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Sequential Tests of Statistical Hypotheses with Confidence Limits *

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Abstract

In this paper, we propose a general method for testing composite hypotheses. Our idea is to use confidence limits to define stopping and decision rules. The requirements of operating characteristic function can be satisfied by adjusting the coefficients of the confidence limits. For common distributions, such adjustment can be done via efficient computation by making use of the monotonicity of the associated operating characteristic function. It is demonstrated that our proposed testing plans can be substantially more efficient than the sequential probability ratio test and its variations.

1 Introduction

A fundamental issue of statistical inference is to test the parameter of a distribution. Let X be a random variable defined in a probability space $(\Omega, \mathscr{F}, \Pr)$. Assume that the distribution of X is determined by a parameter $\theta \in \Theta$, where Θ is the parameter space. A frequent problem is to test hypotheses $\mathscr{H}_0 : \theta \leq \theta_0$ versus $\mathscr{H}_1 : \theta \geq \theta_1$ based on observations $X_1, X_2, \cdots$ of X, where $\theta_0 < \theta_1$ are two parametric values in Θ. To control the risk of making wrong decisions, it is typically required that

$$\Pr\{\text{Reject } \mathscr{H}_0 \mid \theta\} \leq \alpha, \qquad \forall \theta \in \Theta \text{ such that } \theta \leq \theta_0 \tag{1}$$

$$\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta\} \leq \beta, \qquad \forall \theta \in \Theta \text{ such that } \theta \geq \theta_1, \tag{2}$$

where $\alpha, \beta \in (0, 1)$. The probability $\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta\}$ for $\theta \in \Theta$ is referred to as the Operating Characteristic (OC) function. Since there is no requirement imposed on (θ_0, θ_1) for the OC function, such interval (θ_0, θ_1) is referred to as an *indifference zone*.

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A famous solution to the above problem is the Sequential Probability Ratio Test (SPRT) proposed by Abraham Wald [14] during World War II of last century. Various modifications of SPRT have also been proposed (see, [12] and the references therein). A major drawback of SPRT and its variations is that the test plans are efficient only for some parametric values. Another unpleasant feature is that the sample number of the tests may not be bounded. In this paper, to overcome the limitations of SPRT and its variations, we take a different path. Instead of using likelihood ratio as existing testing plans, we use confidence limits to define stopping rules and decision rules. A key idea is to compare confidence limits with the endpoint of indifference zone. We discovered that by adjusting the coefficients of the confidence limits, it is possible to control the risk of making erroneous decisions. We demonstrate that, for testing problems related to common distributions, the adjustment can be done by efficient computation and for such purpose it suffices to evaluate the OC function for the endpoints of the indifference zone.

The remainder of this paper is organized as follows. In Section 2, we proposed our general principle of hypothesis testing in a sequential framework. In Section 3, we discuss the applications of the general principle to test the expectation of a random variable. Our development is motivated by the fact that the means of many important random variables are exactly the parameters of their distributions. In Section 4, we derived multistage testing plans from the fully sequential testing plans. In Section 5, we propose to use confidence limits to design multistage plans for testing multiple hypotheses. In Section 6, we apply the design method based on confidence limits to test hypotheses regarding functions of two binomial proportions. Section 7 is the conclusion. All proofs of theorems are given in Appendices.

Throughout this paper, we shall use the following notations. The notation $\emptyset$ denotes an empty set. The set of real numbers is denoted by $\mathbb{R}$. The set of positive integers is denoted by $\mathbb{N}$. The expectation of a random variable is denoted by $\mathbb{E}[.]$. We use the notation $\Pr\{. \mid \theta\}$ to denote the probability of an event which is defined in terms of random variables parameterized by θ. The parameter θ in $\Pr\{. \mid \theta\}$ may be dropped whenever this can be done without introducing confusion. If Z is parameterized by θ, we denote $\Pr\{Z \leq z \mid \theta\}$ by $F_Z(z, \theta)$ and $\Pr\{Z \geq z \mid \theta\}$ by $G_Z(z, \theta)$ respectively. The support of a random variable Z is denoted by I_Z, i.e., $I_Z = \{Z(\omega) : \omega \in \Omega\}$. The other notations will be made clear as we proceed.

2 Fully Sequential Tests

In this section, we shall consider the problem of testing hypotheses $\mathscr{H}_0 : \theta \leq \theta_0$ versus $\mathscr{H}_1 : \theta \geq \theta_1$ stated in the introduction.

Let $\mathscr{L}(X_1, \cdots, X_n; \delta)$ and $\mathscr{U}(X_1, \cdots, X_n; \delta)$ be confidence limits such that

$$\Pr\{\mathscr{L}(X_1, \cdots, X_n; \delta) < \theta \mid \theta\} \geq 1 - \delta, \qquad \Pr\{\mathscr{U}(X_1, \cdots, X_n; \delta) > \theta \mid \theta\} \geq 1 - \delta \tag{3}$$

for $\delta \in (0, 1)$ and $n = 1, 2, \cdots$. Let ζ be a positive number less than $\min(\frac{1}{\alpha}, \frac{1}{\beta})$. The stopping rule of our testing plan is defined as follows:

Continue sampling until $\mathscr{L}(X_1, \cdots, X_n; \zeta\alpha) \geq \theta_0$ or $\mathscr{U}(X_1, \cdots, X_n; \zeta\beta) \leq \theta_1$ for some n.

Upon termination of sampling, the decision rule of our testing plan is defined as follows:

(i): If $\theta_0 \leq \mathscr{L}(X_1, \cdots, X_n; \zeta\alpha) < \mathscr{U}(X_1, \cdots, X_n; \zeta\beta) \leq \theta_1$ is not satisfied, then accept $\mathscr{H}_0$ when $\mathscr{U}(X_1, \cdots, X_n; \zeta\beta) \leq \theta_1$ and reject $\mathscr{H}_0$ when $\mathscr{L}(X_1, \cdots, X_n; \zeta\alpha) \geq \theta_0$.

(ii): If $\theta_0 \leq \mathscr{L}(X_1, \cdots, X_n; \zeta\alpha) < \mathscr{U}(X_1, \cdots, X_n; \zeta\beta) \leq \theta_1$ is satisfied, then accept $\mathscr{H}_0$ or reject $\mathscr{H}_0$ based on an arbitrarily pre-specified policy. On possible way to specify such policy is to accept $\mathscr{H}_0$ if

$$\frac{f(X_1, \cdots, X_n; \theta_0)}{f(X_1, \cdots, X_n; \theta_1)} \geq \frac{\alpha}{\beta}$$

and reject $\mathscr{H}_0$ otherwise.

Here $f(x_1, \cdots, x_n; \theta)$ denotes the joint probability density function (for the continuous case) or the joint probability mass function (for the discrete case) of $X_1, \cdots, X_n$ parameterized by θ.

Throughout the paper, let $\mathbf{n}$ denote the number of samples at the termination of sampling. Clearly, if $\mathscr{U}(X_1, \cdots, X_n; \zeta\beta) - \mathscr{L}(X_1, \cdots, X_n; \zeta\alpha)$ converges to 0 in probability as $n \to \infty$, then $\Pr\{\mathbf{n} < \infty \mid \theta\} = 1$. As will be seen in the next section, for a wide class of hypothesis testing problems, we can show that the sample sizes are absolutely bounded. Moreover, we can show that the probabilities of committing decision errors for the above testing plan can be adjusted below any prescribed level by choosing a sufficiently small value of $\zeta > 0$.

3 Testing the Expectation of a Random Variable

In many situations, the parameter to be tested is equal to the expectation of the associated random variable. That is, $\theta = \mathbb{E}[X]$. As proposed in the last section, our testing plan is defined in terms of confidence limits. In order to obtain confidence limits and establish the monotonicity of the OC function for the associated testing plan, we need to introduce the concept of *unimodal-likelihood estimator* (ULE). For a random tuple $X_1, \cdots, X_{\mathbf{r}}$ (of deterministic or random length $\mathbf{r}$) parameterized by θ, we say that the estimator $\varphi(X_1, \cdots, X_{\mathbf{r}})$ is a ULE of θ if φ is a multivariate function such that, for any observation $(x_1, \cdots, x_r)$ of $(X_1, \cdots, X_{\mathbf{r}})$, the likelihood function is non-decreasing with respect to θ no greater than $\varphi(x_1, \cdots, x_r)$ and is non-increasing with respect to θ no less than $\varphi(x_1, \cdots, x_r)$. For discrete random variables $X_1, \cdots, X_r$, the associated likelihood function is the joint probability mass function $\Pr\{X_i = x_i,\ i = 1, \cdots, r \mid \theta\}$. For continuous random variables $X_1, \cdots, X_r$, the corresponding likelihood function is, $f_{X_1, \cdots, X_r}(x_1, \cdots, x_r, \theta)$, the joint probability density function of random variable $X_1, \cdots, X_r$. It should be noted that a ULE may not be a maximum-likelihood estimator (MLE). On the other side, a MLE may not be a ULE.

In this section, we shall focus on random variables such that their sample means are ULEs. Let $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, where $X_1, X_2, \cdots$ are i.i.d. samples of X parameterized by $\theta = \mathbb{E}[X]$. Define

$$F_{\overline{X}_n}(z, \theta) = \Pr\left\{\overline{X}_n \leq z \mid \theta\right\}, \qquad G_{\overline{X}_n}(z, \theta) = \Pr\left\{\overline{X}_n \geq z \mid \theta\right\}.$$

Suppose that $\left\{\theta \in \Theta : F_{\overline{X}_n}(z,\theta) \le \delta\right\}$ and $\left\{\theta \in \Theta : G_{\overline{X}_n}(z,\theta) \le \delta\right\}$ are non-empty for $z \in I_{\overline{X}_n}$, $n \in \mathbb{N}$, $\delta \in (0,1)$. Define

$$L(z,\delta) = \max\left\{\theta \in \Theta : G_{\overline{X}_n}(z,\theta) \le \delta\right\}, \qquad U(z,\delta) = \min\left\{\theta \in \Theta : F_{\overline{X}_n}(z,\theta) \le \delta\right\}. \tag{4}$$

Therefore, (3) is satisfied if we define

$$\mathscr{L}(X_1, \cdots, X_n; \delta) = L(\overline{X}_n, \delta), \qquad \mathscr{U}(X_1, \cdots, X_n; \delta) = U(\overline{X}_n, \delta). \tag{5}$$

Suppose that the moment generating function $\mathscr{M}(\rho,\theta) = \mathbb{E}[e^{\rho X}]$ of X exists for any $\rho \in \mathbb{R}$. Define $\mathscr{C}(z,\theta) = \inf_{\rho\in\mathbb{R}} e^{-\rho z}\mathbb{E}[e^{\rho X}]$. For testing plans described in Section 2 with confidence limits defined by (4) and (5), we have the following results.

Theorem 1 *Suppose that the sample mean* $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ *is a ULE of* θ. *Then,*

$$\mathbf{n} < \max\left\{\frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}, \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)}\right\}.$$

Moreover, $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta\}$ *is non-increasing with respect to* $\theta \in \Theta$ *such that* $\theta \notin (\theta_0, \theta_1)$. *Furthermore, both* $\Pr\{$*Reject* $\mathscr{H}_0 \mid \theta_0\}$ *and* $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta_1\}$ *tend to* 0 *as* $\zeta \to 0$.

See Appendix B for a proof. As can be seen from Theorem 1, the risk requirements (1) and (2) can be satisfied by choosing a sufficiently small $\zeta > 0$. For this purpose, it suffices to ensure that (1) and (2) hold for the endpoints of the indifference zone. If X is a Bernoulli random variable, then the confidence limits are exactly the same as Clopper-Pearson [10]. If X is a Poisson random variable, then the confidence limits are exactly the same as [11]. It should be noted that in these particular cases, recursive algorithms can be developed for the evaluation of the OC functions and consequently facilitate the determination of ζ satisfying (1) and (2).

Note that confidence limits (4) and (5) are directly derived from the exact tail probabilities of $\overline{X}_n$. In many cases, simple bounds for the tail probabilities of $\overline{X}_n$ are available. Hence, it is reasonable to hope that the stopping boundary of the testing plans may be simpler if we use confidence limits obtained from the bounds of the tail probabilities of $\overline{X}_n$. Toward this goal, we have the following results regarding confidence limits.

Theorem 2 *Suppose that the sample mean* $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ *is a ULE of* θ. *Suppose that* $\{\theta \in \Theta : [\mathscr{C}(z,\theta)]^n \le \delta,\ \theta \le z\}$ *and* $\{\theta \in \Theta : [\mathscr{C}(z,\theta)]^n \le \delta,\ \theta \ge z\}$ *are nonempty for* $z \in I_{\overline{X}_n}$, $n \in \mathbb{N}$ *and* $\delta \in (0,1)$. *Define*

$$\mathcal{L}(z,\delta) = \max\left\{\theta \in \Theta : [\mathscr{C}(z,\theta)]^n \le \delta,\ \theta \le z\right\}, \qquad \mathcal{U}(z,\delta) = \min\left\{\theta \in \Theta : [\mathscr{C}(z,\theta)]^n \le \delta,\ \theta \ge z\right\}.$$

Then,

$$\Pr\{\mathcal{L}(\overline{X}_n,\delta) < \theta\} \ge 1-\delta, \qquad \Pr\{\mathcal{U}(\overline{X}_n,\delta) > \theta\} \ge 1-\delta, \qquad \Pr\{\mathcal{L}(\overline{X}_n,\delta) < \theta < \mathcal{U}(\overline{X}_n,\delta)\} \ge 1-2\delta.$$

Moreover, both $\mathcal{L}(z,\delta)$ *and* $\mathcal{U}(z,\delta)$ *are non-decreasing with respect to* $z \in I_{\overline{X}_n}$.

See Appendix C for a proof. For testing plans described in Section 2 with confidence limits defined by Theorem 2, we have the following results.

Theorem 3 *Suppose that the sample mean* $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ *is a ULE of* θ*. Then,*

$$\mathbf{n} < \max\left\{ \frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}, \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)} \right\}.$$

Moreover, $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta\}$ *is non-increasing with respect to* $\theta \in \Theta$ *such that* $\theta \notin (\theta_0, \theta_1)$*. Furthermore, both* $\Pr\{$*Reject* $\mathscr{H}_0 \mid \theta_0\}$ *and* $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta_1\}$ *tend to* 0 *as* $\zeta \to 0$*.*

See Appendix D for a proof. Although our proposed testing plans are derived from confidence limits, we can avoid the computation of confidence limits by virtue of the monotonicity of the tail probabilities (or their bounds) of $\overline{X}_n$ with respect to θ. Under the assumption that the sample mean $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ is a ULE of θ, we have the following observations:

(i): $L(\overline{X}_n, \zeta\alpha) \geq \theta_0 \Leftrightarrow G_{\overline{X}_n}(\overline{X}_n, \theta_0) \leq \zeta\alpha$ and $U(\overline{X}_n, \zeta\beta) \leq \theta_1 \Leftrightarrow F_{\overline{X}_n}(\overline{X}_n, \theta_1) \leq \zeta\beta$.

(ii): $\mathcal{L}(\overline{X}_n, \zeta\alpha) \geq \theta_0 \Leftrightarrow \mathscr{C}(\overline{X}_n, \theta_0) \leq \zeta\alpha$, $\overline{X}_n \geq \theta_0$ and $\mathcal{U}(\overline{X}_n, \zeta\beta) \leq \theta_1 \Leftrightarrow \mathscr{C}(\overline{X}_n, \theta_1) \leq \zeta\beta$, $\overline{X}_n \leq \theta_1$.

Here the notation "$\Leftrightarrow$" means "if and only if".

4 Multistage Tests

In the preceding discussion, we have been focusing on fully sequential tests. In many applications, it may not be practical to take one sample at a time. It may be much more efficient to perform testing with samples in groups. This motivates us to adapt the sequential procedure proposed in Section 2 to multistage tests.

Consider hypotheses $\mathscr{H}_0 : \theta \leq \theta_0$ versus $\mathscr{H}_1 : \theta \geq \theta_1$ as before. We shall test such hypotheses with a testing plan of s stages. Let ζ be a positive number less than $\min(\frac{1}{\alpha}, \frac{1}{\beta})$. Let $n_1 < n_2 < \cdots < n_s$ be sample sizes, where n_ℓ is the sample size of the ℓ-th stage. Assume that the largest sample size n_s is no less than the smallest integer n such that $\{\mathscr{L}(X_1, \cdots, X_n; \zeta\alpha) < \theta_0 < \theta_1 < \mathscr{U}(X_1, \cdots, X_n; \zeta\beta)\} = \emptyset$.

The stopping rule of our testing plan is defined as follows:

Continue sampling until $\mathscr{L}(X_1, \cdots, X_{n_\ell}; \zeta\alpha) \geq \theta_0$ or $\mathscr{U}(X_1, \cdots, X_{n_\ell}; \zeta\beta) \leq \theta_1$ for some $\ell \in \{1, \cdots, s\}$.

Upon termination of sampling, the decision rule of our testing plan is defined as follows:

(i): If $\theta_0 \leq \mathscr{L}(X_1, \cdots, X_{n_\ell}; \zeta\alpha) < \mathscr{U}(X_1, \cdots, X_{n_\ell}; \zeta\beta) \leq \theta_1$ is not satisfied, then accept $\mathscr{H}_0$ when $\mathscr{U}(X_1, \cdots, X_{n_\ell}; \zeta\beta) \leq \theta_1$ and reject $\mathscr{H}_0$ when $\mathscr{L}(X_1, \cdots, X_{n_\ell}; \zeta\alpha) \geq \theta_0$.

(ii): If $\theta_0 \leq \mathscr{L}(X_1, \cdots, X_{n_\ell}; \zeta\alpha) < \mathscr{U}(X_1, \cdots, X_{n_\ell}; \zeta\beta) \leq \theta_1$ is satisfied, then accept $\mathscr{H}_0$ or reject $\mathscr{H}_0$ based on an arbitrarily specified policy. On possible option of such policy is to accept $\mathscr{H}_0$ if

$$\frac{f(X_1, \cdots, X_{n_\ell}; \theta_0)}{f(X_1, \cdots, X_{n_\ell}; \theta_1)} \geq \frac{\alpha}{\beta}$$

and reject $\mathscr{H}_0$ otherwise.

Here $f(x_1, \cdots, x_{n_\ell}; \theta)$ denotes the joint probability density function for the continuous case or the joint probability mass function for the discrete case. The confidence limits $\mathscr{L}(X_1, \cdots, X_{n_\ell}, \zeta\alpha)$ and $\mathscr{U}(X_1, \cdots, X_{n_\ell}, \zeta\beta)$ can be defined by (4), (5) or Theorem 2. With regard to such plans for testing $\theta = \mathbb{E}[X]$, we have the following results.

Theorem 4 *Suppose that the sample mean* $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ *is a ULE of* θ. *Then,* $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta\}$ *is non-increasing with respect to* $\theta \in \Theta$ *such that* $\theta \notin (\theta_0, \theta_1)$. *Moreover,* $\Pr\{$*Reject* $\mathscr{H}_0 \mid \theta_0\} \leq s\zeta\alpha$ *and* $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta_1\} \leq s\zeta\beta$. *Furthermore, both* $\Pr\{$*Reject* $\mathscr{H}_0 \mid \theta_0\}$ *and* $\Pr\{$*Accept* $\mathscr{H}_0 \mid \theta_1\}$ *tend to* 0 *as* $\zeta \to 0$.

Theorem 4 can be shown by similar methods as that of Theorems 1 and 3.

5 Multistage Tests of Multiple Hypotheses

In this section, we shall propose a new approach for constructing multistage plans for testing multiple hypotheses based on fixed-sample-size confidence limits. The general problem to be considered is described as follows.

Let X be a random variable defined in a probability space $(\Omega, \mathscr{F}, \Pr)$. Suppose the distribution of X is determined by an unknown parameter θ in a parameter space Θ. In many applications, it is desirable to infer the true value of θ from random samples $X_1, X_2, \cdots$ of X. This topic can be formulated as a general problem of testing m mutually exclusive and exhaustive composite hypotheses:

$$\mathscr{H}_0 : \theta \in \Theta_0, \quad \mathscr{H}_1 : \theta \in \Theta_1, \quad \ldots, \quad \mathscr{H}_{m-1} : \theta \in \Theta_{m-1}, \tag{6}$$

where $\Theta_0 = \{\theta \in \Theta : \theta \leq \theta_1\}$, $\Theta_{m-1} = \{\theta \in \Theta : \theta > \theta_{m-1}\}$ and $\Theta_i = \{\theta \in \Theta : \theta_i < \theta \leq \theta_{i+1}\}$, $i = 1, \cdots, m-2$ with $\theta_1 < \theta_2 < \cdots < \theta_{m-1}$. To control the probabilities of making wrong decisions, it is typically required that, for pre-specified numbers $\delta_i \in (0, 1)$,

$$\Pr\{\text{Accept } \mathscr{H}_i \mid \theta\} \geq 1 - \delta_i, \qquad \forall \theta \in \Theta_i, \quad i = 0, 1, \cdots, m-1 \tag{7}$$

with $\Theta_0 = \{\theta \in \Theta_0 : \theta \leq \theta_1'\}$, $\Theta_{m-1} = \{\theta \in \Theta_{m-1} : \theta \geq \theta_{m-1}''\}$ and $\Theta_i = \{\theta \in \Theta_i : \theta_i'' \leq \theta \leq \theta_{i+1}'\}$, $i = 1, \cdots, m-2$, where θ_i', θ_i'' are parametric values in Θ such that $\theta_1' < \theta_1$, $\theta_{m-1}'' > \theta_{m-1}$ and $\theta_{i-1} < \theta_{i-1}'' < \theta_i' < \theta_i < \theta_i'' < \theta_{i+1}' < \theta_{i+1}$ for $i = 2, \cdots, m-2$. For $i = 0, 1, \cdots, m-1$, $\Pr\{\text{Accept } \mathscr{H}_i \mid \theta\}$ is referred to as an Operating Characteristic (OC) function. Since there is no requirement imposed for controlling the risk of making wrong decisions for θ in $\Theta \setminus \cup_{j=0}^{m-1} \Theta_j =$

$\cup_{i=1}^{m-1}(\theta_i', \theta_i'')$, such a remainder set, $\cup_{i=1}^{m-1}(\theta_i', \theta_i'')$, is referred to as an *indifference zone*. The concept of indifference zone was introduced by Wald [14] for two main reasons. First, when the parameter θ is close to θ_i, the margin between adjacent parameter subsets Θ_{i-1} and Θ_i, it is immaterial to decide whether $\mathscr{H}_{i-1}$ or $\mathscr{H}_i$ should be accepted. Second, the sample size required to make a reliable decision between consecutive hypotheses $\mathscr{H}_{i-1}$ and $\mathscr{H}_i$ becomes increasingly intolerable as θ tends to θ_i. Undoubtedly, the indifference zone should be sufficiently "narrow" so that the consequence of making erroneous decision is practically unimportant when θ lies in it. In general, a testing plan in our proposed framework consists of s stages. For $\ell = 1, \cdots, s$, the number of available samples (i.e., sample size) of the ℓ-th stage is denoted by n_ℓ. For the ℓ-th stage, a decision variable $\boldsymbol{D}_\ell = \mathscr{D}_\ell(X_1, \cdots, X_{n_\ell})$ is defined in terms of samples $X_1, \cdots, X_{n_\ell}$ such that $\boldsymbol{D}_\ell$ assumes $m+1$ possible values $0, 1, \cdots, m$ with the following notion:

(i) Sampling is continued until $\boldsymbol{D}_\ell \neq 0$ for some $\ell \in \{1, \cdots, s\}$.

(ii) The hypothesis $\mathscr{H}_j$ is accepted at the ℓ-th stage if $\boldsymbol{D}_\ell = j+1$ and $\boldsymbol{D}_i = 0$ for $1 \leq i < \ell$.

For practical considerations, we shall only focus on sampling schemes which are closed in the sense that $\Pr\{\boldsymbol{D}_s = 0\} = 0$. For efficiency, a sampling scheme should satisfy the condition that both $\Pr\{\boldsymbol{D}_1 \neq 0\}$ and $\Pr\{\boldsymbol{D}_{s-1} = 0\}$ are greater than zero.

Let $\boldsymbol{l}$ denote the index of stage when the sampling is terminated. Then, the sample number when the sampling is terminated, denoted by $\mathbf{n}$, is equal to $n_{\boldsymbol{l}}$. We shall focus on multistage sampling schemes which can be defined in terms of estimator $\boldsymbol{\varphi}_n = \varphi(X_1, \cdots, X_n)$ such that $\boldsymbol{\varphi}_n$ is a ULE of θ for every n and that $\boldsymbol{\varphi}_n$ converges in probability to θ in the sense that, for any $\varepsilon > 0$ and $\delta \in (0,1)$, $\Pr\{|\boldsymbol{\varphi}_n - \theta| \geq \varepsilon\} < \delta$ provided that n is sufficiently large. Such estimator $\boldsymbol{\varphi}_n$ is referred to as a *Unimodal-likelihood Consistent Estimator* (ULCE) of θ. Assume that, for any $\delta \in (0,1)$ and any positive number n, a lower confidence limit $L(\boldsymbol{\varphi}_n, n, \delta)$ and an upper confidence limit $U(\boldsymbol{\varphi}_n, n, \delta)$ can be constructed in terms of estimator $\boldsymbol{\varphi}_n$ such that $\{L(\boldsymbol{\varphi}_n, n, \delta) \leq \boldsymbol{\varphi}_n \leq U(\boldsymbol{\varphi}_n, n, \delta)\}$ is a sure event and that

$$\Pr\{\theta \leq L(\boldsymbol{\varphi}_n, n, \delta) \mid \theta\} \leq \delta, \qquad \Pr\{\theta \geq U(\boldsymbol{\varphi}_n, n, \delta) \mid \theta\} \leq \delta$$

for any $\theta \in \Theta$. To construct testing plans satisfying the requirement (7), we propose to use fixed-sample-size confidence limits satisfying such assumptions. For the ℓ-th stage, an estimator $\widehat{\boldsymbol{\theta}}_\ell$ for θ can be defined based on samples $X_1, \cdots, X_{n_\ell}$. Consequently, the overall estimator for θ, denoted by $\widehat{\boldsymbol{\theta}}$, is equal to $\widehat{\boldsymbol{\theta}}_{\boldsymbol{l}}$. For the ℓ-th stage, the estimator $\widehat{\boldsymbol{\theta}}_\ell$ is defined as $\boldsymbol{\varphi}_{n_\ell} = \varphi(X_1, \cdots, X_{n_\ell})$. Accordingly, the decision variables $\boldsymbol{D}_\ell$ can be defined in terms of estimator $\widehat{\boldsymbol{\theta}}_\ell = \boldsymbol{\varphi}_{n_\ell}$.

Our general principle for constructing multistage testing plans and their properties can be described by Theorem 5 as follows.

Theorem 5 *Let $\alpha_i = O(\zeta) \in (0,1), \beta_i = O(\zeta) \in (0,1)$ for $i = 1, \cdots, m-1$. Let $\alpha_0 = \alpha_1$, $\beta_m = \beta_{m-1}$ and $\alpha_m = \beta_0 = 0$. Let $\theta_0' = -\infty$ and $\theta_m'' = \infty$. Let $n_1 < n_2 < \cdots < n_s$ be sample sizes such that the largest sample size n_s is no less than the smallest positive integer n guaranteeing that $\{\theta_i' \leq L(\boldsymbol{\varphi}_n, n, \alpha_i) \leq U(\boldsymbol{\varphi}_n, n, \beta_i) \leq \theta_i''\}$ is not an impossible event for $i = 1, \cdots, m-1$. Suppose*

that $\boldsymbol{\varphi}_n$ is a ULCE of θ and that the decision variables $\boldsymbol{D}_\ell$ are defined such that

$$\{\boldsymbol{D}_\ell = i\} \subseteq \{\theta'_{i-1} \le L(\widehat{\boldsymbol{\theta}}_\ell, n_\ell, \alpha_{i-1}) \le U(\widehat{\boldsymbol{\theta}}_\ell, n_\ell, \beta_i) \le \theta''_i\}, \qquad 1 \le i \le m$$

for $\ell = 1, \cdots, s$. The following statements (I)-(VI) hold true for $m \ge 2$.

(I) $\Pr\{$*Reject* $\mathscr{H}_0 \mid \theta\}$ *is non-decreasing with respect to* $\theta \in \Theta_0$.

(II) $\Pr\{$*Reject* $\mathscr{H}_{m-1} \mid \theta\}$ *is non-increasing with respect to* $\theta \in \Theta_{m-1}$.

(III) $\Pr\{$*Reject* $\mathscr{H}_i \mid \theta\} \le s(\max\{\alpha_j : i < j \le m\} + \max\{\beta_j : 0 \le j \le i\})$ *for any* $\theta \in \Theta_i$ *and* $i = 0, 1, \cdots, m-1$.

(IV) For $0 < i \le m-1$, $\Pr\{$*Accept* $\mathscr{H}_i \mid \theta\}$ *is no greater than* $s\alpha_i$ *and is non-decreasing with respect to* $\theta \in \Theta$ *no greater than* θ'_i.

(V) For $0 \le i \le m-2$, $\Pr\{$*Accept* $\mathscr{H}_i \mid \theta\}$ *is no greater than* $s\beta_{i+1}$ *and is non-increasing with respect to* $\theta \in \Theta$ *no less than* θ''_{i+1}.

(VI) Assume that $\mathbb{E}[e^{\rho X}]$ *exists for any* $\rho \in \mathbb{R}$ *and that* $\boldsymbol{\varphi}_n = \frac{\sum_{i=1}^n X_i}{n}$ *is an unbiased and unimodal-likelihood estimator of* θ*, where* $X_1, X_2, \cdots$ *are i.i.d. samples of* X*. Then, for* $i = 0, 1, \cdots, m-1$, $\lim_{\zeta \to 0} \Pr\{$*Reject* $\mathscr{H}_i \mid \theta\} = 0$ *for any* $\theta \in \Theta_i$.

Moreover, the following statements (VII), (VIII) and (IX) hold true for $m \ge 3$.

(VII)

$$\Pr\{\text{Reject } \mathscr{H}_i \mid \theta\} \le \Pr\{\text{Reject } \mathscr{H}_i,\ \widehat{\boldsymbol{\theta}} \le a \mid a\} + \Pr\{\text{Reject } \mathscr{H}_i,\ \widehat{\boldsymbol{\theta}} \ge b \mid b\},$$
$$\Pr\{\text{Reject } \mathscr{H}_i \mid \theta\} \ge \Pr\{\text{Reject } \mathscr{H}_i,\ \widehat{\boldsymbol{\theta}} \le a \mid b\} + \Pr\{\text{Reject } \mathscr{H}_i,\ \widehat{\boldsymbol{\theta}} \ge b \mid a\}$$

for any $\theta \in [a, b] \subseteq \Theta_i$ *and* $1 \le i \le m-2$.

(VIII) $\Pr\{$*Reject* $\mathscr{H}_0$ *and* $\mathscr{H}_{m-1} \mid \theta\}$ *is non-decreasing with respect to* $\theta \in \Theta_0$ *and is non-increasing with respect to* $\theta \in \Theta_{m-1}$.

(IX) $\Pr\{$*Reject* $\mathscr{H}_0$ *and* $\mathscr{H}_{m-1} \mid \theta\}$ *is no greater than* $s \times \max\{\alpha_i : 1 \le i \le m-2\}$ *for* $\theta \in \Theta_0$ *and is no greater than* $s \times \max\{\beta_i : 2 \le i \le m-1\}$ *for* $\theta \in \Theta_{m-1}$.

Theorem 5 asserts that the probabilities of committing decision errors for the proposed testing plan can be adjusted below any prescribed levels by choosing a sufficiently small value of $\zeta > 0$. In situations that the parameter θ to be tested is the expectation of X, for the purpose of simplifying the stopping and decision rules, one can replace the strict confidence limits by their approximations derived from the central limit theorem or by their bounds derived from probabilistic inequalities such as Chernoff bound.

We would like to point out that the testing plan proposed in Theorem 5 can be viewed as a method of constructing a random interval to satisfy certain specifications of coverage probability. Moreover, the properties of the testing plan as described by statements (I)–(IX) can be justified by the general theory of coverage probability of random intervals established in Sections 2.5 and 2.6 of [3]. To see this, we shall first show that the general hypothesis testing problem defined by (6) and (7) can be cast in the framework of constructing random intervals. Consider a multistage sampling scheme with sample sizes $n_1 < n_2 < \cdots < n_s$ for random variable X parameterized

by $\theta \in \Theta$. Let $\widehat{\boldsymbol{\theta}} = \varphi(X_1, \cdots, X_{\mathbf{n}})$, where $\mathbf{n}$ is the number of samples at the termination of the process of drawing samples $X_1, X_2, \cdots$. Let $\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n})$ and $\mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n})$ be functions of $\widehat{\boldsymbol{\theta}}$ and $\mathbf{n}$ such that the following requirements are satisfied:

(i): $\{\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) \leq \widehat{\boldsymbol{\theta}} \leq \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n})\}$ is a sure event.

(ii): The random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}), \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$ has m possible realizations: $(\theta_i', \theta_{i+1}'')$, $i = 0, 1, \cdots, m-1$.

(iii):

$$\Pr\{\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) < \theta < \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) \mid \theta\} > 1 - \delta_i \tag{8}$$

for any $\theta \in \Theta_i$ and $i = 0, 1, \cdots, m-1$.

Clearly, a critical issue in the construction of such random interval is to ensure the coverage requirement (8). Noting that

$$\Pr\{\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) < \theta < \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) \mid \theta\} = \Pr\{(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}), \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n})) \subseteq (\theta_i', \theta_{i+1}'') \mid \theta\}$$

for any $\theta \in \Theta_i$ with $i \in \{0, 1, \cdots, m-1\}$, we propose to control the coverage probability of the random interval by creating a particular inclusion relationship between the confidence intervals and the desired random interval. For $i = 0, 1, \cdots, m-1$, let $(\theta_i', \theta_{i+1}'')$ be referred to as the i-th realization of the random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}), \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$. For $\ell = 1, \cdots, s$ and $i = 0, 1, \cdots, m-1$, let $(L(\widehat{\boldsymbol{\theta}}_\ell, n_\ell, \alpha_i), U(\widehat{\boldsymbol{\theta}}_\ell, n_\ell, \beta_{i+1}))$ be referred to as the i-th confidence interval at the ℓ-th stage, where $\widehat{\boldsymbol{\theta}}_\ell = \boldsymbol{\varphi}_{n_\ell} = \varphi(X_1, \cdots, X_{n_\ell})$ as before. In this setting, we propose the following stopping rule:

Continue the sampling process if there exists no index $i \in \{0, 1, \cdots, m-1\}$ such that the i-th confidence interval at the corresponding stage is included in the i-th realization of the random interval. At the termination of the sampling process, take the following actions:

(a): If there exists a unique index $i \in \{0, 1, \cdots, m-1\}$ such that the i-th confidence interval at the corresponding stage is included in the i-th realization of the random interval, then designate the i-th realization as the outcome of the random interval.

(b): If there exist two consecutive indexes $i-1$ and i in $\{1, \cdots, m-1\}$ such that the $(i-1)$-th and i-th confidence intervals at the corresponding stage are included, respectively, in the $(i-1)$-th and i-th realizations of the random interval, then designate either the $(i-1)$-th or i-th realization as the outcome of the random interval based on an arbitrarily pre-specified policy.

One possible policy to handle the situation described in (b) for the ℓ-th stage is to designate the $(i-1)$-th realization as the outcome of the random interval if $\widehat{\boldsymbol{\theta}}_\ell \leq m_{\ell,i}$ and designate the i-th realization as the outcome of the random interval if $\widehat{\boldsymbol{\theta}}_\ell > m_{\ell,i}$, where $m_{\ell,i} = \frac{1}{2}[\min\{z \in I_{\widehat{\boldsymbol{\theta}}_\ell} : L(z, n_\ell, \alpha_i) \geq \theta_i'\} + \max\{z \in I_{\widehat{\boldsymbol{\theta}}_\ell} : U(z, n_\ell, \beta_i) \leq \theta_i''\}]$ for $\ell = 1, \cdots, s$ and $i = 1, \cdots, m-1$. More

specifically, define

$$\begin{aligned}
&\mathscr{A}_{\ell,i} = \{z \in I_{\widehat{\boldsymbol{\theta}}_\ell} : \mathcal{L}(\ell, z, \alpha_i) \geq \theta_i'\},\\
&\mathscr{B}_{\ell,i} = \{z \in I_{\widehat{\boldsymbol{\theta}}_\ell} : \mathcal{U}(\ell, z, \beta_i) \leq \theta_i''\},\\
&\mathscr{C}_{\ell,i} = \{z \in I_{\widehat{\boldsymbol{\theta}}_\ell} : \theta_i' \leq \mathcal{L}(\ell, z, \alpha_i) \leq \mathcal{U}(\ell, z, \beta_i) \leq \theta_i''\},\\
&f_{\ell,i} = \begin{cases} \max \mathscr{B}_{\ell,i} & \text{if } \mathscr{B}_{\ell,i} \neq \emptyset \text{ and } \mathscr{C}_{\ell,i} = \emptyset,\\ c_{\ell,i} & \text{if } \mathscr{C}_{\ell,i} \neq \emptyset,\\ -\infty & \text{if } \mathscr{B}_{\ell,i} = \emptyset \end{cases}\\
&g_{\ell,i} = \begin{cases} \min \mathscr{A}_{\ell,i} & \text{if } \mathscr{A}_{\ell,i} \neq \emptyset \text{ and } \mathscr{C}_{\ell,i} = \emptyset,\\ c_{\ell,i} & \text{if } \mathscr{C}_{\ell,i} \neq \emptyset,\\ \infty & \text{if } \mathscr{A}_{\ell,i} = \emptyset \end{cases}
\end{aligned}$$

for $\ell = 1, \cdots, s$ and $i = 1, \cdots, m-1$, where $c_{\ell,i}$ is a number such that $\min \mathscr{C}_{\ell,i} \leq c_{\ell,i} \leq \max \mathscr{C}_{\ell,i}$. In particular, we can take $c_{\ell,i} = \frac{1}{2}(\min \mathscr{C}_{\ell,i} + \max \mathscr{C}_{\ell,i})$ or $c_{\ell,i} = \frac{1}{2}(\theta_i' + \theta_i'')$ for $i = 1, \cdots, m-1$. Define $f_{\ell,m} = \infty$, $g_{\ell,0} = -\infty$ and

$$\boldsymbol{D}_\ell = \begin{cases} i & \text{if there exists } i \in \{1, \cdots, m\} \text{ such that } \ g_{\ell,i-1} < \widehat{\boldsymbol{\theta}}_\ell \leq f_{\ell,i} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

for $\ell = 1, \cdots, s$. At the termination of the sampling process, designate the $(i-1)$-th realization as the outcome of the random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}),\ \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$ if $\boldsymbol{D}_\ell = i$.

Under the assumption that $\alpha_i = O(\zeta)$ and $\beta = O(\zeta)$ for $i = 1, \cdots, m-1$ and that the maximum sample size n_s is no less than the minimum integer n guaranteeing that $\{\theta_i' \leq L(\boldsymbol{\varphi}_n, n, \alpha_i) \leq U(\boldsymbol{\varphi}_n, n, \beta_i) \leq \theta_i''\}$ is not an impossible event for $i = 1, \cdots, m-1$, we can show that the coverage requirement (8) associated with the above stopping rule can be satisfied by choosing $\zeta > 0$ to be a sufficiently small number.

Once the random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}),\ \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$ satisfying requirements (i)–(iii) is constructed, the risk requirement (7) can be satisfied by using the random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}),\ \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$ to define a decision rule such that, for $i = 0, 1, \cdots, m-1$, hypothesis $\mathscr{H}_i$ is accepted when the random interval $(\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}), \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n}))$ takes the i-th realization at the termination of the sampling process. Our claim that the risk requirement (7) is satisfied can be justified by virtue of (8) and the observation that

$$\{\text{Accept } \mathscr{H}_i\} = \{\mathscr{L}(\widehat{\boldsymbol{\theta}}, \mathbf{n}) < \theta < \mathscr{U}(\widehat{\boldsymbol{\theta}}, \mathbf{n})\} \tag{10}$$

for any $\theta \in \Theta_i$ and $i = 0, 1, \cdots, m-1$.

It can be seen that the testing plan described in Theorem 5 is proposed by the same idea of constructing the random interval as above. Accordingly, the properties of the testing plan can be shown by virtue of (10) and the theory of coverage probability of random intervals established in Sections 2.5 and 2.6 of [3].

6 Multistage Tests for Functions of Two Binomial Proportions

Testing hypotheses on functions of two binomial proportions is particularly important in prospective comparative studies such as randomized controlled clinical trial. More formally, let X and Y be independent Bernoulli random variables such that $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p_x \in (0, 1)$ and $\Pr\{Y = 1\} = 1 - \Pr\{Y = 0\} = p_y \in (0, 1)$. Let $g(.,.)$ be a bivariate function of p_x and p_y. It is a frequent problem to test hypotheses regarding $g(p_x, p_y)$ based on samples of X and Y. Typical examples of $g(p_x, p_y)$ are $g(p_x, p_y) = p_x - p_y$ and $g(p_x, p_y) = \frac{p_x}{p_y}$, which are respectively referred to as the *difference of population proportions* and *ratio of population proportions*. In the sequel, we shall design multistage plans for testing hypotheses regarding $g(p_x, p_y)$ by virtue of the method proposed in Section 5. The main idea is to make use of confidence limits to define stopping and decision rules. To address the problems of testing hypotheses regarding $g(p_x, p_y)$ in a unified framework, we define $\theta = g(p_x, p_y)$ and consider the general problem of testing m mutually exclusive and exhaustive composite hypotheses defined by (6) and (7). For simplicity of notations, let $\boldsymbol{p} = (p_x, p_y)$ and abbreviate $g(p_x, p_y)$ as $g(\boldsymbol{p})$.

Our strategy is to seek a class of testing plans such that the risk of making wrong decisions can be controlled by the risk tuning parameter $\zeta > 0$, and then apply the risk tuning technique to adjust the risk level to satisfy the requirement (7). Let $\alpha_i = O(\zeta) \in (0, 1)$ and $\beta_i = O(\zeta) \in (0, 1)$ for $i = 1, \cdots, m-1$. Let $\alpha_0 = \alpha_1$ and $\beta_m = \beta_{m-1}$. Let $\theta_0' = -\infty$ and $\theta_m'' = \infty$. Specifically, we seek a class of sampling schemes associated with ζ satisfying the following requirements:

(i) The sample sizes $N_{1,x}(\zeta) < N_{2,x}(\zeta) < \cdots < N_{s,x}(\zeta)$ and $N_{1,y}(\zeta) < N_{2,y}(\zeta) < \cdots < N_{s,y}(\zeta)$ are decreasing functions of the risk tuning parameter $\zeta > 0$. Namely, the sample sizes are increasing as ζ decreases. For simplicity of notations, we abbreviate $N_{\ell,x}(\zeta)$, $N_{\ell,y}(\zeta)$ as $N_{\ell,x}$, $N_{\ell,y}$ respectively and define

$$K_{\ell,x} = \sum_{i=1}^{N_{\ell,x}} X_i, \qquad K_{\ell,y} = \sum_{i=1}^{N_{\ell,y}} Y_i, \qquad \boldsymbol{K}_\ell = (K_{\ell,x}, K_{\ell,y}),$$
$$\widehat{\boldsymbol{p}}_{\ell,x} = \frac{K_{\ell,x}}{N_{\ell,x}}, \qquad \widehat{\boldsymbol{p}}_{\ell,y} = \frac{K_{\ell,y}}{N_{\ell,y}}, \qquad \widehat{\boldsymbol{p}}_\ell = (\widehat{\boldsymbol{p}}_{\ell,x}, \widehat{\boldsymbol{p}}_{\ell,y})$$

for $\ell = 1, \cdots, s$.

(ii) For $\ell = 1, \cdots, s$, one-sided confidence intervals $[\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i),\ \infty)$ and $(-\infty,\ \mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i)]$ for $\theta = g(\boldsymbol{p})$ can be constructed in terms of estimators $\widehat{\boldsymbol{p}}_\ell$, sample sizes $N_{\ell,x}$, $N_{\ell,y}$ and the risk tuning parameter $\zeta > 0$ such that

$$\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \le g(\widehat{\boldsymbol{p}}_\ell), \qquad \Pr\{\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \ge \theta \mid \boldsymbol{p}\} \le \alpha_i, \qquad i = 0, 1, \cdots, m-1$$
$$\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \ge g(\widehat{\boldsymbol{p}}_\ell), \qquad \Pr\{\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta \mid \boldsymbol{p}\} \le \beta_i, \qquad i = 1, \cdots, m$$

for any $\boldsymbol{p} \in \{(p_x, p_y) : p_x \in (0, 1),\ p_y \in (0, 1)\}$.

(iii) The sampling process is continued until $\theta'_{i-1} \le \mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_{i-1}) \le \mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta''_i$ for some $i \in \{1, \cdots, m\}$ at some stage with index $\ell \in \{1, \cdots, s\}$.

(iv) When the sampling process is terminated at the ℓ-th stage, there are only two possible cases (A) and (B) described as follows:

Case (A) : There is a unique $i \in \{1, \cdots, m\}$ such that $\{\theta'_{i-1} \le \mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_{i-1}) \le \mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta''_i\}$.

Case (B) : There is a unique $i \in \{1, \cdots, m-1\}$ such that $\{\theta'_{i-1} \le \mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_{i-1}) \le \mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta''_i\}$ and $\{\theta'_i \le \mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \le \mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_{i+1}) \le \theta''_{i+1}\}$.

The decision is made based on the following policy:

In Case (A), accept hypothesis $\mathscr{H}_i$. In Case (B), accept $\mathscr{H}_i$ if $g(\widehat{\boldsymbol{p}}_\ell) \le \frac{\theta'_i + \theta''_i}{2}$ and accept $\mathscr{H}_{i+1}$ otherwise.

(v) The sampling process is guaranteed to be terminated at or before the s-th stage.

Within this class of sampling schemes, we can apply the bisection risk tuning technique to determine $\zeta > 0$ as large as possible such that the risk requirement (7) is satisfied. In practices, the inequalities "$\Pr\{\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \ge \theta \mid \boldsymbol{p}\} \le \alpha_i$" and "$\Pr\{\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta \mid \boldsymbol{p}\} \le \beta_i$" in requirement (ii) can be respectively relaxed as

$$\Pr\{\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \ge \theta \mid \boldsymbol{p}\} \approx \alpha_i, \qquad \Pr\{\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \le \theta \mid \boldsymbol{p}\} \approx \beta_i$$

provided that the risk of making wrong decisions can be controlled by ζ. The constraints "$\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) \le g(\widehat{\boldsymbol{p}}_\ell)$" and "$\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) \ge g(\widehat{\boldsymbol{p}}_\ell)$" in requirement (ii) ensure that "Case (A)" and "Case (B)" are the only two possible cases at the termination of the sampling process.

As an illustration of the construction of sampling schemes, we consider the problem defined by (6) and (7) of testing m mutually exclusive and exhaustive composite hypotheses regarding $\theta = g(\boldsymbol{p}) = p_x - p_y$. At each stage, one can use the following confidence limits

$$\begin{aligned}
\mathcal{L}(\widehat{\boldsymbol{p}}_\ell, \ell, \alpha_i) &= \mathscr{L}(\widehat{\boldsymbol{p}}_{\ell,x}, \widehat{\boldsymbol{p}}_{\ell,y}, N_{\ell,x}, N_{\ell,y}, \alpha_i), \qquad i = 0, 1, \cdots, m-1, \\
\mathcal{U}(\widehat{\boldsymbol{p}}_\ell, \ell, \beta_i) &= \mathscr{U}(\widehat{\boldsymbol{p}}_{\ell,x}, \widehat{\boldsymbol{p}}_{\ell,y}, N_{\ell,x}, N_{\ell,y}, \beta_i), \qquad i = 1, \cdots, m
\end{aligned}$$

for $\ell = 1, \cdots, s$, where $\mathscr{L}$ and $\mathscr{U}$ are confidence limits investigated in [13], which can be expressed as

$$\begin{aligned}
\mathscr{L}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \delta) &= \widehat{p}_x - \widehat{p}_y - \mathcal{Z}_\delta \sqrt{\frac{l_x(1-l_x)}{N_x} + \frac{u_y(1-u_y)}{N_y}}, \\
\mathscr{U}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \delta) &= \widehat{p}_x - \widehat{p}_y + \mathcal{Z}_\delta \sqrt{\frac{u_x(1-u_x)}{N_x} + \frac{l_y(1-l_y)}{N_y}},
\end{aligned}$$

where

$$\widehat{p}_x \in \left\{ \frac{k}{N_x} : k = 0, 1, \cdots, N_x \right\}, \qquad \widehat{p}_y \in \left\{ \frac{k}{N_y} : k = 0, 1, \cdots, N_y \right\},$$

$$l_x = \frac{c + 2N_x\widehat{p}_x - \sqrt{c^2 + 4cN_x\widehat{p}_x(1-\widehat{p}_x)}}{2(c+N_x)}, \qquad u_x = \frac{c + 2N_x\widehat{p}_x + \sqrt{c^2 + 4cN_x\widehat{p}_x(1-\widehat{p}_x)}}{2(c+N_x)},$$

$$l_y = \frac{c + 2N_y\widehat{p}_y - \sqrt{c^2 + 4cN_y\widehat{p}_y(1-\widehat{p}_y)}}{2(c+N_y)}, \qquad u_y = \frac{c + 2N_y\widehat{p}_y + \sqrt{c^2 + 4cN_y\widehat{p}_y(1-\widehat{p}_y)}}{2(c+N_y)}$$

with $c = \mathcal{Z}_\delta^2$. It can be checked that $l_x,\ u_x$ are the roots for p in the quadratic equation $|\widehat{p}_x - p| = \mathcal{Z}_\delta\sqrt{p(1-p)/N_x}$ and that $l_y,\ u_y$ are the roots for p in the quadratic equation $|\widehat{p}_y - p| = \mathcal{Z}_\delta\sqrt{p(1-p)/N_y}$. Actually, there are many methods to construct confidence intervals satisfying requirement (i) (see, e.g., [1, 13] and the references therein).

For a given value of the risk tuning parameter ζ, we have to determine sample sizes $(N_{\ell,x}, N_{\ell,y})$, $\ell = 1, \cdots, s$ such that the sampling process is guaranteed to be terminated at or before the s-th stage. This can be done as follows.

In applications, the sample sizes for Y can be chosen to be increasing functions of sample sizes of Y. In other words, one can define an increasing function $F(.)$ such that $N_{\ell,y} = F(N_{\ell,x})$ for $\ell = 1, \cdots, s$. Therefore, for a given value of the risk tuning parameter ζ, the maximum sample sizes $(N_{s,x}, N_{s,y})$ can be determined as (N_x, N_y), where $N_y = F(N_x)$ with N_x guaranteeing that $\{(\widehat{p}_x, \widehat{p}_y) : \theta_i' \le \mathscr{L}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \alpha_i) \le \mathscr{U}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \beta_i) \le \theta_i''\}$ is nonempty for $i = 1, \cdots, m-1$. The minimum sample sizes $(N_{1,x}, N_{1,y})$ can be determined as (N_x, N_y), where $N_y = F(N_x)$ with N_x guaranteeing that

$$\bigcup_{i=1}^{m} \{(\widehat{p}_x, \widehat{p}_y) : \theta_{i-1}' \le \mathscr{L}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \alpha_{i-1}) \le \mathscr{U}(\widehat{p}_x, \widehat{p}_y, N_x, N_y, \beta_i) \le \theta_i''\}$$

is nonempty. To reduce the sampling cost, one should choose N_x as small as possible in the determination of $(N_{s,x}, N_{s,y})$ and $(N_{1,x}, N_{1,y})$. Once the sample sizes for the first and last stages are determined, one can choose the sample sizes for other stages. For example, the sample sizes can be chosen as arithmetic or geometric progressions.

Given that a class of testing plans associated with ζ have been constructed such that the risk of making wrong decisions can be made arbitrarily small if ζ is sufficiently small, the bisection risk tuning technique can be applied to determine ζ as large as possible such that the risk requirement (7) can be satisfied. A critical subroutine of the bisection risk tuning method is to apply Adapted Branch and Bound algorithm proposed in Appendix T of [3] to determine whether the probability of rejecting $\mathscr{H}_i$ is no greater than δ_i for all $\boldsymbol{p}$ such that $\theta = g(\boldsymbol{p}) \in \Theta_i$. This requires bounding the probability $\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}$ for $\boldsymbol{p}$ in a rectangular domain $\mathcal{Q} \stackrel{\text{def}}{=} \{(p_x, p_y) : 0 \le \underline{p}_x \le p_x \le \overline{p}_x \le 1,\ 0 \le \underline{p}_y \le p_y \le \overline{p}_y \le 1\}$. From our proposed method of designing multistage testing plans, it can be seen that, for a given value of risking tuning parameter $\zeta > 0$, a testing plan can be described as follows:

At the ℓ-th stage, the sampling process is continued if $\widehat{\boldsymbol{p}}_\ell \in \mathscr{C}_\ell$. Otherwise, the sampling process is terminated and hypothesis $\mathscr{H}_i$ is accepted if $\widehat{\boldsymbol{p}}_\ell \in \mathscr{D}_{i,\ell}$. Here $\mathscr{C}_\ell$ and $\mathscr{D}_{i,\ell},\ i = 1, \cdots, m$ are mutually exclusive and exhaustive subsets of the support of $\widehat{\boldsymbol{p}}_\ell$. In other words, the sampling plan is to continue sampling until $\widehat{\boldsymbol{p}}_\ell \in \mathscr{D}_{i,\ell}$ for some $\ell \in \{1, \cdots, s\}$ and $i \in \{1, \cdots, m\}$ and then accept hypothesis $\mathscr{H}_i$.

With the above understanding of the testing plan, one can see that

$$\{\text{Rejecting } \mathscr{H}_i\} = \cup_{j=1}^{s} \left\{\widehat{\boldsymbol{p}}_\ell \in \mathscr{C}_\ell,\ 1 \le \ell < j;\ \widehat{\boldsymbol{p}}_i \in \mathscr{D}_{i,j}^r\right\}, \tag{11}$$

where

$$\mathscr{D}_{i,j}^r \stackrel{\text{def}}{=} \bigcup_{\substack{1 \le k \le m \\ k \ne i}} \mathscr{D}_{k,j}$$

is the rejection region of hypothesis $\mathscr{H}_i$. To reduce the computational complexity for bounding $\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}$, we invoke the truncation method proposed in [4]. For this purpose, let $\eta \in (0,1)$ and define

$$\begin{aligned}
&a_{\ell,x} = \mathrm{T}_{\mathrm{lb}}(\underline{p}_x, N_{\ell,x}, \eta), \quad b_{\ell,x} = \mathrm{T}_{\mathrm{ub}}(\overline{p}_x, N_{\ell,x}, \eta), \quad a_{\ell,y} = \mathrm{T}_{\mathrm{lb}}(\underline{p}_y, N_{\ell,y}, \eta), \quad b_{\ell,y} = \mathrm{T}_{\mathrm{ub}}(\overline{p}_y, N_{\ell,y}, \eta),\\
&c_{\ell,x} = \mathrm{T}_{\mathrm{lb}}(\overline{p}_x, N_{\ell,x}, \eta), \quad d_{\ell,x} = \mathrm{T}_{\mathrm{ub}}(\underline{p}_x, N_{\ell,x}, \eta), \quad c_{\ell,y} = \mathrm{T}_{\mathrm{lb}}(\overline{p}_y, N_{\ell,y}, \eta), \quad d_{\ell,y} = \mathrm{T}_{\mathrm{ub}}(\underline{p}_y, N_{\ell,y}, \eta)
\end{aligned}$$

for $\ell = 1, \cdots, s$, where $\mathrm{T}_{\mathrm{lb}}(.,.,.)$ and $\mathrm{T}_{\mathrm{ub}}(.,.,.)$ are multivariate functions such that

$$\mathrm{T}_{\mathrm{lb}}(\theta, n, \eta) = \max\left\{0,\ \frac{1}{n}\left\lceil n\theta + \frac{1 - 2\theta - \sqrt{1 + \frac{18n\theta(1-\theta)}{\ln\frac{2}{\eta}}}}{\frac{2}{3n} + \frac{3}{\ln\frac{2}{\eta}}}\right\rceil\right\},$$

$$\mathrm{T}_{\mathrm{ub}}(\theta, n, \eta) = \min\left\{1,\ \frac{1}{n}\left\lfloor n\theta + \frac{1 - 2\theta - \sqrt{1 + \frac{18n\theta(1-\theta)}{\ln\frac{2}{\eta}}}}{\frac{2}{3n} + \frac{3}{\ln\frac{2}{\eta}}}\right\rfloor\right\}$$

for $\theta \in [0,1],\ \eta \in (0,1)$ and $n \in \mathbb{N}$. By virtue of (11), Theorem 3 of [4], and Bonferroni's inequality, we have

$$\begin{aligned}
&\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}\\
\le\ & 2s\eta + \Pr\{a_{\ell,x} \le \widehat{\boldsymbol{p}}_{\ell,x} \le b_{\ell,x},\ a_{\ell,y} \le \widehat{\boldsymbol{p}}_{\ell,y} \le b_{\ell,x},\ \ell = 1, \cdots, s;\ \text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}\\
=\ & 2s\eta + \sum_{j=1}^{s} \Pr\{a_{\ell,x} \le \widehat{\boldsymbol{p}}_{\ell,x} \le b_{\ell,x},\ a_{\ell,y} \le \widehat{\boldsymbol{p}}_{\ell,y} \le b_{\ell,x},\ \ell = 1, \cdots, s;\\
&\qquad \widehat{\boldsymbol{p}}_\ell \in \mathscr{C}_\ell,\ \ell = 1, \cdots, j-1;\ \widehat{\boldsymbol{p}}_j \in \mathscr{D}_{i,j}^r \mid \boldsymbol{p}\}\\
\le\ & 2s\eta + \sum_{j=1}^{s} \Pr\{a_{\ell,x} \le \widehat{\boldsymbol{p}}_{\ell,x} \le b_{\ell,x},\ a_{\ell,y} \le \widehat{\boldsymbol{p}}_{\ell,y} \le b_{\ell,x},\ \ell = 1, \cdots, j;\\
&\qquad \widehat{\boldsymbol{p}}_\ell \in \mathscr{C}_\ell,\ \ell = 1, \cdots, j-1;\ \widehat{\boldsymbol{p}}_j \in \mathscr{D}_{i,j}^r \mid \boldsymbol{p}\}
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\} \ \ge\ & \sum_{j=1}^{s} \Pr\{c_{\ell,x} \le \widehat{\boldsymbol{p}}_{\ell,x} \le d_{\ell,x},\ c_{\ell,y} \le \widehat{\boldsymbol{p}}_{\ell,y} \le d_{\ell,x},\ \ell = 1, \cdots, j;\\
&\qquad \widehat{\boldsymbol{p}}_\ell \in \mathscr{C}_\ell,\ \ell = 1, \cdots, j-1;\ \widehat{\boldsymbol{p}}_j \in \mathscr{D}_{i,j}^r \mid \boldsymbol{p}\}
\end{aligned} \tag{13}$$

for all $\boldsymbol{p} \in \mathcal{Q}$ and $i = 1, \cdots, m$. It can be seen that the bounds of $\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}$ in (12) and (13) can be expressed as summations of probabilistic terms like $\Pr\{\boldsymbol{K}_i \in \mathscr{K}_i,\ 1 \le i \le \ell \mid \boldsymbol{p}\},\ \ell = 1, \cdots, s$, where $\mathscr{K}_\ell$ does not depend on $\boldsymbol{p}$ and is a subset of the support of $\boldsymbol{K}_\ell$ for $\ell = 1, \cdots, s$. Since $\mathscr{K}_\ell$ is independent of $\boldsymbol{p}$, we can employ the recursive technique for bounding $\Pr\{\boldsymbol{K}_i \in \mathscr{K}_i,\ 1 \le i \le \ell \mid \boldsymbol{p}\},\ \ell = 1, \cdots, s$ over $\mathcal{Q}$, which has been established in [3].

Given that the Adapted Branch and Bound algorithm can be used to determine whether $\Pr\{\text{Rejecting } \mathscr{H}_i \mid \boldsymbol{p}\}$ is no greater δ_i for $\boldsymbol{p} \in \{(p_x, p_y) : 0 < p_x < 1,\ 0 < p_y < 1,\ g(p_x, p_y) \in \Theta_i\}$, the bisection risk tuning technique can be applied to determine ζ as large as possible such that the risk requirement (7) can be satisfied.

7 Conclusion

In this paper, we have developed a general approach for testing composite parametric hypotheses. Instead of using likelihood ratio to define testing plans, we use confidence limits to make early stopping possible. For a wide class problems of testing population means, the sample sizes of our proposed testing plans are absolutely bounded. Moreover, our testing plans are more efficient over a wide range of parametric values as compared to SPRT and its variations. Our approach can be applied to test arbitrary number of composite hypotheses.

A Preliminary Results

We need some preliminary results.

Lemma 1 *Let $\mathscr{E}$ be an event determined by random tuple $(X_1, \cdots, X_{\mathbf{r}})$. Let $\varphi(X_1, \cdots, X_{\mathbf{r}})$ be a ULE of θ. Then,*

(i) $\Pr\{\mathscr{E} \mid \theta\}$ *is non-increasing with respect to $\theta \in \Theta$ no less than z provided that $\mathscr{E} \subseteq \{\varphi(X_1, \cdots, X_{\mathbf{r}}) \le z\}$.*

(ii) $\Pr\{\mathscr{E} \mid \theta\}$ *is non-decreasing with respect to $\theta \in \Theta$ no greater than z provided that $\mathscr{E} \subseteq \{\varphi(X_1, \cdots, X_{\mathbf{r}}) \ge z\}$.*

Proof. We first consider the case that $X_1, X_2, \cdots$ are discrete random variables. Let $I_{\mathbf{r}}$ denote the support of $\mathbf{r}$, i.e., $I_{\mathbf{r}} = \{\mathbf{r}(\omega) : \omega \in \Omega\}$. Define $\mathscr{X}_r = \{(X_1(\omega), \cdots, X_r(\omega)) : \omega \in \mathscr{E},\ \mathbf{r}(\omega) = r\}$ for $r \in I_{\mathbf{r}}$. Then,

$$\Pr\{\mathscr{E} \mid \theta\} = \sum_{r \in I_{\mathbf{r}}} \sum_{(x_1, \cdots, x_r) \in \mathscr{X}_r} \Pr\{X_i = x_i,\ i = 1, \cdots, r \mid \theta\}. \tag{14}$$

To show statement (i), using the assumption that $\mathscr{E} \subseteq \{\varphi(X_1, \cdots, X_{\mathbf{r}}) \le z\}$, we have $\varphi(x_1, \cdots, x_r) \le z$ for $(x_1, \cdots, x_r) \in \mathscr{X}_r$ with $r \in I_{\mathbf{r}}$. Since $\varphi(X_1, \cdots, X_{\mathbf{r}})$ is a ULE of θ, we have that $\Pr\{X_i = x_i,\ i = 1, \cdots, r \mid \theta\}$ is non-increasing with respect to $\theta \in \Theta$ no less than z. It follows immediately from (14) that statement (i) is true.

To show statement (ii), using the assumption that $\mathscr{E} \subseteq \{\varphi(X_1, \cdots, X_{\mathbf{r}}) \geq z\}$, we have $\varphi(x_1, \cdots, x_r) \geq z$ for $(x_1, \cdots, x_r) \in \mathscr{X}_r$ with $r \in I_{\mathbf{r}}$. Since $\varphi(X_1, \cdots, X_{\mathbf{r}})$ is a ULE of θ, we have that $\Pr\{X_i = x_i,\ i = 1, \cdots, r \mid \theta\}$ is non-decreasing with respect to $\theta \in \Theta$ no greater than z. It follows immediately from (14) that statement (ii) is true.

For the case that $X_1, X_2, \cdots$ are continuous random variables, we can also show the lemma by modifying the argument for the discrete case. Specially, the summation of likelihood function $\Pr\{X_i = x_i,\ i = 1, \cdots, r \mid \theta\}$ over the set of tuple $(x_1, \cdots, x_r)$ is replaced by the integration of the joint probability density function $f_{X_1, \cdots, X_r}(x_1, \cdots, x_r, \theta)$ over the set of $(x_1, \cdots, x_r)$. This concludes the proof of Lemma 1.

□

Lemma 2 *$\Pr\{F_Z(Z) \leq \alpha\} \leq \alpha$ and $\Pr\{G_Z(Z) \leq \alpha\} \leq \alpha$ for any random variable Z and positive number α, where $F_Z(z) = \Pr\{Z \leq z\}$ and $G_Z(z) = \Pr\{Z \geq z\}$.*

Proof. Let I_Z denote the support of Z. If $\{z \in I_Z : F_Z(z) \leq \alpha\}$ is empty, then, $\{F_Z(Z) \leq \alpha\}$ is an impossible event and thus $\Pr\{F_Z(Z) \leq \alpha\} = 0$. Otherwise, we can define $z^\star = \max\{z \in I_Z : F_Z(z) \leq \alpha\}$. It follows from the definition of $z^\star$ that $F_Z(z^\star) \leq \alpha$. Since $F_Z(z)$ is non-decreasing with respect to z, we have $\{F_Z(Z) \leq \alpha\} = \{Z \leq z^\star\}$. Therefore, $\Pr\{F_Z(Z) \leq \alpha\} = \Pr\{Z \leq z^\star\} = F_Z(z^\star) \leq \alpha$ for any $\alpha > 0$. By a similar method, we can show $\Pr\{G_Z(Z) \leq \alpha\} \leq \alpha$ for any $\alpha > 0$.

□

Lemma 3 *Let X be a random variable parameterized by its mean $\mathbb{E}[X] = \theta \in \Theta$. Suppose that X is a ULE of θ. Let $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, where $X_1, \cdots, X_n$ are i.i.d. samples of random variable X. Then,*

$$\Pr\{\overline{X}_n \leq z\} \leq [\mathscr{C}(z, \theta)]^n, \qquad \forall z \leq \theta$$
$$\Pr\{\overline{X}_n \geq z\} \leq [\mathscr{C}(z, \theta)]^n, \qquad \forall z \geq \theta.$$

Moreover, $\mathscr{C}(z, \theta)$ is non-decreasing with respect to θ no greater than z and is non-increasing with respect to θ no less than z. Similarly, $\mathscr{C}(z, \theta)$ is non-decreasing with respect to z no greater than θ and is non-increasing with respect to z no less than θ.

Proof. By the convexity of function e^x and Jensen's inequality, we have $\inf_{\rho > 0} \mathbb{E}[e^{\rho(X-z)}] \geq \inf_{\rho > 0} e^{\rho \mathbb{E}[X-z]} \geq 1$ for $\theta \geq z$. In view of $\inf_{\rho \leq 0} \mathbb{E}[e^{\rho(X-z)}] \leq 1$, we have $\mathscr{C}(z, \theta) = \inf_{\rho \leq 0} \mathbb{E}[e^{\rho(X-z)}]$ for $\theta \geq z$. Clearly, $\mathscr{C}(z, \theta) = \inf_{\rho \leq 0} e^{-\rho z} \mathbb{E}[e^{\rho X}]$ is non-decreasing with respect to z less than θ. Since X is a ULE of θ, we have that $\mathbb{E}[e^{\rho(X-z)}] = e^{-\rho z} \mathbb{E}[e^{\rho X}] = e^{-\rho z} \int_{u=0}^{\infty} \Pr\{e^{\rho X} > u\} du$ is non-increasing with respect to $\theta \geq z$ for $\rho \leq 0$ and thus $\mathscr{C}(z, \theta)$ is non-increasing with respect to θ greater than z.

Observing that $\inf_{\rho\geq 0}\mathbb{E}[e^{\rho(X-z)}] \leq 1$ and that $\inf_{\rho<0}\mathbb{E}[e^{\rho(X-z)}] \geq \inf_{\rho<0} e^{\rho\mathbb{E}[X-z]} \geq 1$ for $\theta < z$, we have $\mathscr{C}(z,\theta) = \inf_{\rho\geq 0}\mathbb{E}[e^{\rho(X-z)}]$ for $\theta < z$. Clearly, $\mathscr{C}(z,\theta) = \inf_{\rho\geq 0} e^{-\rho z}\mathbb{E}[e^{\rho X}]$ is non-increasing with respect to z greater than θ. Since X is a ULE of θ, we have that $\mathbb{E}[e^{\rho(X-z)}] = e^{-\rho z}\int_{u=0}^{\infty}\Pr\{e^{\rho X} > u\}du$ is non-decreasing with respect to θ for $\rho > 0$ and consequently, $\mathscr{C}(z,\theta)$ is non-decreasing with respect to θ smaller than z.

Making use of the established fact $\inf_{\rho\leq 0}\mathbb{E}[e^{\rho(X-z)}] = \mathscr{C}(z,\theta)$ and the Chernoff bound $\Pr\{\overline{X}_n \leq z\} \leq \left[\inf_{\rho\leq 0}\mathbb{E}[e^{\rho(X-z)}]\right]^n$ (see, [2]), we have $\Pr\{\overline{X}_n \leq z\} \leq [\mathscr{C}(z,\theta)]^n$ for $z \leq \theta$. Making use of the established fact $\inf_{\rho\geq 0}\mathbb{E}[e^{\rho(X-z)}] = \mathscr{C}(z,\theta)$ and the Chernoff bound $\Pr\{\overline{X}_n \geq z\} \leq \left[\inf_{\rho\geq 0}\mathbb{E}[e^{\rho(X-z)}]\right]^n$, we have $\Pr\{\overline{X}_n \geq z\} \leq [\mathscr{C}(z,\theta)]^n$ for $z \geq \theta$. This concludes the proof of Lemma 3. □

B Proof of Theorem 1

Since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 1 that $G_{\overline{X}_n}(z,\theta)$ is non-decreasing with respect to $\theta \in \Theta$. Hence, by the definition of the lower confidence limit, $L(\frac{\theta_0+\theta_1}{2}, \zeta\alpha) \geq \theta_0$ if $G_{\overline{X}_n}(\frac{\theta_0+\theta_1}{2}, \theta_0) \leq \zeta\alpha$. Since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 3 that

$$G_{\overline{X}_n}\left(\frac{\theta_0+\theta_1}{2}, \theta_0\right) = \Pr\left\{\overline{X}_n \geq \frac{\theta_1+\theta_0}{2} \mid \theta_0\right\} \leq \left[\mathscr{C}\left(\frac{\theta_0+\theta_1}{2}, \theta_0\right)\right]^n \leq \zeta\alpha$$

if

$$n \geq \frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}. \tag{15}$$

Observing that $G_{\overline{X}_n}(z,\theta)$ is non-increasing with respect to z, we have that $L(z,\zeta\alpha)$ is non-decreasing with respect to z. This implies that $L(z,\zeta\alpha) \geq L(\frac{\theta_0+\theta_1}{2}, \zeta\alpha) \geq \theta_0$ for $z \geq \frac{\theta_0+\theta_1}{2}$ if (15) holds.

On the other hand, since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 1 that $F_{\overline{X}_n}(z,\theta)$ is non-increasing with respect to θ. Hence, by the definition of the upper confidence limit, $U(\frac{\theta_0+\theta_1}{2}, \zeta\beta) \leq \theta_1$ if $F_{\overline{X}_n}(\frac{\theta_0+\theta_1}{2}, \theta_1) \leq \zeta\beta$. Since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 3 that

$$F_{\overline{X}_n}\left(\frac{\theta_0+\theta_1}{2}, \theta_1\right) = \Pr\left\{\overline{X}_n \leq \frac{\theta_1+\theta_0}{2} \mid \theta_1\right\} \leq \left[\mathscr{C}\left(\frac{\theta_0+\theta_1}{2}, \theta_1\right)\right]^n \leq \zeta\beta$$

if

$$n \geq \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)}. \tag{16}$$

Observing that $F_{\overline{X}_n}(z,\theta)$ is non-decreasing with respect to z, we have that $U(z,\zeta\alpha)$ is non-decreasing with respect to z. This implies that $U(z,\zeta\beta) \leq U(\frac{\theta_0+\theta_1}{2}, \zeta\beta) \leq \theta_1$ for $z \leq \frac{\theta_0+\theta_1}{2}$ if (16) holds.

Let $n_{\max}$ be the smallest integer n such that $\{L(\overline{X}_n, \zeta\alpha) < \theta_0 < \theta_1 < U(\overline{X}_n, \zeta\beta)\} = \emptyset$. From the above analysis, we know that either $U(z,\zeta\beta) \leq \theta_1$ or $L(z,\zeta\alpha) \geq \theta_0$ must be true for any

$z \in I_{\overline{X}_n}$ if (15) and (16) are satisfied. This implies that

$$\mathbf{n} \leq n_{\max} < \max\left\{ \frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}, \ \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)} \right\}. \tag{17}$$

By the definition of the testing plan, we have

$$\begin{aligned} \Pr\{\text{Reject } \mathscr{H}_0 \mid \theta_0\} &\leq \sum_{n=1}^{n_{\max}} \Pr\{L(\overline{X}_n, \zeta\alpha) \geq \theta_0, \ \mathbf{n} = n \mid \theta_0\} \\ &\leq \sum_{n=1}^{n_{\max}} \Pr\{L(\overline{X}_n, \zeta\alpha) \geq \theta_0 \mid \theta_0\} \\ &\leq n_{\max}\, \zeta\alpha \end{aligned}$$

and

$$\begin{aligned} \Pr\{\text{Accept } \mathscr{H}_0 \mid \theta_1\} &\leq \sum_{n=1}^{n_{\max}} \Pr\{U(\overline{X}_n, \zeta\beta) \leq \theta_1, \ \mathbf{n} = n \mid \theta_1\} \\ &\leq \sum_{n=1}^{n_{\max}} \Pr\{U(\overline{X}_n, \zeta\beta) \leq \theta_1 \mid \theta_1\} \\ &\leq n_{\max}\, \zeta\beta. \end{aligned}$$

Making use of (17), we have that both $n_{\max}\zeta\alpha$ and $n_{\max}\zeta\beta$ tend to 0 as $\zeta \to 0$. It follows that $\Pr\{\text{Reject } \mathscr{H}_0 \mid \theta_0\}$ and $\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta_1\}$ tend to 0 as $\zeta \to 0$. From the definition of the testing plan, we have that $\{\text{Reject } \mathscr{H}_0\}$ and $\{\text{Accept } \mathscr{H}_0\}$ are determined by random tuple $(X_1, \cdots, X_{\mathbf{n}})$. Moreover, $\{\text{Reject } \mathscr{H}_0\} \subseteq \{\overline{X}_n \geq \theta_0\}$ and $\{\text{Accept } \mathscr{H}_0\} \subseteq \{\overline{X}_n \leq \theta_1\}$. Hence, by Lemma 1, we have that $\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta\}$ is non-increasing with respect to $\theta \in \Theta$ such that $\theta \notin (\theta_0, \theta_1)$. This completes the proof of the theorem.

C Proof of Theorem 2

By the definition of the lower confidence limit, $\left\{[\mathscr{C}(\overline{X}_n, \mathcal{L}(\overline{X}_n, \delta))]^n \leq \delta, \ \mathcal{L}(\overline{X}_n, \delta) \leq \overline{X}_n\right\}$ is a sure event. It follows from Lemmas 2 and 3 that

$$\begin{aligned} \Pr\{\mathcal{L}(\overline{X}_n, \delta) \geq \theta\} &= \Pr\left\{[\mathscr{C}(\overline{X}_n, \mathcal{L}(\overline{X}_n, \delta))]^n \leq \delta, \ \theta \leq \mathcal{L}(\overline{X}_n, \delta) \leq \overline{X}_n\right\} \\ &\leq \Pr\left\{G_{\overline{X}_n}(\overline{X}_n, \mathcal{L}(\overline{X}_n, \delta)) \leq \delta, \ \theta \leq \mathcal{L}(\overline{X}_n, \delta) \leq \overline{X}_n\right\} \leq \Pr\left\{G_{\overline{X}_n}(\overline{X}_n, \theta) \leq \delta\right\} \leq \delta \end{aligned}$$

for $\theta \in \Theta$. By the definition of the upper confidence limit, $\left\{[\mathscr{C}(\overline{X}_n, \mathcal{U}(\overline{X}_n, \delta))]^n \leq \delta, \ \mathcal{U}(\overline{X}_n, \delta) \geq \overline{X}_n\right\}$ is a sure event. It follows from Lemmas 2 and 3 that

$$\begin{aligned} \Pr\{\mathcal{U}(\overline{X}_n, \delta) \leq \theta\} &= \Pr\left\{[\mathscr{C}(\overline{X}_n, \mathcal{U}(\overline{X}_n, \delta))]^n \leq \delta, \ \overline{X}_n \leq \mathcal{U}(\overline{X}_n, \delta) \leq \theta\right\} \\ &\leq \Pr\left\{F_{\overline{X}_n}(\overline{X}_n, \mathcal{U}(\overline{X}_n, \delta)) \leq \delta, \ \overline{X}_n \leq \mathcal{U}(\overline{X}_n, \delta) \leq \theta\right\} \leq \Pr\left\{F_{\overline{X}_n}(\overline{X}_n, \theta) \leq \delta\right\} \leq \delta \end{aligned}$$

for $\theta \in \Theta$.

Now, we show that $\mathcal{L}(z,\delta)$ is non-decreasing with respect to z. Let $z_1 < z_2$ and $L_1 = \mathcal{L}(z_1,\delta)$, $L_2 = \mathcal{L}(z_2,\delta)$. To show the monotonicity of $\mathcal{L}(z,\delta)$, it suffices to show $L_2 \geq L_1$ for three cases (not necessarily mutually exclusive) as follows:

Case (i): $[\mathscr{C}(z_2,z_2)]^n \leq \delta$;

Case (ii): $[\mathscr{C}(z_1,z_1)]^n \leq \delta$;

Case (iii): $[\mathscr{C}(z_1,z_1)]^n > \delta$ and $[\mathscr{C}(z_2,z_2)]^n > \delta$.

In Case (i), we have $L_2 = z_2 > z_1 \geq L_1$.

In Case (ii), we have $L_1 = z_1$ and $[\mathscr{C}(z_2,z_1)]^n \leq [\mathscr{C}(z_1,z_1)]^n \leq \delta$, which implies that $L_2 \geq z_1 = L_1$.

In Case (iii), it must be true that $[\mathscr{C}(z_1,L_1)]^n = \delta$ and $[\mathscr{C}(z_2,L_2)]^n = \delta$. Suppose, to get a contradiction, that $L_1 > L_2$. Then, $z_1 > L_1 > L_2$ and $\mathscr{C}(z_1,L_2) < \mathscr{C}(z_1,L_1) = \mathscr{C}(z_2,L_2)$, since $\mathscr{C}(z,\theta)$ is non-decreasing with respect to $\theta \leq z$. On the other hand, $\mathscr{C}(z_1,L_2) > \mathscr{C}(z_2,L_2)$, since $z_2 > z_1 > L_2$ and $\mathscr{C}(z,\theta)$ is non-increasing with respect to $z \geq \theta$. This is a contradiction. Thus, it must be true that $L_1 \leq L_2$. This completes the proof that $\mathcal{L}(z,\delta)$ is non-decreasing with respect to z.

In a similar manner, we can show that $\mathcal{U}(z,\delta)$ is non-decreasing with respect to z. This completes the proof of the theorem.

D Proof of Theorem 3

Since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 3 that $\mathcal{C}(z,\theta)$ is non-decreasing with respect to θ no greater than z. Hence, it follows from the definition of the lower confidence limit that $\mathcal{L}(\frac{\theta_0+\theta_1}{2},\zeta\alpha) \geq \theta_0$ if

$$\left[\mathscr{C}\left(\frac{\theta_0+\theta_1}{2},\theta_0\right)\right]^n \leq \zeta\alpha, \qquad \theta_0 \leq \frac{\theta_0+\theta_1}{2},$$

which is guaranteed by

$$n \geq \frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2},\theta_0)}. \tag{18}$$

By Theorem 2, $\mathcal{L}(z,\zeta\alpha)$ is non-decreasing with respect to z. Hence, $\mathcal{L}(z,\zeta\alpha) \geq \mathcal{L}(\frac{\theta_0+\theta_1}{2},\zeta\alpha) \geq \theta_0$ for $z \geq \frac{\theta_0+\theta_1}{2}$ if (18) holds.

On the other hand, since $\overline{X}_n$ is a ULE for θ, it follows from Lemma 3 that $\mathcal{C}(z,\theta)$ is non-increasing with respect to θ no less than z. Hence, it follows from the definition of the upper confidence limit that $\mathcal{U}(\frac{\theta_0+\theta_1}{2},\zeta\beta) \leq \theta_1$ if

$$\left[\mathscr{C}\left(\frac{\theta_0+\theta_1}{2},\theta_1\right)\right]^n \leq \zeta\beta, \qquad \theta_1 \geq \frac{\theta_0+\theta_1}{2},$$

which is ensured by

$$n \geq \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2},\theta_1)}. \tag{19}$$

By Theorem 2, $\mathcal{U}(z, \zeta\beta)$ is non-decreasing with respect to z. Consequently,

$$\mathcal{U}(z, \zeta\beta) \leq \mathcal{U}\left(\frac{\theta_0+\theta_1}{2}, \zeta\beta\right) \leq \theta_1$$

for $z \leq \frac{\theta_0+\theta_1}{2}$ if (19) holds. This shows that either $\mathcal{L}(\overline{X}_n, \zeta\alpha) \geq \theta_0$ or $\mathcal{U}(\overline{X}_n, \zeta\beta) \leq \theta_1$ must be true if

$$n \geq \frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}, \qquad n \geq \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)}.$$

Let $n_{\max}$ be the smallest integer n such that $\{\mathcal{L}(\overline{X}_n, \zeta\alpha) < \theta_0 < \theta_1 < \mathcal{U}(\overline{X}_n, \zeta\beta)\} = \emptyset$. Then,

$$\mathbf{n} \leq n_{\max} < \max\left\{\frac{\ln(\zeta\alpha)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_0)}, \frac{\ln(\zeta\beta)}{\ln \mathscr{C}(\frac{\theta_0+\theta_1}{2}, \theta_1)}\right\}. \tag{20}$$

By the definition of the test plan, we have

$$\Pr\{\text{Reject } \mathscr{H}_0 \mid \theta_0\} \leq \sum_{n=1}^{n_{\max}} \Pr\{\mathcal{L}(\overline{X}_n, \zeta\alpha) \geq \theta_0,\ \mathbf{n} = n \mid \theta_0\} \leq \sum_{n=1}^{n_{\max}} \Pr\{\mathcal{L}(\overline{X}_n, \zeta\alpha) \geq \theta_0 \mid \theta_0\} \leq n_{\max}\, \zeta\alpha$$

and

$$\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta_1\} \leq \sum_{n=1}^{n_{\max}} \Pr\{\mathcal{U}(\overline{X}_n, \zeta\beta) \leq \theta_1,\ \mathbf{n} = n \mid \theta_1\} \leq \sum_{n=1}^{n_{\max}} \Pr\{\mathcal{U}(\overline{X}_n, \zeta\beta) \leq \theta_1 \mid \theta_1\} \leq n_{\max}\, \zeta\beta.$$

Making use of (20), we have that both $n_{\max}\zeta\alpha$ and $n_{\max}\zeta\beta$ tend to 0 as $\zeta \to 0$. It follows that $\Pr\{\text{Reject } \mathscr{H}_0 \mid \theta_0\}$ and $\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta_1\}$ tend to 0 as $\zeta \to 0$.

From the definition of the testing plan, we have that $\{\text{Reject } \mathscr{H}_0\}$ and $\{\text{Accept } \mathscr{H}_0\}$ are determined by random tuple $(X_1, \cdots, X_{\mathbf{n}})$. Moreover, $\{\text{Reject } \mathscr{H}_0\} \subseteq \{\overline{X}_n \geq \theta_0\}$ and $\{\text{Accept } \mathscr{H}_0\} \subseteq \{\overline{X}_n \leq \theta_1\}$. Hence, by Lemma 1, we have that $\Pr\{\text{Accept } \mathscr{H}_0 \mid \theta\}$ is non-increasing with respect to $\theta \in \Theta$ such that $\theta \notin (\theta_0, \theta_1)$. This completes the proof of the theorem.

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