglise) 8
1.2 Weight structures: short reminder 13

2 The Chow weight structure: two constructions and basic properties 17
2.1 Relative Chow motives; the 'basic' construction of w_{Chow} . . . 18
2.2 Functoriality of w_{Chow} 20
2.3 The 'gluing' construction of w_{Chow} over any (excellent separated finite-dimensional) S 24

3 Applications to (co)homology of motives and other matters 30
3.1 The weight complex functor for $DM_c(S)$ 31
3.2 $K_0(DM_c(S))$ and a motivic Euler characteristic 33
3.3 Chow-weight spectral sequences and filtrations 34
3.4 Application to mixed sheaves: the 'characteristic 0' case 37
3.5 Relative weight structures 38
3.6 On mixed complexes of sheaves over finite fields 44

4 Supplements 45
4.1 On adjacent structures 45
4.2 The Chow t-structure for $DM(S)$ 46
4.3 Virtual t-truncations with respect to w_{Chow}; 'pure' cohomology theories 47

Introduction

The goal of this paper is to prove (independently from [Heb11]) that the *Chow weight structure* w_{Chow} (as introduced in [Bon10a] for Voevodsky's motives over a perfect field k) can also be defined for the category $DM_c(S)$ of motives with rational coefficients over any ('reasonable') base scheme S (in [CiD09] where this category was described and studied, $DM_c(S)$ was called the category of Beilinson motives; one can also consider the 'big' category of S-motives $DM(S) \supset DM_c(S)$ here). The *heart* $\underline{Hw}_{Chow}$ of w_{Chow} is 'generated' by the motives of regular schemes that are projective over S (tensored by $\mathbb{Q}(n)[2n]$ for all $n \in \mathbb{Z}$). We also study the functoriality properties of w_{Chow} (they are very similar to the functoriality of weights for mixed complexes of sheaves, as established in §5 of [BBD82]).

As was shown in [Bon10a], the existence of w_{Chow} yields several nice consequences. In particular, there exists a *weight complex* functor $t : DM_c(S) \to K^b(Chow(S))$, as well as *Chow-weight* spectral sequences and filtrations, and *virtual t-truncations* for any (co)homological functor $H : DM_c(S) \to \underline{A}$.

We also relate the weights for S-motives with the 'integral part' of motivic cohomology (as constructed in [Sch00]; cf. §2.4.2 of [Bei85]), and with the weights of mixed complexes of sheaves (as defined in [BBD82] and in [Hub97]). In order to study the latter we introduce a new formalism of *relative weight structures*.

We also obtain that $K_0(DM_c(S)) \cong K_0(Chow(S))$, and define a certain 'motivic Euler characteristic' for S-schemes.

Now we (try to) explain why the concept of a weight structure is important for motives. Recall that weight structures are natural counterparts of t-structures for triangulated categories; they allow 'decomposing' objects of a triangulated $\underline{C}$ into Postnikov towers whose 'factors' belong to the *heart* $\underline{Hw}$ of w. Weight structures were introduced in [Bon10a] (and independently in [Pau08]). They were thoroughly studied and applied to motives (over perfect fields) in [Bon10a]; in [Bon10b] a *Gersten* weight structure for a certain category $\mathfrak{D}_s \supset DM_{gm}^{eff}(k)$ was constructed; see also the survey preprint [Bon09s].

The Chow weight structure yields certain weights for any (co)homology of motives. Note here: 'classical' methods of working with motives often fail (at our present level of knowledge) since they usually depend on (various) 'standard' motivic conjectures. In particular, the 'classical' way to define weights for a motif M is to construct a motif M_s such that $H^i(M_s) \cong W_s H^i(X)$ (for all $i \in \mathbb{Z}$ and a fixed s; here H is either singular or étale cohomology, and $W_s(-)$ denotes the s-th level of the weight filtration for the corresponding mixed Hodge structure or mixed Galois module). It is scarcely possible to do this without constructing a certain motivic t-structure for $DM(-)$ (whose

existence is very much conjectural at the moment). For instance, in order to find such M_s for motives of smooth projective varieties one requires the so-called Chow-Kunneth decompositions; hence this is completely out of reach at our present level of knowledge.

The usage of weight structures (for motives) allows one to avoid these difficulties completely. To this end instead of $H^i(M_s)$ one considers $\mathrm{Im}(H^i(w_{Chow\ge -s-i}M) \to H^i(w_{Chow\ge -s-i-1}M))$ (this is the corresponding virtual t-truncation of H applied to M; see §4.3 below). Here $w_{Chow\ge r}M$ for $r \in \mathbb{Z}$ are certain motives which can be (more or less) explicitly described in terms of M; note in contrast that there are no general conjectures that allow constructing the motivic t-truncations and Chow-Kunneth decompositions explicitly. Whereas this approach is somewhat 'cheating' for pure motives (since it usually gives no new information on their cohomology), it yields several interesting results on mixed motives and their (co)homology. The first paper somewhat related to this approach is [GiS96] (its main result was generalized in [GiS09]); there a weight complex functor that is essentially a (very) partial case of 'our' one was introduced (and related to cohomology with compact support of varieties).

Another example when constructions naturally coming from weight structures yield interesting results is described in §3.3 below.

Now we mention (other) papers on relative motives that are related with the current one.

This text was written independently from the recent article [Heb11] (that appeared somewhat earlier) on the same subject; our proofs are quite distinct from those of ibid. The main results of ibid. are somewhat stronger than our (central) Theorems 2.1.1 and 2.2.1(II). In particular, in Theorem 3.7 of [Heb11] the functoriality properties of w_{Chow} (as constructed in Theorem 3.3 of ibid.) with respect to motivic functors induced by not necessarily quasi-projective morphisms of schemes (and also with respect to tensor products and inner homomorphisms) were established. Quasi-projectivity was not required there since Theorem 3.2 of ibid. yields the necessary orthogonality property for a not necessarily quasi-projective morphism of schemes. Yet (to the opinion of the author) the proof the latter theorem is more complicated than the proof of (the parallel) Lemma 1.1.4 (below). Besides, in loc. cit. we also calculate morphism groups between 'basic' objects of $\underline{Hw}_{Chow}$ (and their shifts). Moreover, we also prove the existence of the Chow weight structure over an arbitrary excellent separated finite-dimensional S (that is not necessarily reasonable in the sense of Definition 1.1.1 below); the main distinction of this existence result from the 'reasonable' case one is that we do not obtain an explicit description of $Obj\underline{Hw}_{Chow}$.

The author should also note that he would have probably not noticed that

the category $Chow(S) = \underline{Hw}_{Chow}$ has a nice description (over a reasonable S) if not for the papers [CoH00] and [GiS09]. In [CoH00] the definition of Chow motives over S was given as a part of a large program of study of relative motives and intersection cohomology of varieties (that relies on several hard 'motivic' conjectures). In [GiS09] certain analogues of (our) Chow motives were used in order to define (a sort of) weight complexes for S-schemes (only for one-dimensional S; cf. §3.1 below). Yet note that these two articles do not treat (any) triangulated categories of 'mixed' motives over S; hence it is difficult to apply them to cohomology of 'general' (finite type) S-schemes.

This paper (also) benefited from [Sch12]. In ibid. a 'mixed motivic' description of Beilinson's 'integral part' of motivic cohomology (as constructed in [Sch00]; see also §2.4.2 of [Bei85]) was proposed. The formulation of the main result of [Sch12] uses the so-called intermediate extensions of mixed motives; so it heavily relies on the (conjectural!) existence of a 'reasonable' motivic t-structure for $DM_c(S)$; note that we describe an alternative construction of this 'part' that does depend on any conjectures (in §3.3 below; our construction vastly generalizes the one Scholl).

Lastly, we note that several of the results of the current paper were applied in [Bon11b], where the existence of the motivic t-structure for $DM_c(S)$ and of certain 'weights' for its heart were reduced to (certain) standard motivic conjectures over universal domains.

Now we list the contents of the paper. More details could be found at the beginnings of sections.

In §1 we recall the basic properties of Beilinson motives and weight structures. Most of the statements of the section are contained in [CiD09] and [Bon10a]; yet we also prove some new results.

In §2 we define the category $Chow(S)$ of Chow motives over S (related definitions can be found in [CoH00], [Heb11], and [GiS09]). By definition, $Chow(S) \subset DM_c(S)$; since $Chow(S)$ is also negative in it and generates it (if S is 'reasonable') we immediately obtain (using Theorem 4.3.2 of [Bon10a]) that there exists a weight structure w_{Chow} on $DM_c(S)$ whose heart is $Chow(S)$. Next we study the 'functoriality' of w_{Chow} (with respect to the functors of the type $f^*, f_*, f^!, f_!$, for f being a quasi-projective or, more generally, a *smoothly embeddable* morphism of schemes). Our functoriality statements are parallel to the 'stabilities' 5.1.14 of [BBD82] (we 'explain' this similarity in the succeeding section). We also prove that Chow motives can be 'lifted from open subschemes up to retracts'; this statement could be called (a certain) 'motivic resolution of singularities'. Next we prove that w_{Chow} can be described 'pointwisely' (cf. §5.1.8 of [BBD82]). Besides, we describe an alternative method for the construction of w_{Chow} (over arbitrary excellent separated finite-dimensional schemes; these don't have to be rea-

sonable). This method uses stratifications and 'gluing' of weight structures; it makes this part of the paper somewhat parallel to the study of weights of mixed complexes of sheaves in §5 of [BBD82].

Section 3 is dedicated to the applications of our main results. The existence of w_{Chow} automatically yields the existence of a conservative exact *weight complex* functor $DM_c(S) \to K^b(Chow(S))$, and the fact that $K_0(DM_c(S)) \cong K_0(Chow(S))$. We also define a certain 'motivic Euler characteristic' for S-schemes.

Next we recall that w_{Chow} yields functorial *Chow-weight* spectral sequences and filtrations. A very partial case of Chow-weight filtrations yields Beilinson's 'integral part' of motivic cohomology. Chow-weight spectral sequences yield the existence of weight filtrations for 'perverse étale homology' of motives over finite type $\mathbb{Q}$-schemes (this is not automatic for mixed perverse sheaves in this setting). We study in more detail the perverse étale homology of motives when $S = X_0$ is a variety over a finite field $\mathbb{F}_q$. It is well-known that mixed complexes of sheaves start to behave better if we extend scalars from $\mathbb{F}_q$ to $\mathbb{F}$ (this is the algebraic closure of $\mathbb{F}_q$) i.e. pass to sheaves over $X = X_0 \times_{\mathrm{Spec}\ \mathbb{F}_q} \mathrm{Spec}\ \mathbb{F}$. We (try to) axiomatize this situation and introduce the concept of a *relative weight structure*. Relative weight structures have several properties that are parallel to properties of 'ordinary' weight structures. The category $D^b_m(X_0)$ (of mixed complexes of sheaves) possesses a relative weight structure whose heart is the class of (pure) complexes of sheaves of weight 0. Besides, the étale realization functor $DM_c(X_0) \to D^b_m(X_0)$ is *weight-exact*.

In §4 we recall the definition of a t-structure *adjacent* to a weight structure. Then we prove the existence of a (Chow) t-structure t_{Chow} for $DM(S)$ that is adjacent to the Chow weight structure for it. We also study the functoriality of t_{Chow} and relate it with *virtual t-truncations* (for cohomological functors from $DM_c(S)$).

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Notation. Ab is the category of abelian groups.

For a category C, $A, B \in ObjC$, we denote by $C(A, B)$ the set of C-morphisms from A into B.

For categories C, D we write $C \subset D$ if C is a full subcategory of D.

For a category C, $X, Y \in ObjC$, we say that X is a *retract* of Y if id_X

can be factorized through Y (if C is triangulated or abelian, then X is a retract of Y whenever X is its direct summand).

For an additive $D \subset C$ the subcategory D is called *Karoubi-closed* in C if it contains all retracts of its objects in C. The full subcategory of C whose objects are all retracts of objects of D (in C) will be called the *Karoubi-closure* of D in C.

$M \in ObjC$ will be called compact if the functor $C(M, -)$ commutes with all small coproducts that exist in C. In this paper (in contrast with the previous ones) we will only consider compact objects in those categories that are closed with respect to arbitrary small coproducts.

$\underline{C}$ below will always denote some triangulated category; usually it will be endowed with a weight structure w (see Definition 1.2.1 below).

We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a functor that preserves the structures of triangulated categories). We will call a covariant (resp. contravariant) additive functor $H : \underline{C} \to \underline{A}$ for an abelian $\underline{A}$ *homological* (resp. *cohomological*) if it converts distinguished triangles into long exact sequences.

For $f \in \underline{C}(X, Y)$, $X, Y \in Obj\underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of f; recall that distinct choices of cones are connected by (non-unique) isomorphisms.

We will often specify a distinguished triangle by two of its morphisms.

For a set of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle ObjD \rangle$. We will call the Karoubi-closure of $\langle C_i \rangle$ in $\underline{C}$ the *triangulated category generated by* C_i.

For $X, Y \in Obj\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset Obj\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by $D^{\perp}$ the class

$$\{Y \in Obj\underline{C} : \ X \perp Y \ \forall X \in D\}.$$

Sometimes we will denote by $D^{\perp}$ the corresponding full subcategory of $\underline{C}$. Dually, ${}^{\perp}D$ is the class $\{Y \in Obj\underline{C} : \ Y \perp X \ \forall X \in D\}$.

We will say that some C_i, $i \in I$, *weakly generate* $\underline{C}$ if for $X \in Obj\underline{C}$ we have: $\underline{C}(C_i[j], X) = \{0\} \ \forall i \in I, \ j \in \mathbb{Z} \implies X = 0$ (i.e. if $\{C_i[j]\}^{\perp}$ contains only zero objects).

$D \subset Obj\underline{C}$ will be called *extension-stable* if for any distinguished triangle $A \to B \to C$ in $\underline{C}$ we have: $A, C \in D \implies B \in D$.

We will call the smallest Karoubi-closed extension-stable subclass of $Obj\underline{C}$ containing D the *envelope* of D.

Below all schemes will be excellent separated of finite Krull dimension. Often our schemes will be *reasonable*; see Definition 1.1.1 below.

Morphisms of schemes by default will be of finite type. We will say that X/S is *smoothly embeddable* if it can be embedded into a smooth X'/S (in particular, such an X is quasi-finite over S). Certainly, any quasi-projective S-scheme is smoothly embeddable (over S).

We will sometimes need certain stratifications of a scheme S. Recall that a stratification α is a presentation of S as $\cup S_l^\alpha$ where S_l^α, $1 \le l \le n$, are pairwise disjoint locally closed subschemes of S. Omitting α, we will denote by $j_l : S_l^\alpha \to S$ the corresponding immersions. We do not demand the closure of each S_l^α to be the union of strata (though we could do this); we will only assume that each S_l^α is open in $\cup_{i \ge l} S_i^\alpha$.

Below we will identify a Zariski point (of a scheme S) with the spectrum of its residue field.

1 Preliminaries: relative motives and weight structures

In §1.1 we recall some of basic properties of Beilinson motives over S (as considered in [CiD09]; we also deduce certain results that were not stated in ibid. explicitly).

In §1.2 we recall some basics of the theory of weight structures (as developed in [Bon10a]); we also prove some new lemmas on the subject.

1.1 Beilinson motives (after Cisinski and Deglise)

We list some of the properties of the triangulated categories of Beilinson motives (this is the version of relative Voevodsky's motives with rational coefficients described by Cisinski and Deglise). Sometimes we will need the following restriction on schemes.

Definition 1.1.1. We will call a separated scheme S *reasonable* if there exists an excellent separated scheme S_0 of dimension lesser than or equal to 2 such that S is of finite type over S_0.

Proposition 1.1.2. *Let X, Y be (excellent separated finite dimensional) schemes; $f : X \to Y$ is a finite type morphism.*

1. *For any X a tensor triangulated $\mathbb{Q}$-linear category $DM(X)$ with the unit object $\mathbb{Q}_X$ is defined; it is closed with respect to arbitrary small coproducts.*

   $DM(X)$ is the category of Beilinson motives over X, as described (and thoroughly studied) in §14–15 of [CiD09].

2. *The (full) subcategory $DM_c(X) \subset DM(X)$ of compact objects is tensor triangulated, and $\mathbb{Q}_X \in ObjDM_c(S)$. $DM_c(X)$ weakly generates $DM(X)$.*

3. *All $DM(X)$ and $DM_c(X)$ are idempotent complete.*

4. *For any f the following functors are defined: $f^* : DM(Y) \leftrightarrows DM(X) : f_*$ and $f_! : DM(X) \leftrightarrows DM(Y) : f^!$; f^* is left adjoint to f_* and $f_!$ is left adjoint to $f^!$.*

   *We call these the **motivic image functors**. Any of them (when f varies) yields a 2-functor from the category of (separated finite-dimensional excellent) schemes with morphisms of finite type to the 2-category of triangulated categories. Besides, all motivic image functors preserve compact objects (i.e. they could be restricted to the subcategories $DM_c(-)$); they also commute with arbitrary (small) coproducts.*

5. *For a Cartesian square of finite type morphisms*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

   we have $g^* f_! \cong f'_! g'^*$ and $g'_* f'^! \cong f^! g_*$.

6. *For any X there exists a Tate object $\mathbb{Q}(1) \in ObjDM_c(X)$; tensoring by it yields an exact Tate twist functor $-(1)$ on $DM(X)$. This functor is an auto-equivalence of $DM(X)$; we will denote the inverse functor by $-(-1)$.*

   Tate twists commute with all motivic image functors mentioned (up to an isomorphism of functors).

   Besides, for $X = \mathbb{P}^1(Y)$ there is a functorial isomorphism $f_!(\mathbb{Q}_{\mathbb{P}^1(Y)}) \cong \mathbb{Q}_Y \bigoplus \mathbb{Q}_Y(-1)[-2]$.

7. *f^* is symmetric monoidal; $f^*(\mathbb{Q}_Y) = \mathbb{Q}_X$.*

8. *$f_* \cong f_!$ if f is proper; $f^!(-) \cong f^*(-)(s)[2s]$ if f is smooth and (everywhere) of relative dimension s.*

   If f is an open immersion, we just have $f^! = f^*$.

9. *If $i : S' \to S$ is an immersion of regular schemes everywhere of codimension d, then $\mathbb{Q}_{S'}(-d)[-2d] \cong i^!(\mathbb{Q}_S)$.*

10. *If $i : Z \to X$ is a closed immersion, $U = X \setminus Z$, $j : U \to X$ is the complementary open immersion, then the motivic image functors yield gluing data for $DM(-)$ (in the sense of §1.4.3 of [BBD82]; see also Definition 8.2.1 of [Bon10a]). That means that (in addition to the adjunctions given by assertion 4) the following statements are valid.*

    (i) $i_* \cong i_!$ is a full embeddings; $j^* = j^!$ is isomorphic to the localization (functor) of $DM(X)$ by $i_*(DM(Z))$.

    (ii) For any $M \in ObjDM(X)$ the pairs of morphisms $j_!j^!(M) \to M \to i_*i^*(M)$ and $i_!i^!(M) \to M \to j_*j^*(M)$ can be completed to distinguished triangles (here the connecting morphisms come from the adjunctions of assertion 4).

    (iii) $i^*j_! = 0$; $i^!j_* = 0$.

    (iv) All of the adjunction transformations $i^*i_* \to 1_{DM(Z)} \to i^!i_!$ and $j^*j_* \to 1_{DM(U)} \to j^!j_!$ are isomorphisms of functors.

11. *For the subcategories $DM_c(-) \subset DM(-)$ the obvious analogue of the previous assertion is fulfilled.*

12. *If f is a finite universal homeomorphism, f^* is an equivalence of categories.*

13. *If S is reasonable (see Definition 1.1.1), $DM_c(S)$ (as a triangulated category) is generated by $\{g_*(\mathbb{Q}_X)(r)\}$, where $g : X \to S$ runs through all projective morphisms (of finite type) such that X is regular, $r \in \mathbb{Z}$.*

14. *Let S be a scheme which is the limit of an essentially affine (filtering) projective system of schemes S_β. Then $DM_c(S)$ is isomorphic to the 2-colimit of the categories $DM_c(S_\beta)$; in these isomorphisms all the connecting functors are given by the corresponding motivic inverse image functors (cf. Remark 1.1.3(2) below).*

15. *If X is regular (everywhere) of dimension d, $i : Z \to X$ is a closed embedding, $p, q \in \mathbb{Z}$, then $DM(X)(\mathbb{Q}_X, i_!i^!(\mathbb{Q}_Z)[p](q)) \cong DM(Z)(\mathbb{Q}_Z, i^!(\mathbb{Q}_X)(q)[p])$ is isomorphic to $Chow_{d-q}(Z, 2q-p)\otimes\mathbb{Q}$ (which we define as $Gr^\gamma_{d-q}K'_{2q-p}(Z)\otimes\mathbb{Q}$). In particular, this morphism group is zero if $p > 2q$.*

Proof. Almost all of these properties of Beilinson motives are stated in (part C of) the Introduction of ibid.; the proofs are mostly contained in §1, §2, §14, and §15 of ibid.

So, we will only prove those assertions that are not stated in ibid. (explicitly).

For (3): Since $DM(X)$ is closed with respect to arbitrary small coproducts, it is idempotent complete by Proposition 1.6.8 of [Nee01]. Since a retract of a compact object is compact also, $DM_c(X)$ is also idempotent complete.

Since $i^! = i^*$ if i is an open immersion, and $i^*(\mathbb{Q}_S) = \mathbb{Q}_{S'}$, it suffices to prove (9) for i being a closed immersion. In this case it is exactly Theorem 4 of [CiD09].

We should also prove (11). Assertion 10 immediately yields everything expect the fact that the (categoric) kernel of $j^* : DM_c(X) \to DM_c(U)$ is contained in $i_*(DM_c(Z))$. So, we should prove that $i_*(ObjDM(Z)) \cap ObjDM_c(X) = i_*(ObjDM_c(Z))$. This is easy, since $i^*i_* \cong 1_{DM(Z)}$ and i_*i^* preserves compact objects.

Assertion 12 is given by Proposition 2.1.9 of of [CiD09] (note that we can apply the result cited by Theorem 14.3.3 of ibid.).

Assertion 13 is immediate from Proposition 15.2.3 of ibid.

It remains to prove (15). Combining (13.4.1.3) and Corollary 14.2.14 of ibid., we obtain that the groups in question are isomorphic to the q-th factor of the γ-filtration of $K^Z_{2q-p}(X) \otimes \mathbb{Q}$ (of the K-theory of X with support in Z). By Theorem 7 of [Sou85], this is the exactly the $d-q$-th factor of the γ-filtration of $K'_{2q-p}(Z) \otimes \mathbb{Q}$.

□

Remark 1.1.3. 1. In [CiD09] for a smooth $f : X \to Y$ the object $f_!f^!(\mathbb{Q}_Y)$ was denoted by $\mathcal{M}_Y(X)$ (cf. also Definition 1.3 of [Sch12]; yet note that in loc. cit. cohomological motives are considered, this interchanges $*$ with ! in the notation for motivic functors). We will not usually need this notation below (yet cf. Remarks 2.1.2(1) and 3.3.2(4)).

2. In [CiD09] the functor f^* was constructed for any morphism f not necessarily of finite type; it preserves compact objects (see Theorem 15.2.1(1) of ibid.). Besides, for such an f and any finite type $g : X' \to X$ we have an isomorphism $f^*g_! \cong g'_!f'^*$ (for the corresponding f' and g'; cf. part 5 of the proposition).

Below the only morphisms of infinite type that we will be interested in are limits of immersions (more precisely, for a Zariski point K of a scheme S we will consider the natural morphism $j_K : K \to S$; cf. Notation).

Now note: if f is a pro-open immersion, then one can define $f^! = f^*$. So, one can also define $j^!_K$ that preserves compact objects (cf. also §2.2.12 of [BBD82]). The system of these functors satisfy the second assertion in part 5 of the proposition (for a finite type g).

3. If f is a finite universal homeomorphism, then f^* is an equivalence of categories by assertion 12 of our proposition. Hence its right adjoint f_* is an

equivalence also. Since $f^!$ is right adjoint to $f_! = f_*$, we conclude that $f_!$ and $f^!$ are equivalences too.

Similarly we obtain that $f^*\mathbb{Q}_Y \cong f^!\mathbb{Q}_Y \cong \mathbb{Q}_X$ and $f_*\mathbb{Q}_X = f_!\mathbb{Q}_X \cong \mathbb{Q}_Y$.

4. Most of the properties of Beilinson motives (as listed above) also hold for various 'sheaf-like' categories. In particular, the methods of the current paper could probably be used in order to prove the existence of the weight structure w constructed in Proposition 2.3(I) of [Bon12] for M. Saito's mixed Hodge modules (see [Sai89]). Yet the properties of mixed Hodge modules listed in §1 of ibid. yield the existence of w immediately.

5. A nice concise exposition of the properties of Beilinson motives (that also follows [CiD09]) can be found in §2 of [Heb11].

The following statements were not proved in [CiD09] explicitly; yet they follow from Proposition 1.1.2 easily. Below we will mostly need assertion I1 in the case when g is projective; note that in this case $g_*(\mathbb{Q}_Y) \cong g_!(\mathbb{Q}_Y)$.

Lemma 1.1.4. *I1. Let Y be a regular scheme everywhere of dimension d; let $f : X \to S$ and $g : Y \to S$ be smoothly embeddable morphisms (see the Notation), $r, b, c \in \mathbb{Z}$.*

Then $DM(S)(f_!(\mathbb{Q}_X)(b)[2b], g_*(\mathbb{Q}_Y)(c)[r+2c]) \cong CH_{d+b-c}(X \times_S Y, -r)$ (cf. Proposition 1.1.2(15) for the definition of the latter). In particular, $f_!(\mathbb{Q}_X)(b)[2b] \perp g_*(\mathbb{Q}_Y)(c)[r+2c]$ if $r > 0$.

2. Let $i : S' \to S$ be an immersion of regular schemes everywhere of codimension d; let g be smooth. Then for $Y' = Y_{S'}$ and $g' = g_{S'}$ we have $i^! g_*(\mathbb{Q}_Y) \cong g'_*(\mathbb{Q}_{Y'})(-d)[-2d]$.

II Let $S = \cup S_l^\alpha$ be a stratification. Then for any $M, N \in ObjDM(S)$ there exists a filtration of $DM(S)(M,N)$ by certain subfactors of $DM(S_l^\alpha)(j_l^*(M), j_l^!(N))$.

Proof. I1. By Proposition 1.1.2(6), we can assume that $b = c = 0$ (to this end we should possibly replace X and Y by $(\mathbb{P}^1)^n(X)$ and $(\mathbb{P}^1)^m(Y)$ for some $n, m \geq 0$).

Next, we have $DM(S)(f_!(\mathbb{Q}_X), g_*(\mathbb{Q}_Y)[r]) \cong DM(Y)(g^* f_!(\mathbb{Q}_X), \mathbb{Q}_Y[r])$ since g^* is left adjoint to g_*. Applying part 5 of loc. cit., we obtain that the group in question is isomorphic to

$$DM(Y)(f'_! g'^*(\mathbb{Q}_X), \mathbb{Q}_Y[r]) = DM(Y)(f'_!(\mathbb{Q}_{X\times_S Y}), \mathbb{Q}_Y[r])$$

(here $f' = f_Y$).

We denote $X \times_S Y$ by Z. Let P be a smooth Y-scheme containing Z as a closed subscheme; we denote by $i : Z \to P$ and $p : P \to Y$ the corresponding morphisms. We can assume that P is everywhere of some dimension d' over Y.

Then we have

$$DM(Y)(f'_!(\mathbb{Q}_Z),\mathbb{Q}_Y[r])=DM(S)(p_!i_!(\mathbb{Q}_Z),\mathbb{Q}_Y[r])\cong DM(P)(i_!(\mathbb{Q}_Z),p^!(\mathbb{Q}_Y)[r])$$

(here we apply the adjunction of $p_!$ with $p^!$). By part 8 of loc. cit., the group in question is isomorphic to $DM(P)(i_!(\mathbb{Q}_Z),p^*(\mathbb{Q}_Y)(d')[r+2d'])\cong DM(P)(i_!(\mathbb{Q}_Z),\mathbb{Q}_P(d')[r+2d'])$. It remains to apply part 15 of loc. cit.

2. $i^!g_*(\mathbb{Q}_Y)\cong g'_*i'^!(\mathbb{Q}_Y)$ by part 5 of loc. cit. (here $i'=i_Y$). Hence using part 9 of loc. cit. we obtain the result.

II We prove the statement by induction on the number of strata. By definition (see the Notation section) S_1^α is open in S, and the remaining S_l^α yield a stratification of $S\setminus S_1^\alpha$. We denote $S\setminus S_1^\alpha$ by Z, the (open) immersion $S_1^\alpha\to S$ by j and the (closed) immersion $Z\to S$ by i.

Now we apply part 10 of loc. cit. We obtain a (long) exact sequence $\dots\to DM(S)(i_*i^*(M),N)\to DM(S)(M,N)\to DM(S)(j_!j^!(M),N)\to\dots$. The adjunctions of functors yield $DM(S)(i_*i^*(M),N)\cong DM(Z)(i^*(M),i^!(N))$ and $DM(S)(j_!j^!(M),N')\cong DM(S_1^\alpha)(j^*(M),j^!(N))$.

Now, by the inductive assumption the group $DM(Z)(i^*(M),i^!(N))$ has a filtration by some subquotients of $DM(S_l^\alpha)(j_l^*(M),j^!(N))$ (for $l\neq 1$). This concludes the proof.

□

1.2 Weight structures: short reminder

Definition 1.2.1. I A pair of subclasses $\underline{C}_{w\le 0},\underline{C}_{w\ge 0}\subset Obj\underline{C}$ will be said to define a weight structure w for $\underline{C}$ if they satisfy the following conditions:

(i) $\underline{C}_{w\ge 0},\underline{C}_{w\le 0}$ are additive and Karoubi-closed in $\underline{C}$ (i.e. contain all $\underline{C}$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**

$\underline{C}_{w\le 0}\subset\underline{C}_{w\le 0}[1]$, $\underline{C}_{w\ge 0}[1]\subset\underline{C}_{w\ge 0}$.

(iii) **Orthogonality.**

$\underline{C}_{w\le 0}\perp\underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**.

For any $M\in Obj\underline{C}$ there exists a distinguished triangle

$$B\to M\to A\xrightarrow{f}B[1] \tag{1}$$

such that $A\in\underline{C}_{w\ge 0}[1]$, $B\in\underline{C}_{w\le 0}$.

II The category $\underline{Hw}\subset\underline{C}$ whose objects are $\underline{C}_{w=0}=\underline{C}_{w\ge 0}\cap\underline{C}_{w\le 0}$, $\underline{Hw}(Z,T)=\underline{C}(Z,T)$ for $Z,T\in\underline{C}_{w=0}$, will be called the *heart* of w.

III $\underline{C}_{w\ge i}$ (resp. $\underline{C}_{w\le i}$, resp. $\underline{C}_{w=i}$) will denote $\underline{C}_{w\ge 0}[i]$ (resp. $\underline{C}_{w\le 0}[i]$, resp. $\underline{C}_{w=0}[i]$).

IV We denote $\underline{C}_{w\ge i}\cap\underline{C}_{w\le j}$ by $\underline{C}_{[i,j]}$ (so it equals $\{0\}$ for $i>j$).

V We will say that $(\underline{C},w)$ is *bounded* if $\cup_{i\in\mathbb{Z}}\underline{C}_{w\le i}=Obj\underline{C}=\cup_{i\in\mathbb{Z}}\underline{C}_{w\ge i}$.

VI Let $\underline{C}$ and $\underline{C}'$ will be triangulated categories endowed with weight structures w and w', respectively; let $F:\underline{C}\to\underline{C}'$ be an exact functor.

F will be called *left weight-exact* (with respect to w,w') if it maps $\underline{C}_{w\le 0}$ to $\underline{C}'_{w'\le 0}$; it will be called *right weight-exact* if it maps $\underline{C}_{w\ge 0}$ to $\underline{C}'_{w'\ge 0}$. F is called *weight-exact* if it is both left and right weight-exact.

VII Let H be a full subcategory of a triangulated $\underline{C}$.

We will say that H is *negative* if $ObjH\perp(\cup_{i>0}Obj(H[i]))$.

VIII We call a category $\frac{A}{B}$ a *factor* of an additive category A by its (full) additive subcategory B if $Obj\left(\frac{A}{B}\right)=ObjA$ and $(\frac{A}{B})(M,N)=A(M,N)/(\sum_{O\in ObjB}A(O,N)\circ A(M,O))$.

IX For an additive B we will consider the category of 'formal coproducts' of objects of B: its objects are (formal) $\coprod_{j\in J}B_j:\ B_j\in ObjB$, and $Mor(\coprod_{l\in L}B_l,\coprod_{j\in J}B'_j)=\prod_{l\in L}(\bigoplus_{j\in J}B(B_l,B'_j))$; here L,J are index sets. We will call the idempotent completion of this category the *big hull* of B.

Remark 1.2.2. 1. If B is a full subcategory of compact objects in an additive C, and C is idempotent complete and closed with respect to arbitrary small coproducts, then there exists a natural full embedding of the big hull of B into C. Note here: if C is triangulated and closed with respect to arbitrary small coproducts, then it is necessarily idempotent complete (see Proposition 1.6.8 of [Nee01]).

2. A simple (and yet useful) example of a weight structure comes from the stupid filtration on the homotopy categories of cohomological complexes $K(B)\supset K^b(B)$ for an arbitrary additive category B. In this case $K(B)_{w\le 0}$ (resp. $K(B)_{w\ge 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees ≥ 0 (resp. ≤ 0). The heart of this weight structure (either for $K(B)$ or for $K^b(B)$) is the the Karoubi-closure of B in the corresponding category. We will use the notation $K(B)_{[i,j]}$ below following Definition 1.2.1(IV).

3. A weight decomposition (of any $M\in Obj\underline{C}$) is (almost) never unique; still we will sometimes denote (any choice of) a pair $(B,A[-1])$ coming from in (1) by $(M^{w\le 0},M^{w\ge 1})$.

$M^{w\le l}\in\underline{C}_{w\le 0}$ (resp. $M^{w\ge l}\in\underline{C}_{w\ge 0}$) will denote $(M[-l])^{w\le 0}$ (resp. $(M[1-l]))^{w\ge 1}$); we will also need $w_{\le l}M=M^{w\le l}[l]$ and $w_{\ge l}M=M^{w\ge l}[l]$.

We will call (any choices of) $w_{\le l}M$, $w_{\ge l}M$, $M^{w\le l}$, and $M^{w\ge l}$ *weight truncations* of M. A certain illustration of this notation (in a more general context of relative weight structure) can be found in Proposition 3.5.3(3) below.

4. In the current paper we use the 'homological convention' for weight structures; it was previously used in [Heb11], [Wil12], [Bon11a], and [Bon12],

whereas in [Bon10a] and in [Bon10b] the 'cohomological convention' was used. In the latter convention the roles of $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ are interchanged i.e. one considers $\underline{C}^{w\le 0}=\underline{C}_{w\ge 0}$ and $\underline{C}^{w\ge 0}=\underline{C}_{w\le 0}$. So, a complex $X\in ObjK(B)$ whose only non-zero term is the fifth one (i.e. $X^5\neq 0$; cf. Definition 3.5.1(5) below) has weight -5 in the homological convention, and has weight 5 in the cohomological convention. Thus the conventions differ by 'signs of weights'; $K(B)_{[i,j]}$ is the class of retracts of complexes concentrated in degrees $[-j.-i]$.

Now we recall those properties of weight structures that will be needed below (and that can be easily formulated). We will not mention more complicated matters (weight spectral sequences and weight complexes) here; instead we will just formulate the corresponding 'motivic' results below.

Proposition 1.2.3. *Let $\underline{C}$ be a triangulated category.*

1. *(C_1,C_2) ($C_1,C_2\subset Obj\underline{C}$) define a weight structure for $\underline{C}$ whenever (C_2^{op},C_1^{op}) define a weight structure for $\underline{C}^{op}$.*

2. *Let w be a weight structure for $\underline{C}$. Then $\underline{C}_{w\le 0}$, $\underline{C}_{w\ge 0}$, and $\underline{C}_{w=0}$ are extension-stable.*

   Besides, for any $M\in\underline{C}_{w\le 0}$ we have $w_{\ge 0}M\in\underline{C}_{w=0}$ (for any choice of $w_{\ge 0}M$).

3. *Let w be a weight structure for $\underline{C}$. Then $\underline{C}_{w\ge 0}=(\underline{C}_{w\le -1})^{\perp}$ and $\underline{C}_{w\le 0}={}^{\perp}\underline{C}_{w\ge 1}$ (see Notation).*

4. *Suppose that v,w are weight structures for $\underline{C}$; let $\underline{C}_{v\le 0}\subset\underline{C}_{w\le 0}$ and $\underline{C}_{v\ge 0}\subset\underline{C}_{w\ge 0}$. Then $v=w$ (i.e. the inclusions are equalities).*

5. *Let w be a bounded weight structure for $\underline{C}$. Then w extends to a bounded weight structure for the idempotent completion $\underline{C}'$ of $\underline{C}$ (i.e. there exists a weight structure w' for $\underline{C}'$ such that the embedding $\underline{C}\to\underline{C}'$ is weight-exact); its heart is the idempotent completion of $\underline{Hw}$.*

6. *Let $H\subset Obj\underline{C}$ be negative; let $\underline{C}$ be idempotent complete. Then there exists a unique weight structure w on the Karoubi-closure T of $\langle H\rangle$ in $\underline{C}$ such that $H\subset T_{w=0}$. Its heart is the envelope (see the Notation) of H in $\underline{C}$; it is the idempotent completion of H if H is additive.*

7. *For the weight structure mentioned in the previous assertion, $T_{w\le 0}$ is the envelope of $\cup_{i\le 0}H[i]$; $T_{w\ge 0}$ is the envelope of $\cup_{i\ge 0}H[i]$.*

8. *A composition of left (resp. right) weight-exact functors is left (resp. right) weight-exact.*

9. *Let $\underline{C}$ and $\underline{D}$ be triangulated categories endowed with weight structures w and v, respectively. Let $F : \underline{C} \leftrightarrows \underline{D} : G$ be adjoint functors. Then F is left weight-exact whenever G is right weight-exact.*

10. *Let $\underline{C}$ and $\underline{D}$ be triangulated categories endowed with weight structures w and v, respectively; let w be bounded. Then an exact functor $F : \underline{C} \to \underline{D}$ is left (resp. right) weight-exact whenever $F(\underline{C}_{w=0}) \subset \underline{D}_{v\le 0}$ (resp. $F(\underline{C}_{w=0}) \subset \underline{D}_{v\ge 0}$).*

11. *Let w be a weight structure for $\underline{C}$; let $\underline{D} \subset \underline{C}$ be a triangulated subcategory of $\underline{C}$. Suppose that w yields a weight structure $w_{\underline{D}}$ for $\underline{D}$ (i.e. $Obj\underline{D} \cap \underline{C}_{w\le 0}$ and $Obj\underline{D} \cap \underline{C}_{w\ge 0}$ give a weight structure for $\underline{D}$).*

    Then w also induces a weight structure on $\underline{C}/\underline{D}$ (the localization i.e. the Verdier quotient of $\underline{C}$ by $\underline{D}$) in the following sense: the Karoubi-closures of $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ (considered as classes of objects of $\underline{C}/\underline{D}$) give a weight structure w' for $\underline{C}/\underline{D}$ (note that $Obj\underline{C} = Obj\underline{C}/\underline{D}$). Besides, there exists a full embedding $\frac{Hw}{Hw_{\underline{D}}} \to \underline{Hw}'$; $\underline{Hw}'$ is the Karoubi-closure of $\frac{Hw}{Hw_{\underline{D}}}$ in $\underline{C}/\underline{D}$.

12. *Suppose that $\underline{D} \subset \underline{C}$ is a full category of compact objects endowed with bounded a weight structure w'. Suppose that $\underline{D}$ weakly generates $\underline{C}$; let $\underline{C}$ admit arbitrary (small) coproducts. Then w' can be extended to a weight structure w for $\underline{C}$. Its heart is the big hull of $\underline{Hw}$ (as defined in Definition 1.2.1(IX)).*

13. *Let $\underline{D} \xrightarrow{i_*} \underline{C} \xrightarrow{j^*} \underline{E}$ be a part of gluing data. This means that $\underline{D}, \underline{C}, \underline{E}$ are triangulated categories, i^* and j^* are exact functors; j^* is a localization functor, i_* is an embedding of the categorical kernel of j^* into $\underline{C}$; i_* possesses both a left adjoint i^* and a right adjoint $i^!$ (see Chapter 9 of [Nee01]; note that this piece of data extends to data similar to that described in Proposition 1.1.2(10)).*

    Then for any pair of weight structures on $\underline{D}$ and $\underline{E}$ (we will denote them by $w_{\underline{D}}$ and $w_{\underline{E}}$, respectively) there exists a weight structure w on $\underline{C}$ such that both i_* and j^* are weight-exact (with respect to the corresponding weight structures). Besides, $i^!$ and j_* are right weight-exact (with respect to the corresponding weight structures); i^* and $j_!$ are left weight-exact. Moreover, $\underline{C}_{w\ge 0} = C_1 = \{M \in Obj\underline{C} :\ i^!(M) \in \underline{D}_{w_{\underline{D}}\ge 0},\ j^*(M) \in \underline{E}_{w_{\underline{E}}\ge 0}\}$ and $\underline{C}_{w\le 0} = C_2 = \{M \in Obj\underline{C} :\ i^*(M) \in \underline{D}_{w_{\underline{D}}\le 0},\ j^*(M) \in \underline{E}_{w_{\underline{E}}\le 0}\}$. Lastly, C_1 (resp. C_2) is the envelope of $j_!(\underline{E}_{w\le 0}) \cup i_*(\underline{D}_{w\le 0})$ (resp. of $j_*(\underline{E}_{w\ge 0}) \cup i_*(\underline{D}_{w\ge 0})$).

14. *In the setting of the previous assertion, if $w_{\underline{D}}$ and $w_{\underline{E}}$ are bounded, then: w bounded also; besides, $\underline{C}_{w\le 0}$ is the envelope of $\{i_*(\underline{D}_{w_{\underline{D}}=l}),\ j_!(\underline{E}_{w_{\underline{E}}=l}),\ l\le 0\}$; $\underline{C}_{w\ge 0}$ is the envelope of $\{i_*(\underline{D}_{w_{\underline{D}}=l}),\ j_*(\underline{E}_{w_{\underline{E}}=l}),\ l\ge 0\}$.*

15. *In the setting of assertion 13, the weight structure w described is the only weight structure for $\underline{C}$ such that both i_* and j^* are weight-exact.*

Proof. Most of the assertions were proved in [Bon10a] (pay attention to Remark 1.2.2(4)!); more precise references to most of the proofs could be found in the proof of Proposition 1.3.3 of [Bon11a].

We only have to prove assertions 9, 10, 14, and 15.

(9) follows immediately from assertion 3 (using the definition of adjoint functors).

(10) is immediate from assertion 7 by assertion 2.

If $w_{\underline{C}}$ and $w_{\underline{D}}$ are bounded, then w also is by definition. The remaining part of assertion 14 is immediate from Remark 8.2.4(1) of [Bon10a] and assertion 7.

(15): Suppose that the assumptions of assertion 13 are fulfilled, and consider some weight structure v for $\underline{C}$ such that i_* and j^* are weight-exact.

Since i_* and j^* are weight-exact, by assertion 9 we obtain: $i^!$ and j_* are right weight-exact; i^* and $j_!$ are left weight-exact (with respect to the corresponding weight structures). Hence the class $\underline{C}_{v\le 0}$ (resp. $\underline{C}_{v\ge 0}$) is contained in C_1 (resp. in C_2) in the notation of assertion 13. Since the couple (C_1, C_2) does yield a weight structure w for $\underline{C}$ (by loc. cit.), by assertion 4 we obtain that $v = w$.

□

Remark 1.2.4. Part 11 of the proposition can be re-formulated as follows. If $i_* : \underline{D} \to \underline{C}$ is an embedding of triangulated categories that is weight-exact (with respect to certain weight structures for $\underline{D}$ and $\underline{C}$), an exact functor $j^* : \underline{C} \to \underline{E}$ is equivalent to the localization of $\underline{C}$ by $i_*(\underline{D})$, then there exists a unique weight structure w' for $\underline{E}$ such that j^* is weight-exact; $\underline{Hw}_{\underline{E}}$ is the Karoubi-closure of $\frac{Hw}{i_*(\underline{Hw}_{\underline{D}})}$ (with respect to the natural functor $\frac{Hw}{i_*(\underline{Hw}_{\underline{D}})} \to \underline{E}$).

2 The Chow weight structure: two constructions and basic properties

In §2.1 we define the category $Chow(S)$ of Chow motives over S (similar definitions can be found in [CoH00], [Heb11], and [GiS09]). By our definition, $Chow(S) \subset DM_c(S)$; since $Chow(S)$ is also negative in it and generates it

(if S is reasonable; here we use the properties of $DM_c(S)$ proved in §1.1) we immediately obtain (by Proposition 1.2.3(7)) that there exists a weight structure on $DM_c(S)$ whose heart is $Chow(S)$.

In §2.2 we study the 'functoriality' of w_{Chow} (with respect to functors of the type $f^*, f_*, f^!$, and $f_!$, for f being a smoothly embeddable morphism of schemes). Our functoriality statements are parallel to the 'stabilities' 5.1.14 of [BBD82]; we will explain this similarity in the next section. We also prove that w_{Chow} can be described 'pointwisely' (similarly to §5.1.8 of [BBD82]), and prove that it is 'continuous' (in a certain sense).

In §2.3 we describe an alternative method for the construction of w_{Chow} for $DM_c(S)$ (for an excellent separated finite-dimensional scheme S that is not necessarily reasonable). This method uses stratifications and 'gluing' of weight structures; this makes this part of the paper very much parallel to the study of weights of mixed complexes of sheaves in §5 of [BBD82]. Actually, this method is the first one developed by the author (it was first proposed in Remark 8.2.4(3) of [Bon10a], that was in its turn inspired by [BBD82]). We prove that this alternative method yields the same result as the method of §2.1 if S is reasonable. This yields some new descriptions of w_{Chow} (in this case); see Remark 2.3.7.

2.1 Relative Chow motives; the 'basic' construction of w_{Chow}

We define $Chow(S)$ as the Karoubi-closure of $\{f_!(\mathbb{Q}_X)(r)[2r]\} = \{f_*(\mathbb{Q}_X)(r)[2r]\}$ in $DM_c(S)$; here $f : X \to S$ runs through all finite type projective morphisms such that X is regular, $r \in \mathbb{Z}$.

Till §2.3 we will assume that all schemes that we consider are reasonable (see Definition 1.1.1).

Theorem 2.1.1. *I There exists a (unique) weight structure w_{Chow} for $DM_c(S)$ whose heart is $Chow(S)$.*

II $w_{Chow}(S)$ can be extended to a weight structure w^{big}_{Chow} for the whole $DM(S)$. $\underline{Hw}^{big}_{Chow}$ is the big hull of $Chow(S)$ (as defined in Definition 1.2.1(IX); see Remark 1.2.2(1)).

Proof. I By Proposition 1.2.3(6) it suffices to verify that $Chow(S)$ is negative and generates $DM_c(S)$. The negativity of $Chow(S)$ is immediate from Lemma 1.1.4(I). $Chow(S)$ generates $DM_c(S)$ by Proposition 1.1.2(13).

II Since $Chow(S)$ generates $DM_c(S)$, and $DM_c(S)$ weakly generates $DM(S)$ (by part 2 of loc. cit.), $Chow(S)$ weakly generates $DM(S)$.

Hence the assertion follows immediately from assertion I and Proposition 1.2.3(12).

□

Remark 2.1.2. 1. In particular, the theorem holds for S being the spectrum of a (not necessarily perfect) field k.

For a perfect k the existence of w_{Chow} was already proved in §6 of [Bon10a]. Note here that $DM_c(\text{Spec } k) \cong DM_{gm}\mathbb{Q}(k)$ for a perfect k (in the notation of loc. cit.; $DM_{gm}\mathbb{Q}(k)$ denotes the category of motives with rational coefficients), whereas $p_!\mathbb{Q}_P(r)[2r]$ yields a Chow motif over k (for any $r \in \mathbb{Z}$ and $p : P \to \text{Spec } k$ being a smooth projective morphism; recall here that the 'ordinary' category of Chow motives over k can be fully embedded into DM_{gm}).

2. Moreover (as was kindly pointed out by the referee) if k^p is perfect closure of an arbitrary field k, then $DM_c(\text{Spec } k^p) \cong DM_c(\text{Spec } k)$; see Proposition 1.1.2(12).

3. Besides, in [Bon09a] a related differential graded 'description' of motives over a characteristic zero k was given. It was generalized in [Lev09] to a description of the category of *smooth motives* over S, when S is a smooth variety over (a characteristic 0 field) k; here the category of smooth motives is the triangulated category generated by motives of smooth projective S-schemes.

Note also: the restriction of w_{Chow} to smooth motives induces a weight structure for this category (that is coherent with the 'description' mentioned).

4. Our results would certainly look nicer if we had a description of the composition of morphisms in $Chow(S)$ (note here that the morphism groups between 'generating objects' of $Chow(S)$ can be immediately computed using Lemma 1.1.4(I)). The author conjectures that this composition is compatible with the ones described §2 of [CoH00] and in §5.2 of [GiS09]. In order to prove this Levine's method could be quite useful, as well as the description of $DM(S)$ in terms of qfh-sheaves (see Theorem 16.1.2 of [CiD09]). Moreover, the methods of [Lev09] could possibly allow giving a 'differential graded' description of the whole $DM_c(S)$ (extending the main result of ibid.).

The author plans to study these matters further.

5. In Theorem 3.2 of [Heb11] an orthogonality property (similar to that in Lemma 1.1.4(I1)) was established for not necessarily smoothly embeddable f and g. This yielded that $\{f_!(\mathbb{Q}_X)(r)[2r]\} \in DM_c(S)_{w=0}$ for any proper (not necessarily projective!) f such that X is regular, $r \in \mathbb{Z}$, and allowed generalizing Theorem 2.2.1(II) (below) to not necessarily smoothly embeddable morphisms.

Note also: in our proof of Lemma 1.1.4(I1) we actually only needed an embedding of $X \times_S Y$ into a regular finite type S-scheme; this also implies a certain generalization of Theorem 2.2.1(II) (in particular, we can extend it

to arbitrary finite type morphisms of schemes that are of finite type over a fixed characteristic 0 field).

6. If S is not reasonable, we still obtain that $Chow(S)$ is negative. Hence, there exists a weight structure on $\langle Chow(S)\rangle$ whose heart is $Chow(S)$ (since $Chow(S)$ is idempotent complete). The problem is that we do not know whether $\langle Chow(S)\rangle$ is the whole $DM_c(S)$ (though this is true for motives over generic points of S, since those are reasonable; cf. also part 1 of this remark).

One can also prove the existence of a certain analogue of the Chow weight structure over a not necessarily reasonable scheme S; see §2.3 below. The main disadvantage of this method is that it does not yield an 'explicit' description of $\underline{Hw}_{Chow}$ (though $\underline{Hw}_{Chow} \supset Chow(S)$; cf. Remark 2.3.7).

2.2 Functoriality of w_{Chow}

Now we study (left and right) weight-exactness of the motivic image functors. These statements are very similar to the properties of pure complexes of constructible sheaves. This is not surprising; cf. §3.6 below. In this paragraph S, X, Y (and hence also Z, U, and all S_l^α) will be reasonable.

Theorem 2.2.1. *I The functor $-(1)[2](=\otimes\mathbb{Q}(1)[2])$ and its inverse $-(-1)[-2]: DM_c(S)\to DM_c(S)$ are weight-exact with respect to w_{Chow} for any S.*

II Let $f: X\to Y$ be a smoothly embeddable morphism of schemes.

1. $f^!$ and f_* are right weight-exact; f^* and $f_!$ are left weight-exact.

2. Suppose moreover that f is smooth. Then f^* and $f^!$ are also weight-exact.

3. Moreover, f^* is weight-exact for f being either (i) a finite universal homeomorphism or (ii) a (filtering) projective limit of smooth morphisms such that the corresponding connecting morphisms are smooth affine. In case (i) $f^!$ is weight-exact also.

III Let $i: Z\to X$ be a closed immersion; let $j: U\to X$ be the complimentary open immersion.

1. $Chow(U)$ is the idempotent completion of the factor (in the sense of Definition 1.2.1(VIII)) of $Chow(X)$ by $i_*(Chow(Z))$.

2. For $M\in ObjDM_c(X)$ we have: $M\in DM_c(X)_{w_{Chow}\ge 0}$ (resp. $M\in DM_c(X)_{w_{Chow}\le 0}$) whenever $j^!(M)\in DM_c(U)_{w_{Chow}\ge 0}$ and $i^!(M)\in DM_c(Z)_{w_{Chow}\ge 0}$ (resp. $j^*(M)\in DM_c(U)_{w_{Chow}\le 0}$ and $i^*(M)\in DM_c(Z)_{w_{Chow}\le 0}$).

IV Let $S=\cup S_l^\alpha$ be a stratification, $j_l: S_l^\alpha\to S$ are the corresponding immersions. Then for $M\in ObjDM_c(X)$ we have: $M\in DM_c(X)_{w_{Chow}\ge 0}$ (resp. $M\in DM_c(X)_{w_{Chow}\le 0}$) whenever $j_l^!(M)\in DM_c(S_l^\alpha)_{w_{Chow}\ge 0}$ (resp. $j_l^*(M)\in DM_c(S_l^\alpha)_{w_{Chow}\le 0}$) for all l.

V 1. For any S we have $\mathbb{Q}_S \in DM_c(S)_{w_{Chow}\le 0}$.
2. If S_{red} is regular, then $\mathbb{Q}_S \in DM_c(S)_{w_{Chow}=0}$.

Proof. I Since w_{Chow} is bounded for any base scheme, in order to prove that a motivic image functor is weight-exact it suffices to prove that it preserves Chow motives; see Proposition 1.2.3(10). The assertion follows immediately.

II Let f be smooth. Then we obtain: $f^*(DM_c(Y)_{w_{Chow}=0}) \subset DM_c(X)_{w_{Chow}=0}$ by Proposition 1.1.2(5). Hence f^* is weight-exact (by the same argument as above). Passing to the limit (using Remark 1.1.3(2)) we prove assertion II3(ii). We also obtain that $f^!$ is weight-exact using assertion I and Proposition 1.1.2(8) i.e. we proved assertion II2. Besides, the adjunctions yield (by Proposition 1.2.3(9)): f_* is right weight-exact, $f_!$ is left weight-exact; i.e. assertion II1 for f is fulfilled.

Now let f be projective. Then $f_!(DM_c(X)_{w_{Chow}=0}) \subset DM_c(Y)_{w_{Chow}=0}$ (since $f_! \circ g_! = (f \circ g)_!$ for any g, and $f_!$ commutes with Tate twists). By Proposition 1.2.3(10) we obtain that $f_! = f_*$ is weight-exact. Hence if f is a finite universal homeomorphism, f^* is weight-exact also (since it is inverse to f_* by Proposition 1.1.2(12)). The same argument can be applied to the functor $f^!$ (see Remark 1.1.3(3)) and we obtain assertion II3(i). Next, using the adjunctions and Proposition 1.2.3(9) again, we obtain that $f^!$ is right weight-exact and f^* is left weight-exact for an arbitrary projective f. So, assertion II1 is fulfilled also in the case when f is projective.

Assertion II1 in the general case follows since any smoothly embeddable morphism (by definition) is a composition of a closed (i.e. projective) immersion with a smooth morphism.

III Since $i_* \cong i_!$ in this case, i_* is weight-exact by assertion II1. j^* is weight-exact by assertion II2.

1. $DM_c(U)$ is the localization of $DM_c(X)$ by $i_*(DM_c(Z))$ by Proposition 1.1.2(11). Hence Proposition 1.2.3(11) yields the result (see Remark 1.2.4).

2. Proposition 1.1.2(11) yields: $w_{Chow}(X)$ is exactly the weight structure obtained by 'gluing $w_{Chow}(Z)$ with $w_{Chow}(U)$' via Proposition 1.2.3(13) (here we use part 15 of loc. cit.). Hence loc. cit. yields the result (note that $j^* = j^!$).

IV The assertion can be easily proved by induction on the number of strata using assertion III2.

V Let S_{red} be regular; denote by v the canonical immersion $S_{red} \to S$. Then $v_*(\mathbb{Q}_{S_{red}}) \in DM_c(S)_{w_{Chow}=0}$ by the definition of w_{Chow}. Now, v^* is an equivalence of categories (by Proposition 1.1.2(12)) that sends $\mathbb{Q}_S$ to $\mathbb{Q}_{S_{red}}$ (see part 7 of loc. cit.). Hence (applying the adjunction) we obtain $v_*(\mathbb{Q}_{S_{red}}) \cong \mathbb{Q}_S$. So, we proved assertion V2.

In order to verify assertion V1 we choose a stratification $S = \cup S_\alpha$ such

that all $S^{\alpha}_{l,red}$ are regular. Since we have $j_l^*(\mathbb{Q}_S) = \mathbb{Q}_{S_l} \in DM_c(S_l)_{w_{Chow}\ge 0}$ (by assertion V2), assertion IV implies the result.

□

Remark 2.2.2. 1. Assertion III1 yields that any $C \in Obj\, Chow(U)$ is a retract of some C' coming from $Chow(X)$. This fact can be easily deduced from Hironaka's resolution of singularities (if we believe that the composition of morphisms in $Chow(-)$ could be described in terms of algebraic cycles; cf. Remark 2.1.2(4)) in the case when X is a variety over a characteristic 0 field. Indeed, then any projective regular U-scheme Y_U possesses a projective regular X-model Y (since one can resolve the singularities of any projective model Y'/X of Y_U by a morphism that is an isomorphism over U). The author does not know any analogues of this argument in the case of a general (reasonable) X (even with alterations instead of modifications, since it does not seem to be known whether there exists an alteration of Y' that is étale over U).

So, assertion III1 could be called (a certain) *motivic resolution of singularities* (over a reasonable X). Certainly, applying the assertion repeatedly one can easily extend it to the case when $X \setminus U$ is not necessarily regular (but U is open in X).

Alternatively, one can note here that for any $C \in ObjChow(U)$ we have $j_!(C) \in \underline{C}_{w\le 0}$ and $j_*(C) \in \underline{C}_{w\ge 0}$. Hence the natural morphism $j_!(C) \to j_*(C)$ can be factorized through $C_X = j_*(C)_{w_{Chow}\le 0} \in ObjChow(X)$ (or through $j_!(C)_{w_{Chow}\ge 0} \in ObjChow(X)$; see Proposition 1.2.3(2)), whereas C is a retract of $j^*(C_X)$.

Actually, any object of $Chow(U)$ comes from $Chow(X)$ itself; see Theorem 1.7 of [Wil12].

2. The following statement is a trivial consequence of part I of the Theorem along with Proposition 1.1.2(8): if $f : X \to S$ is a smooth morphism, then for any $M \in ObjDM_c(S),\ m \in \mathbb{Z}$ we have: $f^*(M) \in DM_c(X)_{w_{Chow}\ge m}$ (resp. $f^*(M) \in DM_c(X)_{w_{Chow}\le m}$) whenever $f^!(M) \in DM_c(X)_{w_{Chow}\ge m}$ (resp. $f^!(M) \in DM_c(X)_{w_{Chow}\le m}$).

Now we prove that positivity and negativity of objects of $DM_c(S)$ (with respect to w_{Chow}) can be 'checked at points'; this is a motivic analogue of §5.1.8 of [BBD82].

Proposition 2.2.3. *Let $\mathcal{S}$ denote the set of (Zariski) points of S; for a $K \in \mathcal{S}$ we will denote the corresponding morphism $K \to S$ by j_K.*

Then $M \in DM_c(S)_{w_{Chow}\ge 0}$ (resp. $M \in DM_c(S)_{w_{Chow}\le 0}$) if and only if for any $K \in \mathcal{S}$ we have $j_K^!(M) \in DM_c(K)_{w_{Chow}\ge 0}$ (resp. $j_K^*(M) \in DM_c(K)_{w_{Chow}\le 0}$); see Remark 1.1.3(2).

Proof. By Theorem 2.2.1(II1) if $M \in DM_c(S)_{w_{Chow}\ge 0}$ (resp. $M \in DM_c(S)_{w_{Chow}\le 0}$) then for any immersion $f : X \to S$ we have $f^!(M) \in DM_c(X)_{w_{Chow}\ge 0}$ (resp. $f^*(M) \in DM_c(X)_{w_{Chow}\le 0}$). Passing to the limits with respect to immersions corresponding to points of S (see Remark 1.1.3(2)) yields one of the implications.

We prove the converse implication by noetherian induction. So, suppose that our assumption is true for motives over any closed subscheme of S, and that for some $M \in ObjDM_c(S)$ we have $j_K^!(M) \in DM_c(K)_{w_{Chow}\ge 0}$ (resp. $j_K^*(M) \in DM_c(K)_{w_{Chow}\le 0}$) for any $K \in \mathcal{S}$.

We should prove that $M \in DM_c(S)_{w_{Chow}\ge 0}$ (resp. $M \in DM_c(S)_{w_{Chow}\le 0}$). By Proposition 1.2.3(3) it suffices to verify: for any $N \in DM_c(S)_{w_{Chow}\le -1}$ (resp. for any $N \in DM_c(S)_{w_{Chow}\ge 1}$), and any $h \in DM_c(S)(N, M)$ (resp. any $h \in DM_c(S)(M, N)$) we have $h = 0$. We fix some N and h.

By the 'only if' part of our assertion (that we have already proved) we have $j_K^*(N) \in DM_c(K)_{w_{Chow}\le -1}$ (resp. $j_K^*(N) \in DM_c(K)_{w_{Chow}\ge 1}$); hence $j_K^*(h) = 0$. By Proposition 1.1.2(14) we obtain that $j^*(h) = 0$ for some open embedding $j : U \to S$, where K is a generic point of U.

Now suppose that $h \ne 0$; let $i : Z \to S$ denote the closed embedding that is complimentary to j. Then Lemma 1.1.4(II) yields that $DM_c(S)(i^*(N), i^!(M)) \ne \{0\}$ (resp. $DM_c(S)(i^*(M), i^!(N)) \ne \{0\}$). Yet $i^*(N) \in DM_c(Z)_{w_{Chow}\le -1}$ (resp. $i^!(N) \in DM_c(Z)_{w_{Chow}\ge 1}$) by Theorem 2.2.1(II), whereas $i^!(M) \in DM_c(Z)_{w_{Chow}\ge 0}$ (resp. $i^*(M) \in DM_c(Z)_{w_{Chow}\le 0}$) by the inductive assumption. The contradiction obtained proves our assertion.

□

Lastly we prove that 'weights are continuous'.

Lemma 2.2.4. *Let K be a generic point of S; denote the morphism $K \to S$ by j_K.*

Let $X \in ObjDM_c(S)$, and suppose that $j_K^*(X) \in DM_c(K)_{w_{Chow}\ge 0}$ (resp. $j_K^*(X) \in DM_c(K)_{w_{Chow}\le 0}$). Then there exists an open immersion $j : U \to S$, $K \in U$, such that $j^*(X) \in DM_c(U)_{w_{Chow}\ge 0}$ (resp. $j^*(X) \in DM_c(U)_{w_{Chow}\le 0}$).

Proof. First consider the case $j_K^*(X) \in DM_c(K)_{w_{Chow}\ge 0}$. We consider a weight decomposition of X: $X \to A \to B \xrightarrow{g} X[1]$. We obtain that $j_K^*(g) = 0$ (since $DM_c(K)_{w_{Chow}\le 0} \perp j_K^*(X)[1]$); hence (by Proposition 1.1.2(14)) there exists an open immersion $j : U \to S$ ($K \in U$) such that $j^*(g) = 0$. Hence $j^*(X)$ is a retract of $j^*(A)$. Since $j^*(A) \in DM_c(U)_{w_{Chow}\ge 0}$ (see Theorem 2.2.1(II2)), and $DM_c(U)_{w_{Chow}\ge 0}$ is Karoubi-closed in $DM_c(U)$, we obtain the result.

The second part of our statement (for the case $j_K^*(X) \in DM_c(K)_{w_{Chow}\le 0}$) can be easily verified using the dual argument (see Proposition 1.2.3(1)).

□

2.3 The 'gluing' construction of w_{Chow} over any (excellent separated finite-dimensional) S

In this paragraph all schemes (including the base scheme S) will be excellent separated finite-dimensional; we do not assume them to be reasonable. Then we can define the Chow weight structure 'locally'. We explain how to do this (using stratifications and gluing of weight structures; we call this approach to constructing w_{Chow} the 'gluing method').

First we describe certain candidates for $DM_c(S)_{w_{Chow}\ge 0}$ and $DM_c(S)_{w_{Chow}\le 0}$ (partially they are motivated by Remark 2.1.2(2)); next we will prove that they yield a weight structure for $DM_c(S)$ indeed.

For a scheme X we will denote by $\mathcal{OP}(X)$ (resp. $\mathcal{ON}(X)$) the envelope (see the Notation) of $p_*(\mathbb{Q}_P)(s)[i+2s](\cong p_!(\mathbb{Q}_P)(s)[i+2s])$ in $DM_c(X)$; here $p: P\to X$ runs through all morphisms to X that can be factorized as $g\circ h$, where $h: P\to X'$ is a smooth projective morphism, X' is a regular scheme, $g: X'\to X$ is a finite universal homeomorphism, $s\in\mathbb{Z}$, whereas $i\ge 0$ (resp. $i\le 0$). We denote $\mathcal{OP}(X)\cap\mathcal{ON}(X)$ by $\mathcal{OZ}(X)$.

Remark 2.3.1. 1. For a morphism $f: Y\to X$ we have (by Proposition 1.1.2(5)): $f^*p_!(\mathbb{Q}_P)\cong p'_!f'^*(\mathbb{Q}_P)=p'_!(\mathbb{Q}_{P_Y})$; if p is projective (as we demanded in the definition of $(\mathcal{OP}(-),\mathcal{ON}(-))$) then we can replace all $-_!$ here with $-_*$. Now suppose that for X'/X as above the reduced scheme Y'_{red} associated to $Y'=X'_Y$ is regular. Then $f^*p_*(\mathbb{Q}_P)(s)[2s]\in\mathcal{OZ}(Y)$. Indeed, consider the diagram

$$\begin{array}{ccccc} P_{Y,red} & \xrightarrow{p_r} & P_Y & \xrightarrow{f_P} & P \\ \downarrow{\scriptstyle h_{Y,red}} & & \downarrow{\scriptstyle h_Y} & & \downarrow{\scriptstyle h} \\ Y'_{red} & \xrightarrow{y'_r} & Y' & \xrightarrow{f'} & X' \\ & & \downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle g} \\ & & Y & \xrightarrow{f} & X \end{array} \qquad (2)$$

where p_r and y'_r are the corresponding nil-immersions. We have $f^*p_*(\mathbb{Q}_P)(s)[2s]\cong p_{Y*}\mathbb{Q}_{P_Y}(s)[2s]\cong p_{Y*}p_{r*}\mathbb{Q}_{P_{Y'_{red}}}(s)[2s]$ (see Remark 1.1.3(3)). We can transform this further into $(g_Y\circ y'_r)_*h_{Y,red*}\mathbb{Q}_{P_{Y_{red}}}(s)[2s]\in\mathcal{OZ}(Y)$.

Moreover, part 9 of Proposition 1.1.2 (that represents $i^!\mathbb{Q}_B$ as a Tate twist $\mathbb{Q}_A$ for an immersion $i : A \to B$ of connected regular schemes) easily yields that $f^!p_*(\mathbb{Q}_P)(s)[2s] \in \mathcal{OZ}(Y)$ if f induces an immersion $Y'_{red} \to X'$ of regular schemes. Indeed, we can assume that Y is connected; hence Y'_{red} and $P_{Y,red}$ are connected also. Denote the codimension of Y'_{red} in X' by c; then $p_r \circ f_P : P_{Y,red} \to P$ is an immersion of regular schemes of codimension c. Then (arguing as above) we obtain: $f^!p_*(\mathbb{Q}_P)(s)[2s] \cong (g_Y \circ y'_r)_* h_{Y,red_*}(p_r \circ f_P)^!\mathbb{Q}_P(s)[2s]$. Using loc.cit., we transform this into $(g_Y \circ y'_r)_* h_{Y,red_*}\mathbb{Q}_{P_{Y_{red}}}(s-c)[2s-2c] \in \mathcal{OZ}(Y)$.

2. Any morphism $p : X \to S$ for a regular X can be factorized through the underlying reduced subscheme S_{red} of S. So, all our descriptions of $w_{Chow}(S)$ (including the ones given below) 'depend' only on S_{red}. This is (certainly) coherent with the (weight-exact) isomorphism $DM_c(S) \to DM_c(S_{red})$ given by Proposition 1.1.2(12).

Besides, in all the statements of this section the reader may assume that $\mathcal{OP}(X) = \mathcal{ON}(X) = \{0\}$ unless X_{red} is regular.

3. Moreover, if X is generically of characteristic 0 then it suffices for our purposes to take $X' = X_{red}$. Indeed, in this case each finite universal homeomorphism with regular domain is generically of the form $X_{red} \to X$; so we can apply the argument used in the proof of Proposition 2.3.4(below) for this alternative version of the definition of $(\mathcal{OP}(-), \mathcal{ON}(-))$ also. Hence in the case when S is a reduced $\mathbb{Q}$-scheme, one can assume that all the morphisms p (that we use for the description of $w_{Chow}(S)$ given below) are smooth projective; this is also true if S is the spectrum of a subring of a number field.

For a stratification $\alpha : S = \cup S_l^\alpha$ we denote by $\mathcal{OP}(\alpha)$ the class $\{M \in ObjDM_c(S) : j_l^!(M) \in \mathcal{OP}(S_l^\alpha), 1 \le l \le n\}$; $\mathcal{ON}(\alpha) = \{M \in ObjDM_c(S) : j_l^*(M) \in \mathcal{ON}(S_l^\alpha), 1 \le l \le n\}$.

We define: $DM_c(S)_{w_{Chow} \ge 0} = \cup_\alpha \mathcal{OP}(\alpha)$, $DM_c(S)_{w_{Chow} \le 0} = \cup_\alpha \mathcal{ON}(\alpha)$; here α runs through all stratifications of S.

Remark 2.3.2. It seems that the unions in the definition of $(DM_c(S)_{w_{Chow} \ge 0}, DM_c(S)_{w_{Chow} \le 0})$ given above are not filtering (if S is not a Spec $\mathbb{Q}$-scheme). In particular, we don't have $\mathcal{OP}(\alpha) \subset \mathcal{OP}(\alpha')$ (and $\mathcal{ON}(\alpha) \subset \mathcal{ON}(\alpha')$) for a general subdivision α' of a stratification α. In order to overcome this difficulty we prove a certain weaker statement instead (see Lemma 2.3.3(3)); it is sufficient for our purposes. In the proof of this result we also treat the question when an element of $\mathcal{OP}(\alpha)$ (or of $\mathcal{ON}(\alpha)$) belongs to $\mathcal{OP}(\alpha')$ (or to $\mathcal{ON}(\alpha')$, respectively) where α' is a subdivision of α.

Lemma 2.3.3. *1. Let δ be a stratification of S; we denote the corresponding immersions $S_l^\delta \to S$ by j_l. Let $M \in ObjDM_c(S)$.*

Suppose that $j_l^!(M) \in DM_c(S_l^\delta)_{w_{Chow}\ge 0}$ *(resp.* $j_l^*(M) \in DM_c(S_l^\delta)_{w_{Chow}\le 0}$*) for all* l*.*

Then $M \in DM_c(S)_{w_{Chow}\ge 0}$ *(resp.* $M \in DM_c(S)_{w_{Chow}\le 0}$*).*

2. $j_*(DM_c(V)_{w_{Chow}\ge 0}) \subset DM_c(S)_{w_{Chow}\ge 0}$ *and* $j_!(DM_c(V)_{w_{Chow}\le 0}) \subset DM_c(S)_{w_{Chow}\le 0}$ *for any immersion* $j : V \to S$*.*

3. For any $M \in DM_c(S)_{w_{Chow}\le 0}$ *and* $N \in DM_c(S)_{w_{Chow}\ge 1}(= DM_c(S)_{w_{Chow}\ge 0}[1])$ *there exists a stratification* α *of* S *such that* $M \in \mathcal{ON}(\alpha)$*,* $N \in \mathcal{OP}(\alpha)[1]$*.*

Proof. 1. We use induction on the number of strata of δ. The 2-functoriality of motivic upper image functors yields: it suffices to prove the statement for δ consisting of two strata.

So, let $S = U \cup Z$, Z and U are disjoint, $U \neq \{0\}$ is open in S; we denote the immersions $U \to S$ and $Z \to S$ by j and i, respectively.

By the assumptions on M, there exist stratifications β of Z and γ of U such that $i^!(M) \in \mathcal{OP}(\beta)$ and $j^!(M) \in \mathcal{OP}(\gamma)$ (resp. $i^*(M) \in \mathcal{ON}(\beta)$ and $j^*(M) \in \mathcal{ON}(\gamma)$).

We 'unify' β with γ and denote the stratification of S obtained by α (for $\#\gamma = \Gamma$ we put $S_l^\alpha = U_l^\gamma$ if $1 \le l \le \Gamma$ and $S_l^\alpha = \beta_{l-\Gamma}^\gamma$ if $l > \Gamma$; note that we really obtain a stratification in our weak sense of this notion this way; see the Notation). Then the 2-functoriality of $-^!$ (resp. of $-^*$) yields that $M \in \mathcal{OP}(\alpha)$ (resp. $M \in \mathcal{ON}(\alpha)$).

2. We choose a stratification δ containing V (as one of the strata). So we assume that $V = S_v^\delta$ for some index v. Then it can be easily seen that $j_l^! j_{v*} = 0 = j_l^* j_{v!}$ for any $l \neq v$ and $j_v^! j_{v*} \cong 1_{DM(V)} \cong j_v^* j_{v!}$ (see Proposition 1.1.2(10)). Hence the result follows from assertion 1.

3. By Remark 2.3.1(1) it suffices to verify: if β, γ are stratifications of S, and $S_{il} \to S_l^\beta$, $S'_{il} \to S_l^\gamma$ are (finite) sets of finite universal homomorphisms, then there exists a common subdivision α of β, γ such that all the (reduced) schemes $(S_{il} \times_S S_m^\alpha)_{red}, (S'_{il} \times_S S_m^\alpha)_{red}$ are regular. To this end it obviously suffices to prove: if $f : Z \to S$ is an immersion, $g_i : T_i \to Z$ are some finite universal homeomorphisms, then there exists a stratification δ of Z such that the schemes $T_{il} = (T_i \times_Z Z_l^\delta)_{red}$ are regular for all i and l.

We prove the latter statement by easy Noetherian induction. Suppose that it is fulfilled for any proper closed subscheme Z' of Z. Since all $(T_i)_{red}$ are generically regular, we can choose a (sufficiently small) open non-empty subscheme Z_1 of Z such that all of $(T_i \times_Z Z_1)_{red}$ are regular.

Next, apply the inductive assumption to the scheme $Z' = Z \setminus Z_1$ and the morphisms $g'_i = g_i \times_Z Z'$; we choose some stratification α' of Z' such that all $T'_{il} = (T_i \times_Z Z'^{\alpha'}_l)_{red}$ are regular. Then it remains to 'unify' Z_1 with α' i.e. we consider the following stratification α: $Z_1^\alpha = Z_1$, and $Z_l^\alpha = Z'^{\alpha'}_{l-1}$ for all $l > 1$.

□

Proposition 2.3.4. *I1. The couple* $(DM_c(S)_{w_{Chow}\ge 0}, DM_c(S)_{w_{Chow}\le 0})$ *yields a bounded weight structure* w_{Chow} *for* $DM_c(S)$.

2. $DM_c(S)_{w_{Chow}\ge 0}$ *(resp.* $DM_c(S)_{w_{Chow}\le 0}$*) is the envelope of* $p_*(\mathbb{Q}_P)(s)[2s+i]$ *(resp. of* $p_!(\mathbb{Q}_P)(s)[2s-i]$*) for* $s\in\mathbb{Z}$, $i\ge 0$, *and* $p: P\to S$ *being the composition of a smooth projective morphism with a finite universal homeomorphism whose base is regular and with an immersion.*

II $w(S)$ *can be extended to a weight structure* w_{Chow}^{big} *for the whole* $DM(S)$.

Proof. I We prove the statement by Noetherian induction. So, we suppose that assertions I1 and I2 are fulfilled for all proper closed subschemes of S. We prove them for S.

We denote the envelopes mentioned in assertion I2 by $(DM_c(S)_{w_{Chow}'\ge 0}, DM_c(S)_{w_{Chow}'\le 0})$. We should prove that w_{Chow} and $w_{Chow}{}'$ yield coinciding weight structures for $DM_c(S)$.

Obviously, $DM_c(S)_{w_{Chow}\le 0}$, $DM_c(S)_{w_{Chow}\ge 0}$, $DM_c(S)_{w_{Chow}'\le 0}$, and $DM_c(S)_{w_{Chow}'\ge 0}$ are Karoubi-closed in $DM_c(S)$, and are semi-invariant with respect to translations (in the appropriate ways).

Now, Lemma 2.3.3(2) yields that $DM_c(S)_{w_{Chow}'\le 0}\subset DM_c(S)_{w_{Chow}\le 0}$ and $DM_c(S)_{w_{Chow}'\ge 0}\subset DM_c(S)_{w_{Chow}\ge 0}$. Hence in order to verify that w_{Chow} and $w_{Chow}{}'$ are indeed weight structures, it suffices to verify:

(i) the orthogonality axiom for w_{Chow}

(ii) any $M\in ObjDM_c(S)$ possesses a weight decomposition with respect to $w_{Chow}{}'$.

Hence these statements along with the boundedness of w_{Chow} imply assertion I1. Besides, Proposition 1.2.3(4) yields that these two statements imply assertion I2 also, whereas in order to prove I1 it suffices to verify the boundedness of $w_{Chow}{}'$ (instead of that of w_{Chow}).

Now we verify (i). For some fixed $M\in DM_c(S)_{w_{Chow}\le 0}$ and $N\in DM_c(S)_{w_{Chow}\ge 1}$ we check that $M\perp N$. By Lemma 2.3.3(3), we can assume that $M\in\mathcal{ON}(\alpha)$, $N\in\mathcal{OP}(\alpha)[1]$ for some stratification α of S. Hence it suffices to prove that $\mathcal{ON}(\alpha)\perp\mathcal{OP}(\alpha)[1]$ for any α. The latter statement is an easy consequence of Lemma 1.1.4 (parts I1 and II).

Now we verify (ii) along with the boundedness of $w_{Chow}{}'$. We choose some generic point K of S, denote by K^p its perfect closure, and by $j_{K^p}: K^p\to S$ the corresponding morphism. We fix some M. Since K^p is a reasonable scheme, we have $j_{K^p}^*(M)\in\langle Chow(K^p)\rangle$ (see Proposition 1.1.2(13)). Moreover, since K^p is perfect, there exist some smooth projective varieties P_i/K^p, $1\le i\le n$ (we denote the corresponding morphisms $P_i\to K^p$ by p_i) and some $s\in\mathbb{Z}$ such that $j_{K^p}^*(M)$ belongs to the triangulated subcategory of

$DM_c(K^p)$ generated by $\{p_{i*}(\mathbb{Q}_{P_i})(s)[2s]\}$. Now we choose some finite universal homeomorphism $K' \to K$ (i.e. a morphism of spectra of fields corresponding to a finite purely inseparable extension) such that P_i are defined (and are smooth projective) over K'. By Proposition 1.1.2(14,12), for the corresponding morphisms $j_{K'} : K' \to S$, $p'_i : P_{K',i} \to K'$ we have: $j_{K'}^*(M)$ belongs to the triangulated subcategory of $DM_c(K')$ generated by $\{p'_{i*}(\mathbb{Q}_{P_{K',i}})(s)[2s]\}$. Applying Zariski's main theorem in the form of Grothendieck, we can choose a finite universal homeomorphism g from a regular scheme U' whose generic fibre is K' to an open $U \subset S$ ($j : U \to S$ will denote the corresponding immersion) and smooth projective $h_i : P_{U',i} \to U'$ such that the fibres of $P_{U',i}$ over K' are isomorphic to $P_{K',i}$. Moreover, by Proposition 1.1.2(14) we can also assume that $(j \circ g)^*(M)$ belongs to the triangulated subcategory of $DM_c(U')$ generated by $\{h_{i*}(\mathbb{Q}_{P_{U',i}})(s)[2s]\}$. Then Remark 1.1.3(3) yields that $j^*(M)$ belongs to the triangulated subcategory D of $DM_c(U)$ generated by $\{(g \circ h_i)_*(\mathbb{Q}_{P_{U',i}})(s)[2s]\}$.

Since id_U yields a stratification of U, the set $\{(g \circ h_i)_*(\mathbb{Q}_{P_{U',i}})(s)[2s]\}$ is negative in $DM_c(U)$ (since $\mathcal{ON}(\alpha) \perp \mathcal{OP}(\alpha)[1]$ for any α, as we have just proved). Therefore (by Proposition 1.2.3(6–7)) there exists a weight structure d for D such that $D_{d\le 0}$ (resp. $D_{d\le 0}$) is the envelope of $\cup_{n\ge 0}\{(g \circ h_i)_*(\mathbb{Q}_{P_{U',i}})(s)[2s+n]\}$ (resp. of $\cup_{n\ge 0}\{(g \circ h_i)_*(\mathbb{Q}_{P_{U',i}})(s)[2s-n]\}$). We also obtain that $D_{d\ge 0} \subset DM_c(U)_{w_{Chow}'\ge 0}$ and $D_{d\le 0} \subset DM_c(U)_{w_{Chow}'\le 0}$.

We denote $S \setminus U$ by Z (Z could be empty); $i : Z \to S$ is the corresponding closed immersion. By the inductive assumption, w_{Chow} and w_{Chow}' yield coinciding bounded weight structures for $DM_c(Z)$.

We have the gluing data $DM_c(Z) \stackrel{i_*}{\to} DM_c(S) \stackrel{j^*}{\to} DM_c(U)$. We can 'restrict it' to a gluing data

$$DM_c(Z) \stackrel{i_*}{\to} j^{*-1}(D) \stackrel{j_0^*}{\to} D$$

(see Proposition 1.2.3(13)), whereas $M \in Obj(j^{*-1}(D))$; here j_0^* is the corresponding restriction of j^*. Hence by loc. cit. there exists a weight structure w' for $j^{*-1}(D)$ such that i_* and j_0^* are weight-exact (with respect to the weight structures mentioned). Hence there exists a weight decomposition $B \to M \to A$ of M with respect to w'. Besides, there exist $m, n \in \mathbb{Z}$ such that $j_0^*(M) \in DM_c(U)_{w_{Chow}'\ge m}$, $j_0^*(M) \in DM_c(U)_{w_{Chow}'\le n}$, $i^!(M) \in DM_c(Z)_{w_{Chow}'\ge m}$, and $i^*(M) \in DM_c(Z)_{w_{Chow}'\le n}$. Hence $A[-1], M[-m] \in DM_c(S)_{w_{Chow}'\ge 0}$; $B, M[-n] \in DM_c(S)_{w_{Chow}'\le 0}$; here we apply Proposition 1.2.3(14). So, we verified (ii) and the boundedness of w_{Chow}'. As was shown above, this finishes the proof of assertion I.

II: immediate from assertion I1; cf. the proof of Theorem 2.1.1.

□

Proposition 2.3.5. *For the version of w_{Chow} constructed in this paragraph, the analogues of all parts of Theorem 2.2.1, as well as of Proposition 2.2.3 and Lemma 2.2.4 are fulfilled.*

Proof. The proof of Theorem 2.2.1(I) carries over to our situation without changes. The same is true for parts II1–II2 of loc. cit. for the case of a smooth f. Lemma 2.3.3(2) yields assertion II1 of Theorem 2.2.1 for the case when f is an immersion. The general case of loc. cit. follows from these two immediately.

The (analogues of) the remaining parts of the Theorem follow from (the analogue of) part II via the same arguments as in §2.2.

□

Corollary 2.3.6. *1. We have $Chow(S) \subset \underline{Hw}_{Chow}(S)$ (see the definition of $Chow(S)$ in §2.1).*

2. For a reasonable S the 'alternative' version of w_{Chow} (constructed above) coincides with the version given by Theorem 2.1.1(I).

Proof. 1. It suffices to verify that $p_*(\mathbb{Q}_P) \in \underline{Hw}_{Chow}(S)$ for any regular P and a projective morphism $p : P \to S$. By the previous proposition, we obtain $\mathbb{Q}_P \in DM_c(P)_{w_{Chow}=0}$; since $p_! \cong p_*$, we obtain the result.

2. Indeed, denote the 'old' version of w_{Chow} by v, and the 'alternative' one by w. The previous assertion along with Proposition 1.2.3(10) yields that $1_{DM_c(S)}$ is weight-exact with respect to v and w. Hence part (4) of loc. cit. yields the result. □

Remark 2.3.7. 1. Proposition 2.3.4 also easily yields that we could have considered larger $\mathcal{OZ}(X)$ (see the beginning of the paragraph): one can take $\mathcal{OZ}(X) = Chow(X)$.

Thus the results of this paragraph (in particular) yield a collection of new descriptions of $w_{Chow}(S)$ for the case of a reasonable S (cf. also Remark 2.3.1(3)).

2. In the first draft of this paper (only) the gluing method of constructing w_{Chow} was used (this approach was first proposed in Remark 8.2.4(3) of [Bon10a], that was in its turn inspired by [BBD82]). Next the author proved part 1 of the Corollary. Then (in order to deduce our main results) it remained to note that $Chow(S)$ generates $DM_c(S)$. Luckily, it was easy to prove the negativity of $Chow(S)$ (without relying on the gluing construction of w_{Chow}; see Lemma 1.1.4(I1)); so the proof was simplified (for a reasonable S; note still that the scheme of the proof of loc. cit. is similar to the chain of arguments that yields the first part of the Corollary). Yet (as we have noted just above) even for a reasonable S the gluing method gives us some 'new'

descriptions of w_{Chow}. The main disadvantage of the gluing method is that it does not yield an explicit description of the whole $DM_c(S)_{w_{Chow}=0}$ (though we can describe it as the intersection of $DM_c(S)_{w_{Chow}\le 0}$ with $DM_c(S)_{w_{Chow}\ge 0}$).

3. Let K be a generic point of S, $j_K : K \to S$ is the corresponding morphism; assume that for $M \in ObjDM_c(S)$ we have $j_K^*(M) \in DM_c(K)^{w_{Chow}=0}$. By Theorem 2.2.1II3(i), for any finite universal homeomorphism $K' \to K$, $j_{K'} : K' \to S$ is the corresponding composition, we also have $j_{K'}^*(M) \in DM_c(K')^{w_{Chow}=0}$. Hence an argument similar to that used in the proof of Proposition 2.3.4 easily yields: there exists an open immersion $j : U \to S$, $K \in U$, such that $j^*(M)$ is a retract of $(g \circ h)_* \mathbb{Q}_P(s)[2s]$, where $h : P \to U'$ is a smooth projective morphism, U' is a regular scheme, $g : U' \to U$ is a finite universal homeomorphism, $s \in \mathbb{Z}$.

4. In [Bon11b] the author reduces the conjecture on the existence of the motivic t-structure for $DM_c(S)$ to the case when S is a universal domain. To this end certain gluing arguments are very useful.

5. Possibly one can use the methods of [Heb11] in order to extend the weight-exactness results for the motivic image functors given by Proposition 2.3.4 to not necessarily smoothly embeddable morphisms.

6. Motives with $\mathbb{Z}$-coefficients are more 'mysterious' than those with $\mathbb{Q}$-ones; yet possibly one can construct Chow weight structure(s) for them also. At least, the author hopes to achieve this for motives over (excellent separated finite-dimensional) Spec $\mathbb{Q}$-schemes (and also for motives with $\mathbb{Z}[\frac{1}{p}]$-coefficients over Spec $\mathbb{F}_p$-schemes; cf. [Bon11a]).

3 Applications to (co)homology of motives and other matters

In §3.1 we study weight complexes for S-motives (and their compatibility with weight-exact motivic image functors).

In §3.2 we prove that $K_0(DM_c(S)) \cong K_0(Chow(S))$ (following [Bon10a]), and define a certain 'motivic Euler characteristic' for (separated finite type) S-schemes.

In §3.3 we consider Chow-weight spectral sequences and filtrations for (co)homology of S-motives (following §2.4 of [Bon10a]). We observe that Chow-weight filtrations yield Beilinson's 'integral part' of motivic cohomology (see §2.4.2 of [Bei85] and [Sch00]).

In §3.4 we verify that Chow-weight spectral sequences yield the existence of weight filtrations for the 'perverse étale homology' of motives over finite type $\mathbb{Q}$-schemes (this is not at all automatic for mixed perverse sheaves in

characteristic 0).

In §3.5 we introduce the notion of relative weight structure. The axiomatics of those was chosen to be an abstract analogue of Proposition 5.1.15 of [BBD82]. Several properties of relative weight structures are parallel to those for 'ordinary' weight structures.

In §3.6 we study the case when $S = X_0$ is a variety over a finite field. In this case the category $D^b_m(X_0, \mathbb{Q}_l)$ of mixed complexes of sheaves possesses a relative weight structure whose heart is the class of pure complexes of sheaves. Since the étale realization of motives preserves weights, we obtain that (Chow)-weight filtrations for some cohomology theories can be described in terms of the category $D^b_m(X_0, \mathbb{Q}_l)$.

In this section we will always assume that our base schemes are reasonable. Yet we also could have used the 'gluing' version of w_{Chow} (and consider any finite-dimensional noetherian S; the main difference is that we would have to put $\underline{Hw}_{Chow}$ instead of $Chow(-)$ everywhere).

3.1 The weight complex functor for $DM_c(S)$

We prove that the weight complex functor (whose 'first ancestor' was defined by Gillet and Soulé) can be defined for $DM_c(S)$.

Proposition 3.1.1. *1. The embedding $Chow(S) \to K^b(Chow(S))$ factorizes through a certain exact conservative weight complex functor $t_S : DM_c(S) \to K^b(Chow(S))$.*

2. For $M \in ObjDM_c(S)$, $i, j \in \mathbb{Z}$, we have $M \in DM_c(S)_{[i,j]}$ (see Definition 1.2.1(IV)) whenever $t(M) \in K(Chow(S))_{[i,j]}$ (see Remark 1.2.2).

3. For schemes X, Y let $F : DM_c(X) \to DM_c(Y)$ be a weight-exact functor of triangulated categories (with respect to the Chow weight structures for these categories; so F could be equal to $i_!$ for a finite type projective morphism $i : X \to Y$, or to j^* for a finite type smooth morphism $j : Y \to X$) that possesses a differential graded enhancement. Denote by $F_{K^b(Chow)}$ the corresponding functor $K^b(Chow(X)) \to K^b(Chow(Y))$. Then there exists a choice of t_X and t_Y that makes the diagram

$$\begin{array}{ccc} DM_c(X) & \xrightarrow{F} & DM_c(Y) \\ \downarrow{\scriptstyle t_X} & & \downarrow{\scriptstyle t_Y} \\ K^b(Chow(X)) & \xrightarrow{F_{K^b(Chow)}} & K^b(Chow(Y)) \end{array}$$

commutative up to an isomorphism of functors.

Proof. 1. By Proposition 5.3.3 of [Bon10a], this follows from the existence of a bounded Chow weight structure for $DM_c(S)$ along with the fact that it admits a differential graded enhancement (see Definition 6.1.2 of ibid.; note also that an alternative 'f-category' version of the weight complex functor was constructed by Beilinson and Schnürer; see §7 of [Sch11]). The latter property of $DM(S)$ can be easily verified since it can be described in terms of the derived category of qfh-sheaves over S; see Theorem 16.1.2 of [CiD09] (and also cf. §6.1 of [BeV08]).

2. Immediate from Theorem 3.3.1(IV) of [Bon10a].

3. We use the notation and definitions of §2 of [Bon09a] (that originate mostly from [BoK90]).

Since $DM_c(X) = \langle Chow(S)\rangle$, we can assume that $DM_c(X) = Tr^+(C_X)$, where C_X is a negative triangulated category such that $H(C_X) = Chow(X)$ (see Remark 2.7.4(2) of ibid.). Replacing $DM_c(Y)$ by an equivalent category, we may also assume (similarly) that $DM_c(Y) = Tr^+(C_Y)$ where C_Y is a negative triangulated category such that $H(C_Y) = Chow(Y)$, and $F = \text{Pre-Tr}(F')$ for some differential graded functor $C_X \to C_Y$. Arguing as in §6.1 of ibid, we obtain that it suffices to apply Tr^+ to the following diagram:

$$\begin{array}{ccc} C_X & \xrightarrow{F'} & C_Y \\ \downarrow & & \downarrow \\ H(C_X) & \xrightarrow{H(F')} & H(C_Y) \end{array}$$

□

Remark 3.1.2. 1. The 'first ancestor' of our weight complex functor was defined by Gillet and Soulé in [GiS96]. Weight complex for a general triangulated category $\underline{C}$ endowed with a weight structure was defined in [Bon10a]. Even in the case when $\underline{C}$ does not admit a differential graded enhancement, one can still define a certain 'weak' version of the weight complex; see §3 of ibid. (and this version does not depend on any choices). It follows that for $M \in ObjDM_c(S)$ the isomorphism class of $t_S(M)$ (in $K^b(Chow(S))$) does not depend on any choices (see ibid.).

2. In [GiS09] a functor h from the category of Deligne-Mumford stacks over S (with morphisms being proper morphisms over S) to the category of complexes over a certain category of K_0-*motives* was constructed; Gillet and Soulé considered base schemes satisfying rather restrictive conditions (mostly, of dimension ≤ 1). We conjecture: for a finite type morphism $p : X \to S$ there is a functorial isomorphism $h(X) \to t(\mathcal{M}_c(X))$, where $\mathcal{M}_c(X) = p_* p^!(\mathbb{Q}_S)$. For S being the spectrum of a characteristic 0 field this

was (essentially) proved in §6.6 of [Bon09a]. Note here: though the category of K_0-*motives* is somewhat 'larger' than $Chow(S)$, it very probably suffices to consider its 'Chow' part (this would be the category $Chow(S)$ considered in [CoH00]).

Note that our definition of a weight complex (for $\mathcal{M}^c(X)$) gives it much more functoriality in X than it was established [GiS09]; we also study its functoriality with respect to S, and relate it with (co)homology (below).

Besides, we can restrict our definition of weight complexes to (motives with compact support of) quotient stacks (cf. Definition 1.2 of [GiS09]). For a finite G, $\#G = n$, acting on a finite type scheme X/S one can take $\mathcal{M}^c(X/G) = a_{G*}\mathcal{M}^c(X) \in ObjDM_c(S)$. Here a_G is the idempotent morphism (correspondence) $\frac{\sum_{g\in G} g}{n} : X \to X$. Certainly, for $G = \{e\}$ we will have $t_S(\mathcal{M}^c(X/G)) = t_S(\mathcal{M}^c(X))$.

3. Theorem 2.1.1 along with the results of [Bon10a] also imply: t_S can be extended to an exact functor $DM(S) \to K(BChow(S))$, where $BChow(S)$ is the big hull of $Chow(S)$ (see Definition 1.2.1(IX)).

4. One can also define exact (and conservative) *higher truncation functors* $t_{S,N}$ from $DM_c(S)$ to certain triangulated $DM_c(S)_N$ for all $N \geq 0$; cf. §6.1 of [Bon09a]. Here $t_{S,0} = t_S$; $DM_c(S)_N$ is constructed using differential graded methods that enable 'killing all morphisms' from $DM_c(S)_{w_{Chow}=0}$ to $DM_c(S)_{w_{Chow}=i}$ for $i < -N$ and preserving such morphisms for $i \geq -N$. $t_{S,N'}$ factorizes through $t_{S,N}$ for any $N' > N$, and the 2-limit of $DM_c(S)_N$ equals $DM_c(S)$. $t_{S,N}$ would also satisfy the analogue of Theorem 6.2.1 of ibid. Yet it seems that $t_S = t_{S,0}$ is the most interesting of these truncation functors.

3.2 $K_0(DM_c(S))$ and a motivic Euler characteristic

Now we calculate $K_0(DM_c(S))$ and study a certain Euler characteristic for (finite type separated) S-schemes.

Proposition 3.2.1. *1. We define $K_0(Chow(S))$ as the groups whose generators are $[M]$, $M \in ObjChow(S)$, and the relations are $[B] = [A] + [C]$ if $A, B, C \in ObjChow(S)$ and $B \cong A \bigoplus C$. For $K_0(DM_c(S))$ we take similar generators and set $[B] = [A] + [C]$ if $A \to B \to C \to A[1]$ is a distinguished triangle.*

Then the embedding $Chow(S) \to DM_c(S)$ yields an isomorphism $K_0(Chow(S)) \cong K_0(DM_c(S))$.

2. For the correspondence $\chi : X \mapsto [p_*p^!(\mathbb{Q}_S)]$ (here $p : X \to S$ is a finite type separated morphism) from the class of finite type separated S-schemes

to $K_0(DM_c(S)) \cong K_0(Chow(S))$ *we have:* $\chi(X \setminus Z) = \chi(X) - \chi(Z)$ *if* Z *is a closed subscheme of* X*.*

Proof. 1. Immediate from (part I of) Theorem 2.1.1 and Proposition 5.3.3(3) of [Bon10a].

2. Denote the immersion $Z \to X$ by i, and the complimentary immersion by j. By Proposition 1.1.2(10) for any $M \in ObjDM_c(X)$ we have a distinguished triangle $i_*i^!(M) \to M \to j_*j^!(M)$ (note that $i_! \cong i_*$ and $j^! = j^*$). Now for $M = p^!(\mathbb{Q}_S)$ this triangle specializes to the triangle $i_*(p \circ i)^!(\mathbb{Q}_S) \to p^!(\mathbb{Q}_S) \to j_*(p \circ j)^!(\mathbb{Q}_S)$. It remains to apply $[p_*(-)]$ and the definition of $K_0(DM_c(S))$ to obtain the result.

□

Remark 3.2.2. 1. Assertion 2 is a vast extension of Corollary 5.13 of [GiS09]. It allows us to define certain motivic Euler characteristics for (finite type separated) S-schemes.

2. We hope that our results would be useful for the theory of motivic integration.

Note in particular: we obtain that any (not necessarily weight-exact!) motivic image functor $DM_c(X) \to DM_c(Y)$ induces a group homomorphism $K_0(Chow(X)) \to K_0(Chow(Y))$.

Besides, in contrast to the 'classical' case (when S is the spectrum of a field) there does not seem to exist a 'reasonable' (tensor) product for $Chow(S)$. Yet $DM_c(S)$ is a tensor triangulated category; hence one can use assertion 1 in order to define a ring structure on $K_0(Chow(S))$.

3.3 Chow-weight spectral sequences and filtrations

Now we discuss (Chow)-weight spectral sequences and filtrations for homology and cohomology of motives. We note that any weight structure yields certain weight spectral sequences for any (co)homology theory; the main distinction of the result below from the general case (i.e. from Theorems 2.3.2 and 2.4.2 of ibid.) is that $T(H, M)$ always converges (since w_{Chow} is bounded). Since below we will be mostly interested in weight filtrations for cohomological functors, we will define them in this situation only; certainly, dualization is absolutely no problem (cf. §2.1 of ibid.)

Proposition 3.3.1. *Let* $\underline{A}$ *be an abelian category.*

I Let $H : DM_c(S) \to \underline{A}$ *be a homological functor; for any* $r \in \mathbb{Z}$ *denote* $H \circ [r]$ *by* H_r*.*

For an $M \in ObjDM_c(S)$ *we denote by* (M^i) *the terms of* $t(M)$ *(so* $M^i \in ObjChow(S)$*; here we can take any possible choice of* $t(M)$*).*

Then the following statements are valid.

1. There exists a (Chow-weight) spectral sequence $T = T(H, M)$ with $E_1^{pq} = H_q(M^p) \implies H_{p+q}(M)$; the differentials for $E_1T(H, M)$ come from $t(M)$.

2. $T(H, M)$ is $DM_c(S)$-functorial in M (and does not depend on any choices) starting from E_2.

II1. Let $H : DM_c(S) \to \underline{A}$ be any contravariant functor. Then for any $m \in \mathbb{Z}$ the object $(W^mH)(M) = \mathrm{Im}(H(w_{Chow\geq m}M) \to H(M))$ does not depend on the choice of $w_{Chow\geq m}M$; it is functorial in M.

We call the filtration of $H(M)$ by $(W^mH)(M)$ its Chow-weight filtration.

2. Let H be cohomological. For any $r \in \mathbb{Z}$ denote $H \circ [-r]$ by H^r.

Then the natural dualization of assertion I is valid. For any $M \in ObjDM_c(S)$ we have a spectral sequence with $E_1^{pq} = H^q(M^{-p})$; it converges to $H^{p+q}(M)$. Moreover, the step of filtration given by $(E_\infty^{l,m-l} : l \geq k)$ on $H^m(X)$ equals $(W^kH^m)(M)$ (for any $k, m \in \mathbb{Z}$). T is functorial in H and M starting from E_2.

Proof. I Immediate from Theorem 2.3.2 of ibid.

II1. This is Proposition 2.1.2(2) of ibid.

2. Immediate from Theorem 2.4.2 of ibid.

□

Remark 3.3.2. 1. We obtain certain *Chow-weight* spectral sequences and filtrations for any (co)homology of motives. In particular, we have them for (rational) étale and motivic (co)homology of motives. For the latter theory, we obtain certain results that cannot be proved using 'classical' (i.e. Deligne's) methods, since the latter heavily rely on the degeneration of (an analogue of) T at E_2. We will conclude this subsection by studying an example of this sort; we obtain a result that (most probably) could not be guessed using the 'mixed motivic intuition'.

2. $T(H, M)$ can be naturally described in terms of the virtual t-truncations of H (starting from E_2); see §2.5 of [Bon10a] and §4.3 below.

3. We obtain that any (co)homology of any $M \in ObjDM_c(S)$ possesses a filtration by subfactors of (co)homology of regular projective S-schemes.

4. Actually, $(W^mH)(M) = \mathrm{Im}(H(w_{Chow\geq m}M) \to H(M))$ does not depend on the choice of $w_{Chow\geq m}M$ and is functorial in M for any contravariant $H : DM_c(S) \to \underline{A}$; see Proposition 2.1.2(2) of [Bon10a].

The functoriality of Chow-weight filtrations has quite interesting consequences.

Proposition 3.3.3. *Let a scheme X be reasonable; adopt the notation of Proposition 1.1.2(10). Let $H : DM_c(X) \to \underline{A}$ be a contravariant func-*

tor. Then for all $N \in DM_c(U)_{w_{Chow}\le 0}$ *consider* $G(N) = (W^0H)(j_!(N)) \subset H(j_!(N))$.

I The following statements are fulfilled.

1. $G(N)$ *is* $DM_c(U)$*-functorial in* N.

2. (For any $N \in DM_c(U)_{w_{Chow}\le 0}$*)* $G(N)$ *is a quotient of* $H(M)$ *for some* $M \in DM_c(X)_{w_{Chow}=0}$.

II Let now $N = j^*(M)(= j^!(M))$ *for* $M \in DM_c(X)_{w_{Chow}=0}$. *Then the following statements are fulfilled.*

1. $G(N) = \operatorname{Im} H(M) \to H(j_!j^!(M))$ *(here we apply* H *to the morphism* $j_!j^!(M) \to M$ *coming from the adjunction* $j_! : DM_c(U) \leftrightarrows DM_c(X) : j^!$*).*

2. Let $H = DM_c(X)(-, \mathbb{Q}_X(q)[p])$ *for some* $q, p \in \mathbb{Z}$, *and* $M = f_*(\mathbb{Q}_R)$ *for a projective* $f : R \to X$, R *is regular. Then* $G(N) \cong \operatorname{Im}(\mathrm{Gr}^p_\gamma(K_{2q-p}(R) \otimes \mathbb{Q}) \to \mathrm{Gr}^p_\gamma(K_{2q-p}(R_U) \otimes \mathbb{Q}))$.

Proof. I. Assertion 1 is an immediate consequence of Remark 3.3.2(4). Next (by Theorem 2.2.1(II1)) $N \in DM_c(U)_{w_{Chow}\le 0}$ implies $j_!(N) \in DM_c(X)_{w_{Chow}\le 0}$. Hence $j_!(N) \in DM_c(X)_{w_{Chow}=0}$ (see Proposition 1.2.3(2)). This yields assertion 2.

II We have $i_*i^*(M) \in DM_c(X)_{w_{Chow}\le 0}$. Hence the distinguished triangle

$$i_*i^*(M) \to j_!(N)[1](= j_!j^!(M)[1]) \to M[1]$$

(given by Proposition 1.1.2(10)) yields a weight decomposition of $j_!(N)[1]$. This yields assertion II1.

Now, Proposition 1.1.2(15) (see also Corollary 14.2.14 of [CiD09]) yields $DM_c(X)(f_*(\mathbb{Q}_R), \mathbb{Q}_X(q)[p]) \cong \mathrm{Gr}^p_\gamma(K_{2q-p}(R) \otimes \mathbb{Q})$. Besides, the adjunction $j_! : DM_c(U) \leftrightarrows DM_c(X) : j^!$ yields that

$$DM_c(X)(j_!j^!(M), \mathbb{Q}_X(q)[p]) \cong DM_c(U)(j^!(M), j^!(\mathbb{Q}_X)(q)[p]) \cong DM_c(U)(f_{U*}(\mathbb{Q}_{R_U}), \mathbb{Q}_U(q)[p]) \cong$$

It remains to compare the rows of the diagram

$$\begin{array}{ccc} H(M) & \longrightarrow & H(j_!j^!(M)) \\ \downarrow & & \downarrow \\ \mathrm{Gr}^p_\gamma(K_{2q-p}(R) \otimes \mathbb{Q}) & \longrightarrow & \mathrm{Gr}^p_\gamma(K_{2q-p}(R_U) \otimes \mathbb{Q}) \end{array}$$

The results of [CiD09] (we need the naturality of the isomorphism given by Corollary 14.2.14 of ibid. and also of Proposition 1.1.2(5)) yield that this diagram can be made commutative by (possibly) modifying the left column by an isomorphism; this finishes the proof of assertion II2.

□

Remark 3.3.4. 1. Thus one may say that $(W^0H)(j_!(N))$ yields the 'integral part' of $H(j_!(N))$: we obtain the subobject of $H^*(j_!(N))$ that 'comes from a nice X-model' of N if the latter exists, and a factor of $H(M)$ for some $M \in DM_c(X)_{w_{Chow}=0}$ in general; cf. [Bei85], [Sch00], and [Sch12]. Note here that one can also consider $N \in Chow(K)$ for K being a generic point of X, since any such N can be lifted to a Chow motif over some U ($K \in U$, U is open in X), by Theorem 2.2.1(III1) combined with Proposition 1.1.2(14); cf. also Remark 1.11 of [Wil12].

Note that this construction enjoys 'the usual' $Chow(K)$-functoriality; this is an easy consequence of Proposition 1.1.2(14) and Remark 2.1.2(1).

2. Hence we proved an alternative to Theorem 1.1.6 of [Sch00]. In more detail: in the setting of our assertion II2 we can take X being the spectrum of a Dedekind domain, and U 'approximating' the spectrum K of its fraction field; then the (corresponding) restriction of $H(j_!(N))$ to Chow motives over K satisfies the conditions of loc. cit. Hence in this partial case our version of the 'integral part' of motivic cohomology coincides with the Scholl's one.

An alternative 'mixed motivic' approach to the construction of this integral part is given by Corollary 1.10 of [Wil12]. It is equivalent to our one for H being the motivic cohomology (as in assertion II2); yet the author does not know how to extend loc. cit. to the case of a general H.

3. Our description of the 'integral part' of cohomology is very short and does not rely on any conjectures (in contrast to the description given in [Sch12]).

4. It could also be interesting to consider $W^lH^*(j_!(N))$ for $l < 0$.

3.4 Application to mixed sheaves: the 'characteristic 0' case

Suppose that S' is a finite type Spec $\mathbb{Z}$-scheme. Denote by $\mathcal{H}^{et}_{\mathbb{Q}_l}$ the étale realization functor $DM_c(S') \to D^bSh^{et}(S', \mathbb{Q}_l)$, where $D^bSh^{et}(S', \mathbb{Q}_l)$ is the Ekedahl's derived category of $\mathbb{Q}_l$-constructible sheaves (cf. §1 of [Hub97]); it could be called étale homology (so, it is covariant). We will assume below that $\mathcal{H}^{et}_{\mathbb{Q}_l}$ converts the motivic image functors into the corresponding functors for $D^bSh^{et}(-, \mathbb{Q}_l)(-)$ (it seems that the existence of such a realization is not fully established in the existing literature; yet a forthcoming paper of Cisinski and Deglise should close this gap).

Now, let S be a finite type (separated) $\mathbb{Q}$-scheme. Presenting it as a projective limit of certain S'^i that are flat of finite type over an open subscheme of Spec $\mathbb{Z}[\frac{1}{l}]$, and using coherent functoriality of motives and étale sheaves, one can construct an exact (covariant) realization functor

$\mathbb{H} : DM_c(S) \to DSH(S)$, where the latter is the category of mixed complexes of $\mathbb{Q}_l$-étale sheaves as considered in [Hub97] (i.e. it is the inductive limit of derived categories of complexes sheaves that are constructible with respect to 'horizontal' stratifications of 'models' of S). Indeed, the arguments of ibid. can be applied here without any problem. The key observation here is that $\mathbb{Q}_S$ is mixed (as an object of $\varinjlim D^b Sh^{et}(S'^i, \mathbb{Q}_l) \supset DSH(S)$), whereas (the étale sheaf) image functors preserve mixedness.

The functoriality properties of $\mathbb{H}$ also yields that it sends Chow motives over S to pure complexes of sheaves (of weight 0; see Definition 3.3 of [Hub97]). Indeed, it suffices to note that $\mathbb{H}$ sends $\mathbb{Q}_X$ for a regular X to an object of $DSH(X)$ of weight 0, whereas $f_! = f_*$ for a projective f preserves weights of sheaves (see the Remark succeeding Definition 3.3 of [Hub97]). We easily obtain the following statement.

Proposition 3.4.1. *Take H^{per} being the perverse étale homology theory i.e. $H_i^{per}(M)$ (for any $M \in ObjDM_c(S)$, $i \in \mathbb{Z}$) is the i-th homology of $\mathbb{H}(M)$ with respect to the perverse t-structure of DSH (see Proposition 3.2 of [Hub97]). Then $H_i^{per}(M)$ have weight filtrations (defined using Definition 3.7 of loc. cit., for all $i \in \mathbb{Z}$) i.e. it has a increasing filtration whose j-th factor is of weight j.*

Proof. The existence (and convergence) of the corresponding Chow-weight spectral sequence $T = T(H^{per}, M)$ (see Proposition 3.3.1)(I1), or Theorem 2.3.2 of [Bon10a]) yields that $H_i^{per}(M)$ possesses a (Chow-weight) filtration whose r-th factor (for any $r \in \mathbb{Z}$) is a subquotient of $H_{i+r}^{per}(M^{-r})$. Now, $H_{i+r}^{per}(M^{-r})$ is of weight $i + r$ (since this is true for $M^{-r}[r + i]$; cf. Theorem 5.4.1 of [BBD82]). By Proposition 3.4 of ibid. we obtain that the same is true for any its subquotient. Shifting the indexing of the Chow-weight filtration, we obtain a filtration of $H_i^{per}(M)$ with the properties required. □

Remark 3.4.2. Note that the existence of weight filtrations for mixed perverse sheaves is not at all automatic (in this setting); see the Warning preceding Proposition 3.4 of loc. cit.

3.5 Relative weight structures

In order to define weights for mixed complexes of sheaves (over a finite field), we have to generalize the definition of a weight structure.

Definition 3.5.1. I Let $F : \underline{C} \to \underline{D}$ be an exact functor (of triangulated categories).

A pair of extension-stable Karoubi-closed subclasses $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset Ob\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a relative weight structure

w for $\underline{C}$ with respect to F (or just an F-weight structure) if they satisfy the following conditions.

(i) **'Semi-invariance' with respect to translations**.

$\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$, $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(ii) **Weak orthogonality**.

$\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[2]$.

(iii) F**-orthogonality**.

F kills all morphisms between $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**.

For any $M \in Obj\underline{C}$ there exists a distinguished triangle

$$B \to M \to A \xrightarrow{f} B[1] \tag{3}$$

such that $B \in \underline{C}_{w\le 0}, A \in \underline{C}_{w\ge 0}[1]$.

II We define $\underline{C}_{w\ge i}$, $\underline{C}_{w\le i}$, $\underline{C}_{w=i}$, $\underline{C}_{[i,j]}$, bounded relative weight structures, and $\underline{C}^b$ similarly to Definition 1.2.1.

We will call the class $\underline{C}_{w=0}$ the heart of w (we will not define the category $\underline{Hw}$).

We will use the same notation for weight truncations with respect to w as the one introduced in Remark 1.2.2. We define weight-exact functors for relative weight structures as in Definition 1.2.1(VI) (i.e. we do not mention the corresponding F's in the definition).

III Let H be a full subcategory of a triangulated $\underline{C}$.

We will say that H is F*-negative* if $ObjH \perp (\cup_{i>1} Obj(H[i]))$ and F kills all morphisms between H and $H[1]$.

Remark 3.5.2. 1. Any weight structure for $\underline{C}$ is a relative weight structure with respect to $F = 1_{\underline{C}}$.

2. An F-weight structure is also a $G \circ F$-weight structure for any exact functor $G : \underline{D} \to \underline{E}$ (for any triangulated $\underline{E}$). In particular, one can always take $F = 0$. Hence we do not lose generality by adding the F-orthogonality axiom to the definition of relative weight structures. Besides, those properties of relative weight structures that do not depend on the choice of F are certainly valid without this axiom. The main reason to put the F-orthogonality axiom together with the weak orthogonality one is that these conditions could be tracked down using similar methods.

3. The weak orthogonality axiom is is a strengthening the higher Hom decomposition condition that was studied in Appendix B of [Pos11]. Respectively, Proposition 2 of loc. cit. generalizes our Proposition 3.5.3(8) considered in the case $F = 0$.

Now we will extend to relative weight structures several properties of weight structures. We will skip those parts of the proofs that do not differ much from the ones in [Bon10a] (for 'usual' weight structures); we will concentrate on the distinctions.

Proposition 3.5.3. *Let $F : \underline{C} \to \underline{D}$ be an exact functor (of triangulated categories).*

In all assertions expect (8) we will also assume that w is a relative weight structure for $\underline{C}$ with respect to F.

1. *(C_1, C_2) ($C_1, C_2 \subset Obj\underline{C}$) define an F-weight structure for $\underline{C}$ whenever (C_2^{op}, C_1^{op}) define a relative weight structure for $\underline{C}^{op}$ with respect to F^{op}; here $F^{op} : \underline{C}^{op} \to \underline{D}^{op}$ is the functor obtained from F by inverting all arrows.*

2. *All $\underline{C}_{[i,j]}$ are extension-stable.*

3. *Let $l \geq m \in \mathbb{Z}$, $M, M' \in Obj\underline{C}$. Let weight decompositions of $M[-m]$ and $M'[-l]$ be fixed; we consider the corresponding triangles $w_{\leq m}M \xrightarrow{b} M \xrightarrow{a} w_{\geq m+1}M$ and $w_{\leq l}M' \xrightarrow{b'} M' \xrightarrow{a'} w_{\geq l+1}M'$.*

   Then for any $g \in \underline{C}(M, M')$ there exists some morphism of distinguished triangles

$$\begin{array}{ccccc} F(w_{\leq m}M) & \xrightarrow{F(b)} & F(M) & \xrightarrow{F(a)} & F(w_{\geq m+1}M) \\ \downarrow & & \downarrow{\scriptstyle F(g)} & & \downarrow \\ F(w_{\leq l}M') & \xrightarrow{F(b')} & F(M') & \xrightarrow{F(a')} & F(w_{\geq l+1}M') \end{array} \tag{4}$$

4. *In addition to the assumptions of the previous assertion, suppose that $l > m$.*

   Then there also exists a morphism of triangles

$$\begin{array}{ccccccc} w_{\leq m}M & \xrightarrow{b} & M & \xrightarrow{a} & w_{\geq m+1}M & \xrightarrow{f} & w_{\leq m}M[1] \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c[1]} \\ w_{\leq l}M' & \xrightarrow{b'} & M' & \xrightarrow{a'} & w_{\geq l+1}M' & \xrightarrow{f'} & w_{\leq l}M'[1] \end{array} \tag{5}$$

   Moreover, (g, a, a') determine $F(d)$ uniquely; (g, b, b') determine $F(c)$ uniquely.

5. *For any $M \in Obj\underline{C}$ any choices of $w_{\geq i}M$ (and of the arrows $a_i : M \to w_{\geq i}M$ for all $i \in \mathbb{Z}$) can be completed to a weight Postnikov tower for M (cf. Definition 1.5.8 of [Bon10a]) i.e. for all $j \in \mathbb{Z}$ we can choose some morphisms $c_j : w_{\geq j}M \to w_{\geq j+1}M$ that are compatible with a_j, a_{j+1}, and for any choice of these c_j we have: $M^j = \mathrm{Cone}(c_{-j}(M))[j-1] \in \underline{C}_{w=0}$.*

6. *For $j_0, j_1 \in \mathbb{Z}$, we can choose a weight Postnikov tower for M such that $w_{\geq j}M = 0$ for $j > j_1$ and $= M$ for $j \leq j_0$ whenever $M \in \underline{C}_{[j_0,j_1]}$.*

   We will call such a weight Postnikov tower a bounded one.

7. *Let w be bounded, G be an exact functor $\underline{C} \to \underline{C}'$; suppose that $\underline{C}'$ is endowed with a relative weight structure (with respect to some exact functor $F' : \underline{C}' \to \underline{D}'$).*

   Then G is left (resp. right) weight-exact whenever $G(\underline{C}_{w=0}) \subset \underline{C}'_{w'\leq 0}$ (resp. $G(\underline{C}_{w=0}) \subset \underline{C}'_{w'\geq 0}$).

8. *Let $H \subset \underline{C}$ be F-negative. Then there exists a bounded weight structure w on $T = \langle H \rangle$ in $\underline{C}$ such that $ObjH \subset T_{w=0}$.*

Proof. Assertions 1 and 2 are immediate from Definition 3.5.1.

The proof of assertions 3 and 4 is similar to that of Proposition 1.5.1(1,2) of [Bon10a]. The axiom (iii) of relative weight structure yields that the composition morphism $F(w_{\leq m}M) \to F(w_{\geq l+1}M')$ vanishes. Hence (an easy) Proposition 1.1.9 of [BBD82] yields the existence of (4).

Similarly, we obtain the existence of (5) if $m < l$. Moreover, any two distinct choices of d (resp. of c) are easily seen (see the proof of loc. cit.) to differ by $s \circ f$ (resp. by $(f' \circ s)[-1]$) for some $s \in \underline{C}(w_{\leq m}M[1], w_{\geq l+1}M')$. Since $F(s) = 0$ (by axiom (iii) of relative weight structures), we conclude the proof of assertion 4.

The argument needed for the proof of assertion 5 is very similar to the one used in the proof Theorem 2.2.1(11) of [Bon10b].

We put $M' = M$, $l = j+1$, $m = j$ in assertion 4; this yields the existence of some set of c_j. Since $\underline{C}_{w\geq 1-j}$ is extension-stable, it contains $\mathrm{Cone}\, c_{-j}$ (for any choice of the latter). Completing the commutative triangle $M \stackrel{a_{-j}}{\to} w_{\geq -j}M \stackrel{c_{-j}}{\to} w_{\geq 1-j}M$ to an octahedral digram (as was drawn in loc. cit.), we obtain that $\mathrm{Cone}\, c_{j-1}$ is also a cone of some morphism $w_{\leq -j-1}M[1] \to w_{\leq -j}M[1]$. Since $\underline{C}_{w\leq 1-j}$ is extension-stable also, we obtain assertion 5.

(6): If $w_{\geq j_1+1}M = 0$ (resp. $w_{\geq j_0}M = M$) then obviously $M \in \underline{C}_{w\geq j_0}$ (resp. $M \in \underline{C}_{w\leq j_1}$). Conversely, if $M \in \underline{C}_{[j_0,j_1]}$, then nothing prevents us from choosing $w_{\geq j}M = 0$ for all $j > j_1$ and $= M$ for all $j \leq j_0$.

(7): Certainly, if G left (resp. right) weight-exact then $G(\underline{C}_{w=0}) \subset \underline{C}'_{w'\le 0}$ (resp. $G(\underline{C}_{w=0}) \subset \underline{C}'_{w'\ge 0}$). Conversely, let $M \in \underline{C}_{w\le 0}$ (resp. $M \in \underline{C}_{w\ge 0}$). By the previous assertion, M possesses a bounded weight Postnikov tower with $M^i = 0$ for $i < 0$ (resp. for $i > 0$). The structure of the tower yields that M belongs to the envelope of $M^i[-i]$; this concludes the proof of the assertion.

The proof of assertion 8 is similar to that of Theorem 4.3.2(II1) of [Bon10a] (also, one can assume that $F = 0$ here). We take the envelope of $H[i]$ for $i \le 0$ (resp. for $i \ge 0$) for $\underline{C}_{w\le 0}$ (resp. for $\underline{C}_{w\ge 0}$; see the Notation). Obviously, $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ are Karoubi-closed, extension-stable, and satisfy the condition (i) of Definition 3.5.1(I). F-orthogonality of H easily yields conditions (ii) and (iii) of loc. cit. It remains to verify that any object of $\underline{C}$ possesses a weight decomposition with respect to w.

We define the notion of complexity for objects of $\underline{C}$. For $M \in ObjH[i]$ we will say that M has complexity ≤ 0. If there exists a distinguished triangle $M \to N \to O$, and M, O are of complexity $\le i$ (they also could have smaller complexity) we will say that the complexity of N is $\le i+1$. Since any object of $\langle H \rangle$ has finite complexity, it suffices to verify: for a distinguished triangle $M \to N \to O$ if M, O possess weight decompositions (with respect to our $(\underline{C}_{w\le 0}, \underline{C}_{w\ge 0})$), then N possesses a weight decomposition also.

By assertion 4, we can complete the morphism $O[-1] \to M$ to a commutative square

$$\begin{array}{ccc} (w_{\le 0}M)[-1] & \longrightarrow & O[-1] \\ \downarrow & & \downarrow \\ w_{\le 0}M & \longrightarrow & M \end{array}$$

Hence by the 3×3-Lemma (i.e. Proposition 1.1.11 of [BBD82]) we can complete the distinguished triangle $M \to N \to O$ to a commutative diagram

$$\begin{array}{ccccc} w_{\le 0}M & \longrightarrow & M & \longrightarrow & w_{\ge 1}M \\ \downarrow & & \downarrow & & \downarrow \\ N' & \longrightarrow & N & \longrightarrow & N'' \\ \downarrow & & \downarrow & & \downarrow \\ w_{\le 0}O & \longrightarrow & O & \longrightarrow & w_{\ge 1}O \end{array} \tag{6}$$

whose rows and columns are distinguished triangles (cf. Lemma 1.5.4 of [Bon10a]). We have $N' \in \underline{C}_{w\le 0}$, $N'' \in \underline{C}_{w\ge 1}$ (by the definition of these classes). Hence N possesses a weight decomposition indeed.

□

Remark 3.5.4. One can also glue relative weight structures similarly to Proposition 1.2.3(13), and define weight structures for 'pure' localizations as in part (11) of loc. cit.

As above, for (any) contravariant $H : \underline{C} \to \underline{A}$ we denote $H \circ [-r]$ by H^r.

Proposition 3.5.5. *I Let H be cohomological, $M \in Obj\underline{C}$. Fix (any choice of) a bounded weight Postnikov tower for M (see Proposition 3.5.3(5)).*

1. There exists a weight spectral sequence T with $E_1^{pq}(T) = H^q(M^{-p}) \Longrightarrow E_\infty^{p+q}(T) = H^{p+q}(M)$ (cf. Proposition 3.3.1(II2)).

2. Denote the step of filtration given by $(E_1^{l,m-l} : l \geq k)$ on $H^m(M)$ by $W^kH^m(M)$. Then

$$W^kH^m(M) = \mathrm{Im}(H^m(w_{\geq k}M) \to H^m(M)). \tag{7}$$

3. Suppose that a not necessarily cohomological $H : \underline{C} \to \underline{A}$ can be factorized through F. Then $W^kH^m(M)$ (defined by the formula (7)) is $\underline{C}$-functorial in M (and does not depend on the choice of the corresponding weight decompositions).

II Let $F : \underline{C} \to \underline{D}$, $F' : \underline{C}' \to \underline{D}$, and $G : \underline{C} \to \underline{C}'$ be exact functors. Let w be an F-weight structure for $\underline{C}$, w' be an F'-weight structure for $\underline{C}'$; suppose that G is weight-exact.

1. G converts w-Postnikov towers into w'-Postnikov towers.

2. For any functor $H' : \underline{C} \to \underline{A}$ suppose that H' can be factorized through F'. Then we have $W^kH'^m(G(-)) = W^kH^m(-)$ (here we define the weight filtration via (7)).

Proof. I 1,2: Immediate from the standard properties of spectral sequences coming from Postnikov towers; see the Exercises after §IV.2 of [GeM03].

3: This is an easy consequence of Proposition 3.5.3(3) (cf. the proof of Proposition 2.1.2 of [Bon10a]). Indeed, the right hand side square in the diagram (4) yields the existence of a commutative diagram

$$\begin{array}{ccc} H^m(w_{\geq k}M') & \longrightarrow & H^m(M') \\ \downarrow & & \downarrow{\scriptstyle H^m(g)} \\ H^m(w_{\geq k}M) & \longrightarrow & H^m(M) \end{array} \tag{8}$$

Now, it suffices to verify that for any $g \in \underline{C}(M, M')$ and for any choice of $w_{\geq k}M$, $w_{\geq k}M'$ we have $H^m(g)(\mathrm{Im}(H^m(w_{\geq k}M') \to H^m(M')) \subset \mathrm{Im}(H^m(w_{\geq k}M) \to H^m(M)$; the latter is an immediate consequence of the existence of (8).

II 1. Obvious.

2. Immediate from assertions II1 and I3.

□

Remark 3.5.6. 1. Suppose that there exist t-structures $t_{\underline{C}}$ for $\underline{C}$ and $t_{\underline{D}}$ for $\underline{D}$ such that F is t-exact. Suppose also that for $M \in \underline{C}^{t=0}$ there exists a choice of $w_{\le 0}M$ and $w_{\ge 1}M$ belonging to $\underline{Ht}$. Then the morphism $F(M) \to F(w_{\ge 1}M)$ is epimorphic in $\underline{Ht}_{\underline{D}}$. It follows: for the functor $H = H_0^{t_{\underline{D}},op} \circ F$ the first level of the weight filtration of $H(F) = F(M)$ is just $F(w_{\ge 1}M)$. Here $H_0^{t_{\underline{D}},op}$ is the zeroth homology with respect to t with values in the category opposite to $\underline{Ht}_{\underline{D}}$ (we invert the arrow in order to make the functor cohomological). So, such weight truncations are 'F-functorial when they exist'; cf. Remark 1.5.2(2) and §8.6 of [Bon10a]. Hence the corresponding weight filtrations are functorial also.

2. Unfortunately, it seems that weight spectral sequences given by the Proposition don't have to be canonical (in general).

3. Without losing any generality, one can assume that $F = 0$ in assertion II2.

3.6 On mixed complexes of sheaves over finite fields

Now let $S = X_0$ be a variety over a finite field $\mathbb{F}_q$; let X denote $X_0 \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$. Let F denote the extension of scalars functor $D^b Sh^{et}(X_0, \mathbb{Q}_l) \to D^b Sh^{et}(X, \mathbb{Q}_l)$. We consider $\mathcal{H}^{et}_{\mathbb{Q}_l} : DM_c(X_0) \to D^b Sh^{et}(X_0, \mathbb{Q}_l)$ (cf. §3.4).

Proposition 3.6.1. *1. The category $DSH = D^b_m(X_0, \mathbb{Q}_l)(\subset D^b Sh^{et}(X_0, \mathbb{Q}_l))$ of mixed complexes of sheaves (see §5.1.5 of [BBD82]) can be endowed with an F-weight structure w_{DSH} such that $DSH_{w_{DSH} \le 0} = D^b_{\le 0}(X_0, \mathbb{Q}_l)$ (resp. $DSH_{w_{DSH} \ge 0} = D^b_{\ge 0}(X_0, \mathbb{Q}_l)$); here we use the notation from §5.1.8 of ibid. The heart of w_{DSH} is the class of pure complexes of sheaves of weight 0.*

2. $\mathcal{H}^{et}_{\mathbb{Q}_l}$ can be factorized through an exact $\mathbb{H} : DM_c(X_0) \to DSH$; $\mathbb{H}$ is a weight-exact functor (with respect to w_{Chow} and w_{DSH}).

Proof. 1. Proposition 5.1.14 of [BBD82] yields all axioms of F-weight structures in our situation expect the existence of weight decompositions. So, by Proposition 3.5.3(8), it suffices to verify that the category of pure complexes of sheaves of weight 0 (note that it is idempotent complete) generates DSH. This is immediate from Theorem 5.3.5 of [BBD82].

2. The same arguments as in §3.4 (actually, a simplification of those, since we don't have to present S as a limit) easily yield the existence of $\mathbb{H}$. We also obtain that it is weight-exact by Proposition 3.5.3(7).

□

Remark 3.6.2. 1. In particular, we obtain that any object M of DSH possesses a weight Postnikov tower whose 'factors' are pure complexes of sheaves.

Besides, we obtain that for any (cohomological) $G : D^bSh^{et}(X, \mathbb{Q}_l) \to \underline{A}$, $H = G \circ F \circ \mathcal{H}^{et}_{\mathbb{Q}_l}$, the Chow-weight filtration (see Remark 3.3.2) $(W^lH)(-)$ can be described as $(W^l_{w_{DSH}} G \circ F)(\mathcal{H}^{et}_{\mathbb{Q}_l}(-))$ (for any $l \in \mathbb{Z}$; see Proposition 3.5.5(II2)).

Possibly, one could prove some more results of this sort via introducing a 'relative' analogue of the notion of *transversal* weight and t-structures (that was introduced in §1 of [Bon12]); cf. also Remark 3.5.6(1).

2. So, it is no surprise that Theorem 2.2.1 is a motivic analogue of the 'stability properties' 5.1.14 of [BBD82].

4 Supplements

In §4.1 we recall the notion of t-structure adjacent to a weight structure (as introduced in §4.4 of [Bon10a]).

In §4.2 we use Theorem 4.5.2 of ibid. to prove the existence of the *Chow t-structure* for $DM(S)$ that is adjacent to the Chow weight structure for it (cf. Theorem 2.1.1(II)); we also establish certain functoriality properties of this t-structure (with respect to the motivic image functors, when S varies).

In §4.3 we recall the notion of *virtual t-truncations* (for cohomological functors from $DM_c(S)$), and relate virtual t-truncations with t_{Chow}.

4.1 On adjacent structures

We recall the notion of adjacent weight and t-structures (that was introduced in §4.4 of [Bon10a]). For t-structures we will use notation and conventions similar to those for weight structures in §1.2 (see also §4.1 of [Bon10a]). In particular, we will denote the heart of t by $\underline{Ht}$ (recall that it is abelian); $Obj\underline{Ht} = \underline{C}^{t=0}$.

We will say that t (for $\underline{C}$) is non-degenerate if $\cap_{n\in\mathbb{Z}}\underline{C}^{t\le n} = \cap_{n\in\mathbb{Z}}\underline{C}^{t\ge n} = \{0\}$.

Definition 4.1.1. We say that a weight structure w is (left) *adjacent* to a t-structure t if $\underline{C}_{w\ge 0} = \underline{C}^{t\le 0}$.

We will also need the following properties of adjacent structures.

Proposition 4.1.2. *I Let $\underline{C}$ be endowed with a weight structure w and also with an adjacent t-structure t.*

1. The functor $\underline{C}(-, \underline{Ht}) : \underline{Ht} \to \mathrm{AddFun}(\underline{Hw}^{op}, Ab)$ that sends $N \in \underline{C}^{t=0}$ to $M \mapsto \underline{C}(M, N)$, $(M \in \underline{C}_{w=0})$, is an exact embedding of $\underline{Ht}$ into the abelian category $\mathrm{AddFun}(\underline{Hw}^{op}, Ab)$.

2. Let t be non-degenerate. Then $\underline{C}^{t=0} = \{M \in Obj\underline{C} : \underline{C}_{w=i} \perp M \ \forall\ i \neq 0\}$.

II Moreover, let a triangulated category $\underline{C}'$ be endowed with a weight structure w' and also with its adjacent t-structure t'. Let $F : \underline{C} \to \underline{C}'$ be an exact functor.

1. F is right weight-exact whenever it is right t-exact (i.e. if $F(\underline{C}^{t\le 0}) \subset \underline{C}'^{t'\le 0}$).

2. Let $G : \underline{C}' \to \underline{C}$ be the right adjoint to F. Then F is right (resp. left) weight-exact with respect to w and w' whenever G is left (resp. right) t-exact with respect to t' and t.

III Let $\underline{D} \subset \underline{C}$ be a full subcategory of compact objects endowed with a weight structure $w_{\underline{D}}$ (we denote its heart by $\underline{Hw}_{\underline{D}}$). Let $\underline{C}$ admit arbitrary (small) coproducts and suppose that $\underline{D}$ weakly generates $\underline{C}$. Then the following statements are valid.

1. For the weight structure w for $\underline{C}$ given by Proposition 1.2.3(12) there exists an adjacent t-structure; it is non-degenerate. $\underline{Ht}$ is isomorphic to $\mathrm{AddFun}(\underline{Hw}_{\underline{D}}^{op}, Ab)$ (via the functor $N \mapsto (M \in \underline{D}^{\underline{Hw}_{\underline{D}}=0} \mapsto \underline{C}(M, N))$).

2. Suppose that $w_{\underline{D}'}$ and $\underline{D}' \subset \underline{C}'$ satisfy the conditions for $w_{\underline{D}}$ and $\underline{D} \subset \underline{C}$ described above; denote the corresponding adjacent structures for $\underline{C}$ by w' and t', respectively.

Let $F : \underline{C} \to \underline{C}'$ be an exact functor that maps $\underline{D}$ into $\underline{D}'$; suppose that is possesses a right adjoint G that maps $\underline{D}'$ in $\underline{D}$. Then the restriction of F to $\underline{D}$ is right (resp. left) weight-exact with respect to $w_{\underline{D}}$ and $w'_{\underline{D}'}$ whenever G is left (resp. right) t-exact with respect to t' and t.

Proof. I These are just parts 4 and 5 of Theorem 4.4.2 of [Bon10a].

II1. Immediate from the definition of adjacent structures.

2. See Remark 4.4.6 of ibid.

III 1. Immediate from Theorem 4.5.2 of ibid.

2. Immediate from the previous assertions by adjunction (we use the description of $\underline{Ht}$). □

4.2 The Chow t-structure for $DM(S)$

Now we study the t-structure adjacent to w_{Chow}^{big}.

Proposition 4.2.1. *I Let S be an excellent separated finite-dimensional scheme.*

1. There exists a non-degenerate t-structure $t_{Chow}(S)$ on $DM(S)$ that is adjacent to w_{Chow}^{big} (the latter is given either by Theorem 2.1.1(II) or by Proposition 2.3.4(II)).

2. $\underline{Ht}_{Chow}(S) \cong \mathrm{AddFun}(Chow(S)^{op}, Ab)$ *(via the functor* $N \mapsto (M \in DM_c(S)_{w_{Chow}=0} \mapsto DM(S)(M,N))$*).*

II Let $f : X \to Y$ *be a smoothly embeddable morphism of (excellent separated finite dimensional) schemes.*

1. $f^!$ *and* f_* *are right* t_{Chow}*-exact (with respect to the corresponding Chow t-structures).*

2. Suppose that f *is smooth. Then* f_* *is (also)* t_{Chow}*-exact.*

Proof. I Immediate from the definition of w_{Chow} and w_{Chow}^{big}, and Proposition 4.1.2(I).

II The assertion follows easily from Proposition 2.3.5 (or from Theorem 2.2.1(II) if X and Y are reasonable); see Proposition 4.1.2(III).

□

Remark 4.2.2. So, for any $N \in ObjDM(S)$ the Chow t-structure for $DM(S)$ allows us to 'slice' the cohomology theory $H : M \mapsto DM(S)(M,N)$, into 'pieces' $H^i : M \to DM(S)(M, t_{Chow=i}N)$; note that $H^i(N[j]) = \{0\}$ for any $N \in ObjChow(S) \subset ObjDM_c(S)$, $j \neq i$ (see Proposition 4.1.2(I2)). One may call these H^i *pure* cohomology theories.

We will describe another (more general) method for slicing a cohomology theory into 'pure pieces' below; yet this reasoning does not demonstrate that the pieces of a representable cohomology theory are representable.

4.3 Virtual t-truncations with respect to w_{Chow}; 'pure' cohomology theories

Now suppose that we are given an arbitrary cohomological functor $H : DM_c(S) \to \underline{A}$, $\underline{A}$ is an abelian category. Virtual t-truncations (defined in §2.5 of [Bon10a] and studied in more detail in §2 of [Bon10b]) allow us to 'slice' H into pure pieces H^i. To this end we only use w_{Chow} (and have no need to put H into some 'category of cohomological functors' $DM_c(S) \to \underline{A}$, and define a t-structure for this category). Virtual t-truncations also yield a functorial description of Chow-weight spectral sequences for cohomological functors (starting from E_2).

Now we just list the main properties of virtual t-truncations (in the case when $(\underline{C}, w) = (DM_c(S), w_{Chow})$; the properties are the same as in the general case).

Proposition 4.3.1. *Let* $H : DM_c(S) \to \underline{A}$ *and* $i \in \mathbb{Z}$ *be fixed.*

1. For any $M \in ObjDM_c(X)$ *there exist unique morphisms* $i_1(M) \in DM_c(S)(w_{Chow \le i-1}M, w_{Chow \le i}M)$ *and* $i_2(M) \in DM_c(S)(w_{Chow \ge i}M, w_{Chow \ge i+1}M)$

that fit into a commutative diagram

$$\begin{array}{ccccc} w_{Chow\le i-1}M & \longrightarrow & M & \longrightarrow & w_{Chow\ge i}M \\ \downarrow{\scriptstyle i_1(M)} & & \downarrow{\scriptstyle id_M} & & \downarrow{\scriptstyle i_2(M)} \\ w_{Chow\le i}M & \longrightarrow & M & \longrightarrow & w_{Chow\ge i+1}M \end{array} \tag{9}$$

here the horizontal arrows are compatible with (arbitrary fixed) weight decompositions of $M[i-j]$ (for $j=0,1$).

2. The correspondences $M \mapsto \operatorname{Im} H(i_1(M))$ and $M \mapsto \operatorname{Im} H(i_2(M))$ yield well-defined cohomological functors $\tau_{\ge 1-i}H$, $\tau_{\le -i-1}H : DM_c(S) \to \underline{A}$ (we call them virtual t-truncations of H).

3. For any $i \in \mathbb{Z}$ the functor $\tau_{\le i}H$ kills $DM_c(S)_{w_{Chow}\le -i-1}$; $\tau_{\ge i}H$ kills $DM_c(S)_{w_{Chow}\ge 1-i}$.

4. H yields naturally an (infinite) sequence of transformations of functors

$$\dots \to (\tau_{\ge i+1}H)\circ[1] \to \tau_{\le i}H \to H \to \tau_{\ge i+1}H \to (\tau_{\le i}H)\circ[-1] \to \dots$$

that yields a long exact sequence when applied to any $M \in ObjDM_c(S)$.

5. For any $j \in \mathbb{Z}$ we have a natural isomorphism $\tau_{\le i}(\tau_{\ge j}H) \cong \tau_{\ge j}(\tau_{\le i}H)$.

6. We have a natural isomorphism $E_2^{-ii}(T(H,M) \cong \tau_{=i}H(=\tau_{\ge i}(\tau_{\le i}H))$ (see Proposition 3.3.1(II2) for the definition of $T(H,M)$ in this case).

7. For $N \in ObjDM(S)$, $H = DM_c(-,N)$ we have: $\tau_{\le i}H \cong (-, t_{Chow\le i}N)$, $\tau_{\ge i}H \cong (-, t_{Chow\ge i}N)$, and $\tau_{=i}H \cong (-, t_{Chow=i}N)$.

Proof. Assertions 1–5 are immediate from Theorem 2.3.1 of [Bon10b]. Assertion 6 is contained in Theorem 2.4.2 of ibid. Assertion 7 follows from Proposition 2.5.4 of ibid. □

Remark 4.3.2. 1. Note that $\tau_{=i}H$ vanishes on $DM_c(S)_{w_{Chow}=j}$ for all $j \ne -i$, so $\tau_{=i}H$ are 'pure' (cf. Remark 4.2.2).

2. One can also describe the whole $T(H,M)$ starting from E_2 in terms of (various) virtual t-truncations of H; see Theorem 2.4.2 of [Bon10b].

3. Our definition of $\tau_{\ge i}H$ and $\tau_{\le i}H$ is compatible with the one of ibid. (if one replaces the cohomological notation for weights in ibid. with our current homological one), i.e. we do not change 'the signs' for virtual t-truncations.

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