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TILTING MODULES OVER TAME HEREDITARY ALGEBRAS

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Abstract. We give a complete classification of the infinite dimensional tilting modules over a tame hereditary algebra R. We start our investigations by considering tilting modules of the form $T = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where $\mathcal{U}$ is a union of tubes, and $R_{\mathcal{U}}$ denotes the universal localization of R at $\mathcal{U}$ in the sense of Schofield and Crawley-Boevey. Here $R_{\mathcal{U}}/R$ is a direct sum of the Prüfer modules corresponding to the tubes in $\mathcal{U}$. Over the Kronecker algebra, large tilting modules are of this form in all but one case, the exception being the Lukas tilting module L whose tilting class $\operatorname{Gen} L$ consists of all modules without indecomposable preprojective summands. Over an arbitrary tame hereditary algebra, T can have finite dimensional summands, but the infinite dimensional part of T is still built up from universal localizations, Prüfer modules and (localizations of) the Lukas tilting module. We also recover the classification of the infinite dimensional cotilting R-modules due to Buan and Krause.

In this paper, we continue our study of tilting modules arising from universal localization started in [5]. More precisely, we consider tilting modules over a ring R that have the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ where $\mathcal{U}$ is a set of finitely presented R-modules of projective dimension one, and $R_{\mathcal{U}}$ denotes the universal localization of R at $\mathcal{U}$ in the sense of Schofield. We have seen in [5] that over certain rings this construction leads to a classification of all tilting modules. For example, over a Dedekind domain, every tilting module is equivalent to a tilting module of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ for some set of simple R-modules $\mathcal{U}$. Aim of this paper is to prove a similar result for finite dimensional tame hereditary algebras.

Universal localizations of a tame hereditary algebra R were already investigated by Crawley-Boevey in [13]. He showed that the normalized defect provides a rank function ρ as studied by Schofield in [33], and that the ρ-torsion modules are precisely the finite dimensional regular modules. He also described the shape of the universal localization $R_{\mathcal{U}}$ at a set $\mathcal{U}$ of quasi-simple modules, proving that there are substantially different situations depending on whether $\mathcal{U}$ does contain a complete clique (that is, all quasi-simples belonging to a certain tube) or not. In particular, $R_{\mathcal{U}}$ will be an infinite dimensional R-module whenever $\mathcal{U}$ contains a complete clique.

We now want to employ these results to give a classification of the large tilting modules over a tame hereditary algebra R. By *large* we mean tilting modules T that are not equivalent to finite dimensional ones, that is, there is no finite dimensional tilting module T' such that $\operatorname{Gen} T = \operatorname{Gen} T'$.

Recall that by a result of Bazzoni and Herbera [8] a large tilting module T is determined up to equivalence by a set of finite dimensional modules $\mathcal{S}$, in the sense that its tilting class $\operatorname{Gen} T$ coincides with the class of modules $X \in \operatorname{Mod} R$ such that $\operatorname{Ext}^1_R(\mathcal{S}, X) = 0$.

The set $\mathcal{S}$ can be chosen to consist of the finite dimensional modules in ${}^{\perp}(T^{\perp})$, and then it turns out that $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ where $\mathbf{p}$ denotes the class of indecomposable preprojective modules, and

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$\mathbf{t}' \subset \mathbf{t}$ is a subset of the class of all finite dimensional indecomposable regular modules (Theorem 2.7).

Notice that, as a consequence, the lattice of large tilting modules has a largest and a smallest element. Indeed, the largest tilting class in $\operatorname{Mod} R$ not generated by a finite dimensional tilting module is the class $\mathbf{p}^\perp$ of modules without indecomposable preprojective summands, which is generated by the Lukas tilting module $\mathbf{L}$ (see [23] and 1.4), while the smallest one is the class of all divisible modules $\mathbf{t}^\perp$, and the corresponding tilting module is the direct sum $\mathbf{W} = \bigoplus_{S\in\mathbb{U}} S[\infty] \oplus G$ of all Prüfer modules and the generic module (see [30] and 1.3), or in other words, it is the tilting module $R_{\mathbf{t}} \oplus R_{\mathbf{t}}/R$ arising form universal localization at the set of all quasi-simple modules.

Moreover, from the description of $\mathcal{S}$ we also deduce that a large tilting module over the Kronecker algebra must have the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ for some set of quasi-simple R-modules $\mathcal{U}$ in all but one case, the exception being the Lukas tilting module $\mathbf{L}$ (Corollary 2.8).

In the general case, the situation is more involved due to the possible presence of finite dimensional summands in T coming from non-homogeneos tubes. On the other hand, there are at most finitely many such indecomposable summands up to isomorphism (Lemma 3.2). This allows to reduce the classification problem to a situation similar to the Kronecker case. More precisely, we show that T is equivalent to a tilting module of the form $Y \oplus M$ where Y is finite dimensional, while M has no finite dimensional indecomposable direct summands and is a tilting module over a suitable universal localization R' of R. Since R' will again be a tame hereditary algebra, this will enable us to conclude that M is either the Lukas tilting module over R', or it arises from universal localization at a union of tubes over R' (Theorem 4.5). Combining this with decomposition results from [31], we will obtain the following classification.

Theorem. (cf. 3.10 and 4.6) *Let $\mathbf{t} = \bigcup_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda$ where the $\mathbf{t}_\lambda$ are the tubes in the Auslander-Reiten quiver of R. Every large tilting module T has a decomposition*

$$T = \bigoplus_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda(T) \oplus \overline{T}$$

where $\overline{T}$ is a torsion-free module, and the $\mathbf{t}_\lambda(T)$ are direct sums of Prüfer modules and finite dimensional regular modules.
The module $\overline{T}$ satisfies one of the following conditions:

- *(i) $\overline{T}$ is a projective generator for the universal localization $R_{\mathcal{V}}$ at a set $\mathcal{V}$ of quasi-simple R-modules containing complete cliques, or*
- *(ii) $\overline{T}$ is equivalent to the Lukas tilting module over the universal localization $R_{\mathcal{U}}$ at a set $\mathcal{U}$ of quasi-simple R-modules containing no complete cliques.*

Moreover, the following statements hold true for a tube $\mathbf{t}_\lambda$ of rank r:

- *(1) $\mathbf{t}_\lambda(T) = 0$ whenever $\mathbf{t}_\lambda \cap {}^\perp(T^\perp) = \emptyset$;*
- *(2) if ${}^\perp(T^\perp)$ contains some rays from $\mathbf{t}_\lambda$, then $\mathbf{t}_\lambda(T)$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s Prüfer modules corresponding to the $s \leq r$ rays from $\mathbf{t}_\lambda$ contained in ${}^\perp(T^\perp)$, and $r - s$ modules from $\mathbf{t}_\lambda$;*
- *(3) if ${}^\perp(T^\perp)$ contains some modules from $\mathbf{t}_\lambda$, but no complete ray, then $\mathbf{t}_\lambda(T)$ is a direct sum of copies of at most $r - 1$ pairwise non-isomorphic modules from $\mathbf{t}_\lambda$.*

Recall that the large cotilting modules over R have been classified by Buan and Krause in [10, 11], given Bazzoni's result [7] that establishes the pure-injectivity of cotilting modules. By using the fact that every cotilting module over a finite dimensional algebra is equivalent to the dual of a tilting module [38], we can now recover this classification. Let us remark that the other direction does not work: one cannot use the classification of cotilting modules for studying the tilting modules, as duals of (large) cotilting modules need not even be tilting, cf. 1.4.

The paper is organized as follows. In Section 1, we collect some preliminaries on infinite dimensional modules, tilting theory, and universal localization. In Section 2, we prove that a large tilting module T is determined by a set $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ as described above, and we settle the cases where $\mathbf{t}' = \emptyset$ (then T is equivalent to the Lukas tilting module) or $\mathbf{t}'$ is a union of tubes (then T arises from universal localization). In Section 3, we show that T has a canonical decomposition as above. The description of the torsion-free part $\overline{T}$ is achieved in Section 4, where we also discuss the cases when T is noetherian over its endomorphism ring or (Σ-)pure-injective. In the Appendix we deal with the classification of cotilting modules.

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1. Preliminaries

Throughout this note, let R be a finite dimensional tame hereditary (w.l.o.g. indecomposable) algebra over an algebraically closed field k. We denote by $\operatorname{Mod} R$ (respectively, $R\operatorname{Mod}$) the category of all right (respectively, left) R-modules and by $\operatorname{mod} R$ (respectively, $R\operatorname{mod}$) the category of finitely generated right (respectively, left) R-modules. Let $D : \operatorname{mod} R \to R\operatorname{mod}$ be the usual duality. Given a subcategory $\mathcal{S} \subset \operatorname{mod} R$, the subcategory of $R\operatorname{mod}$ consisting of the dual modules $D(S)$ with $S \in \mathcal{S}$ will be denoted by $\mathcal{S}^*$.

We adopt terminology and notation from [31, 30]. In particular, we denote by $\mathbf{p}, \mathbf{t}, \mathbf{q}$ the subcategories of $\operatorname{mod} R$ consisting of the indecomposable preprojective, regular, and preinjective modules, respectively. The corresponding subcategories of $R\operatorname{mod}$ will be denoted by ${}_R\mathbf{p}$, ${}_R\mathbf{t}$, ${}_R\mathbf{q}$. An arbitrary R-module will be called *regular* if it has neither preinjective nor preprojective direct summands.

We fix a complete irredundant set of quasi-simple (i.e. simple regular) modules $\mathbb{U}$, and for each $S \in \mathbb{U}$, we denote by $S[m]$ the module of regular length m on the ray

$$S = S[1] \subset S[2] \subset \cdots \subset S[m] \subset S[m+1] \subset \cdots$$

and let $S[\infty] = \varinjlim S[m]$ be the corresponding *Prüfer module*. The *adic module* $S[-\infty]$ corresponding to $S \in \mathbb{U}$ is defined dually as the inverse limit along the coray ending at S.

We denote by $\mathbb{P} = \mathbb{P}_1 k$ the projective line and write $\mathbf{t} = \bigcup_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda$, where the $\mathbf{t}_\lambda$ are the tubes in the Auslander-Reiten quiver of $\operatorname{mod} R$. The tubes in $R\operatorname{mod}$ will be denoted by ${}_R\mathbf{t}_\lambda$. It is well known that almost all tubes are homogeneous, that is, they contain a unique quasi-simple module up to isomorphism. In order to deal with the (at most three) non-homogeneous tubes, we consider the equivalence relation $\sim$ on $\mathbb{U}$ generated by

$$S \sim S' \text{ if } \operatorname{Ext}^1_R(S, S') \neq 0.$$

According to [13], we call the equivalence classes of this relation *cliques*. In other words, two quasi-simple modules belong to the same clique iff they are in the same tube. The order of the clique is the *rank* of the tube.

We will need a combinatorial description of the *extension closure* of a set of quasi-simples $\mathcal{U} \subset \mathbb{U}$, that is, of the smallest subcategory $\mathcal{W} \subset \operatorname{mod} R$ that contains $\mathcal{U}$ and is closed under extensions. Given a tube $\mathbf{t}_\lambda$ of rank $r > 1$ and a module $X \in \mathbf{t}_\lambda$ of regular length $m < r$, we consider the full subquiver $\mathcal{W}_X$ of $\mathbf{t}_\lambda$ which is isomorphic to the Auslander-Reiten-quiver $\Theta(m)$ of the linearly oriented quiver of type $\mathbb{A}_m$ with X corresponding to the projective-injective vertex of $\Theta(m)$. Following [32, 3.3], we call $\mathcal{W}_X$ the *wing* of $\mathbf{t}_\lambda$ with *vertex* X. The following result is straightforward.

Lemma 1.1. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let $\mathcal{U} = \{U_1, \dots, U_m\} \subset \mathbb{U}$ be a set of $m < r$ quasi-simples in $\mathbf{t}_\lambda$ where $U_{i+1} = \tau^- U_i$ for all $1 \leq i < m$. Then the extension closure $\mathcal{W}$ of $\mathcal{U}$ consists of all finite direct sums of modules in the wing $\mathcal{W}_{U_1[m]} = \{U_i[k] \mid 1 \leq i \leq m,\ 1 \leq k \leq m - i + 1\}$. □

Let us introduce some further notation. Let $\mathcal{M} \subset \operatorname{Mod} R$ be a class of modules. Denote by $\operatorname{Add}\mathcal{M}$ (respectively, $\operatorname{add}\mathcal{M}$) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of elements of $\mathcal{M}$. The class consisting of all modules isomorphic to direct summands of products of modules of $\mathcal{M}$ is denoted by $\operatorname{Prod}\mathcal{M}$. The class consisting of the right R-modules which are epimorphic images of arbitrary direct sums of elements in $\mathcal{M}$ is denoted by $\operatorname{Gen}\mathcal{M}$. Dually we define $\operatorname{Cogen}\mathcal{M}$. We further write

$$\mathcal{M}^o = \{X_R \mid \operatorname{Hom}_R(M, X) = 0 \text{ for each } M \in \mathcal{M}\}$$

$$\mathcal{M}^\perp = \{X_R \mid \operatorname{Ext}^1_R(M, X) = 0 \text{ for each } M \in \mathcal{M}\}$$

$$\mathcal{M}^\wedge = \{X_R \mid \operatorname{Ext}^i_R(M, X) = 0 \text{ for each } i \geq 0,\ M \in \mathcal{M}\}$$

$$\mathcal{M}^\intercal = \{X_R \mid \operatorname{Tor}^R_1(M, X) = 0 \text{ for each } M \in \mathcal{M}\}$$

and define dually ${}^o\mathcal{M}$, ${}^\perp\mathcal{M}$, ${}^\wedge\mathcal{M}$, ${}^\intercal\mathcal{M}$. If $\mathcal{M}$ contains a unique module M, then we shall denote these subcategories by $\operatorname{Add} M$, M^o, $M^\perp$, etc.

Finally, we denote by G the *generic module*. It is the unique indecomposable infinite dimensional module which has finite length over its endomorphism ring. In the notation of 1.3 and 1.4, it is the unique indecomposable in $\mathcal{F} \cap \mathcal{D}$, that is, the unique indecomposable torsion-free divisible module, cf. [31, 5.3 and p.408].

We now collect some tools we will freely use when working with infinite dimensional modules.

Lemma 1.2. (1) If $M \in \operatorname{Mod} R$ and X is a finitely generated indecomposable module in $\operatorname{Add} M$, then X is isomorphic to a direct summand of M.

(2) Every finite-dimensional R-module is *endofinite*, that is, it has finite length as a module over its endomorphism ring. Every direct sum of copies of finitely many endofinite modules is endofinite. Every dual $D(M)$ of an endofinite module M is endofinite.

(3) Suppose M is endofinite. Then $\operatorname{Add} M = \operatorname{Prod} M$. In particular, if $M \in X^{\perp}$ for some $X \in \operatorname{Mod} R$, then $\operatorname{Add} M \subset X^{\perp}$.

(4) Every endofinite R-module M is *pure-injective*, that is, pure-exact sequences starting at M split.

(5) Every indecomposable pure-injective R-module is isomorphic to a module in the following list:
- the finitely generated indecomposable modules,
- the Prüfer modules $S[\infty]$, $S \in \mathbb{U}$,
- the adic modules $S[-\infty]$, $S \in \mathbb{U}$,
- the generic module G.

(6) Let M and N be infinite dimensional indecomposable pure-injective modules. Then
- $\operatorname{Ext}^1_R(M, P) \neq 0$ for every $P \in \mathbf{p}$.
- $\operatorname{Ext}^1_R(Q, M) \neq 0$ for every $Q \in \mathbf{q}$.
- $\operatorname{Ext}^1_R(M, N) \neq 0$ if and only if there are $S \sim S'$ such that $M = S[\infty]$ and $N = S'[-\infty]$.

Proof: (1) Since X is a finitely generated module, being (isomorphic to) a direct summand of a direct sum $M^{(I)}$ of copies of M means that X is (isomorphic to) a direct summand in a finite subsum $M^{(I_0)}$. Now the claim follows from the fact that X has a local endomorphism ring.
The first statement in (2) is clear because every finite-dimensional R-module is finitely generated over its endomorphism ring, which is again a finite-dimensional k-algebra. For the other statements on endofinite modules, we refer to [15]. Details on pure-injective modules can be found in [21, Chapter 7]. The classification of the indecomposable pure-injective R-modules is contained in [14]. Statement (6) is shown in [10, 2.5 and 2.7] □

Recall from [12] that a module T is *tilting* provided that $\operatorname{Gen} T = T^{\perp}$, or equivalently, T satisfies
(T1) $\operatorname{proj.dim}(T) \leq 1$;
(T2) $\operatorname{Ext}^1_R(T, T^{(\kappa)}) = 0$ for any cardinal κ;
(T3) There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \operatorname{Add}(T)$.
Note that every tilting module T satisfies $\operatorname{Add} T = T^{\perp} \cap {}^{\perp}(T^{\perp})$. Moreover, T gives rise to a torsion pair with torsion class $T^{\perp}$ and torsion-free class T^{o}. The class $T^{\perp}$ is called a *tilting class*. Tilting modules having that same tilting classes are said to be *equivalent*. *Cotilting modules* and *cotilting classes* are defined dually, and equivalence of cotilting modules is defined correspondingly.
By [18, 5.1.12], two tilting modules T, T' are equivalent if and only if $\operatorname{Add} T = \operatorname{Add} T'$, while two cotilting modules C, C' are equivalent if and only if $\operatorname{Prod} C = \operatorname{Prod} C'$.

Here are some examples of infinite-dimensional tilting or cotilting modules.

Example 1.3. *The Reiten-Ringel tilting module.* It is shown in [30] that the module

$$\mathbf{W} = \bigoplus_{S \in \mathbb{U}} S[\infty] \oplus G$$

is an infinite dimensional tilting module whose tilting class $\operatorname{Gen}\mathbf{W} = \mathbf{W}^{\perp}$ coincides with the class

$$\mathcal{D} = {}^{o}\mathbf{t} = \mathbf{t}^{\perp}$$

of all *divisible* modules, and moreover, $\mathbf{W}$ is a cotilting module whose cotilting class $\operatorname{Cogen}\mathbf{W} = {}^{\perp}\mathbf{W}$ coincides with the class

$$\mathcal{C} = {}^{\perp}\mathbf{q} = \mathbf{q}^{o}$$

of all modules without indecomposable preinjective direct summands. □

Example 1.4. *The Lukas tilting module.* Based on a construction due to F. Lukas [27, 2.1], Kerner and Trlifaj showed in [23] that there is a countably infinitely generated $\mathbf{p}$-filtered tilting module $\mathbf{L} \in \mathbf{p}^{\perp}$ whose tilting class

$$\operatorname{Gen}\mathbf{L} = \mathbf{p}^{\perp} = {}^{o}\mathbf{p}$$

coincides with the class of all modules without indecomposable preprojective direct summands. The corresponding torsion-free class $\mathbf{L}^{o}$ coincides with the class of preprojective modules in the sense of [31, Section 2]. In particular, $\mathbf{L}^{o}$ is contained in the class

$$\mathcal{F} = \mathbf{t}^{o} = {}^{\perp}\mathbf{t}$$

of all *torsion-free* modules, which is a cotilting class with cotilting module $D({}_{R}\mathbf{W})$, cf. [3, Prop.7]. Here ${}_{R}\mathbf{W}$ denotes the Reiten-Ringel tilting module in the category $R\operatorname{Mod}$. The torsion class corresponding to the torsion-free class $\mathcal{F}$ is the class $\operatorname{Gen}\mathbf{t}$ of all *torsion* modules.
Note that the dual $D({}_{R}\mathbf{W})$ of the cotilting module ${}_{R}\mathbf{W}$ is not tilting as it does not satisfy condition (T2). Indeed, G and the adic modules are summands of $D({}_{R}\mathbf{W})$, but no countable direct sum of copies of an adic module belongs to $G^{\perp}$, see [28, Prop.1 and Remark on p.265]. □

Next, let us recall Schofield's notion of universal localization [33, Theorem 4.1].

Theorem 1.5. Let Σ be a set of morphisms between finitely generated projective right R-modules. Then there are a ring R_{Σ} and a morphism of rings $\lambda\colon R \to R_{\Sigma}$ such that

(1) λ is Σ-*inverting,* i.e. if $\alpha\colon P \to Q$ belongs to Σ, then $\alpha \otimes_R 1_{R_\Sigma}\colon P \otimes_R R_{\Sigma} \to Q \otimes_R R_{\Sigma}$ is an isomorphism of right R_{Σ}-modules, and
(2) λ is *universal* Σ-*inverting*, i.e. if S is a ring such that there exists a Σ-inverting morphism $\psi\colon R \to S$, then there exists a unique morphism of rings $\bar{\psi}\colon R_{\Sigma} \to S$ such that $\bar{\psi}\lambda = \psi$.

The morphism $\lambda\colon R \to R_{\Sigma}$ is a ring epimorphism with $\operatorname{Tor}_1^R(R_{\Sigma}, R_{\Sigma}) = 0$. It is called the *universal localization of R at Σ.*

Let now $\mathcal{U}$ be a set of finitely presented right R-modules of projective dimension at most one. For each $U \in \mathcal{U}$, consider a morphism α_U between finitely generated projective right R-modules such that

$$0 \to P \xrightarrow{\alpha_U} Q \to U \to 0$$

We will denote by $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$ the universal localization of R at the set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$, and we will call it the *universal localization of R at $\mathcal{U}$*. Note that $R_{\mathcal{U}}$ does not depend on the choice of Σ.

Example 1.6. *Tilting modules arising from universal localization.* Let now $\mathcal{U} \subset \mathbb{U}$ be a set of quasi-simple modules. Then, as shown in [5, 4.7], the module

$$T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$$

is a tilting module with tilting class $\mathcal{U}^{\perp}$. In particular, if $\mathcal{U} = \mathbb{U}$, then $T_{\mathcal{U}}$ is equivalent to $\mathbf{W}$. □

More generally, if $\mathcal{U}$ is a union of cliques, then $R_{\mathcal{U}}$ is a torsion-free module, and $R_{\mathcal{U}}/R$ is a direct sum of the Prüfer modules corresponding to the quasi-simples in $\mathcal{U}$, as we are going to see below (compare also [38, 2.4]). We first collect some facts on universal localization which we will also need later.

Proposition 1.7. Let $\mathcal{U}$ be a set of quasi-simple modules and let $\mathcal{W}$ be the extension closure of $\mathcal{U}$. Let further $(\mathcal{T}_{\mathcal{U}},\mathcal{U}^o)$ be the torsion pair generated by $\mathcal{U}$, and let t be the corresponding torsion radical. The following statements hold true.

(1) $\mathcal{W}$ is a full exact abelian subcategory of $\operatorname{mod} R$.
(2) $R' = R_{\mathcal{U}}$ coincides with $R_{\mathcal{W}}$, the universal localization of R at $\mathcal{W}$.
(3) The torsion pair $(\mathcal{T}_{\mathcal{U}},\mathcal{U}^o)$ generated by $\mathcal{U}$ coincides with the torsion pair generated by $\mathcal{W}$.
(4) $\mathcal{U}^{\wedge} = \mathcal{W}^{\wedge}$ is the essential image of the restriction functor $\operatorname{Mod} R' \to \operatorname{Mod} R$.
(5) $\mathcal{T}_{\mathcal{U}} = \operatorname{Gen} \mathcal{W} = \{X \in \operatorname{Mod} R \mid X \otimes_R R' = 0\}$.
(6) R'/R is a directed union of finite extensions of modules in $\mathcal{U}$.
(7) For every $A \in \operatorname{Mod} R$ there is a short exact sequence

$$0 \to A/tA \to A \otimes_R R' \to A \otimes_R R'/R \to 0$$

where $A \otimes_R R' \in \mathcal{U}^{\wedge}$ and $A \otimes_R R'/R \in {}^{\wedge}(\mathcal{U}^{\wedge}) = \mathcal{T}_{\mathcal{U}} \cap {}^{\perp}(\mathcal{U}^{\perp})$.

Proof: (1), (2) We adopt Schofield's terminology from [36]. Since $\mathcal{U}$ is a Hom-perpendicular set, $\mathcal{W}$ is well-placed, cf. [36, p.4]. Then $\mathcal{W} = {}^{\wedge}(\mathcal{U}^{\wedge}) \cap \operatorname{mod} R$ is the well-placed closure of $\mathcal{U}$, and $R_{\mathcal{U}} = R_{\mathcal{W}}$, cf. [36, 2.3].
(3), (4) We claim $\mathcal{U}^o = \mathcal{W}^o$. The inclusion '$\supset$' follows from $\mathcal{U} \subset \mathcal{W}$. Conversely, ${}^o(\mathcal{U}^o)$ contains $\mathcal{U}$, and also its extension closure $\mathcal{W}$, hence $\mathcal{U}^o \subset \mathcal{W}^o$. Similarly, we prove $\mathcal{U}^{\perp} = \mathcal{W}^{\perp}$. We then deduce $\mathcal{U}^{\wedge} = \mathcal{W}^{\wedge}$. For the second statement see [1, 1.7].
(5) $\{X \in \operatorname{Mod} R \mid X \otimes_R R' = 0\}$ is closed under extensions, direct sums and epimorphic images, hence it is a torsion class containing $\mathcal{U}$ and thus also $\mathcal{T}_{\mathcal{U}}$, which in turn contains $\operatorname{Gen} \mathcal{W}$. The converse inclusions follow from [35, 5.1 and 5.5].
(6) is a consequence of [33, Theorem 12.6], [34, Theorem 3] and [13, Lemma 4.4]. Another proof can be found in [36, Theorem 2.6].
(7) is contained in [24, page 7]. We give a direct proof for the reader's convenience. Applying $A \otimes_R -$ on the short exact sequence $0 \to R \to R' \to R'/R \to 0$, we obtain an exact sequence $A \to A \otimes_R R' \to A \otimes_R R'/R \to 0$, which gives rise to the short exact sequence $0 \to A/tA \to A \otimes_R R' \to A \otimes_R R'/R \to 0$ because tA is the kernel of the canonical map $A \to A \otimes_R R'$, cf. (5) and [35, 5.5]. Since $A \otimes_R R'$ is an R'-module, it follows from (4) that $A \otimes_R R' \in \mathcal{U}^{\wedge}$.
Let us show that $A \otimes_R R'/R \in {}^{\wedge}(\mathcal{U}^{\wedge})$. Take $M \in \mathcal{U}^{\wedge}$. First of all, note that $A \otimes_R R'/R$ is generated by R'/R, which belongs to $\mathcal{T}_{\mathcal{U}}$ by (5) and (6). Thus $A \otimes_R R'/R \in \mathcal{T}_{\mathcal{U}}$, and since $M \in \mathcal{U}^o$, we have $\operatorname{Hom}_R(A \otimes R'/R, M) = 0$.

Next, note that $A/tA \otimes_R R' \cong A \otimes_R R'$ by (5). Then $\mathrm{Hom}_R(A \otimes_R R', M) \cong \mathrm{Hom}_R(A/tA, M)$ because M is an R'-module by (4), and we have the exact sequence

$$0 \to \mathrm{Ext}^1_R(A \otimes_R R'/R, M) \to \mathrm{Ext}^1_R(A \otimes_R R', M) \xrightarrow{\psi} \mathrm{Ext}^1_R(A/tA, M) \to 0$$

Now, if we prove that ψ is injective, we obtain that $\mathrm{Ext}^1_R(A \otimes_R R'/R, M) = 0$, as desired.
Given an extension

$$\epsilon : 0 \longrightarrow M \longrightarrow X \longrightarrow A \otimes_R R' \longrightarrow 0,$$

its image under ψ is given by pullback

$$\begin{array}{lccccccc} \epsilon : & 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & A \otimes_R R' \longrightarrow 0 \\ & & & \| & & \uparrow & & \uparrow \\ \psi(\epsilon) : & 0 & \longrightarrow & M & \longrightarrow & Z & \longrightarrow & A/tA \longrightarrow 0 \end{array}$$

Observe that X is an R'-module because $\mathcal{U}^\wedge$ is closed under extensions. Therefore we obtain the commutative diagram in $\mathrm{Mod}\, R'$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & A \otimes_R R' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \cong \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & Z \otimes_R R' & \longrightarrow & A/tA \otimes_R R' & \longrightarrow & 0 \end{array}$$

showing that ϵ is uniquely determined by $\psi(\epsilon)$. Thus ψ is injective.
Finally, we show that ${}^\wedge(\mathcal{U}^\wedge) = \mathcal{T}_\mathcal{U} \cap {}^\perp(\mathcal{U}^\perp)$. The inclusion '$\supset$' follows from the definition. For '$\subset$', consider $X \in {}^\wedge(\mathcal{U}^\wedge)$. If $A \in \mathcal{U}^o$, then there is an embedding $0 \to A \to A \otimes_R R' \in \mathcal{U}^\wedge$, hence $0 \to \mathrm{Hom}_R(X, A) \to \mathrm{Hom}_R(X, A \otimes_R R') = 0$, which proves $X \in {}^o(\mathcal{U}^o) = \mathcal{T}_\mathcal{U}$. Moreover, if $A \in \mathcal{U}^\perp$, then by [5, 4.7] there is an epimorphism $R'^{(\alpha)} \to A \to 0$ where $R'^{(\alpha)} \in \mathcal{U}^\wedge$, thus $0 = \mathrm{Ext}^1_R(X, R'^{(\alpha)}) \to \mathrm{Ext}^1_R(X, A) \to 0$, showing $X \in {}^\perp(\mathcal{U}^\perp)$. □

Proposition 1.8. Let $\mathcal{U} \subseteq \mathbb{U}$ be a set of quasi-simple modules. Set $\alpha_U = \dim_k \mathrm{Ext}^1_R(U, R)$ for each $U \in \mathcal{U}$. Then $R_\mathcal{U}$ is torsion-free, and the following statements hold true.

(1) If $\mathcal{U}$ is a union of cliques, then

$$R_\mathcal{U}/R \cong \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\alpha_U)}.$$

In particular, if $\mathcal{U} = \mathbb{U}$, then

$$T_\mathcal{U} = G^{(\alpha)} \oplus \Big(\bigoplus_{U \in \mathbb{U}} U[\infty]^{(\alpha_U)}\Big),$$

where $\alpha = \dim_{\mathrm{End}_R G} G$. In fact, $R_\mathbb{U}$ is a simple artinian ring isomorphic to the ring of $\alpha \times \alpha$-matrices over the division ring $\mathrm{End}_R G$, and G is the unique indecomposable $R_\mathbb{U}$-module. Moreover, $\alpha = -\delta(R)$ equals the negative of the defect $\delta(R)$ of R, cf. [31, 5.6].

(2) Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, let $\mathcal{U} = \{U_1, \ldots, U_m\} \subseteq \mathbb{U}$ be a set of $m < r$ quasi-simples in $\mathbf{t}_\lambda$ where $U_{i+1} = \tau^- U_i$ for all $1 \leq i < m$. Then $R_\mathcal{U}/R$ is a direct sum of modules on the coray ending at U_m. More precisely,

$$R_\mathcal{U}/R \cong U_1[m]^{(\alpha_{U_1})} \oplus U_2[m-1]^{(\alpha_{U_2})} \oplus \cdots \oplus U_m^{(\alpha_{U_m})}.$$

Proof: Let $\mathcal{U} \subseteq \mathbb{U}$. First of all, we show that $R_{\mathcal{U}}/R$ is a torsion regular module, that is, it belongs to $\operatorname{Gen}\mathbf{t}$ and has no summands in $\mathbf{p} \cup \mathbf{q}$. By Proposition 1.7(6), we can write $R_{\mathcal{U}}/R$ as a directed union $\varinjlim N_i$ with the N_i's finite extensions of elements in $\mathcal{U}$. Then $R_{\mathcal{U}}/R$ is a torsion module. Moreover, if $P \in \mathbf{p}$ (respectively, $Q \in \mathbf{q}$) were a direct summand of $R_{\mathcal{U}}/R$, then P (respectively, Q) would be a direct summand of some regular module N_i, a contradiction.

Next, we observe that for any module V in the extension closure of $\mathcal{U}$, applying $\operatorname{Hom}_R(V,-)$ to the exact sequence $0 \longrightarrow R \longrightarrow R_{\mathcal{U}} \longrightarrow R_{\mathcal{U}}/R \longrightarrow 0$ we obtain from Proposition 1.7(4) that

$$\operatorname{Hom}_R(V, R_{\mathcal{U}}/R) \cong \operatorname{Ext}^1_R(V,R). \tag{1}$$

We now prove the statements.

(1) Suppose that $\mathcal{U}$ is a union of cliques. Then $R_{\mathcal{U}}/R$ is divisible. Indeed, if V is a quasi-simple not in $\mathcal{U}$, then $\operatorname{Ext}^1_R(V, R_{\mathcal{U}}/R) = \varinjlim \operatorname{Ext}^1_R(V,N_i) = 0$. On the other hand, if $U \in \mathcal{U}$, then $\operatorname{Ext}^1_R(U, R_{\mathcal{U}}/R) = 0$ because $T_{\mathcal{U}} = R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module with tilting class $\mathcal{U}^\perp$. Since any divisible torsion regular module is a direct sum of Prüfer modules by [31, 4.5, Lemma 3], we infer that $R_{\mathcal{U}}/R$ is a direct sum of Prüfer modules.

Observe that for $U, V \in \mathbb{U}$ we have

$$\operatorname{Ext}^1_R(U[\infty], \tau V) \cong D\operatorname{Hom}_R(V, U[\infty]) = 0 \text{ iff } V \neq U.$$

So, if $V \in \mathbb{U} \setminus \mathcal{U}$, then the fact that $\tau V \in \mathcal{U}^\perp = (R_{\mathcal{U}}/R)^\perp$ implies that the Prüfer module $V[\infty]$ cannot occur as a direct summand of $R_{\mathcal{U}}/R$. Similarly, if $U \in \mathcal{U}$, then clearly $\tau U \notin \mathcal{U}^\perp = (R_{\mathcal{U}}/R)^\perp$, thus the Prüfer module $U[\infty]$ must be a direct summand of $R_{\mathcal{U}}/R$. Therefore

$$R_{\mathcal{U}}/R \cong \bigoplus_{U \in \mathcal{U}} U[\infty]^{(\beta_U)},$$

for some cardinals β_U. Recall now that $\operatorname{End}_R(U) = k$ for $U \in \mathbb{U}$. Then, for a fixed $U \in \mathcal{U}$, the number of direct summands of $R_{\mathcal{U}}/R$ isomorphic to $U[\infty]$ coincides with $\dim_k \operatorname{Hom}_R(U, R_{\mathcal{U}}/R)$, which by (1) equals α_U. Therefore $\beta_U = \alpha_U$ for all $U \in \mathcal{U}$, as desired.

Now, if $\mathcal{U} = \mathbb{U}$, then the fact that $R_{\mathbb{U}}$ is an $R_{\mathbb{U}}$-module yields $\operatorname{Hom}_R(\mathbb{U}, R_{\mathbb{U}}) = \operatorname{Ext}^1_R(\mathbb{U}, R_{\mathbb{U}}) = 0$, that is, $R_{\mathbb{U}}$ is a torsion-free divisible module. Then [31, 5.4 and 5.6] imply that $R_{\mathbb{U}}$ is a direct sum of $\alpha = -\delta(R)$ copies of G. The ring $R_{\mathbb{U}}$ is simple artinian by [13, Lemma 4.4]. The R-module G is an $R_{\mathbb{U}}$-module because it is a torsion-free divisible R-module, and it is the only simple $R_{\mathbb{U}}$-module because it is indecomposable over R and hence over $R_{\mathbb{U}}$. Now $R_{\mathbb{U}} \cong \operatorname{End}_{R_{\mathbb{U}}} R_{\mathbb{U}} \cong \operatorname{End}_R R_{\mathbb{U}} \cong \operatorname{End}_R(G^{(\alpha)}) \cong M_\alpha(\operatorname{End}_R G)$, and $\operatorname{End}_R G$ is a division ring by [31, 5.3]. Finally, $\alpha = \dim_{\operatorname{End}_R G} G$ by the Theorem of Wedderburn-Artin.

In the general case, we have that $R_{\mathcal{U}}$ is torsion-free because $R_{\mathcal{U}} \subset R_{\mathbb{U}}$, cf. 1.10.

(2) Suppose that $\mathcal{U}$ is defined as in (2). Note that the N_i's are finite direct sums of elements in the wing $\mathcal{W}_{U_1[m]}$ of $\mathcal{U}$, see Lemma 1.1. Set $Y = U_1 \oplus \cdots \oplus U_m$. By (1) we get that $\operatorname{Hom}_R(Y, R_{\mathcal{U}}/R) \cong \operatorname{Ext}^1_R(Y,R)$, which implies that $\dim_k \operatorname{Hom}_R(Y, R_{\mathcal{U}}/R) < \infty$. Therefore the directed union $R_{\mathcal{U}}/R = \bigcup N_i$ is finite, which means that $R_{\mathcal{U}}/R$ is a finite direct sum of elements in $\mathcal{W}_{U_1[m]}$. In particular $R_{\mathcal{U}}/R$ and $R_{\mathcal{U}}$ are finite dimensional over k (this is well known, see [13, Theorem 4.2] and [34, Theorem 13] or [17, 10.1]).

From $\bigoplus_{i=1}^m \operatorname{Hom}_R(U_i, R_{\mathcal{U}}/R) \cong \operatorname{Hom}_R(Y, R_{\mathcal{U}}/R) \cong \operatorname{Ext}^1_R(Y,R) \cong \bigoplus_{i=1}^m \operatorname{Ext}^1_R(U_i,R)$ we get that the number of indecomposable direct summands of $R_{\mathcal{U}}/R$ equals $\alpha_{U_1} + \cdots + \alpha_{U_m}$.

Let us consider the modules on the coray ending at $U_m \in \mathcal{U}$. First of all, we infer from (1) that $\mathrm{Hom}_R(U_m, R_{\mathcal{U}}/R) \cong \mathrm{Ext}^1_R(U_m, R)$, which shows that there are α_{U_m} summands isomorphic to U_m. Now we proceed by recurrence.

Let $U_{m-i}[i+1] \in \mathcal{U}$. Suppose that we have proved that $\dim_k \mathrm{Ext}^1_R(U_{m-i+1}[i], R) = \alpha_{U_m} + \cdots + \alpha_{U_{m-i+1}}$. From the exact sequence

$$0 \longrightarrow U_{m-i} \longrightarrow U_{m-i}[i+1] \longrightarrow U_{m-i+1}[i] \longrightarrow 0$$

we obtain $0 \longrightarrow \mathrm{Ext}^1_R(U_{m-i+1}[i], R) \longrightarrow \mathrm{Ext}^1_R(U_{m-i}[i+1], R) \longrightarrow \mathrm{Ext}^1_R(U_{m-i}, R) \longrightarrow 0$, which implies that $\dim_k \mathrm{Ext}^1_R(U_{m-i}[i+1], R) = \alpha_{U_m} + \cdots + \alpha_{U_{m-i+1}} + \alpha_{U_{m-i}}$. Again from (1), it also implies that $\dim_k \mathrm{Hom}_R(U_{m-i}[i+1], R_{\mathcal{U}}/R) = \dim_k \mathrm{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R) + \dim_k \mathrm{Hom}_R(U_{m-i}, R_{\mathcal{U}}/R)$. Now observe that the number of direct summands of $R_{\mathcal{U}}/R$ isomorphic to $U_{m-i}[i+1]$ equals $\dim_k \mathrm{Hom}_R(U_{m-i}[i+1], R_{\mathcal{U}}/R) - \dim_k \mathrm{Hom}_R(U_{m-i+1}[i], R_{\mathcal{U}}/R)$. Thus the foregoing shows that it equals $\alpha_{U_{m-i}}$. □

Remark 1.9. Let the notation be as in Proposition 1.8. By [31, Lemma and Corollary in 6.4], we have $-\delta(R) = \sum_{U \in \mathbb{U} \cap \mathbf{t}_\lambda} \alpha_U$ where $\mathbf{t}_\lambda$ is an arbitrary tube. Thus $\alpha_U \leq -\delta(R)$, with equality exactly when $\mathbf{t}_\lambda$ is a homogeneous tube.

Here are some further results on universal localization that will be needed in Sections 4 and 5.

Proposition 1.10. [13, 34, 17]

(1) Let $\mathcal{Y}$ be a set of quasi-simple modules, and set $\tau\mathcal{Y} = \{\tau V \mid V \in \mathcal{Y}\}$.
   (a) If $S \in \mathbb{U} \setminus (\mathcal{Y} \cup \tau\mathcal{Y})$, then $S \otimes_R R_{\mathcal{Y}} \cong S$.
   (b) For $S \in \tau\mathcal{Y} \setminus \mathcal{Y}$ there exists $2 \leq m \leq r$ such that $S \otimes_R R_{\mathcal{Y}} \cong S[m]$.

(2) Assume that $\mathcal{U} \subset \mathbb{U}$ is a set of quasi-simple R-modules that contains no complete cliques. Then the universal localization $R_{\mathcal{U}}$ is a tame hereditary k-algebra with $\mathrm{rk}\, K_0(R_{\mathcal{U}}) = \mathrm{rk}\, K_0(R) - |\mathcal{U}|$. Moreover, $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathbb{U} \setminus \mathcal{U}\}$ is a complete irredundant set of quasi-simple $R_{\mathcal{U}}$-modules, and $\{S[\infty] \mid S \in \mathbb{U} \setminus \mathcal{U}\}$, is a complete irredundant set of Prüfer $R_{\mathcal{U}}$-modules. We have $(S \otimes_R R_{\mathcal{U}})[\infty] = S[\infty]$ for each $S \in \mathbb{U} \setminus \mathcal{U}$.

   Furthermore,

$$\mathbf{t}_{\mathcal{U}} = \{V \otimes_R R_{\mathcal{U}} \mid V \in \mathbf{t} \text{ with } \mathrm{Hom}_R(V, U) = \mathrm{Hom}_R(U, V) = 0 \text{ for all } U \in \mathcal{U}\}$$

   is a complete irredundant set of finite dimensional indecomposable regular $R_{\mathcal{U}}$-modules.

   In particular, if $\mathbf{t}_\lambda$ is a tube of rank $r > 1$ with quasi-simples U_1, $U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$, and $\mathcal{U} = \{U_2, \ldots, U_{m+1}\}$ for some $m < r$, then the tube $\mathbf{t}_\lambda \otimes R_{\mathcal{U}}$ in the Auslander-Reiten quiver of $R_{\mathcal{U}}$ is given by the quasi-simple $R_{\mathcal{U}}$-modules

$$U_1 \otimes_R R_{\mathcal{U}},\ \tau^-(U_1 \otimes R_{\mathcal{U}}) = U_{m+2} \otimes_R R_{\mathcal{U}},\ \ldots\ , \tau^-(U_{r-1} \otimes_R R_{\mathcal{U}}) = U_r \otimes_R R_{\mathcal{U}}.$$

(3) Assume that $\mathcal{V} \subset \mathbb{U}$ is a set of quasi-simple R-modules that contains a complete clique. Then $R_{\mathcal{V}}$ is a hereditary order. Moreover, $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is a complete irredundant set of simple $R_{\mathcal{V}}$-modules, and $\{S[\infty] \mid S \in \mathbb{U} \setminus \mathcal{V}\}$, is a complete irredundant set of injective envelopes of simple $R_{\mathcal{V}}$-modules. We have $E(S \otimes_R R_{\mathcal{V}}) = S[\infty]$ for each $S \in \mathbb{U} \setminus \mathcal{V}$.

(4) Assume that $\mathcal{U} \subset \mathbb{U}$ and $\mathcal{V} \subseteq \mathbb{U} \setminus \mathcal{U}$. Then $R_{\mathcal{U} \cup \mathcal{V}} = (R_{\mathcal{U}})_{\mathcal{V}'}$ where $\mathcal{V}' = \{V \otimes_R R_{\mathcal{U}} \mid V \in \mathcal{V}\}$. In particular, there is an injective ring epimorphism $R_{\mathcal{U}} \to R_{\mathbb{U}}$.

Proof: (1) (a) If $S \in \mathbb{U} \setminus (\mathcal{Y} \cup \tau\mathcal{Y})$, then $\mathrm{Hom}_R(V,S) = \mathrm{Ext}^1_R(V,S) = \mathrm{Hom}_R(S,\tau V) = 0$ for all $V \in \mathcal{Y}$. That is, S is an $R_{\mathcal{Y}}$-module, and therefore $S \otimes_R R_{\mathcal{Y}} = S$.
(b) Let $S \in \tau\mathcal{Y} \setminus \mathcal{Y}$. Then S belongs to a tube $\mathbf{t}_\lambda$ of rank $r > 1$. Choose the numbering $S = U_1$, $U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$. By assumption there is $2 \leq m \leq r$ such that $U_2, \ldots, U_m \in \mathcal{Y}$ and $U_{m+1} \notin \mathcal{Y}$.
For each $p = 1, \ldots, m-1$, the exact sequence $0 \to S[p] \to S[p+1] \to U_{p+1} \to 0$ induces

$$\cdots \to \mathrm{Tor}^R_1(U_{p+1}, R_{\mathcal{Y}}) \to S[p] \otimes_R R_{\mathcal{Y}} \to S[p+1] \otimes_R R_{\mathcal{Y}} \to U_{p+1} \otimes_R R_{\mathcal{Y}} \to 0.$$

Clearly $\mathrm{Tor}^R_1(U_{p+1}, R_{\mathcal{Y}}) = U_{p+1} \otimes_R R_{\mathcal{Y}} = 0$ as $U_{p+1} \in \mathcal{Y}$. Hence we obtain that $S \otimes_R R_{\mathcal{Y}} \cong S[2] \otimes_R R_{\mathcal{Y}} \cong \ldots \cong S[m] \otimes_R R_{\mathcal{Y}}$. Note that $S[m]$ is an $R_{\mathcal{Y}}$-module because $\mathrm{Hom}_R(\mathcal{Y}, S[m]) = 0$ and $\mathrm{Ext}^1_R(\mathcal{Y}, S[m]) \cong D\,\mathrm{Hom}_R(S[m], \tau\mathcal{Y}) = 0$. Thus $S[m] \otimes_R R_{\mathcal{Y}} \cong S[m]$ as desired.
(2) The first statement is [13, Theorem 4.2(1)]. The shape of the quasi-simple and the regular $R_{\mathcal{U}}$-modules follows from [34, Theorem 10] (cf. [36, Theorem 3.5]), as noted in [13, 2.3, 2.4, Section 4]. See also [17, 10.1]. The statement on $\mathbf{t}_\lambda \otimes R_{\mathcal{U}}$ is shown in [13, Section 4.2].
It remains to prove the statement on the Prüfer modules. Let S be a quasi-simple R-module not in $\mathcal{U}$. Since Prüfer modules are divisible, $S[\infty]$ is a right $R_{\mathcal{U}}$-module. Further, if S belongs to the R-tube $\mathbf{t}_\lambda$, $S[\infty]$ is filtered by the quasi-simple $R_{\mathcal{U}}$-modules $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathbf{t}_\lambda \setminus \mathcal{U}\}$ by (1). Then $\{(S \otimes_R R_{\mathcal{U}})[n] \mid n \in \mathbb{N}\}$ is a ray on the $R_{\mathcal{U}}$-tube $\mathbf{t}_\lambda \otimes R_{\mathcal{U}}$, and $S[\infty] = \varinjlim_n (S \otimes_R R_{\mathcal{U}})[n]$.
(3) By [13, 4.2], $R_{\mathcal{V}}$ is a hereditary order, and by [13, Section 3] (or [5, 6.5]) and [34, Theorem 10], the set $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is an irredundant set of simple $R_{\mathcal{V}}$-modules.
Let $S \in \mathbb{U} \setminus \mathcal{V}$, and suppose that S belongs to the R-tube $\mathbf{t}_\lambda$. By (1), $S[\infty]$ is filtered by the simple $R_{\mathcal{V}}$-modules $U \otimes_R R_{\mathcal{V}}$, where U runs through the quasi-simple modules in $\mathbf{t}_\lambda \setminus \mathcal{V}$. By [34, Theorem 10], there exists an equivalence of categories from the category of bound R-modules M such that

$$\mathrm{Hom}_R(M,V) = \mathrm{Hom}_R(V,M) = 0 \text{ for all } V \in \mathcal{V} \tag{2}$$

to the category of bound $R_{\mathcal{V}}$-modules that restricts to an equivalence from the category of regular R-modules satysfying (2) to the category of torsion $R_{\mathcal{V}}$-modules. Thus $S[\infty]$ is a uniserial $R_{\mathcal{V}}$-module that contains $S \otimes_R R_{\mathcal{V}}$, and the injective envelope $E(S \otimes_R R_{\mathcal{V}})$ of $S \otimes_R R_{\mathcal{V}}$ has to contain $S[\infty]$. But by [19, Theorem 19(c)], $E(S \otimes_R R_{\mathcal{V}})$ is also uniserial and has the same filtration as $S[\infty]$, so they must coincide.
(4) is shown in [13, 2.4] as a consequence of [33] and [34]. □

2. Parametrizing tilting modules.

Tilting classes are in one-to-one-correspondence with certain subcategories of $\mathrm{mod}\,R$. Recall that a subcategory $\mathcal{S} \subset \mathrm{mod}\,R$ is said to be *resolving* provided $\mathcal{S}$ is closed under direct summands, extensions, and kernels of epimorphisms, and R belongs to $\mathcal{S}$. Observe that, since R is hereditary, a subcategory $\mathcal{S} \subset \mathrm{mod}\,R$ is resolving whenever it is closed under direct summands and extensions and contains R, see [4, 1.1].

Bazzoni and Herbera proved in [8] that every tilting class $\mathcal{B} = T^{\perp}$ is determined by a class of finitely presented modules. More precisely, $\mathcal{B} = \mathcal{S}^{\perp}$ where $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. Combining this with [4, Theorem 2.2] and [38, Theorem 4.14] we obtain

Theorem 2.1. (1) The tilting classes in $\operatorname{Mod} R$ correspond bijectively to the resolving subcategories of $\operatorname{mod} R$. The correspondence is given by the mutually inverse assignments

$$\alpha : \mathcal{B} \mapsto {}^{\perp}\mathcal{B} \cap \operatorname{mod} R \quad \text{and} \quad \beta : \mathcal{S} \mapsto \mathcal{S}^{\perp}$$

(2) The cotilting classes in $R\operatorname{Mod}$ correspond bijectively to the resolving subcategories of $\operatorname{mod} R$. The correspondence is given by the mutually inverse assignments

$$\gamma : \mathcal{C} \mapsto {}^{\intercal}\mathcal{C} \cap \operatorname{mod} R \quad \text{and} \quad \delta : \mathcal{S} \mapsto \mathcal{S}^{\intercal} = {}^{\perp}(\mathcal{S}^{*})$$

(3) The above correspondences yield a one-to-one-correspondence between tilting classes in $\operatorname{Mod} R$ and cotilting classes in $R\operatorname{Mod}$.

Remark 2.2. (1) $\alpha, \beta, \gamma, \delta$ are order-reversing: If $\mathcal{B}_1, \mathcal{B}_2$ are two tilting classes with $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\alpha(\mathcal{B}_2) \subset \alpha(\mathcal{B}_1)$, and the analogous property holds for the remaining assignments.
(2) Any resolving subcategory of $\operatorname{mod} R$ is closed under submodules, since it occurs as ${}^{\perp}\mathcal{B} \cap \operatorname{mod} R$ for some class $\mathcal{B} \subset \operatorname{Mod} R$ consisting of modules of injective dimension at most one.
(3) Let $\mathcal{S}$ be a subcategory of $\operatorname{mod} R$ containing R, and assume that $\mathcal{S}$ is closed under predecessors, that is, if $X \in \operatorname{mod} R$ is an indecomposable module with a nonzero map $X \to S$ to a module $S \in \mathcal{S}$, then $X \in \mathcal{S}$. Then it is easy to see that $\mathcal{S}$ is resolving.

In particular we have the following examples:

Example 2.3. The category $\operatorname{add} \mathbf{p}$ is a resolving subcategory of $\operatorname{mod} R$ with $\beta(\operatorname{add} \mathbf{p}) = \mathbf{p}^{\perp} = \operatorname{Gen} \mathbf{L}$, and $\delta(\operatorname{add} \mathbf{p}) = {}^{\perp_R}\mathbf{q} = \operatorname{Cogen}_R \mathbf{W}$. □

Example 2.4. Let $\mathbf{t}'$ be a nonempty union of tubes, and let $\mathcal{U}$ be the set of quasi-simple modules in $\mathbf{t}'$. Then the category $\operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ is a resolving subcategory of $\operatorname{mod} R$ with $\beta(\operatorname{add}(\mathbf{p} \cup \mathbf{t}')) = \mathbf{t}'^{\perp} = \operatorname{Gen} T_{\mathcal{U}}$.
In fact, if $Z \in \mathbf{p}$ and S is quasi-simple, then there is a nonzero map from Z to the ray $\{S[n] \mid n \in \mathbb{N}\}$ defined by S, see [37, XII, 3.6]. So, by the Auslander-Reiten formula we deduce that the modules in $\mathbf{t}'^{\perp}$ cannot have direct summands in $\mathbf{p}$, and therefore $\mathbf{t}'^{\perp} \subset \mathbf{p}^{\perp}$ and $(\operatorname{add}(\mathbf{p} \cup \mathbf{t}'))^{\perp} = \mathbf{t}'^{\perp}$. This implies $\beta(\operatorname{add}(\mathbf{p} \cup \mathbf{t}')) = \mathcal{U}^{\perp} = \operatorname{Gen} T_{\mathcal{U}}$ by Example 1.6.
In particular, $\beta(\operatorname{add}(\mathbf{p} \cup \mathbf{t})) = \mathbf{t}^{\perp} = \operatorname{Gen} \mathbf{W}$. Moreover, with dual arguments one proves that $\delta(\operatorname{add}(\mathbf{p} \cup \mathbf{t})) = {}^{\perp}{}_R\mathbf{t} = {}_R\mathbf{t}^{o}$ is the category of all torsion-free left R-modules. □

We are now going to see that the examples above give a complete list of large tilting modules over the Kronecker algebra.

Lemma 2.5. Let T be a tilting R-module with tilting class $\mathcal{B} = T^{\perp}$, and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. Then $D(T)$ is a cotilting module with cotilting class ${}^{\perp}(\mathcal{S}^{*}) = \{{}_R X \mid D(X) \in \mathcal{B}\}$.

Proof: The well-known Ext-Tor relations yield ${}^{\perp}(\mathcal{S}^*) = \mathcal{S}^{\intercal}$ and ${}^{\perp}D(T) = T^{\intercal} = \{{}_RX \mid D(X) \in \mathcal{B}\}$. Now, if $D(X) \in \mathcal{B}$, and $\mathcal{A} = {}^{\perp}\mathcal{B}$, then $\operatorname{Ext}^1_R(\mathcal{A}, D(X)) = 0$, hence $\operatorname{Tor}^R_1(\mathcal{A}, X) = 0$, and in particular $X \in \mathcal{S}^{\intercal}$. Conversely, since T is a direct limit of modules from $\mathcal{S}$ by [38, 4.4], we have $\mathcal{S}^{\intercal} \subset T^{\intercal}$. So, we have shown ${}^{\perp}D(T) = {}^{\perp}(\mathcal{S}^*)$.
We now deduce that $D(T)$ is a cotilting module. In fact, the conditions (T1) and (T3) for T yield the dual conditions (C1) and (C3) for $D(T)$. Moreover, applying the Ext-Tor relations we obtain that $D(T) \in \mathcal{S}^{\intercal}$ since $T \in \mathcal{S}^{\perp}$. So, $D(T) \in {}^{\perp}D(T)$, and since ${}^{\perp}D(T) = \mathcal{S}^{\intercal}$ is closed under products, we infer $\operatorname{Ext}^1_R(D(T)^{\kappa}, D(T)) = 0$ for any cardinal κ, that is, the dual condition (C2) is also satisfied. □

Lemma 2.6. The following statements are equivalent for a tilting R-module T.

(1) T is equivalent to a finitely generated tilting module.
(2) $D(T)$ is equivalent to a finitely generated cotilting module.
(3) All indecomposable direct summands of $D(T)$ are finitely generated.

Proof: We will freely use the results on endofinite modules collected in Lemma 1.2.
(1) ⇒ (3): Let T' be a finite-dimensional tilting module equivalent to T. Clearly, $\operatorname{Add} T = \operatorname{Add} T'$ implies $\operatorname{Prod} D(T) = \operatorname{Prod} D(T')$. Then the indecomposable direct summands of $D(T)$ belong to $\operatorname{Prod} D(T') = \operatorname{Add} D(T')$, and are therefore isomorphic to indecomposable direct summands of $D(T')$.
(3) ⇒ (2): By a well-known result of Bongartz [9], the number of isoclasses of indecomposable direct summands of $D(T)$ is bounded by the number of isoclasses of simple R-modules, and $D(T)$ is equivalent to a finitely generated cotilting module.
(2) ⇒ (1): Let ${}_RC$ be a finite-dimensional cotilting module equivalent to $D(T)$. Then $D(T)$ belongs to $\operatorname{Prod} C = \operatorname{Add} C$, and is thus isomorphic to a direct sum of copies of a finite number of indecomposable finitely generated modules. In particular, this implies that $D(T)$ is endofinite. But then T is a pure submodule of the endofinite module $D^2(T)$ and is therefore a direct summand of $D^2(T)$ by [15, 4.3]. In particular, also T is isomorphic to a direct sum of copies of a finite number of indecomposable finitely generated modules, which proves (1). □

Theorem 2.7. Let T be a tilting R-module with tilting class $\mathcal{B} = T^{\perp}$, and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. Assume that T is not equivalent to a finitely generated tilting module. Then the following hold true.

(1) T is a regular module whose tilting class contains $\operatorname{Gen}\mathbf{W}$ and is contained in $\operatorname{Gen}\mathbf{L}$.
(2) There is a subset $\mathbf{t}' \subset \mathbf{t}$ such that $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$.
(3) If $\mathbf{t}' = \emptyset$, then T is equivalent to the Lukas tilting module L.
(4) If $\mathbf{t}' \neq \emptyset$ is a union of tubes, and $\mathcal{U}$ is the set of quasi-simple modules in $\mathbf{t}'$, then T is equivalent to $T_{\mathcal{U}}$.

Proof: By assumption and Lemma 2.6, the module ${}_RD(T)$ has an indecomposable direct summand M which is infinite dimensional. Observe that M is pure-injective as it is a summand of a dual module. From Lemma 1.2(6) and Lemma 2.5, we infer that $\mathcal{S}^*$ cannot contain modules from ${}_R\mathbf{p}$, hence $\mathcal{S}$ cannot contain modules from $\mathbf{q}$. Similarly, ${}^{\perp}D(T)$ cannot contain modules from ${}_R\mathbf{q}$, hence $\mathcal{B} = \mathcal{S}^{\perp}$ cannot contain modules from $\mathbf{p}$. But then $\mathcal{B} \subset \mathbf{p}^{\perp} = \operatorname{Gen}\mathbf{L}$, thus $\mathbf{p} \subset {}^{\perp}\mathcal{B} \cap \operatorname{mod} R = \mathcal{S}$. So,

T is a regular module, and we have verified (1) and (2). Now (3) and (4) follow immediately from Examples 2.3 and 2.4. □

Corollary 2.8. Over the Kronecker-algebra, every tilting module is equivalent to a module in the following list:

- the finitely generated tilting modules,
- the Lukas tilting module $\mathbf{L}$,
- the tilting modules of the form $T_{\mathcal{U}}$ for a set of quasi-simples $\mathcal{U}$.

Proof: Assume that T is not equivalent to a finitely generated tilting module. With the notation of Theorem 2.7, we note that $\mathbf{t}'$ can only contain modules from homogeneous tubes. Then, with any regular module $M \in \mathbf{t}'$, the resolving subcategory $\mathcal{S}$ contains also its regular socle S by Remark 2.2(2), and so it contains the whole (homogeneous) tube S belongs to. This shows that $\mathbf{t}'$ is a union of tubes, so the claim follows from Theorem 2.7. □

3. Decomposing tilting modules

Throughout this section, we fix a tilting R-module T with tilting class $\mathcal{B} = T^{\perp}$ and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. We assume that T is not equivalent to a finite dimensional tilting module. We prove a structure result for the modules in $\mathcal{B}$, from which we derive a canonical decomposition for T.

We are going to use two torsion pairs first studied by Ringel in [31]. The first is the split torsion pair whose torsion class is the class $\mathcal{D}$ of the divisible modules. We call a module *reduced* if it belongs to the corresponding torsion-free class $\mathcal{D}^o$. The second is the non-split torsion pair with torsion class $\operatorname{Gen}\mathbf{t}$. Here $(\operatorname{Gen}\mathbf{t})^o = \mathcal{F}$ is the class of all torsion-free modules, cf. 1.3 and 1.4.
We will further need the following canonical decomposition of the regular modules in $\operatorname{Gen}\mathbf{t}$. Write

$$\mathbf{t} = \bigcup_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda$$

where the $\mathbf{t}_\lambda$ are the tubes in the Auslander-Reiten quiver of R, and set $\mathcal{T}_\lambda = \varinjlim \operatorname{add} \mathbf{t}_\lambda$. For $X \in \operatorname{Mod} R$ denote by $\mathbf{t}_\lambda(X)$ the maximal submodule of X belonging to $\mathcal{T}_\lambda$. As shown in [31, 4.5], every regular module $X \in \operatorname{Gen}\mathbf{t}$ has a unique decomposition

$$X = \bigoplus_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda(X).$$

We will say that a Prüfer module $S[\infty]$ (or an adic module $S[-\infty]$) *belongs to a tube* $\mathbf{t}_\lambda$ if S is a quasi-simple module in (the mouth of) $\mathbf{t}_\lambda$.

Let us start by investigating the modules in the tilting class $\mathcal{B} = \operatorname{Gen} T$. Since $\mathcal{S}$ consists of finitely presented modules, the class $\mathcal{B}$ is *definable*, i. e., it is closed under direct limits, direct products, and pure submodules.

Proposition 3.1. Let $X \in \mathcal{B} = \mathcal{S}^{\perp}$. Then
(1) $X = X_{\mathcal{D}} \oplus X_{\mathrm{red}}$ where $X_{\mathcal{D}} \in \mathcal{D}$ is divisible, and X_{red} is reduced.
(2) There is a pure-exact sequence $0 \to X' \to X_{\mathrm{red}} \to \overline{X} \to 0$ where $\overline{X} \in \mathcal{B}$ is torsion-free, and $X' \in \operatorname{Gen}\mathbf{t}$.

(3) $X' = \bigoplus_{\lambda\in\mathbb{P}} X_\lambda$, and for each λ there is a pure-exact sequence $0 \to A_\lambda \to X_\lambda \to Z_\lambda \to 0$ where A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$, and $Z_\lambda \in \mathcal{B}$ is a direct sum of Prüfer modules belonging to the tube $\mathbf{t}_\lambda$.

Proof: For (1) and (2), we refer to [31, 4.7 and 4.1].
(3) Note that the torsion-free class of reduced modules is closed under submodules, and the tilting class $\mathcal{B} = \mathcal{S}^\perp$ is definable, hence closed under pure-submodules. So, we infer from (1) and (2) that X' is a reduced module in $\mathcal{B}$. Since preinjective modules are divisible, it follows that X' has no indecomposable summands from $\mathbf{q}$. Moreover, as shown in 2.7, the module $X' \in \mathcal{B}$ has no indecomposable summands from $\mathbf{p}$. Thus X' is a regular module in $\operatorname{Gen}\mathbf{t}$ and has a decomposition $X' = \bigoplus_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda(X')$ as above by [31, 4.5]. We set $X_\lambda = \mathbf{t}_\lambda(X')$. From [31, Theorem G and 4.8] we know that there is a pure-exact sequence $0 \to A_\lambda \to X_\lambda \to Z_\lambda \to 0$ where A_λ is a direct sum of indecomposable modules of finite length, and Z_λ has no indecomposable direct summand of finite length. Thus Z_λ is regular, and is therefore a direct sum of Prüfer modules. Again, we see that A_λ is a regular module in $\mathcal{B}$, and since $\operatorname{Hom}_R(\mathbf{t}_\nu, \mathcal{T}_\lambda) = \varinjlim \operatorname{Hom}_R(\mathbf{t}_\nu, \mathbf{t}_\lambda) = 0$ for $\nu \neq \lambda$, we infer that A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$. Similarly, we see that $Z_\lambda \in \mathcal{B}$ and that the Prüfer modules occurring as direct summands of Z_λ admit non-zero maps from $\mathbf{t}_\lambda$ and therefore belong to the tube $\mathbf{t}_\lambda$. □

We now turn to the class $\operatorname{Add} T$.

Lemma 3.2. Every finitely generated indecomposable module $X \in \operatorname{Add} T$ is a regular module from a non-homogeneous tube, and its regular length $m < r$ is bounded by the rank r of the tube.

Proof: Notice that X is even (isomorphic to) a direct summand of T (cf. Lemma 1.2), so it follows from Theorem 2.7(1) that X is a regular module, and there exist a tube $\mathbf{t}_\lambda$ and a quasi-simple module $S \in \mathbf{t}_\lambda$ such that $X = S[m]$. Now $0 = \operatorname{Ext}^1_R(X, X) \cong D\operatorname{Hom}(S[m], \tau S[m])$ implies that the tube $\mathbf{t}_\lambda$ has rank $r > 1$.
Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$. Recall that $\operatorname{Hom}_R(S[m], U_i[m-i+1]) \neq 0$ for all $1 \leq i \leq m$, where we suppose that $U_i = U_j$ whenever $i \equiv j \mod r$. Now, if $m \geq r$, we consider the module $S[m-r+1]$. Since it is a submodule of $X \in \mathcal{S}$ and $\mathcal{S}$ is closed under submodules, we have $S[m-r+1] \in \mathcal{S}$. On the other hand, $\operatorname{Ext}^1_R(S[m-r+1], S[m]) = D\operatorname{Hom}_R(S[m], U_r[m-r+1]) \neq 0$, contradicting the fact that $S[m] \in \operatorname{Add} T \subseteq \mathcal{S}^\perp$. So, we conclude that $m < r$. □

Since there are at most finitely many (at most three) non-homogeneous tubes, the Lemma above shows that there are at most finitely many non-isomorphic finitely generated indecomposable modules in $\operatorname{Add} T$. Let us denote by Y the direct sum of a complete irredundant set of such modules.
We can now refine the structure result in Proposition 3.1.

Proposition 3.3. Every module $X \in \operatorname{Add} T$ has a unique direct sum decomposition

$$X = \bigoplus_{\lambda\in\mathbb{P}} \mathbf{t}_\lambda(X) \oplus \overline{X}$$

where $\overline{X}$ is torsion-free and each $\mathbf{t}_\lambda(X)$ has a decomposition in torsion modules with local endomorphism ring. The indecomposable summands of $\mathbf{t}_\lambda(X)$ are isomorphic to direct summands of $\mathbf{t}_\lambda(T)$ and are either modules from $\mathbf{t}_\lambda$ or Prüfer modules belonging to $\mathbf{t}_\lambda$.
Moreover, every torsion (respectively, torsion-free) direct summand of X is a direct summand of the torsion part $\bigoplus_{\lambda\in\mathbb{P}}\mathbf{t}_\lambda(X)$ (respectively, of the torsion-free part $\overline{X}$).

Proof: Let $X \in \operatorname{Add} T$. We know from [31, 4.1] that there is a pure-exact sequence $0 \to X' \to X \to \overline{X} \to 0$ where $\overline{X} \in \mathcal{B}$ is torsion-free, and $X' \in \operatorname{Gen}\mathbf{t}$. Note that $X \in \operatorname{Add} T = {}^{\perp}\mathcal{B} \cap \mathcal{B}$, and ${}^{\perp}\mathcal{B}$ is closed under submodules, while $\mathcal{B}$ is closed under pure submodules because it is a definable class. So, we infer that $X' \in \operatorname{Add} T$ is a regular module in $\operatorname{Gen}\mathbf{t}$, which by [31, 4.5] has a decomposition $X' = \bigoplus_{\lambda\in\mathbb{P}}\mathbf{t}_\lambda(X)$. As in the proof of Proposition 3.1, we deduce from [31, Theorem G and 4.8] that for each λ there is a pure-exact sequence $0 \to A_\lambda \to \mathbf{t}_\lambda(X) \to Z_\lambda \to 0$ where A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$, and Z_λ is a direct sum of Prüfer modules belonging to the tube $\mathbf{t}_\lambda$. Again, we see that $A_\lambda \in \operatorname{Add} T$, which implies by Lemma 3.2 that A_λ has only finitely many non-isomorphic indecomposable direct summands. In particular, this shows that A_λ is endofinite, thus pure-injective (cf. Lemma 1.2), so the pure-exact sequence $0 \to A_\lambda \to \mathbf{t}_\lambda(X) \to Z_\lambda \to 0$ splits, and $\mathbf{t}_\lambda(X)$ is a direct sum of modules in $\mathbf{t}_\lambda \cap \operatorname{Add} T$ and Prüfer modules belonging to $\mathbf{t}_\lambda$. In particular, $\mathbf{t}_\lambda(X)$ has a decomposition in modules with local endomorphism ring.
We infer that $X' = \bigoplus_{\lambda\in\mathbb{P}}\mathbf{t}_\lambda(X)$, being a direct sum of modules isomorphic to indecomposable direct summands of Y or to Prüfer modules, belongs to $\operatorname{Add}(Y \oplus \mathbf{W})$. Now Y is finite dimensional and therefore Σ-pure-injective (that is, every direct sum of copies of Y is pure-injective), and $\mathbf{W}$ is Σ-pure-injective because $\operatorname{Add}\mathbf{W} = \operatorname{Prod}\mathbf{W}$, see [30, 10.1]. Thus X' is pure-injective, and the pure-exact sequence $0 \to X' \to X \to \overline{X} \to 0$ splits, that is, $X = \bigoplus_{\lambda\in\mathbb{P}}\mathbf{t}_\lambda(X) \oplus \overline{X}$ has the stated decomposition. The uniqueness of $\overline{X}$ and the $\mathbf{t}_\lambda(X)$ follows directly from torsion theory.
Now assume that A is a direct summand of $X = X' \oplus \overline{X}$. Then there are morphisms $\iota = (\iota', \overline{\iota}) : A \to X$ and $\pi = (\pi', \overline{\pi}) : X \to A$ such that $1_A = \pi\iota = \pi'\iota' + \overline{\pi}\,\overline{\iota}$, showing that A is a direct summand of X', respectively of $\overline{X}$, if A is torsion, respectively torsion-free.
In particular, each summand A of $\mathbf{t}_\lambda(X)$ belongs to $\operatorname{Add}\mathbf{t}_\lambda(T)$. As $\mathbf{t}_\lambda(T)$ has a decomposition in modules with local endomorphism ring, we deduce from the Theorem of Krull-Remak-Schmidt-Azumaya that A is isomorphic to an indecomposable direct summand of $\mathbf{t}_\lambda(T)$, see e.g. [22, 7.3.4]. □

Our next aim is to investigate the module Y. Of course, $Y = \bigoplus_{\lambda\in\mathbb{P}}\mathbf{t}_\lambda(Y)$ where $\mathbf{t}_\lambda(Y)$ is the direct sum of a complete irredundant set of modules in $\mathbf{t}_\lambda \cap \operatorname{Add} T$. Let us describe the modules $\mathbf{t}_\lambda(Y)$ in more detail.

Lemma 3.4. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let S be a quasi-simple module in $\mathbf{t}_\lambda$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \dots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$.

(1) If $\mathcal{S}$ contains some, but not all modules from the ray $\{S[n] \mid n \in \mathbb{N}\}$, then there is $m < r$ such that $S[m] \in \operatorname{Add} T$. More precisely, if $S[m]$ is the module of maximal regular length in $\mathcal{S} \cap \{S[n] \mid n \in \mathbb{N}\}$, then $S[m] \in \operatorname{Add} T$.
(2) If $S[m] \in \mathbf{t}_\lambda \cap \operatorname{Add} T$, then the rays starting at $U_2, \dots, U_{m+1}$ are not completely contained in $\mathcal{S}$. In fact, $U_2[m],\ U_3[m-1], \dots,\ U_{m+1} \notin \mathcal{S}$.
(3) If $S[m] \in \mathbf{t}_\lambda \cap \operatorname{Add} T$, then $\operatorname{Add} T$ contains precisely m modules belonging to the wing $\mathcal{W}_{S[m]}$.

Proof: (1) Assume that $S[r] \in \mathcal{S}$. We claim that $S[n] \in \mathcal{S}$ for all $n \geq 1$. For $\mathcal{S}$ is closed under submodules, thus $S[l] \in \mathcal{S}$ for all $1 \leq l \leq r$. If $n > r$, write $n = kr + l$ with $r \leq kr < n$ and $1 \leq l \leq r$ and consider the exact sequence $0 \longrightarrow S[kr] \longrightarrow S[n] \longrightarrow S[l] \longrightarrow 0$. Now the claim follows by induction on n since $\mathcal{S}$ is closed under extensions.

Thus there exists $m < r$ such that $S[m] \in \mathcal{S}$ and $S[m+1] \notin \mathcal{S}$. We prove that $S[m] \in \operatorname{Add} T$. For that it is enough to show that $S[m] \in \mathcal{S}^{\perp}$. Take a module $Z \in \mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$, w.l.o.g. assume that Z is indecomposable. If $Z \in \mathbf{p}$, then $\operatorname{Ext}^1_R(Z, S[m]) = D \operatorname{Hom}_R(\tau^- S[m], Z) = 0$. If $Z \in \mathbf{t}'$, we can assume w.l.o.g. that Z belongs to $\mathbf{t}_\lambda$. If $\operatorname{Ext}^1_R(Z, S[m]) = \operatorname{Hom}_R(S[m], \tau Z) \neq 0$, we would have $Z = U_{i+1}[m - i + 1 + l]$ for $1 \leq i \leq m$ and $0 \leq l$. But then the exact sequence $0 \longrightarrow U_1[i] \longrightarrow U_1[m + 1 + l] \longrightarrow U_{i+1}[m - i + 1 + l] \longrightarrow 0$ together with the fact that $\mathcal{S}$ is closed under extensions would imply that $S[m + 1 + l] \in \mathcal{S}$, contradicting the choice of m.

(2) All modules of regular length at most m on the coray ending at U_m are quotients of $S[m]$ and therefore belong to the tilting class $\mathcal{B}$. Hence the modules of regular length at most m on the coray ending at U_{m+1} cannot be in $\mathcal{S}$ by the AR-formula. This yields the claim, because these modules lie on the rays starting at $U_2, \ldots, U_{m+1}$.

(3) We proceed by induction on m. The result clearly holds for $m = 1$. Let $m > 1$. First of all, note that the modules $U_2[m-1], U_3[m-2], \ldots, U_m$ on the coray ending at U_m are in $\mathcal{S}^{\perp}$ because they are epimorphic images of $S[m]$.

Suppose that none of the modules $U_2[m-1], U_3[m-2], \ldots, U_m$ belongs to $\mathcal{S}$. Then no regular module containing any of these modules can belong to $\mathcal{S}$. On the other hand, for $X \in \mathbf{p} \cup \mathbf{t}$ we have $\operatorname{Ext}^1_R(X, S[m-1]) = D \operatorname{Hom}_R(U_2[m-1], X) \neq 0$ if and only if X is a regular module that contains one of the modules $U_2[m-1], U_3[m-2], \ldots, U_m$. Hence $S[m-1]$ belongs to $\mathcal{S}^{\perp}$, and as a submodule of $S[m] \in \mathcal{S}$ it also belongs to $\mathcal{S}$, therefore $S[m-1] \in \operatorname{Add} T$. So $\operatorname{Add} T$ contains precisely m modules in $\mathcal{W}_{S[m]}$: these are $S[m]$ and the $m-1$ modules in $\mathcal{W}_{S[m-1]}$ given by the induction hypothesis.

Suppose now that one of $U_2[m-1], U_3[m-2], \ldots, U_m$ belongs to $\mathcal{S}$. Choose $U_{i+1}[m-i] \in \mathcal{S}$ of maximal regular length. Then $U_{i+1}[m-i] \in \operatorname{Add} T$, and the induction hypothesis implies that $\operatorname{Add} T$ contains precisely $m - i$ modules in $\mathcal{W}_{U_{i+1}[m-i]}$.

Since $U_{i+1}[m-i]$ (and its submodules on the ray starting at U_{i+1}) are in $\mathcal{S}$, no module of regular length at most $m - i$ on the ray starting at U_i can belong to $\mathcal{S}^{\perp}$. This shows that $S[i], S[i+1], \ldots, S[m-1] \notin \mathcal{S}^{\perp}$. We claim that $S[i-1] \in \operatorname{Add} T$. To this end, we note that for $X \in \mathbf{p} \cup \mathbf{t}$ we have $\operatorname{Ext}^1_R(X, S[i-1]) = D \operatorname{Hom}_R(U_2[i-1], X) \neq 0$ if and only if X is regular and contains one of the modules $U_2[i-1], U_3[i-2], \ldots, U_i$ as a submodule. But none of $U_2[i-1], U_3[i-2], \ldots U_i$ can belong to $\mathcal{S}$. Indeed, this follows from the choice of $U_{i+1}[m-i]$, by using that each of the modules $U_2[m-1], U_3[m-2], \ldots, U_i[m-i+1] \notin \mathcal{S}$ can be written as an extension of one of the modules $U_2[i-1], U_3[i-2], \ldots, U_i$ by the module $U_{i+1}[m-i] \in \mathcal{S}$.

So, we infer that $S[i-1] \in \operatorname{Add} T$, and the induction hypothesis implies that $\operatorname{Add} T$ contains precisely $i - 1$ modules in $\mathcal{W}_{S[i-1]}$. We conclude that $\operatorname{Add} T$ contains precisely m modules in $\mathcal{W}_{S[m]}$: these are the $m - i$ modules in $\mathcal{W}_{U_{i+1}[m-i]}$, the $i - 1$ modules in $\mathcal{W}_{S[i-1]}$, and $S[m]$.□

The following result shows that the indecomposable summands of $\mathbf{t}_\lambda(Y)$ are arranged in disjoint wings, and that the union of such wings does not contain all quasi-simples from $\mathbf{t}_\lambda$.

Corollary 3.5. Let X, X' be two finitely generated indecomposable modules in $\operatorname{Add} T$, and let $\mathcal{W}_X, \mathcal{W}_{X'}$ be the corresponding wings. Then either $\mathcal{W}_X \subset \mathcal{W}_{X'}$ or $\mathcal{W}_{X'} \subset \mathcal{W}_X$ or $\mathcal{W}_X \cap \mathcal{W}_{X'} = \emptyset$. Moreover, given a tube $\mathbf{t}_\lambda$ of rank $r > 1$, the quasi-simple modules in the union of all wings $\mathcal{W}_X$ with $X \in \mathbf{t}_\lambda \cap \operatorname{Add} T$ do not form a complete clique, and there are at most $r-1$ isomorphism classes of modules in $\mathbf{t}_\lambda \cap \operatorname{Add} T$.

Proof: We can assume w.l.o.g. that X, X' belong to the same tube $\mathbf{t}_\lambda$. Let S, S' be quasi-simples in $\mathbf{t}_\lambda$ such that $X = S[m]$ and $X' = S'[m']$. Assume that $m \leq m'$, and suppose that $\mathcal{W}_X \not\subseteq \mathcal{W}_{X'}$ and $\mathcal{W}_X \cap \mathcal{W}_{X'} \neq \emptyset$.
We have to consider two cases. In the first case, the coray $\mathfrak{c}_j$ that contains $S'[m']$ meets the ray $\mathfrak{r}_i$ determined by S in a module $S[l] \in \mathfrak{c}_j \cap \mathfrak{r}_i$ with $1 \leq l \leq m$. We even have $l < m$ since otherwise $\mathcal{W}_X \subseteq \mathcal{W}_{X'}$. Then $S[l+1] \in \mathcal{S}$ and $\tau S[l+1] \in \mathcal{S}^\perp$ because $\mathcal{S}$ and $\mathcal{S}^\perp$ are closed under submodules and images respectively. But $\operatorname{Ext}^1_R(S[l+1], \tau S[l+1]) = \operatorname{Hom}_R(\tau S[l+1], \tau S[l+1]) \neq 0$, a contradiction.
In the second case, the coray $\mathfrak{c}_i$ that contains $S[m]$ meets the ray $\mathfrak{r}_j$ determined by S' in a module $S'[l] \in \mathfrak{c}_i \cap \mathfrak{r}_j$, where again $l < m$ (otherwise $\mathcal{W}_X \subseteq \mathcal{W}_{X'}$). Then $S'[l+1] \in \mathcal{S}$ and $\tau S'[l+1] \in \mathcal{S}^\perp$. But $\operatorname{Ext}^1_R(S'[l+1], \tau S'[l+1]) = \operatorname{Hom}_R(S'[l+1], S'[l+1]) \neq 0$, again a contradiction.
For the proof of the second statement, let $U_1 = S, U_2 = \tau^- U_1, \ldots, U_m$ be the quasi-simple modules in $\mathcal{W}_X$. Then it follows from Lemma 3.2 that $m < r$ and from Lemma 3.4(2) that $U_{m+1} \notin \mathcal{S}$ cannot be a submodule of a module $X' \in \operatorname{Add} T$. Thus it cannot belong to any wing $\mathcal{W}_{X'}$ with $X' \in \mathbf{t}_\lambda \cap \operatorname{Add} T$.
Finally, by Lemma 3.4(3) the number of isomorphism classes of modules in $\mathbf{t}_\lambda \cap \operatorname{Add} T$ equals the number of quasi-simple modules in the union of all wings involved, hence it is at most $r-1$. □

Let us now deal with the case that $\mathcal{S}$ contains a complete ray from $\mathbf{t}_\lambda$.

Lemma 3.6. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let S be a quasi-simple module in $\mathbf{t}_\lambda$. Suppose that the ray $\{S[n] \mid n \geq 1\}$ starting at S is completely contained in $\mathcal{S}$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$. The following assertions hold true.

(1) If the ray $\{U_2[n] \mid n \geq 1\}$ starting at U_2 is completely contained in $\mathcal{S}$, then $S[n] \notin \operatorname{Add} T$ for all $n \geq 1$.
(2) If $2 < i \leq r$ is the least number such that the ray $\{U_i[n] \mid n \geq 1\}$ starting at U_i is completely contained in $\mathcal{S}$, then $S[i-2]$ is the module of maximal regular length in $\{S[n] \mid n \geq 1\} \cap \operatorname{Add} T$.
(3) If $\{S[n] \mid n \geq 1\}$ is the only ray of $\mathbf{t}_\lambda$ which is completely contained in $\mathcal{S}$, then $S[r-1] \in \operatorname{Add} T$.

Proof: (1) Clearly $\operatorname{Ext}^1_R(U_2[n], S[n]) = D \operatorname{Hom}_R(S[n], S[n]) \neq 0$.
(2) We have to verify $S[i-2] \in \mathcal{S}^\perp$. Observe that, since $\mathcal{S}$ is closed under submodules, $\operatorname{Ext}^1_R(Z, S[i-2]) = D \operatorname{Hom}_R(U_2[i-2], Z) = 0$ for all $Z \in \mathcal{S}$ if and only if $U_2[i-2], U_3[i-3], \ldots, U_{i-1} \notin \mathcal{S}$. So, assume that one of the modules $U_2[i-2], U_3[i-3], \ldots, U_{i-1}$ belongs to $\mathcal{S}$, say $U_j[i-j] \in \mathcal{S}$ with $2 \leq j \leq i-1$. Since the rays starting at $U_2, \ldots, U_{i-1}$, are not completely contained in $\mathcal{S}$, it follows from Lemma 3.4(1) that $U_j[l] \in \operatorname{Add} T$ for some $l \geq i-j$. As $\mathcal{S}^\perp$ is closed under epimorphic images,

there exists a module in $U_{i-1}[t] \in \mathcal{S}^{\perp}$ on the ray starting at U_{i-1}. But this is a contradiction because $\operatorname{Ext}^1_R(U_i[t], U_{i-1}[t]) = D\operatorname{Hom}_R(U_{i-1}[t], U_{i-1}[t]) \neq 0$.
Hence $S[i-2] \in \operatorname{Add} T$. Moreover, no module of the form $S[i-2+t]$ with $t > 0$ is in $\operatorname{Add} T$, because otherwise its epimorphic image $U_{i-1}[t]$ would be a module in $\mathcal{S}^{\perp}$ on the ray starting at U_{i-1}.
(3) Proceed as in (2) and show that $U_2[r-2], \ldots, U_{r-1} \notin \mathcal{S}$.□

If $\mathcal{S}$ contains some, but not all rays from a tube $\mathbf{t}_\lambda$, then it certainly contains the rays with modules of maximal regular length in $\mathbf{t}_\lambda \cap \operatorname{Add} T$, as we are going to see next.

Lemma 3.7. Let $\mathbf{t}_\lambda$ be a nonhomogeneous tube. Suppose that $\mathcal{S}$ contains a complete ray from $\mathbf{t}_\lambda$. Let $S \in \mathbf{t}_\lambda$ be a quasi-simple such that $S[m]$, $m \geq 1$, is a module of maximal regular length in $\mathbf{t}_\lambda \cap \operatorname{Add} T$. Then the ray $\{S[n] \mid n \geq 1\}$ is completely contained in $\mathcal{S}$.

Proof: Suppose that the ray $\{S'[n] \mid n \geq 1\}$ starting at the quasi-simple module S' is completely contained in $\mathcal{S}$. Choose the numbering $S' = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$.
If there is no other $2 \leq i \leq r$ starting at U_i completely contained in $\mathcal{S}$, then $S'[r-1] \in \operatorname{Add} T$ by Lemma 3.6(3) and then the result holds by Lemmas 3.2 and 3.4(2).
If $i \in \{2, \cdots, r\}$ is the first number such that the ray $\{U_i[n] \mid n \geq 1\}$ starting at U_i is completely contained in $\mathcal{S}$, then $S'[i-2] \in \operatorname{Add} T$. Let U_j, $1 < j < i$, be a simple regular such that $U_j[t]$, $t \geq 1$, belongs to $\operatorname{Add} T$ (and the ray $\{U_j[n] \mid n \geq 1\}$ is not completely contained in $\mathcal{S}$.) Again by Lemma 3.4(2), we know that the rays starting at $U_j, U_{j+1}, \ldots, U_{j+t}$ are not completely contained in $\mathcal{S}$. Thus $i \geq j + t + 1 \geq t + 3$. Thus $i - 2 > t$.
Observe that for each quasi-simple $U \in \mathbf{t}_\lambda$ such that $U[t] \in \operatorname{Add} T$ and the ray $\{U_i[n] \mid n \geq 1\}$ is not completely contained in $\mathcal{S}$, we find a quasi-simples S' and U_i as in the foregoing paragraph. Thus the result follows. □

The following result will be useful when dealing with the torsion-free part $\overline{X}$ in the structure results from Propositions 3.1 and 3.3.

Lemma 3.8. Let $\mathbf{t}_\lambda$ be a tube.

(1) $\mathcal{S}$ contains a complete ray $\{S[n] \mid n \geq 1\}$ from $\mathbf{t}_\lambda$ if and only if $\mathcal{B}$ does not contain any adic module belonging to $\mathbf{t}_\lambda$.
(2) Suppose that $\mathbf{t}_\lambda$ is a tube of rank $r > 1$ such that $\mathcal{S}$ contains no complete ray from $\mathbf{t}_\lambda$. Let $\mathcal{U}$ denote the set of quasi-simple modules in the union of all wings $\mathcal{W}_X$ with $X \in \mathbf{t}_\lambda \cap \operatorname{Add} T$. Then for a quasi-simple module $S \in \mathbf{t}_\lambda$, the adic module $S[-\infty]$ belongs to $\mathcal{B}$ if and only if $S \notin \tau\mathcal{U} = \{\tau U \mid U \in \mathcal{U}\}$. Thus $\mathcal{B}$ contains precisely $r - |\mathcal{U}|$ pairwise non-isomorphic adic modules belonging to $\mathbf{t}_\lambda$.
(3) Let $\mathcal{U}$ be a set of quasi-simple modules in $\mathbf{t}_\lambda$. Every torsion-free module in $\mathcal{B}$ is contained in $\mathcal{U}^{\perp}$ if and only if all adic modules in $\mathcal{B}$ belonging to $\mathbf{t}_\lambda$ are contained in $\mathcal{U}^{\perp}$ (equivalently, every torsion-free module in $\mathcal{B}$ is an $R_{\mathcal{U}}$-module if and only if all the adic modules in $\mathcal{B}$ belonging to $\mathbf{t}_\lambda$ are $R_{\mathcal{U}}$-modules.)

Proof: We start by proving the only-if part of (1). Suppose that $\mathcal{S}$ contains the complete ray $\{S[n] \mid n \geq 1\}$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$, $r \geq 1$, for the quasi-simples in $\mathbf{t}_\lambda$. If $I = U_i[-\infty]$ for some $1 \leq i \leq r$, then $\operatorname{Ext}^1_R(S[\ell], I) \cong D\operatorname{Hom}_R(I, U_r[\ell]) = \operatorname{Hom}_R(U_i[-\infty], U_r[\ell]) \neq 0$ for $\ell = i+1$, hence $I \notin \mathcal{B}$.

Next, we prove (2) and the if-part of (1). First of all, observe that for any quasi-simple $S \in \mathbf{t}_\lambda$ we have $\operatorname{Ext}^1_R(\mathbf{p}, S[-\infty]) = 0$ because $S[-\infty]$ has no non-zero preprojective summands, and $\operatorname{Ext}^1_R(\mathbf{t}_\mu, S[-\infty]) = 0$ for all $\mu \neq \lambda$.

So, if $\mathcal{S} \cap \mathbf{t}_\lambda = \emptyset$, then all adic modules belonging to $\mathbf{t}_\lambda$ are in $\mathcal{B}$ (and indeed, this is the case $\mathcal{U} = \tau\mathcal{U} = \emptyset$ in (2)).

Assume now $\mathcal{S} \cap \mathbf{t}_\lambda \neq \emptyset$, and suppose that $\mathcal{S}$ does not contain a complete ray from $\mathbf{t}_\lambda$. Then we know from Lemma 3.4 that $\mathcal{S} \cap \mathbf{t}_\lambda$ is contained in the union $\mathcal{W}$ of all wings $\mathcal{W}_X$ with $X \in \mathbf{t}_\lambda \cap \operatorname{Add} T$. Moreover, it follows from Corollary 3.5 that $\mathcal{W} = \mathcal{W}_{X_1} \cup \ldots \cup \mathcal{W}_{X_s}$ where the $\mathcal{W}_{X_1}, \ldots, \mathcal{W}_{X_s}$ are pairwise disjoint wings with vertices $X_1, \ldots, X_s \in \operatorname{Add} T$, and that the set $\mathcal{U}$ of all quasi-simples in $\mathcal{W}$ does not form a complete clique. So, there are quasi-simple modules $S \in \mathbf{t}_\lambda \setminus \tau\mathcal{U}$. For such S we have $\operatorname{Ext}^1_R(U, S[-\infty]) = D\operatorname{Hom}_R(S[-\infty], \tau U) = 0$ for all $U \in \mathcal{U}$, so $S[-\infty] \in \mathcal{U}^\perp = \mathcal{W}^\perp$ by Lemma 1.1, and combining this with our first observation, we conclude that $S[-\infty] \in \mathcal{S}^\perp = \mathcal{B}$. On the other hand, if $S \in \tau\mathcal{U}$, then $S = \tau U$ for some quasi-simple U in one of the wings $\mathcal{W}_{X_i}$, and it is easy to see that $\operatorname{Ext}^1_R(X_i, S[-\infty]) = D\operatorname{Hom}_R(S[-\infty], \tau X_i) \neq 0$, which shows $S[-\infty] \notin \mathcal{B}$ and completes the proof of (1) and (2).

(3) First of all, we note that the class of all torsion-free modules $\mathcal{F} = \mathbf{t}^o$, as well as the classes $\mathcal{B}$ and $\mathcal{U}^\perp$, are definable classes. Indeed, $\mathcal{F}$ is clearly closed under direct products and submodules, and it is closed under direct limits since $\mathbf{t}$ consists of finitely presented modules. As for $\mathcal{B} = \mathcal{S}^\perp$ and $\mathcal{U}^\perp$, closure under direct products is clear, and closure under direct limits and pure submodules follows from the fact that $\mathcal{S}$ and $\mathcal{U}$ consist of finitely presented modules.

We are now ready to consider a torsion-free module $X \in \mathcal{B}$. Take the pure-injective envelope I of X, which is again a torsion-free module in $\mathcal{B}$ as definable classes are closed under pure-injective envelopes, see [18, 3.1.10]. Moreover, I is the pure-injective envelope of $\bigoplus_{l \in L} I_l$, where $\{I_l \mid l \in L\}$ is a complete irredundant set of indecomposable summands of I, cf. [21, Chapter 8]. Now the I_l are indecomposable pure-injective torsion-free modules in $\mathcal{B}$, and they are in $\operatorname{Gen}\mathbf{L} = \mathbf{p}^\perp$ by Theorem 2.7(1). We infer from the classification of the indecomposable pure-injective modules reviewed in Lemma 1.2 that I_l is either the generic module G or an adic module. Notice that G is divisible and thus belongs to $\mathcal{U}^\perp$. Moreover, using the Auslander-Reiten formula, it is easy to see that $I_l \in \mathcal{U}^\perp$ if I_l is an adic module belonging to a tube $\mathbf{t}_\mu$ with $\mu \neq \lambda$. So, if we assume that all adic modules I_l belonging to $\mathbf{t}_\lambda$ are contained in $\mathcal{U}^\perp$, then also $\bigoplus_{l \in L} I_l$ and its pure-injective envelope I are in $\mathcal{U}^\perp$, and therefore also the pure submodule X.

Conversely, recall that any adic module is torsion-free.□

Let us now apply our findings to compute the class of modules that are Ext-orthogonal to a ray in a non-homogeneous tube, or in other words, to determine the tilting class $\mathcal{B} = \mathcal{S}^\perp$ when $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ with $\mathbf{t}'$ being a ray. This is a special case that will play an important role in the sequel.

Example 3.9. Let S be a quasi-simple module, and assume $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \{S[n] \mid n \in \mathbb{N}\})$. Then

$$\mathcal{B} = S[\infty]^\perp, \quad \text{and} \quad S[\infty] \in \operatorname{Add} T.$$

Moreover, if S belongs to a tube $\mathbf{t}_\lambda$ of rank $r > 1$, then

$$Y = S \oplus S[2] \oplus \cdots \oplus S[r-1]$$

(and $T \sim S \oplus S[2] \oplus \ldots \oplus S[r-1] \oplus S[\infty] \oplus R_{\mathbf{t}_\lambda}$, as we will see later in 4.7).

Indeed, we show as in Example 2.4 that $\mathcal{S}^\perp = \bigcap_{n\geq 1} S[n]^\perp$. Moreover, as $S[n] \leq S[\infty]$, we have $S[\infty]^\perp \subseteq S[n]^\perp$ for all $n \geq 1$, hence $S[\infty]^\perp \subseteq \mathcal{B}$. For the reverse inclusion, note that $S[\infty]$ is filtered by $S[r]$, thus every $X \in \mathcal{B} \subset S[r]^\perp$ also belongs to $S[\infty]^\perp$ by [18, 3.1.2].
Now we deduce $S[\infty] \in {}^\perp(S[\infty]^\perp) = {}^\perp\mathcal{B}$. Moreover, $S[\infty]$ also belongs to $\mathcal{B}$ as it is divisible and thus satisfies $\operatorname{Ext}^1_R(S[n], S[\infty]) = 0$ for all $n \in \mathbb{N}$. So, we conclude $S[\infty] \in \operatorname{Add} T$.
For the last claim, observe that every finitely generated indecomposable module in $\operatorname{Add} T$ must belong to $\mathcal{B}$, and also to ${}^\perp\mathcal{B} \cap \operatorname{mod} R = \mathcal{S}$ and thus to the ray $\{\, S[n] \mid n \in \mathbb{N} \,\}$. So, we deduce from $\mathbf{t}_\lambda \cap \mathcal{B} = \mathcal{W}_{S[r-1]}$ that $S, S[2], \ldots, S[r-1]$ is a complete irredundant set of the finitely generated indecomposable modules in $\operatorname{Add} T$. □

We are now ready for the main result of this section.

Theorem 3.10. There is a unique direct sum decomposition

$$T = \bigoplus_{\lambda \in \Lambda} \mathbf{t}_\lambda(T) \oplus \overline{T}$$

where $\overline{T}$ is torsion-free, and $\mathbf{t}_\lambda(T)$ is a direct sum of copies of the indecomposable direct summands of $\mathbf{t}_\lambda(Y)$ and of Prüfer modules belonging to $\mathbf{t}_\lambda$. More precisely, the following statements hold true for a tube $\mathbf{t}_\lambda$ of rank r:

(1) $\mathbf{t}_\lambda(T) = 0$ whenever $\mathbf{t}_\lambda \cap \mathcal{S} = \emptyset$;
(2) if $\mathcal{S}$ contains some rays from $\mathbf{t}_\lambda$, then $\mathbf{t}_\lambda(T)$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s Prüfer modules corresponding to the $s \leq r$ rays from $\mathbf{t}_\lambda$ contained in $\mathcal{S}$, and $r - s$ modules from $\mathbf{t}_\lambda$;
(3) if $\mathcal{S}$ contains some modules from $\mathbf{t}_\lambda$, but no complete ray, then $\mathbf{t}_\lambda(T)$ is a direct sum of copies of at most $r - 1$ pairwise non-isomorphic modules from $\mathbf{t}_\lambda$.

Proof: The existence of the decomposition follows from Proposition 3.3. Observe that every indecomposable direct summand of $\mathbf{t}_\lambda(Y)$ lies in $\mathbf{t}_\lambda \cap \operatorname{Add} T$ and therefore occurs as a direct summand in $\mathbf{t}_\lambda(T)$ by Lemma 1.2(1).
We now turn to the additional statements. Note first that the finite dimensional direct summands of $\mathbf{t}_\lambda(T)$ are contained in $\mathbf{t}_\lambda \cap \mathcal{S}$, and further, recall that ${}^\perp\mathcal{B}$ is closed under submodules, so with every Prüfer module $S[\infty]$ it contains also the corresponding ray $\{S[n] \mid n \in \mathbb{N}\}$. This proves (1) and shows that $\mathbf{t}_\lambda(T)$ cannot contain infinite dimensional direct summands if $\mathbf{t}_\lambda \cap \mathcal{S}$ does not contain complete rays. Thus, in case (3), the non-isomorphic indecomposable summands of $\mathbf{t}_\lambda(T)$ are precisely the indecomposable direct summands of $\mathbf{t}_\lambda(Y)$, and by Corollary 3.5 their number is at most $r - 1$.
It remains to prove (2). Assume that $\mathbf{t}_\lambda \cap \mathcal{S}$ contains a complete ray $\{S[n] \mid n \in \mathbb{N}\}$. Then $\mathcal{B} \subset \bigcap\limits_{n\geq 1} S[n]^\perp = S[\infty]^\perp$ by Example 3.9, so $S[\infty] \in {}^\perp\mathcal{B}$. Further, using that $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ for some $\mathbf{t}' \subseteq \mathbf{t}$, we see that $S[\infty]$ lies in $\mathcal{B}$, hence in $\operatorname{Add} T$. We then infer from Proposition 3.3 that $S[\infty]$ is a direct summand in $\mathbf{t}_\lambda(T)$.

Now we determine the remaining indecomposable summands of $\mathbf{t}_\lambda(T)$. If $\mathcal{S}$ contains the whole tube $\mathbf{t}_\lambda$, then it follows from Lemma 3.6(1) that $\mathbf{t}_\lambda(T)$ has no finite dimensional summands, and therefore it is a direct sum of all Prüfer modules belonging to $\mathbf{t}_\lambda$. If $\mathbf{t}_\lambda$ has rank $r > 1$, and $\{S[n] \mid n \geq 1\}$ is the only complete ray contained in $\mathcal{S}$, we get from Lemma 3.6(3) that $S[r-1] \in \operatorname{Add} T$. Now Lemma 3.4(3) implies that there are exactly $r-1$ indecomposable summands of $\mathbf{t}_\lambda(T)$ belonging to $\mathcal{W}_{S[r-1]}$. There are no further finite dimensional indecomposable summands in $\mathbf{t}_\lambda(T)$ by Corollary 3.5, and no further infinite dimensional indecomposable summands by the considerations above.

Finally, suppose that there are $1 < s < r$ complete rays contained in $\mathbf{t}_\lambda \cap \mathcal{S}$. We have seen above that the Prüfer modules determined by these rays are the only infinite dimensional indecomposable summands of $\mathbf{t}_\lambda(T)$. As in the foregoing case, we obtain the remaining (finite dimensional) indecomposable summands by employing Lemma 3.6(2). □

Remark 3.11. There seems to be an asymmetry between case (2) and (3) in Theorem 3.10 above: the number s of pairwise non-isomorphic indecomposable summands of $\mathbf{t}_\lambda(T)$ equals the rank r of $\mathbf{t}_\lambda$ when $\mathcal{S}$ contains some rays from $\mathbf{t}_\lambda$, but is smaller than r otherwise. Note however that in the latter case s coincides with the number of quasi-simple modules in the union of all wings $\mathcal{W}_X$ with vertex $X \in \operatorname{Add} T \cap \mathbf{t}_\lambda$. So, the "missing" summands are somehow "replaced" by the $r-s$ adic modules in $\mathcal{B}$ established by Lemma 3.8(2). This aspect will become more clear in the Appendix.

4. Classifying tilting modules

Let again T be a tilting R-module with tilting class $\mathcal{B} = T^\perp$, and $\mathcal{S} = {}^\perp\mathcal{B} \cap \operatorname{mod} R$. We assume that T is not equivalent to a finitely generated tilting module, nor to the Lukas tilting module. Then we know from Theorem 2.7 that there is a non-empty subset $\mathbf{t}' \subset \mathbf{t}$ such that $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$. We have seen in Theorem 2.7 that T is equivalent to a tilting module arising from universal localization in case $\mathbf{t}'$ is a union of tubes. We now discuss the general case.

Recall that we denoted by Y the direct sum of a complete irredundant set of the finitely generated indecomposable modules in $\operatorname{Add} T$. Thus Y is a finite dimensional direct summand of T by Lemmas 3.2 and 1.2(1).

Our aim is to reduce the classification problem to the situation considered in Theorem 2.7. To this end, we will show that T is equivalent to a tilting module of the form $Y \oplus M$, while M has no finite dimensional indecomposable direct summands and is a tilting module over a suitable universal localization of R. We will prove this step by step, by considering the finitely many non-homogeneous tubes $\mathbf{t}_\lambda$ where $\mathbf{t}_\lambda \cap \operatorname{Add} T \neq \emptyset$.

We first give a general criterion for constructing a tilting module of the desired form.

Lemma 4.1. Let $Y' \in \operatorname{Add} T$, and let M be a module satisfying condition (T2). Then $Y' \oplus M$ is a tilting module equivalent to T if and only if the following hold true.

(a) $\mathcal{B} \subset M^\perp$,
(b) $M \in \mathcal{B}$,
(c) $T \in \operatorname{Add}(Y' \oplus M)$.

Proof: For the only-if-part, note that $\operatorname{Add}(Y' \oplus M) = \operatorname{Add} T \subset \mathcal{B} = (Y' \oplus M)^{\perp}$, which immediately yields (a), (b), (c).
For the if-part, we show that $Y' \oplus M$ is tilting. Condition (T1) is trivially verified. In order to check (T2), let α be a cardinal. Then $\operatorname{Ext}^1_R((Y' \oplus M), (Y' \oplus M)^{(\alpha)}) \cong \operatorname{Ext}^1_R(Y', Y'^{(\alpha)}) \oplus \operatorname{Ext}^1_R(M, M^{(\alpha)}) \oplus \operatorname{Ext}^1_R(M, Y'^{(\alpha)}) \oplus \operatorname{Ext}^1_R(Y', M^{(\alpha)})$. Now the first term vanishes since $Y' \in \operatorname{Add} T$, the second by assumption on M, the third term vanishes by (a) since $\mathcal{B}$ is closed under direct sums and therefore $Y'^{(\alpha)} \in \mathcal{B}$, and the last term vanishes because $M^{(\alpha)} \in \mathcal{B} = T^{\perp} \subset Y'^{\perp}$ by property (b). Finally, condition (T3) is satisfied by property (c).
So, $Y' \oplus M$ is a tilting module with $\operatorname{Add} T \subset \operatorname{Add}(Y' \oplus M)$, thus $(Y' \oplus M)^{\perp} \subset T^{\perp}$. Conversely, $T^{\perp} = \mathcal{B} \subset Y'^{\perp} \cap M^{\perp} = (Y' \oplus M)^{\perp}$, showing that $Y' \oplus M$ is equivalent to T. □

Now we proceed with our reduction. Given a non-homogeneous tube $\mathbf{t}_\lambda$ where $\mathbf{t}_\lambda \cap \operatorname{Add} T \neq \emptyset$, we want to replace T by an equivalent tilting module of the form $\mathbf{t}_\lambda(Y) \oplus M$ where M is a tilting module over a suitable universal localization $R' = R_{\mathcal{U}}$ of R. To this end, we replace the resolving subcategory $\mathcal{S}$ by its localization $\mathcal{S}' = \{A \otimes_R R' \mid A \in \mathcal{S}\}$ and choose M to be a tilting R'-module with tilting class $\mathcal{S}'^{\perp}$. The existence of M is granted by [18, 5.2.2]. We formulate criteria that will allow to perform the replacement.

Proposition 4.2. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let $\mathcal{U}$ be a set of $m < r$ quasi-simples in $\mathbf{t}_\lambda$ with extension closure $\mathcal{W}$. Denote $R' = R_{\mathcal{U}}$, and assume that M is an R'-tilting module with tilting class

$$\mathcal{B}' = \{X \in \operatorname{Mod} R' \mid \operatorname{Ext}^1_{R'}(A \otimes_R R', X) = 0 \text{ for all } A \in \mathcal{S}\}$$

such that

(i) $\mathbf{t}' \subset \mathcal{W} \cup \mathcal{W}^o$,
(ii) $\operatorname{Add}(\mathbf{t}_\lambda \cap \mathcal{B}) \subset M^{\perp}$,
(iii) every adic module in $\mathcal{B}$ belonging to $\mathbf{t}_\lambda$ is contained in $\mathcal{U}^{\perp}$.

Then $\mathbf{t}_\lambda(Y) \oplus M$ is a tilting module equivalent to T.

Proof: As $\mathbf{t}_\lambda(Y) \in \operatorname{Add} T$ and M satisfies (T2), we only have to verify the conditions in Lemma 4.1.
(a) We prove $\mathcal{B} \subset M^{\perp}$ in two steps.
Step 1: We show $\mathcal{B} \cap \mathcal{U}^{\perp} \subset M^{\perp}$. Take $X \in \mathcal{B} \cap \mathcal{U}^{\perp}$. We claim $M \in {}^{\perp}X$. Since the R'-tilting module M is filtered by the modules in $\mathcal{S}' = \{A \otimes_R R' \mid A \in \mathcal{S}\}$ by [38, Lemma 4.5], it suffices to show that $\mathcal{S}' \subset {}^{\perp}X$. So, let $A \in \mathcal{S}$, w.l.o.g. A indecomposable. Then $A \in \mathbf{p} \subset \mathcal{W}^o$ or $A \in \mathbf{t}' \subset \mathcal{W} \cup \mathcal{W}^o$ by (i). If $A \in \mathcal{W}$, then $A \otimes_R R' = 0$ by Propostion 1.7(5), so we can assume w.l.o.g. $A \in \mathcal{W}^o$. Then we know from Proposition 1.7(7) that there is a short exact sequence $0 \to A \to A \otimes_R R' \to A \otimes_R R'/R \to 0$ where the two outer terms $A \in \mathcal{S}$ and $A \otimes_R R'/R \in {}^{\perp}(\mathcal{U}^{\perp})$ belong to ${}^{\perp}X$, so we infer that the middle-term $A \otimes_R R'$ belongs to ${}^{\perp}X$ as well.
Step 2: We now consider an arbitrary $X \in \mathcal{B}$ and apply the structure result in 3.1. Since the divisible module $X_{\mathcal{D}}$ belongs to $\mathcal{B} \cap \mathcal{U}^{\perp} \subset M^{\perp}$, and $M^{\perp}$ is closed under extensions, it is enough to show that X' and $\overline{X}$ are in $M^{\perp}$. Observe first that X' and $\overline{X}$ are in $\mathcal{B}$ since $\mathcal{B}$ is closed under pure submodules and epimorphic images. Furthermore, we know from Lemma 3.8 and condition (iii) that the torsion-free module $\overline{X} \in \mathcal{B}$ is contained in $\mathcal{U}^{\perp}$. So, we conclude from Step 1 that $\overline{X} \in M^{\perp}$.
Now let us turn to $X' = \bigoplus_{\mu \in \mathbb{P}} X_\mu$. Since $\mathcal{B} \cap \mathcal{U}^{\perp}$ is closed under direct sums, we have $\bigoplus_{\mu \neq \lambda} X_\mu \in \mathcal{B} \cap \mathcal{U}^{\perp} \subset M^{\perp}$, so we only have to consider X_λ. Recall that there is a pure-exact sequence $0 \to$

$A_\lambda \to X_\lambda \to Z_\lambda \to 0$ where A_λ is a direct sum of modules in $\mathbf{t}_\lambda \cap \mathcal{B}$, and $Z_\lambda \in \mathcal{B}$ is a direct sum of Prüfer modules belonging to the tube $\mathbf{t}_\lambda$. Then Z_λ is divisible and therefore in $M^\perp$ by Step 1, and $A_\lambda \in M^\perp$ by (ii), thus also $X_\lambda \in M^\perp$.

(b) We now prove $M \in \mathcal{B}$. Let $A \in \mathcal{S}$, and assume w.l.o.g. that A is indecomposable. As in (a) we infer from (i) that $A \in \mathcal{W} \cup \mathcal{W}^o$. If $A \in \mathcal{W}$, then $\operatorname{Ext}^1_R(A, M) = 0$ because M is an R'-module and thus belongs to $\mathcal{W}^\wedge$ by Proposition 1.7(4). If $A \in \mathcal{W}^o$, then we know from Proposition 1.7(7) that A embeds in $A \otimes_R R' \in {}^\perp M$, hence $A \in {}^\perp M$, and the claim is verified.

(c) Finally, we check that $T \in \operatorname{Add}(\mathbf{t}_\lambda(Y) \oplus M)$. By Theorem 3.10 there is a decomposition $T = \bigoplus_{\mu \in \mathbb{P}} \mathbf{t}_\mu(T) \oplus \overline{T}$ where $\overline{T}$ is torsion-free, and each $\mathbf{t}_\mu(T)$ is a direct sum of copies of $\mathbf{t}_\mu(Y)$ and Prüfer modules belonging to $\mathbf{t}_\mu$. Moreover, a Prüfer module $S[\infty]$ occurs as a direct summand in $\mathbf{t}_\mu(T)$ if and only if $\mathbf{t}_\mu \cap \mathcal{S}$ contains the complete ray $\{S[n] \mid n \geq 1\}$, again by Theorem 3.10. Observe that complete rays in $\mathcal{S}$ are contained in $\mathcal{W}^o$ by (i). So, we deduce that the Prüfer modules occurring as direct summands in $\mathbf{t}_\lambda(T)$ do not belong to quasi-simples in $\mathcal{U}$ and are therefore contained in $\mathcal{U}^o$, and even in $\mathcal{U}^\wedge$ as they are divisible modules. Thus $\mathbf{t}_\lambda(T)$ is the direct sum of a module in $\operatorname{Add} \mathbf{t}_\lambda(Y)$ with a module in $\mathcal{U}^\wedge$. Of course, also the $\mathbf{t}_\mu(T)$ with $\mu \neq \lambda$ belong to $\mathcal{U}^\wedge$. Finally, the torsion-free module $\overline{T}$ is contained $\mathcal{U}^o$, and even in $\mathcal{U}^\wedge$ by Lemma 3.8 and condition (iii). So, our claim will be proven once we show that $\operatorname{Add} T \cap \mathcal{U}^\wedge \subset \operatorname{Add} M$.
Let us thus consider $X \in \operatorname{Add} T \cap \mathcal{U}^\wedge$. First of all, $X \in \mathcal{B} \subset M^\perp$ by (a). Moreover, X is an R'-module, hence $\operatorname{Ext}^1_{R'}(M, X) = \operatorname{Ext}^1_R(M, X) = 0$. Therefore X belongs to the R'-tilting class $\mathcal{B}'$, and there is an exact sequence $0 \to M_1 \to M_0 \xrightarrow{f} X \to 0$ with $M_0, M_1 \in \operatorname{Add} M$ by [18, 5.1.8(d)]. Since $X \in M^\perp$, the exact sequence splits and $X \in \operatorname{Add} M$.
Now the proof of the Proposition is complete. □

In order to specify the set $\mathcal{U}$ at which we will localize, we have to distinguish two cases, depending on whether $\mathcal{S} \cap \mathbf{t}_\lambda$ contains a complete ray or not.

Proposition 4.3. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let $S[m]$ be a module of maximal regular length m in $\mathbf{t}_\lambda \cap \operatorname{Add} T$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$, and suppose that the ray $\{S[n] \mid n \in \mathbb{N}\}$ starting at U_1 is not completely contained in $\mathcal{S}$. Set $\mathcal{U} = \{U_1, \ldots, U_m\}$, denote $R' = R_\mathcal{U}$, and let M be an R'-tilting module with tilting class $\mathcal{B}' = \{X \in \operatorname{Mod} R' \mid \operatorname{Ext}^1_{R'}(A \otimes_R R', X) = 0 \text{ for all } A \in \mathcal{S}\}$. Then $\mathbf{t}_\lambda(Y) \oplus M$ is a tilting R-module equivalent to T.

Proof: We denote by $\mathcal{W}$ the extension closure of $\mathcal{U}$ and recall from Lemma 1.1 that $\mathcal{W}$ consists of all finite direct sums of modules in $\mathcal{W}_{S[m]}$. We verify conditions (i)-(iii) in Proposition 4.2.
(i) We claim $\mathbf{t}' \subset \mathcal{W} \cup \mathcal{W}^o$. Indeed, if $A \in \mathbf{t}'$ lies on a ray which is completely contained in $\mathcal{S}$, then, from the assumption and Lemma 3.4(2), it must be in $\mathcal{U}^o$, which coincides with $\mathcal{W}^o$ by Lemma 1.7(3). If, on the other hand, $A \in \mathbf{t}'$ lies on a ray $\{S'[n] \mid n \in \mathbb{N}\}$ which is not completely contained in $\mathcal{S}$, then by Lemma 3.4(1) this ray must contain a module $X' = S'[\ell] \in \operatorname{Add} T$ and $A = S'[k]$ with $k \leq \ell$ belongs to the wing $\mathcal{W}_{X'}$. By Lemma 3.5 and by the maximality of m, the wings $\mathcal{W}$ and $\mathcal{W}_{X'}$ are either disjoint or $\mathcal{W}_{X'} \subset \mathcal{W}$. In the latter case $A \in \mathcal{W}$. In the first case, we see that A does not lie on a ray starting at $\mathcal{U}$, hence $A \in \mathcal{W}^o$.

(ii) Let us now verify $\mathrm{Add}(\mathbf{t}_\lambda \cap \mathcal{B}) \subset M^\perp$. Choose $A \in \mathrm{Add}(\mathbf{t}_\lambda \cap \mathcal{B})$. By [39] there is an indecomposable decomposition of the form

$$A = \bigoplus_{j\in J} W_j^{(\alpha_j)} \oplus \bigoplus_{i\in I} X_i^{(\alpha_i)}$$

where $\{W_j \mid j \in J\}$ is a complete irredundant set of modules in $\mathcal{W}_{S[m]}$, and $\{X_i \mid i \in I\}$ is a complete irredundant set of modules in $\mathbf{t}_\lambda \cap \mathcal{B} \setminus \mathcal{W}_{S[m]}$. Note that the index set J is finite.
First of all, we prove that $\mathcal{U} \subset M^\perp$. Indeed, for $1 \le i < m$ we have $\mathrm{Ext}^1_R(M, U_i) \cong D\mathrm{Hom}_R(U_{i+1}, M) = 0$ since M is an R'-module. Moreover, $U_m \in \mathcal{B}$ because it is a quotient of $S[m] \in \mathrm{Add}\, T$, and $U_m \in \mathcal{U}^\perp$ because $\mathrm{Ext}^1_R(U_i, U_m) \cong D\mathrm{Hom}_R(U_{m+1}, U_i) = 0$ for all $1 \le i \le m$. Now we infer as in Step 1 of the proof of Proposition 4.2 that $\mathcal{U}^\perp \cap \mathcal{B} \subset M^\perp$, so $U_m \in M^\perp$ as well. Hence $\mathcal{U} \subset M^\perp$, thus also $\mathcal{W}_{S[m]} \subset M^\perp$, yielding by Lemma 1.2(2) and (3) that $\bigoplus_{j\in J} W_j^{(\alpha_j)} \in M^\perp$.
Next, we consider $X \in (\mathbf{t}_\lambda \cap \mathcal{B}) \setminus \mathcal{W}_{S[m]}$. Note that $S[m] \in \mathrm{Add}\, T$ implies $S[\ell] \in \mathcal{S}$ for all $\ell \le m$, thus $0 = \mathrm{Ext}^1_R(S[\ell], X) \cong D\mathrm{Hom}_R(X, U_r[\ell])$. Then X cannot lie on a coray ending at $U_r, U_1, \ldots, U_{m-1}$. Hence $\mathrm{Ext}^1_R(U_j, X) \cong D\mathrm{Hom}_R(X, \tau U_j) = 0$ for all $1 \le j \le m$, which shows that $X \in \mathcal{U}^\perp$. Therefore $\bigoplus_{i\in I} X_i^{(\alpha_i)} \in \mathcal{B} \cap \mathcal{U}^\perp$ as $\mathcal{B}$ and $\mathcal{U}^\perp$ are closed under direct sums. Now we infer as in Step 1 of the proof of Proposition 4.2 that $\mathcal{U}^\perp \cap \mathcal{B} \subset M^\perp$, and we conclude that $A \in M^\perp$ as desired.
(iii) Finally, we check that every adic module in $\mathcal{B}$ belonging to $\mathbf{t}_\lambda$ is contained in $\mathcal{U}^\perp$. So suppose that $I = U_i[-\infty]$ for some $1 \le i \le r$. As in (ii), we see that $0 = \mathrm{Ext}^1_R(S[\ell], I) \cong D\mathrm{Hom}_R(I, U_r[\ell])$ for all $\ell \le m$, hence $i \notin \{r, 1, \ldots, m-1\}$. This implies $\mathrm{Ext}^1_R(U_j, I) \cong D\mathrm{Hom}_R(I, \tau U_j) = 0$ for all $1 \le j \le m$, that is, $I \in \mathcal{U}^\perp$. □

Proposition 4.4. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let $S[m]$ be a module of maximal regular length m in $\mathbf{t}_\lambda \cap \mathrm{Add}\, T$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$, and suppose that the ray $\{S[n] \mid n \in \mathbb{N}\}$ starting at U_1 is completely contained in $\mathcal{S}$. Set $\mathcal{U} = \{U_2, \ldots, U_{m+1}\}$, denote $R' = R_\mathcal{U}$, and let M be an R'-tilting module with tilting class $\mathcal{B}' = \{X \in \mathrm{Mod}\, R' \mid \mathrm{Ext}^1_{R'}(A \otimes_R R', X) = 0 \text{ for all } A \in \mathcal{S}\}$. Then $\mathbf{t}_\lambda(Y) \oplus M$ is a tilting R-module equivalent to T.

Proof: Again, we denote by $\mathcal{W}$ the extension closure of $\mathcal{U}$ and recall from Lemma 1.1 that $\mathcal{W}$ consists of all finite direct sums of modules in $\mathcal{W}_{U_2[m]}$. Let us verify conditions (i)-(iii) in Proposition 4.2.
(i) We claim $\mathbf{t}' \subset \mathcal{W} \cup \mathcal{W}^o$. Indeed, from Lemma 3.4(2), if $A \in \mathbf{t}'$ lies on a ray which is completely contained in $\mathcal{S}$, then it must be in $\mathcal{U}^o$, which coincides with $\mathcal{W}^o$ by Proposition 1.7. Assume now that $A \in \mathbf{t}'$ lies on a ray $\{S'[n] \mid n \in \mathbb{N}\}$ which is not completely contained in $\mathcal{S}$. Of course, $S' \ne U_1$, but also $S' \ne U_{m+1}$, because $U_{m+1} \notin \mathcal{S}$ by Lemma 3.4(2). Moreover, by Lemma 3.4(1) this ray must contain a module $X' = S'[\ell] \in \mathrm{Add}\, T$ and $A = S'[k]$ with $k \le \ell$ belongs to the wing $\mathcal{W}_{X'}$. By Lemma 3.5 and by the maximality of m, the wings $\mathcal{W}_{S[m]}$ and $\mathcal{W}_{X'}$ are either disjoint or $\mathcal{W}_{X'} \subset \mathcal{W}_{S[m]}$. In the latter case $A \in \mathcal{W}_{S[m]}$, and since $S' \ne U_1$, it follows $A \in \mathcal{W}$. In the first case, we have $S' \notin \{U_1, \ldots, U_m\}$ and $S' \ne U_{m+1}$, so we conclude that $A \in \mathcal{W}^o$.
(ii) In order to verify $\mathrm{Add}(\mathbf{t}_\lambda \cap \mathcal{B}) \subset M^\perp$, we first observe that no module $U_r[n] = \tau S[n]$ on the ray starting at U_r can belong to $\mathcal{B}$. Consider now a module $X \in \mathbf{t}_\lambda \cap \mathcal{B}$, and assume $X \notin \mathcal{U}^\perp$. Then there is $2 \le i \le m+1$ such that $0 \ne \mathrm{Ext}^1_R(U_i, X) \cong D\mathrm{Hom}_R(X, U_{i-1})$, hence X lies on one of the corays ending at $U_1, \ldots, U_m$. But then X must belong to the wing $\mathcal{W}_{S[m]}$, because otherwise there

is an epimorphism from X to a module $U_r[n]$ with $2 \leq n \leq m+1$, which would imply $U_r[n] \in \mathcal{B}$. Now recall that the R'-module M belongs to $\mathcal{U}^\wedge$, hence $\mathrm{Ext}^1_R(M, U_i) \cong D\mathrm{Hom}_R(U_{i+1}, M) = 0$ for all $1 \leq i \leq m$, which proves $\{U_1, \ldots, U_m\} \subset M^\perp$ and therefore $X \in \mathcal{W}_{S[m]} \subset M^\perp$.
Now the claim follows as in the proof of Proposition 4.3.
(iii) is trivially satisfied, because the assumption on the ray $\{S[n] \mid n \in \mathbb{N}\}$ implies by Lemma 3.8 that $\mathcal{B}$ does not contain any adic module belonging to $\mathbf{t}_\lambda$. □

Now we are in a position to prove our main result.

Theorem 4.5. Every tilting module T is equivalent to a module in the following list:

- the finitely generated tilting modules,
- tilting modules of the form $Y \oplus (\mathbf{L} \otimes R_\mathcal{U})$ where Y is a finitely generated regular module, and $\mathcal{U}$ is a set of quasi-simple modules that contains no complete cliques,
- tilting modules of the form $Y \oplus R_\mathcal{V} \oplus R_\mathcal{V}/R_\mathcal{U}$ where Y is a finitely generated regular module, and $\mathcal{U} \subset \mathcal{V}$ are sets of quasi-simple modules such that $\mathcal{U}$ contains no complete clique and $\mathcal{V}$ contains complete cliques.

Proof: Let T be a tilting module which is not equivalent to a finitely generated tilting module.
Step 1: Assume that $\mathrm{Add}\, T$ does not contain finitely generated modules. Then $\mathbf{t}'$ is empty or a union of tubes. Indeed, if $\mathcal{S}$ contained some, but not all modules from a ray $\{S[n] \mid n \in \mathbb{N}\}$ in a tube $\mathbf{t}_\lambda$ of rank $r > 1$, then by Lemma 3.4(1) there would be $m < r$ such that $S[m] \in \mathrm{Add}\, T$, a contradiction. Hence $\mathbf{t}'$ is a union of rays. Assume now that there is a tube $\mathbf{t}_\lambda$ of rank $r > 1$ such that $\mathcal{S}$ contains some, but not all rays from $\mathbf{t}_\lambda$. Choose a numbering $U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$ such that $U_r \notin \mathcal{S}$, and suppose that m is maximal with respect to the fact that the ray starting at U_m is contained in $\mathcal{S}$. Then $\mathrm{Ext}^1_R(U_i[n], U_m) \cong D\mathrm{Hom}_R(U_{m+1}, U_i[n]) = 0$ for all $1 \leq i \leq m < r$, showing that $U_m \in \mathcal{B}$. But then even $U_m \in \mathrm{Add}\, T$ since $U_m \in \mathcal{S}$, again a contradiction.
Thus in this case the statement (with $Y = 0$ and $\mathcal{U} = \emptyset$) follows immediately from Theorem 2.7.
Step 2: Assume now that $\mathrm{Add}\, T$ contains some finitely generated indecomposable module. Recall from Lemma 3.2 that there are only finitely many such modules up to isomorphism, and all of them are regular modules from some non-homogeneous tube. So, let us consider a tube $\mathbf{t}_\lambda$ of rank $r > 1$, and let $S[m]$ be a module of maximal regular length m in $\mathbf{t}_\lambda \cap \mathrm{Add}\, T$. Choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$. We have to distinguish two cases depending on whether the ray $\{S[n] \mid n \geq 1\}$ starting at S is completely contained in $\mathcal{S}$ or not.
Suppose first that the ray $\{S[n] \mid n \geq 1\}$ starting at U_1 is not completely contained in $\mathcal{S}$. Set $\mathcal{U} = \{U_1, \ldots, U_m\}$, and let M be an $R_\mathcal{U}$-tilting module with tilting class ${\mathcal{S}'}^\perp$ where $\mathcal{S}' = \{A \otimes_R R_\mathcal{U} \mid A \in \mathcal{S}\}$. Then $\mathbf{t}_\lambda(Y) \oplus M$ is a tilting R-module equivalent to T by Proposition 4.3. Now recall that $\mathrm{Mod}\, R_\mathcal{U}$ is a full subcategory of $\mathrm{Mod}\, R$ closed under direct summands, and further, every finitely generated $R_\mathcal{U}$-module is also finitely generated over R. As $\mathrm{Add}(\mathbf{t}_\lambda(Y) \oplus M) = \mathrm{Add}\, T$, we deduce that the finitely generated indecomposable $R_\mathcal{U}$-modules in $\mathrm{Add}_{R_\mathcal{U}} M$ form a proper subset of the set of all finitely generated indecomposable R-modules in $\mathrm{Add}\, T$. Indeed, the subset is proper, because if Z is one of the m modules in $\mathcal{W}_{S[m]} \cap \mathrm{Add}\, T$, then $Z \otimes_R R_\mathcal{U} = 0$ by Proposition 1.7(5).
Suppose now that the ray $\{S[n] \mid n \geq 1\}$ starting at S is completely contained in $\mathcal{S}$. Set $\mathcal{U} = \{U_2, \ldots, U_{m+1}\}$ and let M be an $R_\mathcal{U}$-tilting module with tilting class ${\mathcal{S}'}^\perp$ where $\mathcal{S}' = \{A \otimes_R R_\mathcal{U} \mid$

$A \in \mathcal{S}\}$. Then $\mathbf{t}_\lambda(Y) \oplus M$ is a tilting R-module equivalent to T by Proposition 4.4. As above, we observe that the finitely generated indecomposable $R_\mathcal{U}$-modules in $\operatorname{Add}_{R_\mathcal{U}} M$ form a subset of the set of all finitely generated indecomposable R-modules in $\operatorname{Add} T$. We claim that also in this case the subset is proper. Indeed, if $\{S[n] \mid n \geq 1\}$ is the only ray of $\mathbf{t}_\lambda$ which is completely contained in $\mathcal{S}$, then we infer from Lemma 3.6(3) and Proposition 1.10 that $m = r-1$ and $\mathbf{t}_\lambda \otimes R_\mathcal{U}$ is a homogeneous tube in the Auslander-Reiten quiver of $R_\mathcal{U}$, thus $\operatorname{Add}_{R_\mathcal{U}} M$ cannot contain modules from $\mathbf{t}_\lambda \otimes R_\mathcal{U}$. If, on the other hand, there are further rays of $\mathbf{t}_\lambda$ which are completely contained in $\mathcal{S}$, we consider the least number $1 < i \leq r$ such that the ray $\{U_i[n] \mid n \geq 1\}$ starting at U_i is completely contained in $\mathcal{S}$. We have $i > m+1$ by Lemma 3.4(2), and $i \leq m+2$ by Lemma 3.6(2) and the maximality of m, so $i = m+2$, and we infer from Proposition 1.10 that the rays starting at $S \otimes R_\mathcal{U}$ and at $U_{m+2} \otimes R_\mathcal{U}$ are two consecutive rays in the tube $\mathbf{t}_\lambda \otimes R_\mathcal{U}$ which are completely contained in $\mathcal{S}'$. Then we know from Lemma 3.6(1) that $\operatorname{Add}_{R_\mathcal{U}} M$ cannot contain modules of the form $S[n] \otimes R_\mathcal{U}$, $n \in \mathbb{N}$. Hence in both cases M is a tilting module over the finite-dimensional k-algebra $R_\mathcal{U}$ such that $\operatorname{Add}_{R_\mathcal{U}} M$ has strictly less finitely generated indecomposable $R_\mathcal{U}$-modules (or equivalently, R-modules) than $\operatorname{Add}_R T$, and T is equivalent to $\mathbf{t}_\lambda(Y) \oplus M$ as R-modules.

Step 3: Now we apply Propositions 4.3 and 4.4 repeatedly until we obtain that T is equivalent to $Y \oplus M$ where M is a tilting module over a universal localization $R_\mathcal{U}$ at a set of quasi-simple modules and $\operatorname{Add}_{R_\mathcal{U}} M$ does not contain finitely generated $R_\mathcal{U}$-modules. Note that $\mathcal{U}$ is a union of quasi-simples from different tubes, and it does not contain a complete clique by Corollary 3.5. Thus $R_\mathcal{U}$ is a tame hereditary algebra, and Step 1 yields that M is equivalent either to the Lukas tilting module over $R_\mathcal{U}$, or to a tilting module of the form $(R_\mathcal{U})_{\mathcal{V}'} \oplus (R_\mathcal{U})_{\mathcal{V}'}/R_\mathcal{U}$ for a set $\mathcal{V}'$ of quasi-simple $R_\mathcal{U}$-modules which is a union of cliques over $R_\mathcal{U}$. In the first case we know from [3, Theorem 6] that M is equivalent to $\mathbf{L} \otimes_R R_\mathcal{U}$. In the second case we apply Proposition 1.10. We have $\mathcal{V}' = \{V \otimes R_\mathcal{U} \mid V \in \widehat{\mathcal{V}}\}$ where $\widehat{\mathcal{V}}$ is a set of quasi-simple R-modules with the property that for any λ either $\widehat{\mathcal{V}} \cap \mathbf{t}_\lambda$ is empty, or it is the complement of $\mathcal{U} \cap \mathbf{t}_\lambda$. Then $(R_\mathcal{U})_{\mathcal{V}'} \cong R_\mathcal{V}$ where $\mathcal{V} = \mathcal{U} \cup \widehat{\mathcal{V}}$ is a set of quasi-simple R-modules containing complete cliques. Thus the proof is complete.□

Corollary 4.6. Let T be a tilting R-module which is not equivalent to a finite dimensional tilting module, and let

$$T = \bigoplus_{\lambda \in \mathbb{P}} \mathbf{t}_\lambda(T) \oplus \overline{T}$$

be a decomposition as in Theorem 3.10. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \operatorname{mod} R$. The following hold true.

(1) If $\mathcal{S}$ contains no complete rays, then T is equivalent to a tilting module of the form $Y \oplus (\mathbf{L} \otimes R_\mathcal{U})$ where Y is a finitely generated regular module, and $\mathcal{U}$ is a set of quasi-simple modules that contains no complete cliques. Moreover,
   (a) $\mathcal{F} \cap \mathcal{B}$ consists of the torsion-free $R_\mathcal{U}$-modules with no direct summand from $\mathbf{p} \otimes_R R_\mathcal{U}$.
   (b) $\overline{T}$ is equivalent to the Lukas tilting module over $R_\mathcal{U}$.

(2) If $\mathcal{S}$ contains some rays, then T is equivalent to a tilting module of the form $Y \oplus R_\mathcal{V} \oplus R_\mathcal{V}/R_\mathcal{U}$ where Y is a finitely generated regular module, and $\mathcal{U} \subset \mathcal{V}$ are sets of quasi-simple modules such that $\mathcal{U}$ contains no complete clique and $\mathcal{V}$ contains complete cliques. Moreover,
   (a) $\mathcal{F} \cap \mathcal{B}$ consists of the torsion-free $R_\mathcal{V}$-modules.
   (b) $\overline{T}$ is a projective generator for $R_\mathcal{V}$.

Proof: Following the proof of Theorem 4.5, we see that $\mathcal{S}$ contains no complete rays (respectively, does contain some ray) if and only if T is equivalent to a tilting module as in (1) (respectively, (2)). Observe further that, given a subset $\mathcal{Y} \subset \mathbb{U}$, an $R_{\mathcal{Y}}$-module X, and a quasi-simple S, we have

$$(*) \qquad \mathrm{Hom}_R(S, X) \cong \mathrm{Hom}_{R_{\mathcal{Y}}}(S \otimes_R R_{\mathcal{Y}}, X).$$

In case (1), $R_{\mathcal{U}}$ is a tame hereditary algebra with preprojective component $\mathbf{p} \otimes_R R_{\mathcal{U}}$, and $\{S \otimes_R R_{\mathcal{U}} \mid S \in \mathbb{U} \setminus \mathcal{U}\}$ is a complete irredundant set of quasi-simple $R_{\mathcal{U}}$-modules, cf. Proposition 1.10(1) and [17, 10.1]. So, $(*)$ shows that an $R_{\mathcal{U}}$-module is torsion-free over $R_{\mathcal{U}}$ if and only if it is torsion-free over R.

Now assume that $X \in \mathcal{F} \cap \mathcal{B}$. Then X is generated by $\mathbf{L} \otimes_R R_{\mathcal{U}}$, thus $X \in \mathcal{U}^{\perp}$ because the same holds true for the $R_{\mathcal{U}}$-module $\mathbf{L} \otimes R_{\mathcal{U}}$. Hence X is an $R_{\mathcal{U}}$-module which is generated by $\mathbf{L} \otimes_R R_{\mathcal{U}}$. Since $\mathbf{L} \otimes_R R_{\mathcal{U}}$ is equivalent to the Lukas tilting module over $R_{\mathcal{U}}$ [3, Theorem 6], it follows that X has no direct summand in $\mathbf{p} \otimes_R R_{\mathcal{U}}$. Conversely, if X is a torsion-free $R_{\mathcal{U}}$-module which has no direct summand from $\mathbf{p} \otimes_R R_{\mathcal{U}}$, then it is generated by the Lukas tilting module over $R_{\mathcal{U}}$, whence $X \in \mathrm{Gen}\,(\mathbf{L} \otimes_R R_{\mathcal{U}}) \subset \mathcal{B}$.

For assertion (b), we have to prove that

$$\mathrm{Gen}\,\overline{T}_{R_{\mathcal{U}}} = \{X \in \mathrm{Mod}R_{\mathcal{U}} \mid \mathrm{Ext}^1_{R_{\mathcal{U}}}(\overline{T}, X) = 0\} = \mathrm{Gen}\,(\mathbf{L} \otimes_R R_{\mathcal{U}})_{R_{\mathcal{U}}}$$

First of all, note that $\overline{T}$ is an $R_{\mathcal{U}}$-module by (a), and that the functors $\mathrm{Ext}^1_{R_{\mathcal{U}}}$ and Ext^1_R coincide on $\mathrm{Mod}R_{\mathcal{U}}$ by [33, 4.7]. Thus $\mathrm{Ext}^1_{R_{\mathcal{U}}}(\overline{T}, \overline{T}^{(\alpha)}) = 0$ for every cardinal α, and $\mathrm{Gen}\,\overline{T}_{R_{\mathcal{U}}} \subset \{X \in \mathrm{Mod}R_{\mathcal{U}} \mid \mathrm{Ext}^1_{R_{\mathcal{U}}}(\overline{T}, X) = 0\}$. Next, assume that $X \in \mathrm{Mod}R_{\mathcal{U}}$ such that $\mathrm{Ext}^1_{R_{\mathcal{U}}}(\overline{T}, X) = 0$. Then $X \in \mathcal{U}^{\perp} \subset Y^{\perp}$, because it follows from the construction of $\mathcal{U}$ in the proof of Theorem 4.5 that Y belongs to the extension closure of $\mathcal{U}$. Now, $T = \bigoplus_{\lambda \in \mathbb{P}} \mathbf{t}_\lambda(T) \oplus \overline{T} = \bigoplus_{\lambda \in \mathbb{P}} \mathbf{t}_\lambda(Y) \oplus \overline{T} = Y \oplus \overline{T}$, cf. Theorem 3.10(3). So, we infer that $X \in T^{\perp} = Y^{\perp} \cap (\mathbf{L} \otimes R_{\mathcal{U}})^{\perp} \subset \mathrm{Gen}\,(\mathbf{L} \otimes_R R_{\mathcal{U}})_{R_{\mathcal{U}}}$. For the remaining part observe that $\mathrm{Add}(\mathbf{L} \otimes_R R_{\mathcal{U}}) = \mathrm{Add}\,\overline{T}$ by Proposition 3.3, hence $\mathrm{Gen}\,(\mathbf{L} \otimes_R R_{\mathcal{U}})_{R_{\mathcal{U}}} = \mathrm{Gen}\,\overline{T}_{R_{\mathcal{U}}}$, which completes the proof of (b).

We now turn to case (2). Here $R_{\mathcal{V}}$ is a hereditary order in $R_{\mathbb{U}}$ by [13, 4.2], and $\{S \otimes_R R_{\mathcal{V}} \mid S \in \mathbb{U} \setminus \mathcal{V}\}$ is a complete irredundant set of simple $R_{\mathcal{V}}$-modules, cf. Proposition 1.10(2). Moreover, by definition an $R_{\mathcal{V}}$-module X is torsion-free if its torsion submodule $\{x \in X \mid xs = 0 \text{ for some regular element } s \in R_{\mathcal{V}}\}$ is zero, or equivalently, if the canonical map $X \to X \otimes_R R_{\mathbb{U}}$ is an embedding.

If $X \in \mathcal{F} \cap \mathcal{B}$, then $X \in \mathrm{Gen}\,R_{\mathcal{V}}$. Thus $X \in \mathcal{V}^{\perp}$ because the same holds true for $R_{\mathcal{V}}$. Hence X is an $R_{\mathcal{V}}$-module, which is torsion-free as $X \hookrightarrow X \otimes_R R_{\mathbb{U}}$ by Proposition 1.7. For the converse, having $\mathrm{Gen}\,R_{\mathcal{V}} \subset \mathcal{B}$, it is enough to show that any torsion-free $R_{\mathcal{V}}$-module is also torsion-free over R. This is clear in case $\mathcal{V} = \mathbb{U}$, so we can assume w.l.o.g. that $\mathcal{V}$ is properly contained in $\mathbb{U}$. Then, as is well known, all simple $R_{\mathcal{V}}$-modules are torsion, so $(*)$ yields the claim, and (a) is verified.

For assertion (b), we show as in case (1) that $\overline{T}$ is an $R_{\mathcal{V}}$-module such that $\mathrm{Add}\,\overline{T} = \mathrm{Add}\,R_{\mathcal{V}}$. □

We remark that the projective $R_{\mathcal{V}}$-modules are well understood, see for example [26] and [36, §4].

Example 4.7. Let $\mathbf{t}_\lambda$ be a tube of rank $r > 1$, and let S be a quasi-simple module in $\mathbf{t}_\lambda$. If T is a tilting module with $\mathcal{S} = \mathrm{add}\,(\mathbf{p} \cup \{S[n] \mid n \in \mathbb{N}\})$, then

$$T \sim S \oplus S[2] \oplus \ldots \oplus S[r-1] \oplus S[\infty] \oplus R_{\mathbf{t}_\lambda}.$$

Indeed, we have already computed $Y = \mathbf{t}_\lambda(T) = S \oplus S[2] \oplus \ldots \oplus S[r-1]$ in 3.9. So, following the proof of Theorem 4.5, we choose the numbering $S = U_1, U_2 = \tau^- U_1, \ldots, U_r = \tau^- U_{r-1}$ for the quasi-simples in $\mathbf{t}_\lambda$, and we set $\mathcal{U} = \{U_2, \ldots, U_r\}$ and $R' = R_\mathcal{U}$. Then $T \sim S \oplus S[2] \oplus \ldots \oplus S[r-1] \oplus M$ where M is a tilting R'-module whose tilting class ${\mathcal{S}'}^\perp$ is given by $\mathcal{S}' = \{A \otimes R' \mid A \in \mathcal{S}\} = \operatorname{add}(\{P \otimes R' \mid P \in \mathbf{p}\} \cup \{S[n] \otimes R' \mid n \in \mathbb{N}\})$. Note that $\{P \otimes R' \mid P \in \mathbf{p}\}$ is the preprojective component of R' by [17, 10.1], and $\{S[n] \otimes R' \mid n \in \mathbb{N}\}$ is a homogeneous tube with mouth $\mathcal{V}' = \{S \otimes R'\}$. Hence $M \cong R'_{\mathcal{V}'} \oplus R'_{\mathcal{V}'}/R' \cong R_\mathcal{V} \oplus R_\mathcal{V}/R_\mathcal{U}$ where $\mathcal{V} = \mathcal{U} \cup \{S\} = \{U_1, \ldots, U_r\}$. We conclude that $R_\mathcal{V} = R_{\mathbf{t}_\lambda}$. Moreover, we deduce from Proposition 1.8 that $R_\mathcal{V}/R_\mathcal{U} \cong (R_{\mathbf{t}_\lambda}/R)/(R_\mathcal{U}/R)$ is a direct sum of copies of $S[\infty]$. This proves the claim. □

We now turn to the tilting modules arising from ring epimorphisms studied in [5].

Corollary 4.8. Let T be a tilting R-module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^\perp$ and $\mathcal{S} = {}^\perp\mathcal{B} \cap \operatorname{mod} R$. The following statements are equivalent.

(1) There exists an injective ring epimorphism $\lambda\colon R \to S$ such that $\operatorname{Tor}_1^R(S,S) = 0$ and $S \oplus S/R$ is a tilting R-module equivalent to T.
(2) T is equivalent to a tilting module $T_\mathcal{U} = R_\mathcal{U} \oplus R_\mathcal{U}/R$ with $\mathcal{U} \subseteq \mathbb{U}$.

Moreover, under these conditions, $\mathcal{S}$ must contain some rays.

Proof: In [24, Theorem 6.1] it is proved that λ as in (1) can be chosen as a universal localization of R. We will give a different proof for that and also show that T is equivalent to $T_\mathcal{U}$ as stated.
By Proposition 3.3, the modules $S, S/R \in \operatorname{Add} T$ both are direct sums of their torsion part and their torsion-free part. We denote by $\overline{S}$ the torsion-free part of S and observe that

$$\mathcal{B} = \operatorname{Gen} T = \operatorname{Gen} S_R = (S/R)^\perp, \text{ and in particular, } \mathcal{F} \cap \mathcal{B} \subseteq \operatorname{Gen} \overline{S}.$$

Suppose that $\mathcal{S}$ contains no complete ray. Then T is equivalent to a tilting R-module of the form $Y \oplus (\mathbf{L} \otimes_R R_\mathcal{U})$ as in Corollary 4.6(1). Since $\mathbf{L} \otimes_R R_\mathcal{U} \in \mathcal{F} \cap \mathcal{B}$, we have $\overline{S} \neq 0$. Moreover, it follows from Proposition 3.3 that any torsion-free module in $\operatorname{Add} T$ belongs to $\operatorname{Add}(\mathbf{L} \otimes_R R_\mathcal{U})$, and any torsion module in $\operatorname{Add} T$ belongs to $\operatorname{Add} Y$. Now $\mathbf{L} \otimes_R R_\mathcal{U}$ is equivalent to the Lukas tilting module over the tame hereditary algebra $R_\mathcal{U}$ by [3, Theorem 6], and we know from [27, Lemma 3.3(a)] that $\operatorname{Hom}_{R_\mathcal{U}}(A,B) \neq 0$ for any two nonzero $A, B \in \operatorname{Add}(\mathbf{L} \otimes_R R_\mathcal{U})$. This shows that any torsion-free module $0 \neq A \in \operatorname{Add} T$ satisfies $\operatorname{Hom}_R(A, \overline{S}) \neq 0$. Note that $\operatorname{Hom}_R(S/R, S) = 0$, see for example [5, 2.6]. So, we infer that S/R is a torsion module, hence $S/R \in \operatorname{Add} Y$. In particular, it follows that $S/R^\perp = Y^\perp$. By a well known result of Bongartz [9], the finite dimensional module Y can be completed to a finite dimensional tilting module T' with tilting class $\operatorname{Gen} T' = Y^\perp$. But then $\operatorname{Gen} T = (S/R)^\perp = \operatorname{Gen} T'$, contradicting the assumption that T is not equivalent to a finite-dimensional tilting R-module.

So, $\mathcal{S}$ contains some rays, and T must be equivalent to a tilting module of the form $Y \oplus R_\mathcal{V} \oplus R_\mathcal{V}/R_\mathcal{U}$ as in Corollary 4.6(2). Since $R_\mathcal{V} \in \mathcal{F} \cap \mathcal{B}$, we have $\overline{S} \neq 0$. Moreover, it follows from Proposition 3.3 that any torsion-free module $A \in \operatorname{Add} T$ belongs to $\operatorname{Add} R_\mathcal{V}$ and is therefore a projective $R_\mathcal{V}$-module. So $0 \neq A \in \mathcal{F} \cap \mathcal{B} \subseteq \operatorname{Gen} \overline{S}$ implies $A \in \operatorname{Add} \overline{S}$ and $\operatorname{Hom}_R(A, \overline{S}) \neq 0$. Again, from $\operatorname{Hom}_R(S/R, S) = 0$ we infer that S/R is a torsion module, hence a direct sum of Prüfer modules and finite-dimensional torsion modules. Observe that if $S[\infty] \in \operatorname{Add} T$ belongs to a tube of rank $r > 1$, then it is filtered

by $S[r]$, which belongs to $\{S[n] \mid n \geq 1\} \subseteq \mathcal{S}$ by (the proof of) Theorem 3.10. Thus S/R is filtered by non-projectives in $\mathcal{S}$, and we can assume that λ is a universal localization by [1, Corollary 2.5].

We then know from [36, 2.3] that T is equivalent to $R_{\mathbf{E}} \oplus R_{\mathbf{E}}/R$ for some full exact abelian subcategory $\mathbf{E}$ of $\operatorname{mod} R$ which is closed under extensions. By [36, 2.6] and [5, 4.12 and 4.13] we have $T^{\perp} = \mathbf{E}^{\perp}$, hence $\mathcal{S} = {}^{\perp}(\mathbf{E}^{\perp}) \cap \operatorname{mod} R$ contains $\mathbf{E}$, and from the bijection between resolving subcategories of $\operatorname{mod} R$ and tilting classes given in Theorem 2.1 we infer that $\mathcal{S}$ is the resolving closure of $\mathbf{E}$. In particular, it follows that $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{E})$. So, the set $\mathbf{t}' = \mathcal{S} \cap \mathbf{t}$ coincides with $\mathbf{E} \cap \mathbf{t}$ and is therefore closed under cokernels.

We claim that T is equivalent to $T_{\mathcal{U}}$ where $\mathcal{U}$ is the set of quasi-simple modules in $\mathbf{t}'$. Indeed, $\operatorname{Gen} T = \mathbf{t}'^{\perp}$ as $\mathbf{t}'$ contains a complete ray, cf. Example 2.4. We now verify $\mathbf{t}'^{\perp} = \mathcal{U}^{\perp}$. Take $S[m] \in \mathbf{t}'$. Since $\mathcal{S}$ is closed under submodules, all $S[n]$ with $n \leq m$ are in $\mathbf{t}'$ as well, and so are the cokernels of the inclusions $S[n] \hookrightarrow S[n']$ for $n < n' \leq m$. Thus $\mathbf{t}'$ contains the wing $\mathcal{W}_{S[m]}$, and $\mathcal{U}$ contains the quasi-simples from that wing. But then $\mathcal{U}^{\perp} \subset S[m]^{\perp}$, and the proof is complete.□

We know from [3, Corollary 9] that the Lukas tilting module $\mathbf{L}$ is noetherian over its endomorphism ring. The following result generalizes this.

Corollary 4.9. Let T be a tilting R-module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^{\perp}$ and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. The following statements are equivalent.

(1) T is noetherian over its endomorphism ring.
(2) T is equivalent to a tilting module $Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ as in Corollary 4.6(1).
(3) $\mathcal{S}$ contains no complete rays.

Proof: We know from [2, 9.9] that T is noetherian over its endomorphism ring if and only if $D(T)$ is Σ-pure-injective. For (2)⇒(1), we proceed as in the proof of [3, Corollary 9]. Suppose that T is equivalent to $Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$. Then $\operatorname{Add} T = \operatorname{Add}(Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}}))$, and $D(T) \in \operatorname{Prod}(D(Y) \oplus D(\mathbf{L} \otimes_R R_{\mathcal{U}}))$. Since Y is finite dimensional, Y and $D(Y)$ are (right and left, respectively) endofinite modules. Moreover, by Lemma 2.5, the dual of the Lukas tilting $R_{\mathcal{U}}$-module $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is a cotilting $R_{\mathcal{U}}$-module whose cotilting class is the class of $R_{\mathcal{U}}$-modules without preinjective summands. Then $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is equivalent to the Reiten-Ringel cotilting module over $R_{\mathcal{U}}$ and therefore it is a Σ-pure-injective $R_{\mathcal{U}}$-module. By [16, 1.36], $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is also Σ-pure-injective over R. Hence $D(T)$ is Σ-pure-injective, and the claim is proven.

For the remaining implications, we show that T is not noetherian over its endomorphism ring whenever it is equivalent to a tilting module $Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ as in Corollary 4.6(1). Indeed, in the latter case, the indecomposable direct summands of the torsion part of $Y \oplus R_{\mathcal{V}} \oplus R_{\mathcal{V}}/R_{\mathcal{U}}$ are direct summands of T. Observe that $R_{\mathcal{V}}/R_{\mathcal{U}}$ is a non-trivial direct sum of Prüfer modules by Proposition 1.8. Hence $D(T)$ has an adic module as a direct summand. Since adic modules are not Σ-pure-injective modules, $D(T)$ is not. □

Let us now describe the dual property.

Corollary 4.10. Let T be a tilting R-module which is not equivalent to a finite dimensional tilting module. Set $\mathcal{B} = T^{\perp}$ and $\mathcal{S} = {}^{\perp}\mathcal{B} \cap \operatorname{mod} R$. The following statements are equivalent.

(1) T is (Σ-)pure-injective.

(2) $G \in \operatorname{Add} T$.
(3) $\mathcal{S}$ contains a complete ray from each tube.
(4) T is a cotilting right R-module with ${}^{\perp}T = {}^{\perp}\mathcal{B}$
(5) There exists a cotilting left R-module C such that $D(C)$ is a tilting module equivalent to T.

Proof: (1)⇒(2) and (3): Recall that $T = Y \oplus Z \oplus \overline{T}$ where Y is a finite-dimensional module, Z is a direct sum of Prüfer modules, and $\overline{T}$ is a non-zero torsion-free module. Now Y is endofinite, hence Σ-pure-injective, and $Z \in \operatorname{Add} \mathbf{W}$ is (Σ-)pure-injective by [30, 10.1]. So, we have that T is (Σ-)pure-injective if and only if so is $\overline{T}$.
By Corollary 4.6, either $\overline{T}$ is equivalent to the Lukas tilting module over the tame hereditary algebra $R_{\mathcal{U}}$, where $\mathcal{U}$ is a set of quasi-simple modules that contains no complete cliques, or $\overline{T}$ is a projective generator for $R_{\mathcal{V}}$, where $\mathcal{V}$ is a set of quasi-simple modules that contains complete cliques.
In the first case, we know from [3, Proposition 7 and Example 8] that $\operatorname{Add} \overline{T}$ does not contain indecomposable pure-injective $R_{\mathcal{U}}$-modules, and therefore $\overline{T}$ is not a pure-injective $R_{\mathcal{U}}$-module. By [21, 8.62], an $R_{\mathcal{U}}$-module is pure-injective over $R_{\mathcal{U}}$ if and only if it is pure-injective over R. So, we conclude that T is not pure-injective.
Let us consider the second case. If $\mathcal{V}$ is properly contained in $\mathbb{U}$, then $R_{\mathcal{V}}$ is a hereditary order in $R_{\mathbb{U}}$ which is not simple artinian, and from the classification of the indecomposable pure-injective $R_{\mathcal{V}}$-modules in [29, 3.3] we know that no projective $R_{\mathcal{V}}$-module can be pure-injective. So, $\overline{T}$ can only be pure-injective if $\mathcal{V} = \mathbb{U}$ and $\overline{T} \in \operatorname{Add} G$.
In particular, we see that $\operatorname{Add} T$ contains G, but does not contain adic modules. On the other hand, the class $\mathcal{F}$ of all torsion-free modules coincides with ${}^{\perp}(G^{\perp})$ by [3, Proposition 7] and is therefore contained in ${}^{\perp}(T^{\perp})$. We infer that all adic modules are in ${}^{\perp}(T^{\perp}) \setminus \mathcal{B}$. By Lemma 3.8 it follows that $\mathcal{S}$ contains a complete ray from each tube.
(3)⇒(2): The assumption means by Lemma 3.8 that $\mathcal{B}$ does not contain any adic module. Then the pure-injective envelope I of $\overline{T}$, which is contained in $\mathcal{F} \cap \mathcal{B}$, is a direct sum of copies of G. Since G is Σ-pure-injective, all pure submodules of I are direct summands. Hence $\overline{T} \in \operatorname{Add} G$, and $G \in \operatorname{Add} T$.
(2)⇒(1): If $G \in \operatorname{Add} T$, then $\overline{T} \in G^{\perp}$ is Σ-pure-injective by [3, Proposition 7].
(1)⇒(4): Since T is Σ-pure-injective, any module in $\operatorname{Add} T$ is also Σ-pure-injective, and thus every pure embedding into a module in $\operatorname{Add} T = \mathcal{B} \cap {}^{\perp}\mathcal{B}$ splits. Hence [6, Corollary 2.3] implies that ${}^{\perp}\mathcal{B}$ is closed under direct limits. Now (4) follows from [6, Corollary 3.3].
(4)⇒(1): Every cotilting module over an arbitrary ring is pure-injective by [11].
(3)⇒(5): By the all the foregoing, we can suppose that T is equivalent to $X = Y \oplus G \oplus \bigoplus_{S \in \mathcal{Y}} S[\infty]$ where Y is a finite-dimensional module and $\mathcal{Y} \subseteq \mathbb{U}$. Note that, for each $\lambda \in \mathbb{P}$, $\mathbf{t}_{\lambda}(X)$ has precisely r_{λ}, the rank of $\mathbf{t}_{\lambda}$, pairwise non-isomorphic direct summands. Observe that DX is a cotilting module. By [10, 3.9], (the r_{λ}) nonisomorphic direct summands of DX that belong to ${}_R\mathbf{t}_{\lambda}$ are precisely the duals of the nonisomorphic direct summands of $\mathbf{t}_{\lambda}(X)$. Hence, again by [10, 3.9], DX is equivalent to $C = DY \oplus G \oplus \bigoplus_{S \in \mathcal{Y}} (DS)[-\infty]$.
Condition (4) implies that $\operatorname{Add} T = \operatorname{Prod} T$ by [6, Corollary 3.3]. Therefore DC is a tilting module equivalent to T.
(5)⇒(1): The dual of any left R-module is a pure-injective right R-module. □

5. Appendix: The classification of cotilting modules

Combining work of Buan and Krause [10, 3.9] with some combinatorial arguments form [11] and with Bazzoni's result [7] stating that every cotilting module over an arbitrary ring is pure-injective, one obtains a classification of cotilting modules over tame hereditary algebras which we recall below. We now recover this classification by an elementary proof that only uses the results from Sections 2 and 3.

Theorem 5.1. Let C be a cotilting left R-module with an indecomposable direct summand which is not finitely generated. The following hold true:

(I) Each indecomposable direct summand of C is either generic or of the form $S[n]$ for some quasi-simple left R-module S and some $n \in \mathbb{N} \cup \{\infty, -\infty\}$.

(II) For each tube ${}_R\mathbf{t}_\lambda$, $\lambda \in \mathbb{P}$, let $\mathcal{I}_\lambda$ be the set of non-isomorphic indecomposable direct summands of C which are of the form $S[n]$ for some $n \in \mathbb{N} \cup \{\infty, -\infty\}$ and quasi-simple $S \in {}_R\mathbf{t}_\lambda$. Then the number of elements in $\mathcal{I}_\lambda$ equals the rank of ${}_R\mathbf{t}_\lambda$.

Proof: Let us fix a cotilting left R-module C having an indecomposable direct summand which is not finitely generated. We know from Theorem 2.1 that the cotilting class ${}^\perp C$ is of the form ${}^\perp(\mathcal{S}^*)$ where

$$\mathcal{S} = {}^\intercal({}^\perp C) \cap \operatorname{mod} R$$

is a resolving subcategory of $\operatorname{mod} R$. Furthermore, if T is a tilting module with tilting class $\mathcal{S}^\perp$, then we know from Lemma 2.5 that $D(T)$ is a cotilting module equivalent to C. More precisely, denoting as before $\mathcal{B} = T^\perp$, we have

$${}^\perp C = {}^\perp(\mathcal{S}^*) = \{{}_R X \mid D(X) \in \mathcal{B}\}.$$

Moreover,

$$({}^\perp C)^\perp \cap R\text{mod} = \mathcal{S}^*$$

because every finitely generated left R-module X is of the form $X = D(W)$ for some $W \in \operatorname{mod} R$, and the condition $D(W) \in ({}^\perp C)^\perp$ means by the Ext-Tor-relations that $W \in {}^\intercal({}^\perp C) \cap \operatorname{mod} R = \mathcal{S}$.

Recall that the modules in $\operatorname{Prod} C = \operatorname{Prod} D(T)$ are pure-injective. In particular, this implies that ${}^\perp C$ is closed under direct limits. Since ${}^\perp C$ is also closed under submodules, it follows that ${}^\perp C = \varinjlim({}^\perp C \cap R\text{mod})$, see [10, 1.1]. If I is a pure-injective module left R-module, we thus have

$$(\sharp) \qquad I \in ({}^\perp C)^\perp \text{ if and only if } \operatorname{Ext}^1_R(A, I) = 0 \text{ for all } A \in {}^\perp C \cap R\text{mod}.$$

Step 1: We compute the indecomposable modules in $\operatorname{Prod} C$. First of all, we have

(0) $\operatorname{Prod} C \cap R\text{mod} = \{D(W) \mid W \in \operatorname{Add} T \cap \operatorname{mod} R\}$.

In fact, if X is a finitely generated left R-module of the form $X = D(W)$ with $W \in \operatorname{mod} R$, then by the observations above, the condition $X \in \operatorname{Prod} C = {}^\perp C \cap ({}^\perp C)^\perp$ means that $W \in \mathcal{S} \cap \mathcal{B} = \operatorname{Add} T \cap \operatorname{mod} R$, so (0) is verified.

Recall that there are at most finitely many non-isomorphic finitely generated indecomposable right R-modules in $\operatorname{Add} T$. As before, we denote by Y the direct sum of a complete irredundant set of such modules. Then $D(Y)$ is the direct sum of a complete irredundant set of finitely generated indecomposable left R-modules in $\operatorname{Prod} C$.

Next, we compute the adics and the Prüfer modules in $\operatorname{Prod} C$. Observe that adic and Prüfer modules are dual to each other. So, Lemma 3.8, Theorem 3.10 and Remark 3.11 yield

$$D(T) \cong \prod_{\lambda\in\Lambda} D(\mathbf{t}_\lambda(T)) \oplus D(\overline{T})$$

where $D(\overline{T})$ is divisible without finite dimensional direct summands, hence a direct sum of Prüfer modules and copies of G, and $D(\mathbf{t}_\lambda(T))$ is a direct product of copies of the indecomposable direct summands of $D(\mathbf{t}_\lambda(Y))$ and of adic modules belonging to the corresponding tube ${}_R\mathbf{t}_\lambda$ in $R\text{mod}$. More precisely, the following statements hold true for a tube ${}_R\mathbf{t}_\lambda$ of rank r:

(1) $D(\mathbf{t}_\lambda(T)) = 0$ whenever ${}_R\mathbf{t}_\lambda \cap \mathcal{S}^* = \emptyset$;
(2) if $\mathcal{S}^*$ contains some corays from ${}_R\mathbf{t}_\lambda$, then ${}^\perp C$ does not contain any Prüfer module belonging to ${}_R\mathbf{t}_\lambda$, and $D(\mathbf{t}_\lambda(T))$ has precisely r pairwise non-isomorphic indecomposable summands: these are the s adic modules corresponding to the $s \leq r$ corays from ${}_R\mathbf{t}_\lambda$ contained in $\mathcal{S}^*$, and $r - s$ modules from ${}_R\mathbf{t}_\lambda$;
(3) if $\mathcal{S}^*$ contains s modules from ${}_R\mathbf{t}_\lambda$, but no complete coray, then ${}^\perp C$ contains precisely $r - s$ pairwise non-isomorphic Prüfer modules belonging to ${}_R\mathbf{t}_\lambda$, and $D(\mathbf{t}_\lambda(T))$ is a direct sum of copies of s pairwise non-isomorphic modules from ${}_R\mathbf{t}_\lambda$.

It remains to show that an indecomposable module belongs to $\operatorname{Prod} C$ if and only if it is isomorphic to a module in the following list:

- the indecomposable summands of $D(\mathbf{t}_\lambda(T))$, $\lambda \in \mathbb{P}$,
- the Prüfer modules in ${}^\perp C$,
- the generic left R-module ${}_RG$.

For the if-part, we verify that all these modules belong to $\operatorname{Prod} C$. This is clear for the indecomposable summands of $D(\mathbf{t}_\lambda(T))$, $\lambda \in \mathbb{P}$. For the other modules, recall first from Theorem 2.7 that $\mathcal{B} \subset \mathbf{p}^\perp$, and $\mathcal{S} = \operatorname{add}(\mathbf{p} \cup \mathbf{t}')$ for some subset $\mathbf{t}' \subset \mathbf{t}$. Then ${}^\perp C \subset {}^\perp{}_R\mathbf{q}$, and $\mathcal{S}^* = \operatorname{add}({}_R\mathbf{q} \cup \mathbf{t}'')$ for some subset $\mathbf{t}'' \subset {}_R\mathbf{t}$. Thus every $A \in {}^\perp C \cap R\text{mod}$ belongs to ${}_R\mathbf{p} \cup {}_R\mathbf{t}$, and $\operatorname{Ext}^1_R(A, I) = 0$ for any divisible module I without indecomposable preprojective summands. In particular, we deduce from (♯) that all Prüfer modules and the generic module ${}_RG$ belong to $({}^\perp C)^\perp$. Furthermore, since ${}_RG$ is a torsion-free module without indecomposable preinjective summands, we also have ${}_RG \in {}^\perp(\mathcal{S}^*) = {}^\perp C$. This shows that all modules in our list belong to $\operatorname{Prod} C$.
For the only-if-part, let X be an indecomposable module in $\operatorname{Prod} C$. Then X is pure-injective, and we can assume w.l.o.g. that X is neither a Prüfer module nor generic. If X is finite dimensional, then by (0) it is isomorphic to a finite dimensional indecomposable summand of $D(\mathbf{t}_\lambda(T))$ for some $\lambda \in \mathbb{P}$. If $X = S[-\infty]$ is an adic module, then the class $({}^\perp C)^\perp$, being closed under epimorphic images, must contain the whole coray ending at S. So X is the adic module corresponding to a coray in $\mathcal{S}^*$, and by (2) it is isomorphic to an indecomposable summand of $D(\mathbf{t}_\lambda(T))$ for some $\lambda \in \mathbb{P}$. This completes the proof of the claim.

Step 2: Now statement (I) in the Theorem follows immediately from Step 1. For statement (II), we fix a tube ${}_R\mathbf{t}_\lambda$ of rank r and let $\mathcal{I}_\lambda$ be the set of non-isomorphic indecomposable direct summands of C which are adic, Prüfer or finite dimensional modules belonging to ${}_R\mathbf{t}_\lambda$. We have to verify that $\mathcal{I}_\lambda$ has precisely r elements. By (2) and (3) in Step 1, we need only to show that all Prüfer modules in $\operatorname{Prod} C$ belonging to ${}_R\mathbf{t}_\lambda$ and all indecomposable summands of $D(\mathbf{t}_\lambda(T))$ occur as direct summands

of C and therefore are elements of $\mathcal{I}_\lambda$. Note that this is clear for the finite dimensional summands, as it is well known that a finite dimensional indecomposable module arises as a direct summand of a product of modules $\prod C_j$ if and only if it is a direct summand of one of the factors C_j. For the other modules, we distinguish two cases.

(i) Suppose first that $\mathcal{S}^*$ contains $s > 0$ corays from ${}_R\mathbf{t}_\lambda$. By (2) we have only to consider the s adic modules corresponding to these corays. Let X be one of these adic modules. Then there is a quasi-simple right R-module $S \in \mathbf{t}_\lambda$ such that $X = D(S[\infty])$ and $\mathcal{S}$ contains the complete ray starting at S. Choose the numbering $S = U_1,\ U_2 = \tau^- U_1,\ \ldots,\ U_r = \tau^-[U_{r-1}]$ for the quasi-simple modules in $\mathbf{t}_\lambda$. Then we have the numbering $D(U_r) = \tau[D(U_{r-1})],\ \ldots,\ D(U_2) = \tau[D(U_1)],\ D(U_1) = D(S)$ for the quasi-simple modules in ${}_R\mathbf{t}_\lambda$.
Let m be the greatest positive integer such that $S[m] \in \operatorname{Add} T$, or $m = 0$ if $S[m] \notin \operatorname{Add} T$ for all $m \geq 1$. Consider $A = D(U_r[m+1]) \in {}_R\mathbf{t}_\lambda$. Since $S = \tau^- U_r$, we have $\operatorname{Ext}^1_R(A, X) = \operatorname{Ext}^1_R(S[\infty], U_r[m+1]) = D\operatorname{Hom}_R(S[m+1], S[\infty]) \neq 0$. But this means that $A \notin {}^\perp C$ as $X \in \operatorname{Prod} C$. Now ${}^\perp C = \bigcap_N {}^\perp N$ where N runs through all indecomposable direct summands of C by [10, 2.2]. Thus there must be an indecomposable direct summand N of C with $\operatorname{Ext}^1_R(A, N) \neq 0$, and of course, N cannot be divisible, nor can it belong to a tube ${}_R\mathbf{t}\mu$ with $\mu \neq \lambda$, so N is a finite dimensional or an adic module belonging to ${}_R\mathbf{t}_\lambda$.
Note that $D\operatorname{Hom}_R(N, \tau A) \cong \operatorname{Ext}^1_R(A, N) \neq 0$, and $\tau A \cong D(S[m+1])$ lies on the coray ending at $D(S)$. Moreover, $U_2[m], U_3[m-1], \ldots, U_{m+1} \notin \mathcal{S}$ by Lemma 3.4(2), hence $D(U_2[m]), D(U_3[m-1]), \ldots, D(U_{m+1}) \notin \mathcal{S}^*$. Since the finite dimensional quotients of N lie in $({}^\perp C)^\perp \cap R\text{mod} = \mathcal{S}^*$, we deduce that N does neither lie on one of the corays ending at $D(U_2), D(U_3), \ldots, D(U_{m+1})$ nor it is an adic module determined by one of these corays. Further, N does not lie on the coray ending at $D(S)$ by choice of m. It follows that $X = D(S)[-\infty] = N$ is the desired direct summand of C.

(ii) Suppose now that $\mathcal{S}^*$ contains no complete coray from ${}_R\mathbf{t}_\lambda$. By (3) we have only to consider the $r - s$ Prüfer modules in $\operatorname{Prod} C$ belonging to ${}_R\mathbf{t}_\lambda$. Let $X = S[\infty]$ be one of these Prüfer left R-modules. Take the greatest positive integer m such that $S[m] \in \operatorname{Prod} C$, or $m = 0$ if $S[m] \notin \operatorname{Prod} T$ for all $m \geq 1$. Then $A = S[m+1] \in {}_R\mathbf{t}_\lambda$ is cogenerated by C, so there must be an indecomposable direct summand N of C with $\operatorname{Hom}_R(A, N) \neq 0$. Of course, N cannot be torsion-free, nor can it belong to a tube ${}_R\mathbf{t}\mu$ with $\mu \neq \lambda$, so N is a finite dimensional or a Prüfer module belonging to ${}_R\mathbf{t}_\lambda$. Choose the numbering $S = U_1,\ U_2 = \tau^- U_1, \ldots,\ U_r = \tau^-[U_{r-1}]$ for the quasi-simple modules in ${}_R\mathbf{t}_\lambda$. As in Lemma 3.4(2), we show that $U_2[m], U_3[m-1], \ldots, U_{m+1} \notin {}^\perp C$. Since the finite dimensional submodules of N lie in ${}^\perp C$, we deduce that N does neither lie on one of the rays starting at $U_2, U_3, \ldots, U_{m+1}$ nor it is a Prüfer module determined by one these rays. Further, N does not lie on the ray starting at S by choice of m. It follows that $X = S[\infty] = N$ is the desired direct summand of C. □

Remark 5.2. Assume that ${}_R\mathbf{t}_\lambda$ is a tube of rank r having no complete coray in $\mathcal{S}^*$ and having precisely $s \geq 0$ non-isomorphic indecomposable modules in $\operatorname{Prod} C$. As we have seen above, the set $\mathcal{I}_\lambda$ contains $r - s$ Prüfer modules. They arise as duals of the $r - s$ adic modules in $\mathcal{B}$ established by Lemma 3.8(2), see also Remark 3.11. We will now give an alternative explanation for the occurrence of these Prüfer modules by using Proposition 1.10.

Let the notation be as above. According to Theorem 4.5 and Corollary 4.6, we distinguish two cases.

(1) $\mathcal{S}^*$ contains no complete corays. Then, up to equivalence, $T = Y \oplus (\mathbf{L} \otimes_R R_{\mathcal{U}})$ as in Corollary 4.6(1). By [3, Theorem 6], $\mathbf{L} \otimes_R R_{\mathcal{U}}$ is equivalent to the Lukas tilting module over $R_{\mathcal{U}}$, so $D(\mathbf{L} \otimes_R R_{\mathcal{U}})$ is a cotilting $R_{\mathcal{U}}$-module equivalent to the Reiten-Ringel tilting $R_{\mathcal{U}}$-module $\mathbf{W}_{\mathcal{U}}$. Hence $\operatorname{Prod}_{R_{\mathcal{U}}} D(\mathbf{L} \otimes_R R_{\mathcal{U}}) = \operatorname{Prod}_{R_{\mathcal{U}}} \mathbf{W}_{\mathcal{U}} = \operatorname{Add}_{R_{\mathcal{U}}} \mathbf{W}_{\mathcal{U}}$, and $\operatorname{Prod} D(T) = \operatorname{Add}(D(Y) \oplus \mathbf{W}_{\mathcal{U}})$. Therefore any module in $\operatorname{Prod} C$ is a direct sum of indcomposable direct summands of $D(Y)$ and of Prüfer $R_{\mathcal{U}}$-modules.

By assumption, there are precisely $s \geq 0$ non-isomorphic indecomposable modules in $\mathbf{t}_\lambda \cap \operatorname{Add} T$ (in fact, in $\operatorname{Add} \mathbf{t}_\lambda(Y)$), whose duals give the indecomposables in ${}_R\mathbf{t}_\lambda \cap \operatorname{Prod} C$. By construction, $\mathcal{U} \cap \mathbf{t}_\lambda$ has s elements. Hence the $R_{\mathcal{U}}$-tube $\mathbf{t}_\lambda \otimes R_{\mathcal{U}}$ has $r - s$ quasi-simples, and $\operatorname{Prod} C$ has precisely $r - s$ Prüfer left $R_{\mathcal{U}}$-modules belonging to this tube.

(2) $\mathcal{S}^*$ contains some rays. Then, up to equivalence, $T = Y \oplus R_{\mathcal{V}}/R_{\mathcal{U}} \oplus R_{\mathcal{V}}$ as in Corollary 4.6(2). Thus $\operatorname{Prod} C = \operatorname{Prod}(D(Y) \oplus D(R_{\mathcal{V}}/R_{\mathcal{U}}) \oplus D(R_{\mathcal{V}}))$, and the Prüfer modules in $\operatorname{Prod} C$ are all in $\operatorname{Prod} D(R_{\mathcal{V}})$ because there are no nonzero morphism from a Prüfer module neither to a torsion-free module nor to a regular module.

By assumption, $\mathbf{t}_\lambda \cap \mathcal{S}$ does not contain a complete ray, and according to the construction, $\mathcal{V}$ cannot contain all quasi-simple R-modules in $\mathbf{t}_\lambda$. More precisely, $\mathbf{t}_\lambda(T)$ has s pairwise non-isomorphic indecomposable direct summands, whose duals give the indecomposables in ${}_R\mathbf{t}_\lambda \cap \operatorname{Prod} C$. They are arranged in disjoint wings from $\mathbf{t}_\lambda$, and the quasi-simple modules in $\mathcal{V} \cap \mathbf{t}_\lambda$ are precisely the quasi-simples in these wings. So, there are exactly s quasi-simple modules in $\mathcal{V} \cap \mathbf{t}_\lambda$. Each of the remaining $r - s$ quasi-simple modules $S \in \mathbf{t}_\lambda \setminus \mathcal{V}$ gives rise to a simple $R_{\mathcal{V}}$-module $S \otimes_R R_{\mathcal{V}}$ with projective presentation $0 \to \mathfrak{m} \to R_{\mathcal{V}} \to S \otimes_R R_{\mathcal{V}} \to 0$ for some maximal right ideal $\mathfrak{m}$. Applying D, we obtain the exact sequence $0 \to D(S \otimes_R R_{\mathcal{V}}) \to D(R_{\mathcal{V}}) \to D(\mathfrak{m}) \to 0$.

Observe that $D(R_{\mathcal{V}})$ is an injective left $R_{\mathcal{V}}$-module [25, Corollary 3.6C] that contains the simple left $R_{\mathcal{V}}$-module $D(S \otimes_R R_{\mathcal{V}})$. Thus the injective envelope ${}_RS[\infty]$ of $D(S \otimes_R R_{\mathcal{V}})$ is a direct summand of $D(R_{\mathcal{V}})$. We conclude that $\operatorname{Prod} C$ has precisely $r - s$ Prüfer left $R_{\mathcal{V}}$-modules belonging to this tube.

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