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Geometric approach to stable homotopy groups of spheres, III. Abelian, cyclic and quaternionic structure for mappings with singularities

P.M. Akhmet'ev

Abstract

Collection of PL-mappings admitting a relative abelian, cyclic, quaternionic, bicyclic, and quaternionic-cyclic structures are constructed.

Introduction

A map with a target in an Euclidean space is assumed PL, if a smoothness conditions do not mentioned. A generic PL-map is a PL-map, such that each pair of hyperplanes spanned by the images of corresponding pair of simplexes are transversal. A critical point is a point, such that the restriction of the map on an arbitrary neighborhood of this point is not an embedding. We do not assume extra conditions for a generic PL-map in critical points.

Let us consider the groups $\mathbb{Z}/2^{[s]}$, this group was defined in the introduction of [A2] as a subgroup of the group $\mathbb{Z}/2 \int \Sigma(2^{s-1})$, and the corresponding cobordism groups of immersions (see [A2, Diagram (21)]). In [A2, Diagram (20)] subgroups $\mathbf{I}_b \times \dot{\mathbf{I}}_b$, $\mathbf{E}_{b\times b}$, $\mathbf{J}_a \times \dot{\mathbf{J}}_a$, $\mathbf{Q} \times \mathbb{Z}/4$ of the groups $\mathbb{Z}/2^{[s]}$, $2 \leq s \leq 5$, are defined and the following definitions were considered: abelian structure (Definition 5), $\mathbf{E}_{b\times b}$–structure (Definition 14), $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure (bicyclic structure) (Definition 16), and quaternionic–cyclic structure (Definition 23) for corresponding framed immersions. These notions are used in Theorems 8, Lemmas 15 and 17, Theorem 25 to prove the Main Theorem in section 5. Analogously, in [A1] the following definitions were considered: cyclic structure (Definition 18) and quaternionic structure (Definition 20). These notions were used in Propositions 28 and 29 and in Theorem 12 and in the Main Corollary in section 2.

The definitions of abelian, $\mathbf{E}_{b\times\dot{b}}$–structure, $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure, and quaternionic–cyclic structure of $\mathbb{Z}/2^{[s-1]}$–framed immersions, $s \geq 2$, are introduced to weaken the condition of a reduction of classifying mappings of the self-intersection $\mathbb{Z}/2^{[s]}$–framed immersions of the considered framed immersion, see [A2, Definitions 4, 13, 13, 22] correspondingly. Analogously, for the notion of cyclic and quaternionic reductions see Definitions 14 and 15 in [A1]. In the present part of the paper these notions were not considered, the analogous relative notions were considered, and I will recall them.

The definitions of abelian, cyclic, and quaternionic structure of framed immersions admit relative analogs for formal PL–mappings with singularities of the standard projective (see [A2, Definition 10]), standard $\mathbb{Z}/4$–lens (see [A1, Definition 25]), and for PL–mappings with singularities of the standard $\mathbf{Q}$-lens spaces (see [A1, Definition 27]). The definitions of $\mathbf{E}_{b\times\dot{b}}$–structure and $\mathbf{J}_a\times\dot{\mathbf{J}}_a$–structure of framed immersions also admit relative analogs for formal PL–mappings with singularities of the standard skeleton of the corresponding Eilenberg-Mac Lane spaces (see [A2, Definition 29, 31]). The definitions of quaternionic–cyclic structure also admit relative analogs, this analogous definition is formulated for PL–mappings with singularities of the standard skeleton of the corresponding Eilenberg-Mac Lane space (see [A2,Definition 36]).

The existence of (a relative) abelian structure is formulated in Lemma 7 of [A2], for convenience this lemma is reformulated below as Lemma 1. (In the statement of this lemma below we re-denote the integer k' by k.)

Lemma 1. *A. For dimensional restrictions*

$$n-k\equiv 1 \pmod 2, n-3k-4>0, \quad n\equiv 0 \pmod 2. \tag{1}$$

there exists a generic PL*-mapping* $d:\mathbb{R}\mathrm{P}^{n-k}\to\mathbb{R}^n$ *(with singularities), for which the pair* $(d,d^{(2)}$*, where* $d^{(2)}$ *is the formal extension of* d *admits an abelian structure (see [A2, Definition 10]).*

B. For the dimensional restrictions

$$n-k\equiv -1 \pmod 4, \quad k\geq 7, \quad n\equiv 0 \pmod 2 \tag{2}$$

there exists a pair $(d,d^{(2)})$*, where* $d:\mathbb{R}\mathrm{P}^{n-k}\to\mathbb{R}^n$ *is a generic* PL*–mapping (with singularities)* $d^{(2)}$ *is a formal* PL*-mapping, such that the pair admits an abelian structure (in the sense of [A2, Definition 10]).*

The existence of (a relative) cyclic structure is formulated in Lemma 32 of [A1], this lemma is reformulated below as Lemma 2.

Lemma 2. *A. For the dimensional restrictions*

$$n-k\equiv 1 \pmod 2, n-3k-10>0, \quad n\equiv 0 \pmod 2 \tag{3}$$

there exists a generic PL*-mapping* $d:\mathbb{RP}^{n-k}\to\mathbb{R}^n$ *(with singularities) with a marked closed component of the self-intersection, for which the formal extension* $d^{(2)}$ *admits a cyclic structure (in the sense of Definition [A1, Definition 25]).*

B. For the dimensional restrictions

$$k\geq 10, \quad n-k\equiv 0 \pmod 8 \tag{4}$$

there exists a formal mapping $d^{(2)}$ *with holonomic self-intersection along a marked closed component* N_a *of the formal self-intersection of* $d^{(2)}$ *(in the sense of [A1, Definition 24]), which admits a cyclic structure (in the sense of [A1, Definition 25]).*

The existence of (a relative) quaternionic structure is formulated in [A1, Lemma 33], this lemma is reformulated below as Lemma 3 (in this lemma we re-denote the mapping c by d_1).

Lemma 3. *For* $n=4k+(2^\sigma-1)$, $n=2^\ell-1$, $\ell\geq 7$, $\sigma=\left[\frac{\ell-1}{2}\right]$, *then there exists a generic* PL*-mapping* $d_1:S^{n-2k}/\mathbf{i}\to\mathbb{R}^n$ *with singularities admitting a quaternionic structure in the sense of [A1, Definition* 25*].*

The existence of a relative $\mathbf{E}_{b\times b}$–structure in the sense of [A2, Definition 29] is formulated in [A2, Proposition 30].

The existence of a relative $\mathbf{J}_a\times\dot{\mathbf{J}}_a$–structure in the sense of [A2, Definition 31] is formulated in [A2, Proposition 32]. The existence of a relative $\mathbf{Q}\times\mathbb{Z}/4$–structure in the sense of [A2, Definition 36] is formulated in [A2, Lemma 37].

In this part of the paper we shall prove all the results formulated above from a unified point of view. The possibility of such an approach in the case of cyclic structure was discovered by Prof. A.V.Chernavsky at the end of the last century, and by Dr. S.A.Melikhov (2005) in the case of quaternionic structure. Preliminary results for cyclic and $\mathbf{E}_{b\times b}$–structure in the case of weaker restrictions on the codimension of the immersion, are given in the papers [Akh1], [Akh2].

Let us formulate a number of remarks, which seem to be of interest.

1. It is not, in general, possible to formulate the notion of abelian structure (and analogous notions considered above) in terms of the reduction of a classifying mapping to the classifying subspace of a corresponding

abelian subgroup. For example, in the case $n = 62$ there is, as proved in [M], an obstruction to the reduction of the classifying mapping for the self-intersection manifold of an immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$ into classifying subspace $K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1) \subset K(\mathbb{Z}/2^{[2]}, 1)$ of the abelian subgroup.

2. For the construction of cyclic and quaternionic structure for immersions (relative cyclic and quaternionic structures for PL–mappings with singularities) only double self-intersection points of immersions (of PL–mappings) are considered. Alternatively, in the paper [E] (this paper, as was noted in [A1],[A2], is the foundation of our construction) self-intersection points of an arbitrary multiplicity were considered. In particular, it is interesting to define and to study a quaternionic structure, related with quadruple points manifolds of skew-framed immersions.

3. The construction of quaternionic structure in Lemma 3 does not require the Massey embedding $S^3/\mathbf{Q} \subset \mathbb{R}^4$ [M], see also [Me]. Such an embedding was known earlier to W.Hantzsche [He]. By means of such an approach, it might be possible to weaken the dimensional restrictions in Lemma 3. For example, the Massey embedding allows to generalize Lemma 3 for maps in the range $\frac{4}{5}$ (for maps $M^m \to \mathbb{R}^n$, $\frac{m}{n} \leq \frac{4}{5}$). This means that one may consider an extra two quadratic extensions of the quaternionic group as the structure group of framing of immersions.

Note that in [A1] the cases $n = 15$, $n = 31$ and $n = 63$ were not considered. Additional arguments, in particular, might yield a proof of the last cases in the Adams Theorem on Hopf invariants, and clarify the remaining case in dimension 126 not covered by the Hill-Hopkins-Ravenel Theorem on Kervaire invariants.

1 Auxiliary mappings

We start by construction of auxiliary mappings. In Lemma 1 this is axillary mapping c for the mapping $d0$; in Lemma 2 there are axillary mappings $\hat{c}$, c for the mapping d; in Lemma 3 there are axillary mappings c_1, $\tilde{c}_1$ for the mapping $\tilde{d}_1$.

The transformation in Lemma 2 to the required mapping d from the mapping c is given by an approximation, which is constructed in Lemma 9, Lemma 2 is proved by means of Lemma 8.

To proof the mentioned lemmas and propositions we introduce on the singular set of auxiliary mappings the coordinate system called *angle-momentum*. By means of this coordinate system in Lemmas 4,5. The configuration space in Lemma 4 is defined as finite-dimensional resolution spaces for the singularity of the mapping c. In Lemma 5 the resolution spaces are

much simpler, because the mapping under investigation is close to stable.

Construction of an axillary mapping $c : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^n$ in Lemma 1

Let a positive integer parameter k and a positive integer n be given as in Lemma 1. Denote by J the standard $(n-k)$–dimensional sphere of codimension k in $\mathbb{R}^n$, which is represented as the join of $\frac{n-k+1}{2} = r$ copies of the circle S^1. We denote the standard embedding of J into $\mathbb{R}^n$ by

$$i_J : J \subset \mathbb{R}^n. \tag{5}$$

A mapping $p' : S^{n-k} \to J$ is obtained as a result of taking the join of r copies of the standard double covering $S^1 \to \mathbb{R}\mathrm{P}^1$. The standard antipodal action $\mathbf{I}_d \times S^{n-k} \to S^{n-k}$ (here and below for notations of the group $\mathbf{I}_d$ etc. see the first part of the section 2 in [A1]) commutes with the mapping p. Hence, there results a mapping with ramification $p' : \mathbb{R}\mathrm{P}^{n-k} \to J$. The required mapping $c : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^n$ is defined by means of the following composition: $i_J \circ p'$.

Construction of axillary mappings $c : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^n$, $\hat{c} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$ in Lemma 2

Let a positive integer parameter k and a positive integer n be given as in Lemma 2. The mapping $p : S^{n-k} \to J$ is well defined as the join of r copies of the standard 4-sheeted coverings $S^1 \to S^1/\mathbf{i}$. The standard action $\mathbf{I}_a \times S^{n-k} \to S^{n-k}$ commutes with the mapping p. Thus, the map $\hat{p} : S^{n-k}/\mathbf{i} \to J$ is well defined and the map $p : \mathbb{R}\mathrm{P}^{n-k} \to J$ is well defined as the composition $\hat{p} \circ \pi : \mathbb{R}\mathrm{P}^{n-k} \to J$ of the standard double covering $\pi : \mathbb{R}\mathrm{P}^{n-k} \to S^{n-k}/\mathbf{i}$ with the map $\hat{p}$.

The required mapping c is defined by the formula

$$i_J \circ p : \mathbb{R}\mathrm{P}^{n-k} \to J \subset \mathbb{R}^n. \tag{6}$$

The required mapping $\hat{c}$ is defined by the formula

$$i_J \circ \hat{p} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n. \tag{7}$$

Construction of axillary mappings $c_1 : S^{n-2k+2^{\sigma-1}}/\mathbf{i} \to \mathbb{R}^n$, $\tilde{c}_1 : S^{n-2k}/\mathbf{i} \to \mathbb{R}^n$

Let a positive integer parameter k and a positive integer n are given as in Lemma 3. Let us denote by J_1 a $(n-2k+2^{\sigma-1})$–dimensional polyhedron

(the equation $n-2k+2^{\sigma-1}=\frac{n-1}{2}+2^{\sigma}$ is satisfied), this polyhedron is defined as the join of

$$\frac{n+1}{2^{\sigma+1}}+1=r_1 \tag{8}$$

copies of the standard quaternionic lens space $S^{2^{\sigma}-1}/\mathbf{Q}$. Below we shall used the following notation $n_\sigma=2^\sigma-1$, as in [A1] and $m_\sigma=2^\sigma-2$, as in [A2]). An embedding $i_{\mathbf{Q}}: S^{n_\sigma}/\mathbf{Q}\subset\mathbb{R}^{n_\sigma-3}$ is well defined. Assuming $n=4k+2^\sigma-1$, $\ell\geq 7$ the embedding $J_1\subset\mathbb{R}^n$, as the join of r_1 copies of the embedding $i_{\mathbf{Q}}$, is well defined; let us denote this embedding by $i_{J_1}: J_1\subset\mathbb{R}^n$ (comp. with the mapping in [Lemma 35, A2].

The mapping $p_1': S^{n-2k+n_{\sigma-1}-1}\to J_1$ is well defined as the join of r_1 copies of the standard coverings $S^{n_\sigma}\to S^{n_\sigma}/\mathbf{Q}$. The action $\mathbf{Q}\times S^{n-2k+n_{\sigma-1}-1}\to S^{n-2k+n_{\sigma-1}-1}$ is well defined as the standard diagonal action, given by (23)-(25) in [A1], this action commutes with the mapping p_1'.

Thus, the map $\hat{p}_1: S^{n-2k+n_{\sigma-1}-1}/\mathbf{Q}\to J_1$ is well defined and the map

$$p_1\cong\hat{p}_1\circ\pi_1: S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}\to J_1, \tag{9}$$

as the composition of the standard double covering $\pi_1: S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}\to S^{n-2k+n_{\sigma-1}-1}/\mathbf{Q}$ with the map $\hat{p}_1$.

Define the required mapping c_1 as the composition $i_{J_1}\circ p_1: S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}\to S^{n-2k+n_{\sigma-1}-1}/\mathbf{Q}\to J_1\subset\mathbb{R}^n$. Consider the submanifold $i: S^{n-2k}/\mathbf{i}\subset S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}$, this submanifold is in general position with respect to strata of the manifold $S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}$, the strata are determined by the join structure. Define the mapping

$$\tilde{p}_1\cong\hat{p}_1\circ\pi_1\circ i: S^{n-2k}/\mathbf{i}\subset S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}\to J_1. \tag{10}$$

Define the required mapping $\tilde{c}_1$ as the composition $S^{n-2k}/\mathbf{i}\subset S^{n-2k+n_{\sigma-1}-1}/\mathbf{i}\xrightarrow{c_1}\mathbb{R}^n$.

2 Configuration spaces and singularities

Subspaces and factorspaces of the 2-configuration space for $\mathbb{RP}^{n-k}$, related with the axillary mapping c in Lemma 1

The space Γ, its double covering $\bar{\Gamma}$, and the structural mapping $\eta_\Gamma:\Gamma\to K(\mathbf{D},1)$ were defined in [Section 2, A1]. The space Γ is a manifold with boundary. Denote the interior of this manifold by $\Gamma_\circ$. The restriction of the structural map η_Γ on $\Gamma_\circ$ will be denoted by $\eta_{\Gamma_\circ}:\Gamma_\circ\to K(\mathbf{D},1)$.

Denote by $K_\circ \subset \Gamma_\circ$ the polyhedron of double-point singularities of the map $p : \mathbb{R}\mathrm{P}^{n-k} \to J$, this polyhedron is defined by the formula $\{[(x,y)] \in \Gamma_\circ, p(x) = p(y), x \neq y\}$ (see [Formula (39),A1]). This polyhedron is equipped with a structural mapping

$$\eta_{K_\circ} : K_\circ \to K(\mathbf{D}, 1), \tag{11}$$

which is induced by the restriction of the structural mapping $\eta_{\Gamma_\circ}$ (see (38) in [A1] and below) to the subspace $K_\circ$.

Consider the manifold, which is defined by the compactification of the open manifold $\Gamma_\circ$ by means of diagonal component Σ_{diag} (the blowing up of the diagonal is not considered). Denote the closure of $Cl(K_\circ)$ of the polyhedron $K_\circ$ in this manifold with singularities by K. Denote by Q_{diag} the space $Cl(K_\circ) \setminus K_\circ$. Obviously, $Q_{diag} \subset K$. Let us call this subspace the boundary of the polyhedron K.

Note that the structural mapping of $\eta_{K_\circ}$ does not extended from $K_\circ$ to the component Q_{diag} of the boundary. The mapping $\kappa_{diag} : Q_{diag} \to K(\mathbf{I}_d, 1)$ is well defined. Denote by $U(Q_{diag})_\circ \subset K_\circ$ a small regular deleted neighborhood of Q_{diag}.

The restriction of the structure mapping $\eta_{K_\circ}$ on a regular deleted neighborhood $UQ_{diag\circ}$ is given by the composition of the mapping $\eta_{UQ_{diag\circ}} : UQ_{diag\circ} \to K(\mathbf{I}_b, 1)$ and the mapping $i_{\mathbf{I}_b,\mathbf{D}} : K(\mathbf{I}_b, 1) \to K(\mathbf{D}, 1)$. Homotopy classes of the mappings η_{diag} and $\eta_{UQ_{diag\circ}}$ are related by the equation:

$$\eta_{diag} \circ proj_{diag} = p_{\mathbf{I}_b,\mathbf{I}_d} \circ \eta_{UQ_{diag\circ}}.$$

Subspaces and factorspaces of the 2-configuration space for $\mathbb{R}\mathrm{P}^{n-k}$, related with the axillary mappings c, $\hat{c}$ in Lemma 2

In the construction above the space Γ, the subspace $\Gamma_\circ \subset \Gamma$, its double coverings $\bar{\Gamma}$, $\bar{\Gamma}_\circ$ were defined. The structural mapping $\eta_\Gamma : \Gamma \to K(\mathbf{D}, 1)$ and its restriction $\eta_{\Gamma_\circ} : \Gamma_\circ \to K(\mathbf{D}, 1)$ on $\Gamma_\circ$ also were defined. Denote by

$$\Sigma_\circ \subset \Gamma_\circ \tag{12}$$

the polyhedron of double-points singularities of the map $p : \mathbb{R}\mathrm{P}^{n-k} \to J$, this polyhedron is defined by the formula $\{[(x,y)] \in \Gamma_\circ, p(x) = p(y), x \neq y\}$. This polyhedron is equipped with a structural mapping $\eta_{\Sigma_{0\circ}} : \Sigma_\circ \to K(\mathbf{D}, 1)$, which is induced by the restriction of the structural mapping $\eta_{\Gamma_\circ}$ on the subspace $\Sigma_\circ$.

The standard free involution $\mathbf{i} : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}\mathrm{P}^{n-k}$ is well defined. This involution permutes points in each fiber of the standard double covering

$\mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i}$. The space $\bar{\Gamma}_\circ$ admits an involution (with fixed points)

$$T_{\mathbf{i}\circ} : \bar{\Gamma}_\circ \to \bar{\Gamma}_\circ, \tag{13}$$

which is defined as the restriction of an involution $\mathbf{i} \times \mathbf{i} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$, constructed by the involution $\mathbf{i}$ on each factor, on the subspace $\bar{\Gamma}_\circ \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$. On the quotient $\bar{\Gamma}_\circ/T = \Gamma_\circ$ of $\Gamma_\circ$ by the another involution T, which permutes the coordinates, the factorinvolution $T_{\mathbf{i}\circ} : \Gamma_\circ \to \Gamma_\circ$ is well defined.

Let us denote by $\Sigma_{antidiag} \subset \Gamma_\circ$ a subspace, called the antidiagonal, which is formed by all antipodal pairs $\{[(x,y)] \in \Gamma_\circ : x, y \in \mathbb{RP}^{n-k}, x \neq y, \mathbf{i}(x) = y\}$. It is easy to verify that the antidiagonal $\Sigma_{antidiag} \subset \Gamma_\circ$ is the set of fixed points for the involution $T_{\mathbf{i}\circ}$.

The subpolyhedron $\Sigma_\circ \subset \Gamma_\circ$ of multiple-points of the map p is represented by a union $\Sigma_\circ = \Sigma_{antidiag} \cup K_\circ$, where $K_\circ$ is an open subpolyhedron contains all points of $\Sigma_\circ$ outside the antidiagonal. The subpolyhedron $K_\circ \subset \Gamma_{K_\circ}$ is invariant under the involution $T_{\mathbf{i}\circ}$.

Define the restriction of the involution $T_{\mathbf{i}\circ}|_{K_\circ}$ by $T_{K_\circ}$. The considered restriction is a free involution. Denote the factorspace $K_\circ/T_{K_\circ}$ by $\hat{K}_\circ$. The restriction of the structure mapping $\eta_{\Gamma_\circ} : \Gamma_\circ \to K(\mathbf{D}, 1)$ on $K_\circ$ denote by $\eta_{K_\circ}$.

Consider the manifold with singularities, which is defined by the closure of open manifold $\Gamma_\circ$, this space is obtained by the completion of the open manifold $\Gamma_\circ$ by means of diagonal and antidiagonal components Σ_{diag} and $\Sigma_{antidiag}$ (the blow-up of the diagonal and anti-diagonal is not assumed). Denote the closure of $Cl(K_\circ)$ of the polyhedron $K_\circ$ (respectively, the closure of the polyhedron $Cl(\hat{K}_\circ)$ polyhedron $\hat{K}_(\circ)$) in this manifold with singularities by K (respectively, by $\hat{K}$).

Denote by $Q_{antidiag}$ the space $\Sigma_{antidiag} \cap K$, denote by Q_{diag} the space $\partial\Gamma_{diag} \cap K$. Obviously, $Q_{diag} \subset K$, $Q_{antidiag} \subset K$. We shall call these subspaces the components of the boundary of the polyhedron K. Similarly, we denote by $\hat{Q}_{diag}$, $\hat{Q}_{antidiag}$ the components of the boundary of the polyhedron $\hat{K}$.

Note that the structural mapping of $\eta_{K_\circ}$ is extended from $K_\circ$ to the component $Q_{antidiag}$ of the boundary. Denote this extension by $\eta_{Q_{antidiag}} : Q_{antidiag} \to K(\mathbf{D}, 1)$. The mapping $\eta_{Q_{antidiag}}$ is represented by the composition $\eta_{antidiag} : Q_{antidiag} \to K(\mathbf{I}_a, 1)$ and the inclusion $i_{\mathbf{I}_a,\mathbf{D}} : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$.

Note that the mapping η_K is not expendable to boundary component Q_{diag}. The mapping $\kappa_{diag} : Q_{diag} \to K(\mathbf{I}_d, 1)$ is well defined. Let us denote by $U(Q_{diag})_\circ \subset K_\circ$ a small regular deleted neighborhood of Q_{diag}. The projection $proj_{diag} : U(Q_{diag})_\circ \to Q_{diag}$ of the regular deleted neighborhood

to Q_{diag}. The restriction of the structural mapping $\eta_{K_\circ}$ to the neighborhood $U(Q_{diag})_\circ$ is represented by a composition of the map $\eta_{U(Q_{diag})_\circ} : U(Q_{diag})_\circ \to K(\mathbf{I}_b, 1)$ and the maps $i_{\mathbf{I}_b,\mathbf{D}} : K(\mathbf{I}_b, 1) \to K(\mathbf{D}, 1)$. Homotopy classes of maps $\eta_K|_{Q_{diag}}$ and $\eta_{U(Q_{diag})_\circ}$ satisfy the equation:

$$\kappa_{diag} \circ proj_{diag} = p_{\mathbf{I}_b,\mathbf{I}_d} \circ \eta_{U(Q_{diag})_\circ}.$$

Let us investigate the polyhedron of singularities of an axillary mapping $\hat{c}$.

$$\begin{array}{ccccccc} & & & & \mathbf{I}_{b\times\dot{b}} & & \\ & & \nearrow & & \cap & & \\ \mathbf{I}_d & \subset & \mathbf{I}_a & \subset & \mathbf{D} & \subset & \mathbf{H}. \\ & & \searrow & & \cap & & \\ & & & & \mathbf{I}_c & & \end{array} \tag{14}$$

In this diagram, the inclusion $\mathbf{D} \subset \mathbf{H}$ is a central quadratic extension of $\mathbf{D}$ by the element $\mathbf{i}$ (of the order 4), for which $\mathbf{i}^2$ coincides with the generator -1 of the subgroup $\mathbf{I}_d \subset \mathbf{D}$. The abelian groups $\mathbf{H}_a, \mathbf{H}_{b\times\dot{b}}, \mathbf{H}_c, \mathbf{H}_d$ are the subgroups in $\mathbf{H}$, this groups are the quadratic extensions of the corresponding subgroups $\mathbf{I}_a, \mathbf{I}_b \times \dot{\mathbf{I}}_b, \mathbf{I}_c, \mathbf{I}_d$ by means of the element $\mathbf{i}$. Note that the groups $\mathbf{H}_{b\times\dot{b}}$ and $\mathbf{E}_{b\times\dot{b}}$ (see above [formula (84), A2]) are isomorphic.

The difference between the considered groups $\mathbf{H}_{b\times\dot{b}}$ and $\mathbf{E}_{b\times\dot{b}}$ are the following: the representation of $\mathbf{H}_{b\times\dot{b}} \to \mathbb{Z}^{[3]}$ (see below [Example 16, A1]) and $\mathbf{E}_{b\times\dot{b}} \to \mathbb{Z}/2^{[3]}$ (see [Diagram (85), A2]) are different. The kernel of the epimomorphism

$$\mathbf{H}_{b\times\dot{b}} \to \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2, \tag{15}$$

where $\mathbb{Z}/2^{[3]} \to \mathbb{Z}/2$ corresponds to the subgroup [(19),A2] of the index 2, contains an element $\mathbf{i} \in \mathbf{H}_d \subset \mathbf{H}_{b\times\dot{b}}$ of the order 4 (comp. with Diagram (17) below, in which $\mathbf{H}_d = \mathbf{H}_c \cap \mathbf{H}_{b\times\dot{b}}$). The kernel of the homomorphism $\mathbf{E}_{b\times\dot{b}} \to \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2$ coincides with the subgroup $\mathbf{I}_{b\times\dot{b}} \subset \mathbf{H}_{b\times\dot{b}}$, which is an elementary 2-group.

The induced automorphism $\chi^{[3]} : \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2^{[3]}$ of the group $\mathbf{H}_{b\times\dot{b}}$, re-denoted by

$$\hat{\chi}^{[2]} : \mathbf{H}_{b\times\dot{b}} \to \mathbf{H}_{b\times\dot{b}}, \tag{16}$$

is defined by the formula $\hat{\chi}^{[2]}(\mathbf{i}) = \mathbf{i}$, where $\mathbf{i} \in \mathbf{H}_d$–is the generator.

The following natural mapping $\eta_{\hat{K}_\circ} : \hat{K}_\circ \to K(\mathbf{H}, 1)$, which corresponds to the mapping of canonical 2-sheeted covering, is well-defined:

$$\begin{array}{ccc} \bar{K}_\circ & \stackrel{\bar{r}}{\longrightarrow} & \tilde{K}_\circ \\ \downarrow & & \downarrow \\ K_\circ & \stackrel{r}{\longrightarrow} & \hat{K}_\circ \end{array} \quad \longrightarrow \quad \begin{array}{ccc} K(\mathbf{I}_c,1) & \longrightarrow & K(\mathbf{H}_c,1) \\ \downarrow & & \downarrow \\ K(\mathbf{D},1) & \longrightarrow & K(\mathbf{H},1). \end{array} \tag{17}$$

Horizontal maps between the spaces of the diagrams we re-denote for brevity by $\bar{\eta}, \check{\eta}, \eta, \hat{\eta}$, respectively.

Subspaces and factorspaces of the 2-configuration space for $S^{n-2k}/\mathbf{i}$, related with the axillary mapping c_1

The space Γ_1, its double covering $\bar{\Gamma}_1$, and the structural map $\eta_{\Gamma_1} : \Gamma_1 \to K(\mathbf{H},1)$ was defined ([Formula (53) and below, A1]. The space Γ_1 is a manifold with boundary. Denote the interior of this manifold by $\Gamma_{1\circ}$. The restriction of the structural map η_{Γ_1} to $\Gamma_{1\circ}$ will be denoted by $\eta_{\Gamma_{1\circ}} : \Gamma_{1\circ} \to K(\mathbf{H},1)$.

Denote by $\Sigma_{1\circ} \subset \Gamma_{1\ circ}$ the polyhedron of double-points singularities of the map $p : S^{n-2k} \to J_1$, this polyhedron is obtained by the blowing up of the polyhedron $\{[(x,y)] \in \Gamma_{1\circ}, p(x) = p(y), x \neq y\}$. This polyhedron is equipped with a structural mapping

$$\zeta_{\Sigma_{1\circ}} : \Sigma_{1\circ} \to K(\mathbf{H},1), \tag{18}$$

which is induced by the restriction of the structural mapping $\zeta_{\Gamma_{1\circ}}$ on the subspace $\Sigma_{1\circ}$.

The subpolyhedron $\Sigma_{1\circ} \subset \Gamma_{1\circ}$ of multiple-points of the map p_1 is represented by a union $\Sigma_{1\circ} = \Sigma_{antidiag} \cup K_{1\circ}$, where $K_{1\circ}$ is an open subpolyhedron, this subpolyhedron contains all points of $\Sigma_{1\circ}$ outside the antidiagonal. Let us denote the restriction of the structural mapping $\zeta_{\Gamma_{1\circ}} : \Gamma_{1\circ} \to K(\mathbf{H},1)$ on $\Gamma_{K_{1\circ}}$ and on $K_{1\circ}$ by $\zeta_{\Gamma_{1\circ}}$ and by $\zeta_{K_{1\circ}}$ respectively.

Consider the manifold with singularities Γ_1, which is defined by the closure of open manifold $\Gamma_{1\circ}$, this space is obtained by the completion of the open manifold $\Gamma_{1\circ}$ by means of diagonal and antidiagonal components Σ_{diag} and $\Sigma_{antidiag}$. Denote the closure of $Cl(K_{1\circ})$ of the polyhedron $K_{1\circ}$ in Γ_1 (respectively, the closure of the polyhedron $Cl(\hat{K}_{1\circ})$ polyhedron $\hat{K}_{1\circ}$ in $\hat{\Gamma}_1$) by K_1 (respectively, by $\hat{K}_1$). Denote by $Q_{antidiag}$ the space $\Sigma_{antidiag} \cap K_1$, denote by Q_{diag} the space $\partial\Gamma_{diag} \cap K_1$. Obviously, $Q_{diag} \subset K_1$, $Q_{antidiag} \subset K_1$. We shall call these subspaces the components of the boundary of the polyhedron K_1.

Note that the structural mapping of $\zeta_{K_{1\circ}}$ is extended from $K_{1\circ}$ to the component $Q_{antidiag}$ of the boundary. Denote this extension by $\zeta_{Q_{antidiag}}$:

$Q_{antidiag} \to K(\mathbf{H},1)$. The mapping $\zeta_{Q_{antidiag}}$ is the composition $\zeta_{antidiag} : Q_{antidiag} \to K(\mathbf{Q},1)$ and the inclusion $i_{\mathbf{Q},\mathbf{H}} : K(\mathbf{Q},1) \subset K(\mathbf{H},1)$.

Note that the mapping ζ_{K_1} is not expendable to boundary component Q_{diag}. The mapping $\zeta_{diag} : Q_{diag} \to K(\mathbf{I}_a,1)$ is well defined. Let us denote by $U(Q_{diag})_\circ \subset K_{1\circ}$ a small regular deleted neighborhood of Q_{diag}. The projection $proj_{diag} : U(Q_{diag})_\circ \to Q_{diag}$ of the regular deleted neighborhood to Q_{diag} to the central manifold is well defined.

The restriction of the structural mapping $\zeta_{K_{1\circ}}$ to the neighborhood $U(Q_{diag})_\circ$ is represented by a composition of the map $\zeta_{UQ_{diag\circ}} : UQ_{diag\circ} \to K(\mathbf{H}_{b\times\dot{b}},1)$ and the maps $i_{\mathbf{H}_{b\times\dot{b}},\mathbf{H}} : K(\mathbf{H}_{b\times\dot{b}},1) \to K(\mathbf{H},1)$.

Homotopy classes of maps ζ_{diag} and $\zeta_{UQ_{diag\circ}}$ are related by the equation:

$$\zeta_{diag} \circ proj_{diag} = p_{\mathbf{H}_{b\times\dot{b}},\mathbf{I}_a} \circ \zeta_{UQ_{diag\circ}}.$$

3 Resolution spaces for singularities

Resolution spaces for polyhedra $\Sigma_\circ$ and $\hat{K}_\circ$

We construct a space $R\Sigma_\circ$, which we call the resolution space of the polyhedron $\Sigma_\circ$, which is defined by the formula (12).

The space $R\Sigma_\circ$ contains two components, which is denoted by $R\Sigma_a$, $RK_{b\times\dot{b}\circ}$:

$$R\Sigma_a \cup RK_{b\times\dot{b}\circ} = R\Sigma_\circ. \tag{19}$$

The space $R\Sigma_a$ is a closed polyhedron, for which the structured mapping

$$\phi_a : R\Sigma_a \to K(\mathbf{I}_a,1) \tag{20}$$

is well-defined. The mapping (20) is included into the following commutative diagram:

$$\begin{array}{ccc} \Sigma_\circ & \xleftarrow{pr} & R\Sigma_a \\ \downarrow \eta_\circ & & \downarrow \phi_a \\ K(\mathbf{D},1) & \supset & K(\mathbf{I}_a,1). \end{array} \tag{21}$$

The space $RK_{b\times\dot{b}\circ}$ is a 2-sheeted covering space of the covering $Rr_{b\times\dot{b}} : RK_{b\times\dot{b}\circ} \to R\hat{K}_{b\times\dot{b}\circ}$.

$$\begin{array}{ccccccc} K_\circ & \xleftarrow{pr} & RK_{b\times\dot{b}\circ} & \xrightarrow{Rr_{b\times\dot{b}}} & R\hat{K}_{b\times\dot{b}\circ} & \xrightarrow{p\hat{r}} & \hat{K}_\circ \\ & & \downarrow \hat{\phi}_{b\times\dot{b}} & & \downarrow \phi_{b\times\dot{b}} & & \\ & & K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) & \subset & K(\mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}} \mathbb{Z}, 1). & & \end{array} \tag{22}$$

The group $\mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}}$, which is used in Diagram (22) is defined analogously to the group $(\mathbf{E}_{b\times\dot{b}}) \int_{\chi^{[3]}} \mathbb{Z}$, [Formula (68), A2], using the automorphism (involution) (16).

Denote $(p\hat{r})^{-1}(\hat{U}Q_{diag\circ})$ by $R\hat{Q}_{diag\circ}$. The following inclusion $R\hat{Q}_{diag\circ} \subset R\hat{K}_{b\times\dot{b}\circ}$ is well-defined.

Let us denote by $RQ_{diag\circ}$ the boundary of the corresponding 2-sheeted covering space over $R\hat{Q}_{diag\circ}$. The following diagram is well-defined.

$$\begin{array}{ccc} R\hat{Q}_{diag\circ} & \xrightarrow{p\hat{r}} & U\hat{Q}_{diag\circ} \\ \hat{\phi}_{b\times\dot{b}} \downarrow & & \hat{\eta}_{diag\circ} \downarrow \\ K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) & \supset & K(\mathbf{H}_{b\times\dot{b}}, 1). \end{array} \tag{23}$$

To prove the main result of the section we will use the following lemma.

Lemma 4. *There exists a space $R\Sigma_\circ$, which is satisfies the equation (19).*

The component $R\Sigma_a$ is equipped by the mapping (20), which is included into the commutative diagram (21).

The component $RK_{b\times\dot{b}\circ}$ is the total space of a regular 2-sheeted covering over the space $R\hat{K}_{b\times\dot{b}\circ}$ such that the commutative diagram (22) is well-defined. Moreover, the commutative diagram (23), which determines boundary conditions, is well-defined.

Resolution space for the polyhedron Σ_1

We shall define a space $R\Sigma_{1\circ}$, which we call resolution space of the polyhedron Σ_1. The space $R\Sigma_{1\circ}$ contains two components, which is denoted by $R\Sigma_{\mathbf{Q}}$, $RK_{\mathbf{E}_{b\times\dot{b}}\circ}$, as follows:

$$R\Sigma_{\mathbf{Q}} \cup RK_{\mathbf{H}_{b\times\dot{b}}\circ} = R\Sigma_{1\circ}. \tag{24}$$

Let us consider the following diagrams:

$$\begin{array}{ccc} R\Sigma_{\mathbf{Q}} \cup RK_{\mathbf{H}_{b\times b}\circ} & \stackrel{pr_1}{\longrightarrow} & \Sigma_1 \\ \phi_1 \downarrow & & \\ K(\mathbf{Q},1) \cup K(\mathbf{H}_{b\times b},1), & & \end{array} \tag{25}$$

$$\begin{array}{ccc} RQ_{diag} & \stackrel{pr_1}{\longrightarrow} & Q_{diag} \\ \phi_1 \searrow & & \swarrow \zeta_{diag} \\ & K(\mathbf{H}_{b\times b},1), & \end{array} \tag{26}$$

in which $RQ_{diag} = (pr_1)^{-1}(Q_{diag})$.

The following lemma is analogous to Lemma 4

Lemma 5. *There exists a space RK_1, which is satisfies the equation (24), an which is included in the commutative diagram (25). Moreover, the commutative diagrams (26) determines boundary conditions.*

4 Preliminary steps toward Lemma 2. The proof with the dimensional restriction (3)

Let us recall that the polyhedron J is PL homeomorphic to the standard sphere S^{n-k}. Consider the embedding (5). Decomposes this embedding into the following composition of the standard embeddings: $i_1 : J \subset J \times \mathbb{R}^{k-5} \subset \mathbb{R}^{n-5}$, $i_2 : \mathbb{R}^{n-5} \subset \mathbb{R}^{n-1}$, $i_3 : \mathbb{R}^{n-1} \subset \mathbb{R}^n$.

Consider the mapping $\hat{c} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$, which is given by the formula (7). Let us represents this mapping by the composition of the mapping $\hat{c}'_1 : S^{n-k}/\mathbf{i} \to J \times \mathbb{R}^{k-5}$, the inclusion $i_2 : J \times \mathbb{R}^{k-5} \subset \mathbb{R}^{n-1}$, and the standard inclusion $i_3 : \mathbb{R}^{n-1} \subset \mathbb{R}^n$.

Define the mapping $\hat{c}_1 : S^{n-k}/\mathbf{i} \to \mathbb{R}^{n-5}$ as a result by a special C^1–small PL–deformation of the mapping $\hat{c}'_1$.

Denote by $U_{J,1} \subset \mathbb{R}^{n-5}$ the regular neighborhood of the embedded sphere $J \subset J \times \mathbb{R}^{k-5} \subset \mathbb{R}^{n-5}$. Denote by $proj_J : U_{J,1} \to J$ the orthogonal projection of a smallest neighborhood onto the central sphere J. The PL–deformation $\hat{c}'_1 \mapsto \hat{c}_1$ is defined as a vertical deformation with respect to the orthogonal projection $proj_J$. Let us denote the caliber of this deformation by $\hat{\varepsilon}$.

Consider the mapping $c_1 = p \circ \hat{c}_1 : \mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i} \to \mathbb{R}^{n-5}$ and define a mapping $c'_1 : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-5}$ as the result of an additional C^1-small deformation $c_1 \mapsto c'_1$, which is vertical with respect to the projection $proj_J$, and

which has the caliber ε, much smaller then the caliber $\hat{\varepsilon}$ of the deformation $\hat{c}'_1 \mapsto \hat{c}_1$.

Let us denote the self-intersection polyhedron of the mapping c'_1, and the open subplolyhedron the of regular self-intersection points of this map by

$$N'_\circ \subset N'. \tag{27}$$

By dimensional reasons, the mapping c'_1 has no self-intersection points of the multiplicity 3 and more. Because the codimension $codim(\Sigma(c'_1)) = k-5$, using the condition (3) we get: $2codim(N') > n-k$.

Because the deformation $c_1 \mapsto c'_1$ is vertical, the polyhedron $N'_\circ$ is a subpolyhedron in the polyhedron $\Sigma_\circ$. Denote by

$$N'_{b\times\dot{b}\circ} \subset N'_\circ \tag{28}$$

an open polyhedron, which is defined by the inverse image of the subpolyhedron (39) (see below) by the standard inclusion $N'_\circ \subset \Sigma_\circ$.

Because $\varepsilon << \hat{\varepsilon}$, the subpolyhedron (28) is equipped by the involution, which is induced from the involution (13) by the standard inclusion. This involution is a free involution, because the polyhedron (28) does not intersects the antidiagonal. Let us denote by $\hat{N}'_{b\times\dot{b}\circ}$ the quotient of the polyhedron $N'_{b\times\dot{b}\circ}$ with respect to this involution. The associated 2-sheeted covering denote by

$$N'_{b\times\dot{b}\circ} \to \hat{N}'_{b\times\dot{b}\circ}. \tag{29}$$

The following commutative diagrams are well defined:

$$\begin{array}{ccc} N'_\circ & \supset & U(N'_{diag\circ}) \\ \downarrow \eta'_\circ & & \downarrow \eta'_{diag\circ} \\ K(\mathbf{D},1) & \supset & K(\mathbf{I}_{b\times\dot{b}},1), \end{array} \tag{30}$$

$$\begin{array}{ccc} \hat{N}'_{b\times\dot{b}\circ} & \supset & U(\hat{N}'_{diag\circ}) \\ \downarrow \hat{\eta}'_\circ & & \downarrow \hat{\eta}'_{diag\circ} \\ K(\mathbf{H},1) & \supset & K(\mathbf{H}_{b\times\dot{b}},1). \end{array} \tag{31}$$

Below we shall define the required mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ as the result of a special deformation $i_2 \circ c'_1 \mapsto d$. The deformation $i_2 \circ c'_1 \mapsto d$, generally speaking, is not a vertical deformation with respect to the orthogonal

projection $proj_J \circ (\mathbb{R}^n \to \mathbb{R}^{n-5})$. Let us denote by $N_\circ$ an open polyhedron of self-intersection points of the mapping d. The following subpolyhedra are well defined: $N_{b\times\dot{b}\circ} \subset N_\circ$, $\hat{N}_{b\times\dot{b}\circ}$. Properties of the mapping d is described in Lemma 6, the mapping d itself is constructed in Lemma 8.

Lemma 6. *There exists a C^0–small* PL*-deformation $i_2 \circ c_1' \mapsto d$, $d : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-1}$, such that for the polyhedron $N_\circ$ is decomposed into the union of two subpolyhedra:*

$$N_\circ = N_a \cup N_{b\times\dot{b}\circ}, \tag{32}$$

where N_a is closed.

The restriction of the structure mapping $\eta_\circ$ on the closed subpolyhedron N_a admits a reduction, given by a mapping $\mu_a : N_a \to K(\mathbf{I}_a, 1)$:

$$\eta_a = i_a \circ \mu_a : N_a \to K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1). \tag{33}$$

The restriction of the structured map $\eta_{b\times\dot{b}\circ}$ to the component $N_{b\times\dot{b}\circ}$ is a 2-sheeted covering mapping over a mapping $\hat{\eta}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}, 1)$. The mapping $\hat{\eta}_{b\times\dot{b}\circ}$ admits a reduction by a mapping $\hat{\mu}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$:

$$\hat{\eta}_{b\times\dot{b}\circ} = \hat{\Phi}^{[2]} \circ \hat{\mu}_{b\times\dot{b}\circ} : N_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{H}, 1), \tag{34}$$

where $\hat{\Phi}^{[2]} : K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{H}, 1)$ is a natural mapping (see an analogous [Diagram (85),A2]).

A sketch of the proof of Lemma 6

The deformation $i_2 \circ c_1' \mapsto d$ will be defined, such that the polyhedron (101) admits a resolution mapping:

$$t_a \cup t_{b\times\dot{b}\circ} : N_a \cup N_{b\times\dot{b}\circ} \to RK_a \cup RK_{b\times\dot{b}\circ}.$$

The following properties are well-defined: The mapping t_a induces the following mapping $\mu_a = \phi_a \circ t_a : N_a \to K(\mathbf{I}_a, 1)$, which is the required mapping. The mapping t_a induces the following mapping $\mu_{b\times\dot{b}\circ} = \phi_{b\times\dot{b}\circ} \circ t_{b\times\dot{b}\circ} : N_{b\times\dot{b}\circ} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$. This mapping is a 2-sheeted mapping over the second required mapping $\hat{\mu}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$. Let us consider the proof more precisely.

5 Coordinate system angle-momentum on the spaces of singularities and construction of the resolution spaces

The complex stratification of polyhedra J, Σ, $\Sigma_\circ$ by means of the coordinate system angle - momentum

Let us order lens spaces, which form the join, by the integers from 1 up to r and let us denote by $J(k_1,\dots,k_s)\subset J$ the subjoin, formed by a selected set of circles (one-dimensional lens spaces) $S^1/\mathbf{i}$ with indexes $1\le k_1<\dots<k_s\le r$, $0\ge s\ge r$. The stratification above is induced from the standard stratification of the open faces of the standard r-dimensional simplex δ^r under the natural projection $J\to\delta^r$. The preimages of vertexes of a simplex are the lens spaces $J(j)\subset J$, $J(j)\approx S^1/\mathbf{i}$, $1\le j\le r$, generating the join.

Define the space $J^{[s]}$ as a subspace of J, obtained by the union of all subspaces $J(k_1,\dots,k_s)\subset J$.

Thus, the following stratification

$$J^{(r)}\subset\dots\subset J^{(1)}\subset J^{(0)}, \tag{35}$$

of the space J is well-defined. For the considered stratum a number $r-s$ of missed coordinates to the full set of coordinates is called the deep of the stratum.

Let us introduce the following denotation:

$$J^{[i]}=J^{(i)}\setminus J^{(i+1)}. \tag{36}$$

Denote the maximum open cell of the space $\hat{p}^{-1}(J(k_1,\dots,k_s))$ by $\hat{U}(k_1,\dots,k_s)\subset S^{n-k}/\mathbf{i}$. This open cell is called an elementary stratum of the depth $(r-s)$. A point at an elementary stratum $U(k_1,\dots,k_s)\subset S^{n-k}/\mathbf{i}$ is defined by a set of coordinates $(\check{x}_{k_1},\dots,\check{x}_{k_s},\lambda)$, where $\check{x}_{k_i}\in S^1$ is a coordinate on the 1-sphere (circle), covering lens space with the number k_i, $\lambda=(l_{k_1},\dots,l_{k_s})$ is a barycentric coordinate on the corresponding $(s-1)$-dimensional simplex of the join. Thus if the two sets of coordinates are identified under the transformation of the cyclic $\mathbf{I}_a$-covering by means of the generator, which is common to the entire set of coordinates, then these sets define the same point on $S^{n-k}/\mathbf{i}$. Points on elementary stratum $\hat{U}(k_1,\dots,k_s)$ belong in the union of simplexes with vertexes belong to the lens spaces of the join with corresponding coordinates. Each elementary strata $\hat{U}(k_1,\dots,k_s)$ is a base space of the double covering $U(k_1,\dots,k_s)\to\hat{U}(k_1,\dots,k_s)$, which is induced from the double covering $\mathbb{R}\mathrm{P}^{n-k}\to S^{n-k}/\mathbf{i}$ by the inclusion $\hat{U}(k_1,\dots,k_s)\subset S^{n-k}/\mathbf{i}$.

The polyhedron $\Sigma_\circ$ is split into the union of open subsets (elementary strata), these elementary strata are defined as the connected components of the inverse images of elementary strata (36). Denote these elementary strata by

$$K^{[r-s]}(k_1,\dots,k_s), \qquad 1\le s\le r. \tag{37}$$

Let us describe an elementary stratum $K^{[r-s]}(k_1,\dots,k_s)$ by means of the coordinate system. To simplify the notation let us consider the case $s=r$. Suppose that for a pair of points $(x_1,\, x_2)$, defining a point on $K^{[0]}(1,\dots,r)$, the following pair of points $(\check{x}_1,\check{x}_2)$ on the covering space S^{n-k} is fixed, and the pair $(\check{x}_1,\check{x}_2)$ is mapped to the pair $(x_1,\, x_2)$ by means of the projection of $S^{n-k}\to\mathbb{RP}^{n-k}$. Accordingly to the construction above, we denote by $(\check{x}_{1,i},\check{x}_{2,i})$, $i=1,\dots,r$ a set of spherical coordinates of each point. Each such coordinate with the number i defines a point on 1-dimensional sphere (circle) S^1_i with the same number i, which covers the corresponding circle $J(i)\subset J$ of the join. Note that the pair of coordinates with the common number determines the pair of points in a common layer of the standard cyclic $\mathbf{I}_a$-covering $S^1\to S^1/\mathbf{i}$.

The collection of coordinates $(\check{x}_{1,i},\check{x}_{2,i})$ are considered up to independent changes to the antipodal. In addition, the points in the pair (x_1,x_2) does not admit a natural order and the lift of the point in K to a pair of points $(\bar{x}_1,\bar{x}_2)$ on the sphere S^{n-k}, is well determined up to 8 different possibilities. (The order of the group $\mathbf{D}$ is equal to 8.)

An analogous construction holds for points on deeper elementary strata $K^{[r-s]}(k_1,\dots,k_s)$, $1\le s\le r$.

The coordinate description of elementary strata of the polyhedra $K_\circ\subset\Sigma_\circ$

Let $x\in K^{[r-s]}(k_1,\dots,k_s)$ be a point on an elementary stratum. Consider the sets of spherical coordinates $\check{x}_{1,i}$ и $\check{x}_{2,i}$, $k_1\le i\le k_s$ of the point x. For each i the following cases: a pair of i-th coordinates coincides; antipodal, the second coordinate is obtained from first by the transformation by means of the generator (or by the minus generator) of the cyclic cover. Associate to an ordered pair of coordinates $\check{x}_{1,k_i}$ and $\check{x}_{2,k_i}$, $1\le i\le s$ the residue $v_{k_i}=\check{x}_{1,k_i}(\check{x}_{2,k_i})^{-1}$ of a value $+1$, -1, $+\mathbf{i}$ or $-\mathbf{i}$, respectively. It is easy to check that the collection of residues $\{v_{k_i}\}$ is changed by the following transformation. When the collection of coordinates of a point is changed to the antipodal collection, say, the collection of coordinates of the point x_2 is changed to the antipodal collection, the set of values of residues of the

new pair $(\bar{x}_1, \bar{x}_2)$ on the spherical covering is obtained from the initial set of residues by changing of the signs:

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(-\check{x}_{1,k_i}, \check{x}_{2,k_i})\}, \quad \{v_{k_i}\} \mapsto \{-v_{k_i}\},$$

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(\check{x}_{1,k_i}, -\check{x}_{2,k_i})\}, \quad \{v_{k_i}\} \mapsto \{-v_{k_i}\}.$$

The residues of the renumbered pair of points change by the inversion:

$$\{(\check{x}_{1,k_i}, \check{x}_{2,k_i})\} \mapsto \{(\check{x}_{2,k_i}, \check{x}_{1,k_i})\}, \quad \{v_{k_i}\} \mapsto \{\bar{v}_{k_i}\},$$

where $v \mapsto \bar{v}$ means the complex conjugation. Obviously, the set of residues does not change, if we choose another point on the same elementary stratum of the space $K_\circ$.

Elementary strata of the space $K(k_1, \ldots, k_s)$, in accordance with sets of residues, are divided into 3 types: $\mathbf{I}_a, \mathbf{I}_{b\times\dot{b}}, \mathbf{I}_d$. If among the set of residues are only residues $\{+\mathbf{i}, -\mathbf{i}\}$ (respectively, only residues $\{+1, -1\}$), we shall speak about the elementary stratum of the type $\mathbf{I}_a$ (respectively of the type $\mathbf{I}_{b\times\dot{b}}$). If among the residues are residues from the both set $\{+\mathbf{i}, -\mathbf{i}\}$ and $\{+1, -1\}$, we shall speak about elementary stratum of the type $\mathbf{I}_d$. It is easy to verify that the restriction of the structure mapping $\eta : K_{0\circ} \to K(\mathbf{D}, 1)$ on an elementary stratum of the type $\mathbf{I}_a, \mathbf{I}_{b\times\dot{b}}, \mathbf{I}_d$ is represented by the composition of a map in the space $K(\mathbf{I}_a, 1)$ (respectively in the space $K(\mathbf{I}_{b\times\dot{b}}, 1)$ or $K(\mathbf{I}_d, 1)$) with the map $i_a : K(\mathbf{I}_a, 1) \to K(\mathbf{D}, 1)$ (respectively, with the map $i_{b\times\dot{b}} : K(\mathbf{I}_{b\times\dot{b}}, 1) \to K(\mathbf{D}, 1)$ or $i_d : K(\mathbf{I}_d, 1) \to K(\mathbf{D}, 1)$). For the first two types of strata the reduction of the structural mapping (up to homotopy) is not well defined, but is defined only up to a composition with the conjugation in the subgroups $\mathbf{I}_a$, $\mathbf{I}_{b\times\dot{b}}$.

The polyhedron $\Sigma_\circ$ contains the polyhedron $K_\circ$ and $\Sigma_\circ \setminus K_\circ$ consists of antidiagonal elementary strata. For an arbitrary elementary antidiagonal stratum $K(k_1, \ldots, k_s)$ the residue of the each angle coordinate is equal to $+\mathbf{i}$. A antidiagonal stratum is an elementary stratum of the type $\mathbf{I}_a$. The polyhedron Σ is derived from $\Sigma_\circ$ by the joining of all diagonal strata (on each diagonal strata the residue of an arbitrary angle coordinate is equal $+1$), which is in the boundary of the polyhedron. It is easy to verify that $\Sigma \setminus \Sigma_\circ$ contains all elementary diagonal strata of the deep greater, or equal, then 1.

Define the following open subpolyhedra

$$K_{a\circ} \subset K_\circ \subset \Sigma_\circ, \tag{38}$$

$$K_{b\times\dot{b}\circ} \subset K_\circ \subset \Sigma_\circ, \tag{39}$$

$$K_{d\circ} \subset K_\circ \subset \Sigma_\circ \tag{40}$$

as the unions of all elementary strata of the corresponding type.

The following polyhedron

$$\hat{K}_{b\times b\circ} \subset \hat{K}_\circ \tag{41}$$

is defined as the base of 2-sheeted covering over the polyhedron (39). The description of (39) by means of the coordinates is obvious and is omitted.

Description of the structural map $\eta_\circ : \Sigma_\circ \to K(\mathbf{D}, 1)$, by means of the coordinate system

Let $x = [(x_1, x_2)]$ be a marked a point on $K_\circ$, on a maximal elementary stratum. Consider closed path $\lambda : S^1 \to K_\circ$, with the initial and ending points in this marked point, intersecting the singular strata of the depth 1 in a general position in a finite set of points. Let $(\check{x}_1, \check{x}_2)$ be the two spherical preimages of the point x. Define another pair $(\check{x}'_1, \check{x}'_2)$ of spherical preimages of x, which will be called coordinates, obtained in result of the natural transformation of the coordinates $(\check{x}_1, \check{x}_2)$ along the path λ.

At regular points of the path λ the family of pairs of spherical preimages in the one-parameter family is changing continuously, that uniquely identifies the inverse images of the end point of the path by the initial data. When crossing the path with the strata of depth 1, the corresponding pair of spherical coordinates with the number l is discontinuous. Since all the other coordinates remain regular, the extension of regular coordinates along the path at a critical moment time is uniquely determined. For a given point x on elementary stratum of the depth 0 of the spaces $K_\circ$ the choice of at least one pair of spherical coordinates is uniquely determines the choice of spherical coordinates with the rest numbers. Consequently, the continuation of the spherical coordinates along a path is uniquely defined in a neighborhood of a singular point of the path.

The transformation of the ordered pair $(\check{x}_1, \check{x}_2)$ to the ordered pair $(\check{x}'_1, \check{x}'_2)$ defines an element the group $\mathbf{D}$. This element does not depend on the choice of the path l in the class of equivalent paths, modulo homotopy relation in the group $\pi_1(\Sigma_\circ, x)$. Thus, the homomorphism $\pi_1(\Sigma_\circ, x) \to \mathbf{D}$ is well defined and the induced map

$$\eta_\circ : \Sigma_\circ \to K(\mathbf{D}, 1) \tag{42}$$

coincides with structural mapping, which was determined earlier. It is easy to verify that the restriction of the structural mapping $\eta_\circ$ on the connected

components of a single elementary stratum $K_\circ(1,\dots,r)$ is homotopic to a map with the image in the subspeces $K(\mathbf{I}_a,1)$, $K(\mathbf{I}_{b\times\dot{b}},1)$, $K(\mathbf{I}_d,1)$, which corresponds to the type and subtype elementary stratum.

Coordinate description of the canonical covering over an elementary stratum

Consider an elementary stratum $K^{[r-s]}(k_1,\dots,k_s) \subset K_\circ^{(r-s)}$ of the depth $(r-s)$. Denote by

$$\pi : K^{[r-s]}(k_1,\dots,k_s) \to K(\mathbb{Z}/2,1) \tag{43}$$

the classifying map, that is responsible for the permutation of a pair of points around a closed path on this elementary stratum. This mapping is called the *classified* mapping for the corresponding 2-sheeted covering.

The mapping π coincides with the composition

$$K^{[r-s]}(k_1,\dots,k_s) \xrightarrow{\eta} K(\mathbf{D},1) \xrightarrow{p} K(\mathbb{Z}/2,1),$$

where $K(\mathbf{D},1) \xrightarrow{p} K(\mathbb{Z}/2,1)$ be the map of the classifying spaces, which is induced by the epimorphism $\mathbf{D} \to \mathbb{Z}/2$ with kernel $\mathbf{I}_c \subset \mathbf{D}$. The canonical 2-sheeted covering, which is associated with the mapping π let us denote by

$$\bar{K}^{[r-s]}(k_1,\dots,k_s) \to K^{[r-s]}(k_1,\dots,k_s). \tag{44}$$

With the mapping (43) the following equivariant mapping is associated:

$$\bar{\pi} : \bar{K}^{[r-s]}(k_1,\dots,k_s) \to S^\infty, \tag{45}$$

where the involution in the image is the standard antipodal involution. This mapping is a 2-sheeted covering over the mapping (43).

For an elementary strata of the type $\mathbf{I}_{b\times\dot{b}}$ with the mapping (45) the following equivariant mapping is associated:

$$\tilde{\pi} : \tilde{K}^{[r-s]}(k_1,\dots,k_s) \to S^\infty, \tag{46}$$

where the mapping $\tilde{K}^{[r-s]}(k_1,\dots,k_s) \subset \tilde{K}(\mathbf{H}_{b\times\dot{b}},1)$, (46) is a 2-sheeted covering over the mapping (45).

Lemma 7. *The restriction of the map (45) to the canonical 2-sheeted covering over an elementary strata of an arbitrary type is homotopic to the following composition*

$$\bar{\pi} : \bar{K}^{[r-s]}(k_1,\dots,k_s) \to S^1 \subset S^\infty. \tag{47}$$

The restriction of the equivariant map (46) to the canonical 2-sheeted covering over an elementary strata of the type $\mathbf{H}_{b\times\dot{b}}$ is homotopic to the following composition

$$\tilde{\pi} : \tilde{K}^{[r-s]}(k_1, \ldots, k_s) \to S^1 \subset S^\infty, \tag{48}$$

where $S^1 \subset S^\infty$ is the equivariant embedding of the standard 1-dimensional skeleton of the classifying space.

Proof of Lemma 7

Let us prove the lemma by means of explicit formulas for the mappings (47) (48). An arbitrary point $[(x_1, x_2)] \in \hat{K}^{[r-s,i]}(k_1, \ldots, k_s)$, or $[(x_1, x_2)] \in K^{[r-s,i]}(k_1, \ldots, k_s)$ is determined by the equivalence class of the collection of angle coordinates and the momentum coordinate. The structure mapping $\eta_\circ$, $\hat{\eta}_{b\times\dot{b}\circ}$ is determined by a transformation of angle coordinates. Let us define the mappings (47), (48) by the corresponding transformation of the *marked* pair of the angle coordinates. Below the prescribed pair of the angle coordinates for an elementary stratum of each arbitrary type is defined.

Assume that a point $[(\hat{x}_1, \hat{x}_2)] \in \hat{K}^{[r-s]}(k_1, \ldots, k_s)$ is belong to the stratum of the type $\mathbf{H}_{b\times\dot{b}}$. Because the residue of the prescribed pair of the angle coordinates is well-defined, a non-ordered pair of the angle coordinates with the residue -1 it is convenient to denote by $[(\check{x}_{1,-}, \check{x}_{2,-})]$, a pair of the angle coordinates with the residue $+1$ denote by $[(\check{x}_{1,+}, \check{x}_{2,+})]$.

The each coordinate $\check{x}_{1,-}$, $\check{x}_{2,-}$, $\check{x}_{1,+}$, $\check{x}_{2,+}$ determines the corresponding point on S^1. It is not difficult to check, that $\check{x}_{1,+} = \check{x}_{2,+}$, $\check{x}_{1,-} = -\check{x}_{2,-}$. Therefore the mapping $(\hat{x}_1, \hat{x}_2) \mapsto (\check{x}_{1,-}^{-1}\check{x}_{1,+}, \check{x}_{2,-}^{-1}\check{x}_{2,+})$ transforms the points of an ordered pair into the antipodal points on S^1. The changing of a pair of the angle coordinates to an equivalent pair, which keeps the order of the points of the pair, does not change the equivariant mapping. The changing of the order of points in the pair transforms the equivariant mapping to the antipodal mapping. The constructed equivariant mapping is the required equivariant mapping (47) for the stratum of the type $\mathbf{H}_{b\times\dot{b}}$.

Assume a point $[(x_1, x_2)] \in K^{[r-s,i]}(k_1, \ldots, k_s)$ belongs to an elementary stratum of the type $\mathbf{I}_a$ (including the case, when a stratum is antidiagonal). The mapping (47) is determined by a transformation of the prescribed pair of the angle coordinates with the residue $+\mathbf{i}$, which we denote (and the same time introduce an order of the pair) as $(\check{x}_{1,+\mathbf{i}}, \mathbf{i}\check{x}_{1,+\mathbf{i}})$. The mapping $(x_1, x_2) \mapsto (\check{x}_{1,+\mathbf{i}}^2, -\check{x}_{1,+\mathbf{i}}^2)$ transforms the points of the ordered pair into an antipodal points on S^1. This mapping is the required mapping (47) for the elementary stratum of the type $\mathbf{I}_a$.

Assume a point $(x_1, x_2) \in K^{[r-s]}(k_1, \ldots, k_s)$ belongs to an elementary stratum of the type $\mathbf{I}_d$. The mapping (47) is determined by a transformation of the prescribed pair of the angle coordinates with the residue $+\mathbf{i}$, which we denote by $[(\check{x}_{1,+\mathbf{i}}, \mathbf{i}\check{x}_{1,+\mathbf{i}})]$. The mapping $(x_1, x_2) \mapsto (\check{x}_{1,+\mathbf{i}})^2, -\check{x}_{1,+\mathbf{i}})^2$ transforms the points of the ordered pair into an antipodal points on S^1. This mapping is the required mapping (47) for the elementary stratum of the type $\mathbf{I}_d$. Let us denote that the constructed mapping (47) on each elementary stratum of the type $\mathbf{I}_d$ is homotopic to the constant mapping.

Lemma 7 is proved.

Prescribed coordinate system and marked pair of the angle coordinates on an elementary stratum of the polyhedron $\hat{K}_{b\times\dot{b}\circ}$

Let us recall that the space $\hat{K}_{\mathbf{I}_{b\times\dot{b}}\circ}$ is the union of closures $Cl(\hat{K}^{[r-s,i]}(k_1, \ldots, k_s))$, $0 \leq s \leq r$ of elementary strata of the stratification (37) (closures are considered in the space $\hat{K}_\circ$). The collection of coordinates is fixed by an ordering of the spherical preimages $(\check{x_1}, \check{x_2})$ of the marked point. On each elementary stratum $\hat{\alpha}$ of the type $\mathbf{H}_{b\times\dot{b}}$ let us fix the prescribed coordinate system $\Omega(\hat{\alpha})$ as follows. (In the case an equivalent class of the prescribed coordinate system of an elementary stratum depends no of an order of the preimages.)

Let us call a coordinate system a prescribed coordinate system if,

–assuming the number of the angle coordinates is odd, the product of residues is equal to $+1$;

–assume that the number of the angle coordinate is even, the number of residues $+1$ is greater then the number of residues -1, if the the numbers of residues $+1$ and -1 coincide, the residue with the smallest number is equal to $+1$.

The angle coordinate of the prescribed system with the residue $+1$ of the smallest number is called the marked coordinate on $\hat{K}^{[r-s,i]}(k_1, \ldots, k_s)$.

Prescribed coordinate system and marked pair of the angle coordinates on an elementary stratum of the polyhedron $K_{\mathbf{I}_a\circ}$

Let us recall that the space $\hat{K}_{\mathbf{I}_a\circ}$ is the union of closures $Cl(K^{[r-s,i]}(k_1, \ldots, k_s))$, $0 \leq s \leq r$ of elementary strata of the stratification (37) (closures are considered in the space $K_\circ$). On each elementary stratum α of the type $\mathbf{I}_a$ residues are $+\mathbf{i}$, or $-\mathbf{i}$. Let us define the prescribed coordinate system $\Omega(\alpha)$ as follows.

Let us call a coordinate system is the prescribed coordinate system if,

–assuming the number of the angle coordinates is odd, the product of residues is equal to $+\mathbf{i}$;

–assume that the number of the angle coordinate is even, the number of residues $+\mathbf{i}$ is greater then the number of residues $-\mathbf{i}$, if the the numbers of residues $+\mathbf{i}$ and $-\mathbf{i}$ coincide, the residue with the smallest number is equal to $+\mathbf{i}$.

The angle coordinate of the prescribed system with the residue $+\mathbf{i}$ of the smallest number is called the marked coordinate on $K^{[r-s,i]}(k_1,\dots,k_s)$.

Prescribed coordinate system and marked pair of the angle coordinates on an elementary stratum of the polyhedron $K_{\mathbf{I}_d\circ}$

On each elementary stratum α of the type $\mathbf{I}_d$ residues are $\{+\mathbf{i},-\mathbf{i},+1,-1\}.\{+\mathbf{i},-\mathbf{i}\}$. Let us fix the prescribed coordinate system $\Omega(\alpha)$ as follows.

Let us call a coordinate system is the prescribed coordinate system if,

–assuming the number of the angle coordinates with imaginary residues is odd, the product of imaginary residues is equal to $+\mathbf{i}$;

–assume that the number of the angle coordinate with imaginary residues is even, the number of residues $+\mathbf{i}$ is greater then the number of residues $-\mathbf{i}$, if the the numbers of residues $+\mathbf{i}$ and $-\mathbf{i}$ coincide, the imaginary residue with the smallest number is equal to $+\mathbf{i}$.

The angle coordinate of the prescribed system with the residue $+\mathbf{i}$ of the smallest number is called the marked coordinate on $K^{[r-s,i]}(k_1,\dots,k_s)$.

Let us recall that the space $K_{\mathbf{I}_a\circ}$ is the union of closures $Cl(K^{[r-s,i]}(k_1,\dots,k_s))$, $0\le s\le r$ of elementary strata of the stratification (37) On each elementary stratum let us fix the coordinate system as follows.

Assume the number of the angle coordinates is odd. Let us call a coordinate system is the prescribed coordinate system, if the sum of residues of angle coordinates are equal to $+\mathbf{i}$. Assume that the number of the angle coordinate is even. Let us fixes the prescribed coordinate system arbitrarily, namely, such that the residue of the pair of coordinates with the smallest number is equal to $+\mathbf{i}$.

Admissible pair of neighbor strata

Let β be an elementary stratum (a connected component of the space $K^{[r-s,i]}(k_1,\dots,k_s)$), let α be an elementary stratum, $\alpha\subset Cl(\beta)\subset Cl(K(k_1,\dots,k_s))$, $\beta\ne\alpha$. In this case we shall write $\alpha\prec\beta$.

For an arbitrary $\beta \subset K^{[r-s,i]}(k_1,\dots,k_s)$ of the type $\mathbf{I}_a$ (correspondingly, of the type $\mathbf{I}_d$), let us consider an arbitrary α, $\alpha \prec \beta$ of the same type. Analogously, for an arbitrary $\hat{\beta} \subset \hat{K}^{[r-s,i]}(k_1,\dots,k_s)$ of the type $\mathbf{H}_{b\times\dot{b}}$, let us consider an arbitrary $\hat{\alpha}$, $\hat{\alpha} \prec \hat{\beta}$ of the same type.

Let us consider the prescribed coordinate system $\Omega(\beta)$ on β and take the restriction of this coordinate system to α. Assume that the considered restriction system is prescribed on α. Then we shall call that the pair (α,β) is admissible. In the case α and β are of different types, we shall call that the pair (α,β) is admissible.

Assume that a pair (α,β) is not admissible. Take a point $b \in \beta \subset K(k_1,\dots,k_s)$ and a point $a \in \alpha$, which is closet to b on $Cl(K(k_1,\dots,k_s))$. The restriction of the prescribed coordinate system $\Omega(\beta)|_a$ is transformed to the prescribed system $\Omega(\alpha)|_a$ by one of the following transformation, which is listed below for the strata of the each type.

A non-admissibility of a pair of strata (α,β) of the type $\mathbf{I}_a$ means that the transformation of $\Omega(\beta)|_a$ into $\Omega(\alpha)|_a$ is one of the following:

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_2},\check{x_1}), \tag{49}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_2},-\check{x_1}), \tag{50}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_1},\check{x_2}), \tag{51}$$

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_1},-\check{x_2}). \tag{52}$$

A non-admissibility of a pair of strata (α,β) of the type $\mathbf{I}_d$ means that the transformation of $\Omega(\beta)|_a$ into $\Omega(\alpha)|_a$ is one of the following:

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_2},\check{x_1}), \tag{53}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_2},-\check{x_1}), \tag{54}$$

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_2},\check{x_1}), \tag{55}$$

$$(\check{x_1},\check{x_2}) \mapsto (\check{x_2},-\check{x_1}). \tag{56}$$

A non-admissibility of a pair of strata $(\hat{\alpha},\hat{\beta})$ of the type $\mathbf{H}_{b\times\dot{b}}$ means that the transformation of $\Omega(\hat{\beta})|_a$ into $\Omega(\hat{\alpha})|_a$ is one of the following:

$$(\check{x_1},\check{x_2}) \mapsto (-\check{x_2},\check{x_1}), \tag{57}$$

$$(\check{x_1}, \check{x_2}) \mapsto (\check{x_2}, -\check{x_1}), \tag{58}$$

$$(\check{x_1}, \check{x_2}) \mapsto (-\check{x_1}, \check{x_2}), \tag{59}$$

$$(\check{x_1}, \check{x_2}) \mapsto (\check{x_1}, -\check{x_2}), \tag{60}$$

$$(\check{x_1}, \check{x_2}) \mapsto (-\mathbf{i}\check{x_2}, \mathbf{i}\check{x_1}), \tag{61}$$

$$(\check{x_1}, \check{x_2}) \mapsto (\mathbf{i}\check{x_2}, -\mathbf{i}\check{x_1}), \tag{62}$$

$$(\check{x_1}, \check{x_2}) \mapsto (-\mathbf{i}\check{x_1}, \mathbf{i}\check{x_2}), \tag{63}$$

$$(\check{x_1}, \check{x_2}) \mapsto (\mathbf{i}\check{x_1}, -\mathbf{i}\check{x_2}). \tag{64}$$

The space $Y_\circ$

Let α, β be elementary strata of $\Sigma_\circ$. Assume that $\alpha \prec \beta$ and define the elementary ε-cone of a smallest stratum α into β as an open neighborhood, which is defined as the open cone of a small height ε, $\varepsilon << 1$, over the interior of the closure of the union of all lower-dimensional ε-cones, which are inside $Cl(\beta)$. The structure of an elementary ε-cone corresponds to the Euclidean structure in the r-simplex, given by the corresponding momenta coordinates. The elementary cone of the strata α in β denote by $Con'(\alpha, \beta; \varepsilon) \subset \beta$.

Assume the following condition is satisfied:

$$\varepsilon' << \varepsilon << 1. \tag{65}$$

For an arbitrary elementary ε-cone $Con'(\alpha, \beta; \varepsilon)$ define an elementary $(\varepsilon, \varepsilon')$–cone (which is open in $\Sigma_\circ$), which is denoted by $Con(\alpha, \beta; \varepsilon, \varepsilon') \subset \Sigma_\circ$. The coordinates of points on the elementary $(\varepsilon, \varepsilon')$–cone corresponds to regular coordinates of β, and to singular coordinates, which are the last coordinates (which are satisfied restriction, see below). Singular momenta coordinates on $\beta \subset Con(\alpha, \beta; \varepsilon, \varepsilon')$ itself belong to the corresponding face of the standard simplex of the momenta coordinates, singular angle coordinates are degenerated. Let us assume that the system of momenta coordinates on β is defined, such that the singular momenta coordinates are trivial. Then in $Con(\alpha, \beta; \varepsilon, \varepsilon')$ an arbitrary singular momentum coordinate belongs to the interval $(0, \varepsilon')$. By the construction, the following inclusion is well-defined: $Con'(\alpha, \beta; \varepsilon) \subset Con(\alpha, \beta; \varepsilon, \varepsilon')$.

For each pair α, β, of non-admissible strata, $\alpha \prec \beta$, let us consider the elementary $(\varepsilon, \varepsilon')$–cone.

Define $Y_\circ(\varepsilon, \varepsilon')$ as the result of the elimination from $\Sigma_\circ$ of conical $(\varepsilon, \varepsilon')$–neighborhoods of all pairs of non-admissible strata. By the construction $Y_\circ(\varepsilon, \varepsilon)$ is derived from $\Sigma_\circ$ by the elimination of an open polyhedron. Therefore the polyhedron $Y_\circ(\varepsilon)$ is closed in $\Sigma_\circ$.

Define the space $Y_\circ$ by the formula:

$$Y_\circ = \varinjlim(\varepsilon, \varepsilon')Y_\circ(\varepsilon, \varepsilon'), \quad \varepsilon, \varepsilon' \to 0, \tag{66}$$

where the limit is taken over inclusions $Y_\circ(\varepsilon, \varepsilon') \subset Y_\circ(\bar{\varepsilon}, \bar{\varepsilon}')$, which are satisfy the condition $\varepsilon > \bar{\varepsilon}$, $\varepsilon' > \bar{\varepsilon}'$ and satisfy the inequalities (65). Evidently, the limit is formal, because the homotopy type is totally defined by a suitable finite space.

Consider the polyhedron $\Sigma_\circ \subset \Gamma_\circ$ and denote by $X_\circ(\varepsilon, \varepsilon', \varepsilon'')$ an ε''–regular neighborhood of the polyhedron $Y_\circ(\varepsilon) \subset \Sigma_\circ \subset \Gamma_\circ$, where

$$\varepsilon'' << \varepsilon'. \tag{67}$$

Evidently, that the inclusion

$$Y_\circ(\varepsilon, \varepsilon') \subset X_\circ(\varepsilon, \varepsilon', \varepsilon''), \tag{68}$$

which is a homotopy equivalence, is well-defined. The thicknesses ε of elementary cones, the thickness ε' of cylindric elementary cones, and the thickness of the regular neighborhood of the subspace (68), which satisfy the inequality (65), (67) are important, because the homotopy type of the space (68) depends no of those parameters. Below we shall write for short:

$$Y_\circ \subset \Sigma_\circ, \quad Y_\circ \subset X_\circ. \tag{69}$$

The spaces $\hat{Y}_{b\times\dot{b}\circ}$, $Y_{d\circ}$

Define the spaces

$$Y_{b\times\dot{b}\circ} \subset K_\circ, \tag{70}$$

$$Y_{d\circ} \subset K_\circ. \tag{71}$$

are subspaces in (69), which contain only strata of the type $\mathbf{I}_{b\times\dot{b}}$, or of the type $\mathbf{I}_d$ correspondingly. The 2-sheeted covering

$$Y_{b\times\dot{b}\circ} \to \hat{Y}_{b\times\dot{b}\circ}, \quad \hat{Y}_{b\times\dot{b}\circ} \subset \hat{K}_\circ, \tag{72}$$

is well-defined.

The subspaces $R\Sigma_a$, $R\hat{K}_{b\times\dot{b}\circ}$ in Lemma 4

Define the subspace

$$Y_a \subset Y_\circ, \tag{73}$$

which consists of strata of the type $\mathbf{I}_a$. Define the space $R\Sigma$ as the space Y_a. (Below in the proof of Statement B of the lemma the space (73) will be used, the space $R\Sigma$ will be defined another way).

Define the space $R\hat{K}_{b\times\dot{b}\circ}$ as the space (72), i.e. by the formula:

$$R\hat{K}_{b\times\dot{b}\circ} = \hat{Y}_{b\times\dot{b}\circ}. \tag{74}$$

This space is the base of a 2-sheeted covering, denote the covering space by $RK_{b\times\dot{b}}$.

All mappings, which are included into the diagram (22), in particular, the mappings pr, $p\hat{r}$, are standardly defined.

Resolution mappings $\phi_a : R\Sigma_a \to K(\mathbf{I}_a, 1)$, $\hat{\phi}_{b\times\dot{b}} : \hat{R}\Sigma_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$ and Proof of Lemma 4

Let us consider the restriction

$$\hat{\eta}_\circ|_{R\hat{K}_{b\times\dot{b}\circ}} : R\hat{K}_{b\times\dot{b}\circ} \to K(\mathbf{H}, 1) \tag{75}$$

of the structure mapping to the subpolyhedron (72). By the construction the following mapping is well defined:

$$\hat{\eta}_{b\times\dot{b}\circ} : R\hat{K}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}}, 1), \tag{76}$$

for which the following equation is satisfied: $\hat{\eta}_\circ|_{R\hat{K}_{b\times\dot{b}\circ}} = i_{b\times\dot{b}} \circ \hat{\eta}_{b\times\dot{b}\circ}$, where $i_{b\times\dot{b}} : K_{b\times\dot{b}} \subset K(\mathbf{H}, 1)$ is the standard inclusion. The mapping (76), generally speaking, does not satisfy the boundary conditions, which is given by the diagram (23).

Let us decompose the space $R\hat{Q}_{diag\circ}$ into 2 components, for the first component boundary conditions for the mapping (76) are satisfied up to the composition with the mapping $K(\mathbf{H}_{b\times\dot{b}}, 1) \to K(\mathbf{H}_{b\times\dot{b}}, 1)$, which is induced by the automorphism $\chi^{[2]} : \mathbf{H}_{b\times\dot{b}} \to \mathbf{H}_{b\times\dot{b}}$. Let us change the mapping (76) by the mapping

$$\hat{\phi}_{b\times\dot{b}} : \hat{R}\Sigma_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1), \tag{77}$$

for which the boundary conditions, which are given by the diagram (23), are satisfied.

Let us consider the restriction

$$\eta_\circ|_{Y_a} : Y_a \to K(\mathbf{D}, 1), \tag{78}$$

(recall, that $R\Sigma_a = Y_a$) of the structure mapping to the subpolyhedron (73). By the construction of the subpolyhedron a reduction of the mapping (78) by the mapping

$$\phi_{Y_a} : Y_a \to K(\mathbf{I}_a, 1) \tag{79}$$

is well-defined, and the following equation is satisfied: $\eta_\circ|_{Y_a} = i_{\mathbf{I}_a} \circ \phi_{Y_a}$. Lemma 4 is proved.

The last step of the proof of Lemma 6; the deformation $i_2 \circ c'_1 \mapsto d$

Consider the sphere $i_J : J \subset \mathbb{R}^n$ inside the hyperspace of the codimension 5: $i_1 : J \subset J \times \mathbb{R}^{n-k-5} \subset \mathbb{R}^{n-5}$, $i_3 \circ i_2 : \mathbb{R}^{n-5} \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$.

Denote the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^{n-5}$ by $proj$. Let us consider the space $\mathbb{R}^n$ as the Cartesian product: $\mathbb{R}^1 \times \mathbb{C}^2 \times \mathbb{R}^{n-5} \cong \mathbb{R}^n$, and therefore, let us fix in the fiber of the projection $proj$ the subspace of the (real) codimension 4, which is equipped by the standard complex structure, and also let us fix the real line, which is the orthogonal complement to $i_3(\mathbb{R}^{n-1})$).

Let us recall that by $proj_J$ the orthogonal projection of the neighborhood $U_{J,1}$ of the embedded sphere $J \subset J \times \mathbb{R}^{n-k-5} \subset \mathbb{R}^{n-5}$ to the sphere itself $i_1(J) \subset J \times \mathbb{R}^{n-k-5} \subset \mathbb{R}^{n-5}$ is defined.

Recall that the deformation $i_1 \circ c \mapsto c'$ was constructed as the vertical deformation along the fibers of the orthogonal projection $proj_J$ inside the subspace $\mathbb{R}^{n-5}$. Let us consider the standard inclusion $i_2 : \mathbb{R}^{n-5} \subset \mathbb{R}^{n-1}$. In the following diagram a deformation $\tau : i_2 \circ c' \mapsto d'$, where $d' : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^{n-1}$ is defined. (Non-formally speaking, the deformation $c' \mapsto d'$ is defined inside the subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$.) For the mapping d', analogously to the polyhedra (27), (28), (29), the following polyhedron is well-defined:

$$N(d')_\circ, \tag{80}$$

which contains a subpolyhedron of the self-intersection (generally speaking, non-closed):

$$N(d')_{b \times \dot{b}\circ} \subset N(d')_\circ. \tag{81}$$

Moreover, the regular 2-sheeted covering

$$N(d')_{b\times \dot{b}\circ} \to \hat{N}(d')_{b\times \dot{b}\circ} \tag{82}$$

is well-defined.

Lemma 8. *Assume that the dimensional condition (3) is satisfied. Then there exists an arbitrary small C^0–deformation $\tau : c' \mapsto d'$ (a support of the deformation is inside the subspace $\mathbb{R}^{n-1}$), for which the following conditions are satisfied:*

—the polyhedron (80) of the self-intersection points of the mapping d' is contained for suitable $0 < \varepsilon'' << \varepsilon' << \varepsilon << 1$ in a regular neighborhood $X_\circ(\varepsilon, \varepsilon', \varepsilon'')$ of the subpolyhedron $Y_\circ$, which is defined by the formula (69);

—the polyhedron (80) contains the subpolyhedron (81), which is the total space of the 2-sheeted covering (82).

Proof of Lemma 8[1]

Below the stratification (37) of the polyhedron $\Sigma_\circ$, which is agree with the stratification (35) with respect to the projection $proj_1$ is well-defined. Denote by $T(\Sigma_\circ^{[s]})$ the tangent space to the corresponding strata of the deep s of the stratification (36), assuming that the considered stratum of $\Sigma_\circ^{[s]}$ is projected by $proj_1$.

In each point $y \in N2_\circ^{[s]}$ define a complex vector space $R(y)^{[s]}$, orthogonal to $T(N2_\circ^{[s]})$. The real dimension of the space $R(y)^{[s]}$ is equal to $2(s+2)$. The space $R(y)^{[s]}$ is the direct sum $R_1(y) \oplus R_2^{[s]}(y)$, where $R_1(y)$ is the complex vector space of real dimension 4, parallel to the subspace $\mathbb{C}^2$ in the fiber of the projection $proj$, $R_2^{[2]}(y)$ is the complex vector space of real dimension $2s$ which is parallel to the tangent space $T(J)$ of the sphere (this tangent space is embedded into the image of the projection $proj_J$) and orthogonal to $T(\Sigma_\circ^{[s]})$. For an arbitrary point $y \in \Sigma_\circ^{[s]}$ the complex vector spaces $R_1(y)$, $R_2^{[s]}(y)$ are well-defined.

There is the standard base of the space $R_2^{[2]}(y)$, associated with the collection of tangent spaces to the strata of the stratification (35) of the sphere J (the number of the basis vector corresponds to the deep of the stratum).

At a point $y \in \Sigma_\circ^{[s]}$ let us define a complex linear subspace $E(y)^{[s]} \subset R(y)_2^{[s]}$. The collection of the subspaces $\{E(y)^{[s]}\}$, indexed by points y satisfies the following condition:

[1]This proof was discussed at the topological seminair by Prof. O.Saeki (2006) and by Prof. S.A.Bogaty

The space $E(y)^{[s]}$ in defined by the common linear combinations of the vectors of the prescribed basis of the space $R(y)^{[s]}$, independently of a choice of the point y at the given stratum α of the deep s of the stratification (37). The collection $\{E(y)^{[s]}\}$ is defined by the induction over the deep s of strata independently of a point $y \in \alpha$. Therefore instead of $E(y)^{[s]}$ we will write $E^{[s]}(\alpha)$ or $E(\alpha)$.

Define the space $E(x)^{[s]}$, assuming, that spaces over the upper strata in the family are well defined. Let y be a point on the stratum α of the deep s, and therefore this point is not in a deepest stratum. Let $\{x_i\}$ be a collection of s points, $s > i \geq 0$, $\{x_i \in \beta_i\}$, $0 \leq i \leq s-1$, satisfied the following condition: the stratum β_i contains the stratum β_{i+1} in its boundary, e.t.c., the stratum β_{s-1} contains the stratum α in its boundary. (In the previous construction the stratum α is denoted by β_s, where we allow the case $i = s$.)

Let us define an index ind, depended of various sequences $\{x_i\}$. For a given sequence this index contains s different values, parameterized by an integer parameter j, $0 < j \leq s$. The space $E(y)^{[s]}$ is transversal to an each space of the following collection of complex vector spaces $\{F(y; ind)\}$, the real dimension of an each space in the collection satisfies the equation $\dim(F(y; ind)) = 2(s+1)$ (i.e. $F(y; ind) \subset R^{[s]}$ is a complex vector space of the complex codimension 1):

$$F(y; ind) = \oplus_{i=0}^{j-1} E(x_i)^{[i]} \oplus E(\alpha, \beta_{j-1}), \quad j = s, \ldots, 1. \tag{83}$$

In this formula the complex vector space $E(\alpha, \beta_{j-1})$ of real dimension $2(s-j+1)$ is uniquely defined for $0 < j \leq s$ as a complex vector subspace at the point y is in the orthogonal complement to α inside the tangent space to β_{j-1}. In the case $j = 1$ the complex linear space $E(\beta_0)^{[0]}$ coincides with a linear subspace in $R_1(\beta_0)$, this space is a common for all maximal strata of the deep 0, and is denoted by $E^{[0]}$.

The proof of the existence of the collection of the linear vector spaces satisfied the prescribed conditions can be proved by the dimension arguments. This proof is omitted.

Let us re-denote the map f' by f_0. Let us consider a sequence of infinitesimal small real numbers $\delta_0, \ldots, \delta_r$ (recall, r is the maximal deep of strata in $T(J)$) such that $\delta_{i+1} = o(\delta_i)$. Let us define a sequence of infinitesimal C^0–small PL deformations

$$f_0 \mapsto f_1 \mapsto \ldots \mapsto f_r \tag{84}$$

with a support outside a small regular neighborhood $U(Q_{diag})$ of the diagonal. The caliber of the deformation $f_{i-1} \mapsto f_i$ is given by the infinitesimal small

real number δ_i. This sequence of infinitesimal small deformations one could considered like a sequence of PL–deformations, each next deformation is sufficiently C^0–small, and properties of the mapping from the previous step of the construction are preserved.

At the i-th step of the construction the deformation is defined with a support on strata of the deep i, and this deformation is fixed on the strata of the deep $i+1$. The parameters ε'', ε' are defined such that $\varepsilon'' << \varepsilon' << \delta_r$ (see the equation (65), in which $\varepsilon \approx \delta_r$ and the inequality above the inclusion (68)).

In the formula below the parameter ε'' is omitted for short. Let α be an arbitrary stratum of the deep i and let the following chain of strata $\alpha \prec \beta_{i-1} \prec \dots \beta_1 \prec \beta_0$ be defined. The chain of strata induces the following inclusions

$$C(\alpha, \beta_{i-1}; \delta_i, \varepsilon') \supset \dots \supset C(\alpha, \beta_0; \delta_i, \varepsilon').$$

Near a stratum α the deformation is defined in the union of elementary cones $Con'(\alpha, \beta_{i-1}; \delta_i) \cup_{j=i-1,\dots,1} Con'(\beta_j, \beta_{j-1}; \delta_i)$ along the vector space $E(\alpha) \oplus E(\beta_{i-1}) \oplus \dots E(\beta_1) \oplus E(\beta_0)$. The deformation on elementary strata β, β', which contain in the boundary the common deepest stratum β'', $\alpha \prec \beta''$ are agree on β''.

As the result, the mapping f_r has only self-intersection points, which are in regular neighborhood of the polyhedron $X_\circ(\varepsilon, \varepsilon', \varepsilon')$, which is defined by the formula (68), in which $\varepsilon_1 = \varepsilon'$ (ε'' is omitted). Outside a neighborhood of the diagonal, namely, in a neighborhood of the polyhedron (28) of the self-intersection, the deformation is given by formula, which keep the symmetry, given by the involution of 2-sheeted covering (29). Therefore the second condition in the statement of the lemma 2 is satisfied, is the first condition is satisfied. Below we construct the deformation, which is satisfied the first property.

At the first step the deformation $f_0 \mapsto f_1$ is a deformation with the support in a small neighborhood of $\Sigma^{[0]}_\circ$, this deformation is fixed on $\Sigma^{(1)}_\circ$. This deformation is vertical with respect to the linear complex space $E^{[0]}$ over each point at elementary strata $\alpha \subset K^{[0]}(k_1, \dots k_r)$ of the space $\Sigma^{[0]}_\circ$. To construct this deformation it is sufficient to consider only maximal strata of (37)). Let us consider a real linear bundle, associated with the canonical 2-sheeted covering over the considered stratum. Denote this bundle by l_α : $D(l_\alpha) \to \alpha$.

By Lemma 7 a fiberwise monomorphism $\psi_\alpha : D(l_\alpha) \subset \alpha \times E^{[0]}_\alpha$ (below we will mention this fiberwise monomorphism as a $\mathbb{Z}/2$-section) into the trivial linear complex bundle with the structure group S^1 over α is constructed by explicit formula for an arbitrary type of the strata α. The monomorphism ψ_α

determines an infinitesimal C^1–small deformation $f_0 \mapsto f_1$ (this deformation is called the reprojectivization of self-intersection points along the fiber ψ over α).

Then this deformation is extended by the linearity to the C^1–small deformation of the space $\mathbb{R}\mathrm{P}^{n-k}$, fixed on the subpolyhedron $\Sigma_\circ^{[0]}$.

The C^1–deformation $f_1 \mapsto f_2$ on $\Sigma_\circ^{(1)}$ is constructed analogously to the previous step along the prescribed complex linear space over each stratum of the deep 1. Let us describe the C^0–joining of the map f_1 with the map f_2 let us describe the deformation $f_1 \mapsto f_2$ (recall that the mapping f_2 is defined yet on $\Sigma_\circ^{(1)}$). Consider a pair of strata α of the deep 1, and β of the deep 0. There are the following possibility: α is a non-admissible, or, an admissible in $Cl(\beta)$. In the first case there are the following subclasses, which correspond to the formula (49)-(52).

Define the deformation $f_1 \mapsto f_2$ in each conical neighborhood $C(\alpha, \beta; \delta_1, \varepsilon')$ in maximal strata β of the deep 0, assuming that strata α of the deep 1 are non-admissible in $Cl(\beta)$ (if α is admissible in $Cl(\beta)$ the deformation could be arbitrary). Below we write $C(\alpha, \beta; \delta_1)$ instead of $C(\alpha, \beta; \delta_1, \varepsilon')$ for short. In the considered case the conical neighborhood $C(\alpha, \beta; \delta_1)$ coincides with the cone $Con(\alpha, \beta; \delta_1)$. The mapping f_2 outside the union of all cones for pairs of non-admissible strata, generally speaking, could have self-intersections inside $X(\varepsilon, \varepsilon_1)$.

Let us consider over $Con(\alpha, \beta)_\circ$ the trivial complex 2-line bundle $E_\alpha^{[1]} \oplus E^{[0]}$. Consider $\mathbb{Z}/2$–sections, which are constructed in Lemma 7, and which are associated with the canonical covering over the corresponding component of the boundary $U(\alpha, \beta)_\circ$ in the strata $E_\alpha^{[1]}$, $E^{[0]}$. This sections are extended to a common $\mathbb{Z}/2$-section of the bundle $E_\alpha^{[1]} \oplus E^{[0]}$ over $C'(\alpha, \beta)_\circ$ by linearity.

Let us define the deformation $f_1 \mapsto f_2$ near each conical neighborhood $C(\alpha, \beta; \delta_1)$ of the greater stratum β of the deep 0 in the case of non-admissible pair $\alpha \prec \beta$. For a maximal stratum of the deep 0 the conical neighborhood $C(\alpha, \beta; \delta_1)$ coincides with the cone $Con(\alpha, \beta; \delta_1)$. For an admissible pair $\alpha \prec \beta$ the deformation inside the prescribed linear space could be arbitrary. The mapping f_2 outside the union of all cones of admissible pairs, generally speaking, could have self-intersection points inside $X(\varepsilon, \varepsilon_1)$.

The joining of the mapping f_2 with the mapping f_1 by means of the deformation $f_1 \mapsto f_2$ on β is defined by means of the prescribed $\mathbb{Z}/2$-sections over the collection of elementary cones $C'(\alpha, \beta)_\circ$ for each pair of strata $\alpha \prec \beta$. The mapping f_1 in $C'(\alpha, \beta)_\circ$ is deformed along the space $E^{[0]} \oplus E^{[1]}(\alpha)$, using the standard formulas. By this formula deformations over each non-admissible pair of strata are well-defined.

Assume that a pair of strata $\alpha \prec \beta$ is non-admissible. Then the mapping

f_2 has no self-intersection on β near α. The mapping f_2 on strata of the deeps 0 and 1 is well-defined.

Let us construct a C^0–deformation of the mapping f_2 into the mapping f_3. Supports of the deformation is in the union of elementary cones of pairs of strata of the deep 0 and 1, of the deep 0 and 2 and of the deep 1 and 2. The deformation is defined such that in a cone $(\alpha, \beta; \delta_2)$ of each non-admissible pair of strata there are no self-intersection points.

The main observation (analogous observations are used at the next steps of the construction) is the following. Assume that for a stratum α (of the deep 2) we have $\alpha \prec \beta$, $\alpha \prec \beta'$, and the both pairs are non-admissible.

Assume that $C(\alpha, \beta; \delta_2) \cap C(\alpha, \beta'; \delta_2) = \emptyset$ and, therefore, if $\alpha \prec \beta_1 \prec \beta$, $\alpha \prec \beta_1 \prec \beta'$ then one of the following two cases is only possible:

– a one of the pair, namely, the pair $\alpha \prec \beta_1$ is admissible (self-intersection points on β_1 near α could be possible);

– the pairs $\beta_1 \prec \beta$, $\beta_1 \prec \beta'$ are admissible (self-intersection points on β_1 near β and on β_1 near β' could be possible).

Assume that $C(\alpha, \beta; \delta_2) \cap C(\alpha, \beta'; \delta_2) \neq \emptyset$ and, therefore, (up to a re-denotation we have $\alpha \prec \beta' \prec \beta$ then one of the following two cases is only possible:

– the pair $\alpha \prec \beta'$ is admissible (self-intersection points on β' near α could be possible);

– the pair $\beta' \prec \beta$ is admissible (self-intersection points on β near β' could be possible).

Let us classify non-admissible pairs as following: , a pair of strata β_0 of the deep 0 and α of the deep 2 (Case 1) a pair of strata β_1 of the deep 1 and α of the deep 2 (Case 2).

Let us consider Case 1. Assume β_0 is an arbitrary stratum of the deep 0, such that $\alpha \prec \beta_0$, and the pair (α, β_0) is non-admissible. The deformation $f_2 \mapsto f_3$ along $E(\alpha) \oplus E(\beta_1) \oplus E(\beta)$ on $C(\alpha, \beta_1; \delta_2)$ by the standard formula is well-defined. There are no self-intersection points of f_3 in $C(\alpha, \beta_0; \varepsilon)$ (but self-intersection points in $C(\beta_1, \beta_0; \varepsilon)$, where $\alpha \prec \beta_1 \prec \beta_0$ could be possible.

Let us consider Case 2. Assume β_1 is an arbitrary stratum of the deep 1, such that $\alpha \prec \beta_1 \prec \beta_0$, and the pair (α, β_1) is non-admissible. Assume that the pair (β_1, β_0) is admissible. The deformation $f_2 \mapsto f_3$ along $E(\alpha) \oplus E(\beta_1) \oplus E(\beta)$ on $C(\alpha, \beta_1; \delta_2)$ by the standard formula is well-defined. There are no self-intersection points of f_3 in $C(\alpha, \beta_0; \varepsilon)$ (but self-intersection points in $C(\beta_1, \beta_0; \varepsilon)$ could be possible.

Assume that $\alpha \prec \beta_1 \prec \beta_0$, the pair the pair (α, β_1) is non-admissible and the pair (β_1, β_0) is non-admissible. Therefore the pair (α, β_0) is admissible. Let us define the deformation $f_2 \mapsto f_3$ along $E(\alpha) \oplus E(\beta_1) \oplus E(\beta)$ on $C(\alpha, \beta_1; \delta_2) \cup C(\beta_1, \beta; \delta_2)$.

Assume $y \in \alpha$ is an arbitrary point. Let us consider the sphere $S(y,\varepsilon)$ with the center at y of the radius ε, which contains no points of deepest strata inside. Let us investigate the intersection $S(y,\varepsilon) \cap Im(f_1)$. By the construction of f_1, the space $S(y,\varepsilon) \cap Im(f_1)$ is inside the subspace $(T(\alpha) \oplus T(\alpha,\beta) \oplus E^{[0]}) \cap S(y,\varepsilon)$, where

—$T(\alpha)$ is the tangent space to α,

—$T(\alpha,\beta)$ is an orthogonal complement of α in β_0,

—$E^{[0]}$ is a linear space, along which the deformation $f_0 \mapsto f_1$ on β at the first step of the construction is defined.

The space $E^{[2]}(\alpha) \cap S(y,\varepsilon)$ has no intersection with $S(y,\varepsilon) \cap Im(f_1)$.

The constant δ_2, which was used at the previous step of the construction, should be satisfies the following property. Let us consider the intersection $A = S(y,\varepsilon) \cap Im(f_2(C(\alpha,\beta;\delta_2) \cap C(\beta_1,\beta;\delta_2)))$ of the sphere with the image of the neighborhood by the mapping f_2. The space A is inside a sufficiently small neighborhood (the sphere $S(y,\varepsilon)$ has to be considered as the standard unite sphere with the standard metric; the constant δ_2 is well-defined using thicknesses of the conical neighborhoods of the strata β_1 of the deep 1 inside the stratum β_0 of the deep 0) of the space

$$T(\alpha) \oplus E(\alpha,\beta_1) \oplus E^{[1]}(\beta_1) \oplus E^{[0]}; \tag{85}$$

—$T(\alpha,\beta_1)$ is the orthogonal complement of α in β_1,

—$E^{[1]}(\beta_1)$ is the prescribed linear space, along this space the deformation $f_1 \mapsto f_2$ on β_1 at the previous step of the construction is defined.

The deformation $f_2 \mapsto f_3$ on α is defined along the linear space $E^{[2]}(\alpha)$. This space is transversal to the tangent space of the image of $Im(f_2)$. The deformation $f_2 \mapsto f_3$ is defined by linear formula, which joins the prescribed $\mathbb{Z}/2$-sections along the momenta coordinates. The deformation $f_2 \mapsto f_3$ is well defined.

By the construction, the mapping f_3 has no self-intersection points in neighborhoods of an arbitrary non-admissible pair of strata of the deeps 0 and 1 and of the deeps 0 and 2. Evidently, the first property of the lemma is well defined, exept strata of the deep 3 and more.

Assume that the mapping f_{i-1} is well defined. Let us consider the mapping f_{i-1} near strata of the deep i and let us consider the deformation $f_{i-1} \mapsto f_i$ with a support in the union of all elementary cones of strata of the deep i with analogous properties.

An axillary C^0-deformation $f_{i-1} \mapsto f_i$ is constructed first on $\Sigma_\circ^{[i]}$, fixed on $\Sigma_\circ^{(i+1)}$. This deformation is defined along the collection of the line spaces $E(\alpha)^{[i]}$ for an arbitrary $\alpha \subset \Sigma_\circ^{[i]}$ as at the first step of the construction. Then

the deformation is extended by linear formula to the required C^0–deformation on $\mathbb{RP}^{n-k}$, which is the identity on $\Sigma_\circ^{(i+1)}$.

Let α be a stratum of the deep i and let a pair of strata $\alpha \prec \beta_0$ be non-admissible. Let us consider all chains of strata such that $\alpha \prec \beta_j \prec \dots \beta_1 \prec \beta_0$ and all the pairs $(\alpha, \beta_j), \dots (\beta_1, \beta_0)$ are non-admissible. The mapping f_i is defined such way that this mapping has no self-intersection points near α into the elementary cone of an arbitrary pair of neighbor strata of the chain. The deformation $f_{i-1} \mapsto f_i$ is well-defined. The mapping f_i, $i = r$, satisfies the conditions of the lemma. Lemma 8 is proved.

Proof of Lemma 6

Let us consider the mapping $d' : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-1}$, which is constructed in Lemma 8. Consider the self-intersection polyhedron of the mapping d', which is given by the formula (80). This polyhedron contains the following subpolyhedra

$$N(d')_a \subset N(d')_\circ, \tag{86}$$

$$N(d')_d \subset N(d')_\circ. \tag{87}$$

The subpolyhedra (86), (87) is naturally mapped into the polyhedra (40),(38) correspondingly, by the standard mapping $N(d')_\circ \subset Y_\circ \longrightarrow \Sigma_\circ$ (see. (69)).) The canonical 2-sheeted covering $\bar{Y}_{d\circ} \to Y_{d\circ}$ is trivial, because all the translations between elementary strata belongs to the subgroup $\mathbf{I}_c \subset \mathbf{D}$. In particular, the canonical covering over the subpolyhedron (87) is also the trivial covering.

The image of d' is inside the standard subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. Consider an additional vertical deformation $d' \mapsto d$, which is vertical with respect to the last coordinate of $\mathbb{R}^n$, for which all the self-intersection points in (87) are canceled. The self-intersection points of d is divided into two subpolyhedra, respectively to the required formula (101). The required mappings $\hat{\mu}_{b\times\hat{b}\circ}$, μ_a are induced by the mappings, which is constructed in Lemma 4. Lemma 6 is proved.

Proof of Lemma 2A

Let us prove two conditions in the statement of [Lemma 27, A1]. Condition 1 is, obviously, well proved, namely, the restriction of the mapping $\eta_\circ$ to the marked component N_a admits a cyclic reduction, given by μ_a.

Let us prove Condition 2 in [Lemma 27, A1], which is formulated for the component $N_{b\times\dot{b}\circ}$. For the convenience let us write-down this condition:

$$0 = (p_{\mathbf{I}_c,\mathbf{I}_d} \circ \bar{\eta})_*([\bar{N}_{b\times\dot{b}}]) \in H_{n-2k}(K(\mathbf{I}_d,1);\mathbb{Z}/2). \tag{88}$$

Assume that the polyhedron $N_{b\times\dot{b}\circ}$ is closed (let us remain that in this case the lower index $\circ$ in omitted) and the mapping η admits a reduction

$$\eta_{b\times\dot{b}} : N_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}},1). \tag{89}$$

In this case the formula (88) is satisfied, because the composition

$$\bar{\eta}_{b\times\dot{b}} : \bar{N}_{b\times\dot{b}} \to K(\mathbf{I}_d,1)$$

is the composition of a mapping $N_{b\times\dot{b}} \to K(\mathbf{I}_d,1)$ with the standard 2-sheeted covering

$$\bar{N}_{b\times\dot{b}} \to N_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}},1) \to K(\mathbf{I}_d,1),$$

where the mapping $K(\mathbf{I}_{b\times\dot{b}},1) \to K(\mathbf{I}_d,1)$ is induced by the homomorphism $\mathbf{I}_{b\times\dot{b}} \to \mathbf{I}_d$ with the kernel $\mathbf{I}_b \subset \mathbf{I}_{b\times\dot{b}}$.

Assume that the polyhedron $N_{b\times\dot{b}\circ}$ is not closed, and the mapping $\eta_\circ$ admits a reduction (89) with the prescribed boundary conditions. The formula (88) is rewritten as follows:

$$0 = (p_{\mathbf{I}_c,\mathbf{I}_d} \circ \bar{\eta}_{b\times\dot{b}\circ,\circ})_*([C\bar{N}_{b\times\dot{b}\circ}]) \in H_{n-2k}(K(\mathbf{I}_d,1);\mathbb{Z}/2). \tag{90}$$

The difference between the formulas (90) and (88) is following: if the polyhedron $N_{b\times\dot{b}\circ}$ is non-closed, then the polyhedron $\bar{N}_{b\times\dot{b}\circ}$ is also non-closed. Therefore the polyhedron $\bar{N}_{b\times\dot{b}\circ}$ have to be compactified into a closed by a gluing of the cone of the canonical 2-sheeted cover $\bar{N}_{b\times\dot{b}\circ} \to N_{b\times\dot{b}\circ}$ over the boundary. The result is a closed polyhedron, which is denoted in the formula (90) by $\bar{N}_{b\times\dot{b}\circ}$. The polyhedron $\bar{N}_{b\times\dot{b}\circ}$ is the covering space of the 2-sheeted covering $\bar{N}_{b\times\dot{b}\circ} \to CN_{b\times\dot{b}\circ}$, which corresponds to the subgroup $\mathbf{I}_{\dot{b}} \subset \mathbf{I}_{b\times\dot{b}}$ of the index 2. Therefore, as in the previous case, the cycle $p_{\mathbf{I}_c,\mathbf{I}_d} \circ \bar{\eta}_{b\times\dot{b}\circ} : C\bar{N}_{b\times\dot{b}\circ} \to K(\mathbf{I}_d,1)$ is a boundary.

Let us consider a general case: the polyhedron $N_{b\times\dot{b}\circ}$ is non-closed and the mapping $\eta_\circ$ admits a reduction

$$\eta_{b\times\dot{b}\circ} : N_{b\times\dot{b}\circ} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1)$$

with prescribed boundary conditions.

By the assumption the following mapping

$$\hat{\eta}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1)$$

is well-defined. Consider the 2-sheeted covering over the structure mapping, which we denote by

$$\tilde{\eta}_{b\times\dot{b}\circ} : C\widetilde{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_d \times \mathbb{Z}, 1).$$

Let us recall, that respectively to the diagram (17), the 2-sheeted covering mapping $\tilde{\eta}_{b\times\dot{b}\circ}$ over $\eta_{b\times\dot{b}\circ}$ is totally defined by the subgroup of the index 2:

$$\mathbf{H}_d \times \mathbb{Z} \subset \mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}} \mathbb{Z}. \tag{91}$$

The formula (90) is equivalent to the following condition: the homology class

$$(p_{\mathbf{H}_d\times\mathbb{Z},\mathbf{H}_d} \circ \tilde{\eta}_{b\times\dot{b}\circ})_*([C\widetilde{N}_{b\times\dot{b}\circ}]) \in H_{n-2k}(K(\mathbf{H}_d, 1); \mathbb{Z}) \tag{92}$$

is even.

By the representation $\mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[3]}$ the universal 4-bundle over $K(\mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}} \mathbb{Z}, 1)$ is well-defined, denote this bundle by $\hat{\tau}_{b\times\dot{b}}$. The bundle

$$\hat{\eta}^*_{b\times\dot{b}\circ}(\hat{\tau}_{b\times\dot{b}}) \tag{93}$$

over $\hat{N}_{b\times\dot{b}\circ}$ is well-defined.

Denote by

$$\widehat{NN}_\circ \subset \hat{N}_{b\times\dot{b}\circ} \tag{94}$$

the 3-dimensional subpolyhedron, generally speaking, with boundary, as a homology Euler class of the Whitney sum of $\frac{n-2k-3}{4}$ copies of the bundle (93). The condition (92) is equivalent to the following: the homology class

$$(p_{\mathbf{H}_d\times\mathbb{Z},\mathbf{H}_d} \circ \tilde{\eta}_{b\times\dot{b}\circ})_*([C\widetilde{\widehat{NN}}_\circ]) \in H_3(K(\mathbf{H}_d, 1); \mathbb{Z}) \tag{95}$$

is even.

Consider the mapping $\widehat{NN}_\circ \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbb{Z}, 1)$. Without loss of the generality, the inverse image by this mapping of the marked point of $S^1 = K(\mathbb{Z}, 1)$ is a closed 2-dimensional subpolyhedron, denoted by

$$\widehat{LL} \subset \widehat{NN}_\circ. \tag{96}$$

This polyhedron is PL–homeomorphic to an oriented surface, which is equipped with a mapping

$$\hat{f} : \widehat{LL} \longrightarrow K(\mathbf{H}_{b\times\dot{b}}, 1). \tag{97}$$

Let us use the following isomorphism: $H_2(K(\mathbf{H}_{b\times\dot{b}},1);\mathbb{Z})=\mathbb{Z}/2$.

Let us prove that there exists a closed oriented 3-manifold $\widehat{NN}$, its submanifold as in the formula (96) and a mapping

$$\hat{F}:\widehat{NN}\to K(\mathbf{H}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1), \tag{98}$$

for which the following two conditions are satisfied:

–1. The image of the fundamental class by the mapping (97) determines the generator of the group $H_2(K(\mathbf{H}_{b\times\dot{b}},1);\mathbb{Z})$.

–2. The image of the fundamental class by the mapping

$$\tilde{F}:\widetilde{NN}\to K(\mathbf{H}_d\times\mathbb{Z},1)\to K(\mathbf{H}_d,1)=K(\mathbb{Z}/4,1)$$

is an even (or the trivial) element in the group $H_3(\mathbf{H}_d;\mathbb{Z})$.

Let us consider 2-torus $\widehat{LL}$, which is the the 2-skeleton of the standard cell decomposition of the space $(\mathbb{R}\mathrm{P}^\infty\times\mathbb{R}\mathrm{P}^\infty)/T_{\mathbf{i}}\supset(\mathbb{R}\mathrm{P}^1\times\mathbb{R}\mathrm{P}^1)/T_{\mathbf{i}}=\widetilde{LL}$, where $T_{\mathbf{i}}:\mathbb{R}\mathrm{P}^\infty\times\mathbb{R}\mathrm{P}^\infty\to\mathbb{R}\mathrm{P}^\infty\times\mathbb{R}\mathrm{P}^\infty$ is the diagonal involution, which is defined by the standard involution $\mathbf{i}:\mathbb{R}\mathrm{P}^\infty\to\mathbb{R}\mathrm{P}^\infty$. We may visualized the space $K(\mathbf{H}_d,1)$ as the space $(\mathbb{R}\mathrm{P}^\infty\times\mathbb{R}\mathrm{P}^\infty)/T_{\mathbf{i}}\setminus diag(\mathbb{R}\mathrm{P}^\infty)$. By this construction the involution $\hat{\chi}^{[2]}:K(\mathbf{H}_d,1)\to K(\mathbf{H}_d,1)$, which corresponds to the automorphism (16) is defined by the formula: $x\times y\mapsto y\times x$.

Define the (orientation preserving) involution $\hat{\chi}:\widehat{LL}\to\widehat{LL}$, which permutes the factors and reverses the diagonal. Define the mapping $\hat{f}:\widehat{LL}\to K(\mathbf{H}_{b\times\dot{b}},1)$ (97), which transforms the diagonal generator $\mathbf{i}\in H_1(\widehat{LL};\mathbb{Z})$ to the element $ab\in\mathbf{E}_{b\times\dot{b}}$ (this element is represented by the sum of the diagonal loop with the generic loop of the first factor). Obviously, the mapping $\hat{f}$ commutes up to homotopies with the involutions $\hat{\chi}$, $\hat{\chi}^{[2]}$ in the source and target spaces of the mapping $\hat{f}$. Let us call the considered property Gluing Condition.

Let us define the manifold $\widehat{NN}$ as an oriented 3-manifold by the cylinder of the involution $\hat{\chi}:\widehat{LL}\to\widehat{LL}$. The mapping (98) is well-defined by a fibered family over S^1 of mappings of 2-tori in the space $K(\mathbf{H}_{b\times\dot{b}},1)$ (the source and the target space of (98) is the total spaces of fibrations over S^1). By Gluing Condition the mapping (98) is well-defined. This mapping satisfies Condition 1.

Let us check Condition 2. Consider the following composition:

$$p_{\mathbf{H}_d,\mathbb{Z}/2}\circ\tilde{F}:\widetilde{NN}\to K(\mathbf{H}_d\times\mathbb{Z},1)\to K(\mathbf{H}_d,1)\to K(\mathbb{Z}/2,1), \tag{99}$$

where the mapping $p_{\mathbf{H}_d,\mathbb{Z}/2}:K(\mathbf{H}_d,1)\to K(\mathbb{Z}/2,1)$ is induced by the epimorphism $\mathbf{H}_d\to\mathbb{Z}/2$ with the kernel $\mathbf{I}_d\subset\mathbf{H}_d$. It is well-known, that the

cellular mapping $p_{\mathbf{H}_d,\mathbb{Z}/2}$ transforms the standard 3-skeleton $S^3/\mathbf{i} \subset K(\mathbf{H}_d,1)$ into the standard 3-skeleton $\mathbb{R}\mathrm{P}^3 \subset K(\mathbb{Z}/2,1)$ with degree 2.

Assuming Condition 2 is not satisfied and the mapping (98) determines the generic homology class, then the mapping (99) is not homotopic to zero. Assume that the mapping (99) is cellular. Then the image of this mapping coincides with the standard 3-skeleton $\mathbb{R}\mathrm{P}^3 \subset K(\mathbb{Z}/2,1)$ and the degree of the mapping (99) is equal to 2 modulo 4.

The mapping (99) is a 2-sheeted covering over the mapping

$$\widehat{NN} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1) \to K(\mathbb{Z}/2\times\mathbb{Z},1) \to K(\mathbb{Z}/2,1). \qquad (100)$$

By the construction, the mapping (100) is homotopic to a mapping into the standard 2-skeleton $\mathbb{R}\mathrm{P}^2 \subset K(\mathbb{Z}/2,1)$. This implies that image of the fundamental class by the mapping (100), and by the mapping (99) is the trivial homology class. This prove that the degree of the mapping (99) is equal to 0 modulo 4. The mapping $\hat{F}$ satisfies Condition 2.

To prove Condition (95) we may assume that the image of the fundamental class by the mapping (97) is the trivial homology class. Therefore it is sufficiently to prove Condition (95), assuming, that the surface $\hat{L}L$ is empty. In this case the mapping $\hat{\eta}_{b\times\dot{b}\circ}$ admits a reduction into the subspace $K(\mathbf{H}_{b\times\dot{b}},1) \subset K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1)$. Condition (95) is reformulated analogously to Condition (90), which was proved above. Condition 2 from [Lemma 27, A1] is proved. Lemma 2A is proved.

6 Proof of Lemma 2, assuming the dimensional restriction (4)

Let us re-denote the embeddings i_1, i_2, i_3, which was used in the proof of Lemma A, as follows:

$$i_1 : J \subset J\times\mathbb{R}^{k-7} \subset \mathbb{R}^{n-7}, \quad i_2 : \mathbb{R}^{n-7} \subset \mathbb{R}^{n-3},$$

$$i_3 : \mathbb{R}^{n-3} \subset \mathbb{R}^n, \quad c_1' : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^{n-7}.$$

(Compare denotations in the following lemma with denotations in Lemma 6.)

Lemma 9. *There exists a C^0–small* PL*-deformation $(i_2\circ c_1')^{(2)} \mapsto d^{(2)}$ of the formal (holonomic) mapping $(i_2\circ c_1')^{(2)}$, which is the extension of the mapping $i_2\circ c_1'$, into a formal mapping $d^{(2)}$, such that the pair $(d, d^{(2)})$, $d : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^n$,*

has a holonomic self-intersection in the sense of [Definition 9,A2], and such that the polyhedron $N_\circ$ of the formal self-intersection of $d^{(2)}$ is decomposed into the union of two subpolyhedra:

$$N_\circ = N_a \cup N_{b\times\dot{b}\circ}, \tag{101}$$

where N_a is closed.

The restriction of the structure mapping $\eta_\circ$ on the subpolyhedron N_a admits a reduction by a mapping $\mu_a : N_a \to K(\mathbf{I}_a, 1)$:

$$\eta_a = i_a \circ \mu_a : N_a \to K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1). \tag{102}$$

The restriction of the structured map $\eta_\circ$ to the component $N_{b\times\dot{b}\circ}$ is a 2-sheeted covering over the mapping $\hat{\eta}_\circ : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}, 1)$, which admits a reduction by a mapping $\hat{\mu}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$:

$$\hat{\eta}_{b\times\dot{b}\circ} = \hat{\Phi}^{[2]} \circ \hat{\mu}_{b\times\dot{b}\circ} : \hat{N}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{H}, 1), \tag{103}$$

where $\hat{\Phi}^{[2]} : K(\mathbf{H}_{b\times\dot{b}} \int_{\hat{\chi}^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{H}, 1)$ is the standard mapping (the group $\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}$ is the quadratic extension of the group $\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}$, the mapping $\Phi^{[2]}$, which is given by the formula [A2,(44)], is the 2-sheeted covering over the mapping $\hat{\Phi}^{[2]}$).

A sketch of the proof of Lemma 9

The proof is analogous to the proof of Lemma 6. The classifying space $RK_a \cup RK_{b\times\dot{b}\circ}$ in Lemma 4 is defined by a different way (the statement of the lemma remains).

Let us consider the mapping $c'_1 : \mathbb{R}\mathrm{P}^{n-k} \to \mathbb{R}^{n-7}$. Denote by $L' \subset N'$ the subpolyhedron of the self-intersection points of the mapping c'_1, such that at least one of the preimages $\{x \in \bar{N}', y \in \bar{N}'\}$ of a self-intersection point $[(x, y)]$ is a critical point. The subpolyhedron of the critical point $\partial N'$ is included into L', denote the subpolyhedron $\partial N' \subset L'$ by $\partial L'$. Let us define the following open subpolyhedron $L' \setminus \partial L'$ by

$$L'_\circ, \quad L'_\circ \subset N'_\circ. \tag{104}$$

Denote by

$$M'_\circ \subset L'_\circ \tag{105}$$

the subpolyhedron in $L'_\circ$, which consists of self-intersection points, which both are critical. Let us introduce the following denotation:

$$LL_\circ = L'_\circ \setminus M'_\circ. \tag{106}$$

Below the universal space, which is a classification space of points of the subpolyhedron $LL'_\circ \subset L'_\circ \subset N'_\circ$ is defined.

The configuration space of a boundary singularities

Consider an ordered pair of points $(x, y) \in (\mathbb{R}\mathrm{P}^{n-k})^2$, $x \neq y$, which satisfy the additional condition $p(x) = p(y) \in J$, where p is a mapping, which is given by the formula (6). Assume that a point x is a critical point of the mapping p. Let us consider an ordered pair of points (x_1, x_2), each point of the pair is infinitesimal closed to the point x, which is satisfies the following condition:

$$x_1 \neq x_2, \quad x_1 \neq x, \quad x_2 \neq x, \quad p(x_1) = p(x_2). \tag{107}$$

Assume that the pair (x_1, x_2), satisfies the condition (107) and there exists a point y', which is infinitesimal closed to the point y, $x \neq y$ (generally speaking, the case $y' = y$ is possible), and the following condition is satisfied:

$$p(y') = p(x_1) = p(x_2). \tag{108}$$

Define a space $Z'_{3\circ}$ as the space of 5-tuple of points $((x_1, x_2), y', x, y)$, where $x \neq y$, $p(x) = p(y)$; each point of the ordered pair (x_1, x_2) is infinitesimal closed to the point x and satisfies the condition (107), and the point y' is infinitesimal closed to y, $x \neq y$, moreover, the triple $((x_1, x_2), y')$ satisfies the condition (108).

The following mapping

$$F'_{3\circ} : Z'_{3\circ} \to \Sigma_\circ, \tag{109}$$

which transforms a 5-tuple of point $((x_1, x_2), y', x, y) \in Z'_{3\circ}$ into the non-ordered pair of points $[(x_1, y')] \in \Sigma_\circ$ is well-defined.

Define a space $Z_{4\circ}$ as the space of non-ordered pairs of ordered triples of points $[(((x_1, x_2), x), ((y_1, y_2), y))]$, where each point of the ordered pair (x_1, x_2) is infinitesimal closed to the point x, each point of the non-ordered pair (y_1, y_2) is infinitesimal closed to the point y, $x \neq y$. Additionally, the 5-tuple of points $((x_1, x_2), y_1, x, y)$ $((y_1, y_2), x_1, y, x)$ satisfies the conditions (107), (108), i.e. determines a point of $Z'_{3\circ}$, and, moreover, the following condition is satisfied:

$$p(y_1) = p(x_1). \tag{110}$$

This condition and the equality (108) imply the equality

$$p(x_1) = p(x_2) = p(y_1) = p(y_2).$$

The polyhedron $Z_{4\circ}$ is the base of the standard 2-sheeted covering. This condition is re-formulated as follows: the two-valued mapping

$$[(((x_1, x_2), x), ((y_1, y_2), y))] \mapsto (((x_1, x_2), x), ((y_1, y_2), y)) \tag{111}$$

is the inverse mapping to the regular 2-sheeted covering, which is denoted by

$$p_{Z_{4\circ}} : \bar{Z}_{4\circ} \to Z_{4\circ}. \tag{112}$$

Let us define a two-valued mapping

$$F_{4\circ} : Z_{4\circ} \to Z'_{3\circ}, \tag{113}$$

which maps a non-ordered pair of points $[(((x_1, x_2), x), ((y_1, y_2), y))] \in Z_{4\circ}$ into two 5-tuples of points $(((x_1, x_2), y_1, x, y), ((y_1, y_2), x_1, x, y))$. The mapping (113) is the inverse mapping to the regular 2-sheeted covering, this condition follows from the regularity condition for the 2-valued mapping (111). The following mapping

$$FF_{4\circ} = F'_{3\circ} \circ F_{4\circ} : Z_{4\circ} \to \Sigma_\circ, \tag{114}$$

which transforms a non-ordered pair $[(((x_1, x_2), x), ((y_1, y_2), y))] \in Z_{4\circ}$ into the non-ordered pair of points $[(x_1, y_1)] \in \Sigma_\circ$, is well-defined.

Let us define a space $Z_{3\circ}$ as the space, which is obtain as the completion of the space $Z'_{3\circ}$ by degenerate 5-tuples of points $((x_1, x_2), y', x, y)$, which are satisfy the condition $x_1 = x_2 = x$, $x \neq y$. The mapping $F'_{3\circ}$ is extended to the mapping

$$F_{3\circ} : Z_{3\circ} \to \Sigma_\circ \tag{115}$$

by continuity.

The mapping $F_{4\circ}$ is naturally extended to the mapping $F_{4\circ} : Z_{4\circ} \to Z'_{3\circ} \subset Z_{3\circ}$, which is denoted the same. Let us extends the mapping (114) into the mapping $FF_{4\circ}$ (which is denoted the same) and define the mapping $G_{4\circ}$ by the composition:

$$G_{4\circ} = \eta_\circ \circ FF_{4\circ} = \eta_\circ \circ F_{3\circ} \circ F_{4\circ} : Z_{4\circ} \to Z_{3\circ} \to \Sigma_\circ \to K(\mathbf{D}, 1). \tag{116}$$

All the spaces, considered above, are 2-sheeted coverings over the corresponding spaces, which are denoted by $\hat{}$. The 2-sheeted coverings are constructed by the quadratic extensions of the corresponding groups, as in the diagram (17).

All the spaces, defined above, are equipped with a natural stratification, which is induced from the stratification (37). These spaces contains regular neighborhoods of boundaries, the boundary arise by the limit $x = y$.

Structure mappings of configuration spaces of the boundary singularities

In this section we shall prove two lemmas about the space of boundary singularities $Z_{3\circ}$, $\hat{Z}_{3\circ}$, $Z_{4\circ}$, $\hat{Z}_{4\circ}$, which are constructed above.

Lemma 10. *1. The space $Z_{4\circ}$ is naturally divided into 3 components (possibly, non-connected):*

$$Z_{4\circ} = Z_{4,a\circ} \cup Z_{4,b\times\dot{b}\circ} \cup Z_{4,d\circ}. \tag{117}$$

2. The restriction of the mapping (116) to the components (117) admit the following reductions:

$$G_{a\circ} : Z_{4,a\circ} \to K(\mathbf{I}_a, 1), \tag{118}$$

$$G_{b\times\dot{b}\circ} : Z_{4,b\times\dot{b}\circ} \to K(\mathbf{I}_{b\times\dot{b}}, 1), \tag{119}$$

$$G_{d\circ} : Z_{4,d\circ} \to K(\mathbf{I}_d, 1). \tag{120}$$

3. The mapping (119) is a 2-sheeted covering over the mapping

$$\hat{G}_{b\times\dot{b}\circ} : \hat{Z}_{4,b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}}, 1). \tag{121}$$

4. The restriction of the mapping (118) on the regular neighborhood of the boundary of the space $Z_{4,a\circ}$ is homotopic to a mapping into the subspace $K(\mathbf{I}_d, 1) \subset K(\mathbf{I}_a, 1)$.

Proof of Lemma 10

Let us prove Property 1. Assume that $w \in Z_{4\circ}$ is a regular point in a maximal stratum, which is denoted by β (by a regular point we mean a point inside a maximal regular stratum; to describe maximal strata of the space $Z_{4\circ}$ $(r-1)$ angle coordinates are needed). Let us consider a 6-tuple $[(((x_1, x_2), x), ((y_1, y_2), y))]$ of points, which represents a point w. The point $FF_{4\circ}(w)$ is represented by the non-ordered pair $[(x_1, x_2)]$, where the mapping $FF_{4\circ}$ is defined by the formula (114). The point $[(x, y)]$ is inside a stratum of the deep 1 of the space $\Sigma_\circ$, see (37).

Let us consider the collection of the angle coordinates $C = \{(\check{x}_i, \check{y}_i)\}$ of the point $FF_{4\circ}(w) = [(x, y)]$. Let us call this collection the collection of pairs of principal coordinates. The number of pairs of principal coordinates is odd (is equal to $(r-1)$). The full collection $C' = \{(\check{x}_{1,i}, \check{y}_{1,i})\}$ of r pairs of angle coordinates of the point $[(x_1, y_1)]$ is obtain by a completion of the collection $C = \{(\check{x}_i, \check{y}_i)\}$ by means of the last pair of angle coordinates, which is called the axillary pair. Let us consider the collection $V' = \{v_i\}$ of residues, which is constructed by means of the full collection of the coordinates. In the case the product of residues $\prod_{i=1}^{i=r} v_i$ of the full collection of coordinates is equal to $\pm\mathbf{i}$, we shall call the point w is of the type $\mathbf{I}_d$. If the considered product of residues is equal to -1, we shall call the point w is of the type $\mathbf{I}_a$. If the considered product of residues is equal to $+1$, we shall call the point w is of the type $\mathbf{I}_{b\times b}$.

Let us prove that a type of a regular point w determines the type of the connected component of the decomposition (117), which contains the point w. It is sufficiently to prove the following: if a point w' belongs to a neighbor maximal stratum β' to the stratum β of the space $Z_{4\circ}$ then the point w' is of the same type as the point w. Let us denote by α a stratum of the deep 2, which contains to the closures of the both strata β, β'. Denote by i_β the number of the disappeared angle coordinates of the stratum β on α, by $i_{\beta'}$ the number of diapered angle coordinate of the stratum β' on α. Consider two pairs of angle coordinates with the numbers $i_\beta, i_{\beta'}$ of the point $FF_{4\circ}(w)$: $(\check{x}_{i_\alpha}, \check{y}_{i_\alpha})$ (this is the pair of principle coordinates) and $(\check{x}_{i_{\alpha'}}, \check{y}_{i_{\alpha'}})$ (this is the pair of axillary coordinates). Consider two pairs of angle coordinates with the numbers $i_\beta, i_{\beta'}$ of the point $FF_{4\circ}(w')$: $(\check{x}'_{i_\alpha}, \check{y}'_{i_\alpha})$ and $(\check{x}'_{i_{\alpha'}}, \check{y}'_{i_{\alpha'}})$.

Without loss of a generality, we may take two closed points w, w', such that the following equations are satisfied:

$$\check{x}_{i_\beta} = \check{x}'_{i_\beta}, \quad \check{x}_{i_{\beta'}} = \check{x}'_{i_{\beta'}};$$

$$\check{y}_{i_\beta} = \check{y}'_{i_\beta}, \quad \check{y}_{i_{\beta'}} = \check{y}'_{i_{\beta'}}.$$

Let us take the difference between the sum of the first two equalities the sum of the second two equalities correspondingly. The resulting equality proves that the points w, w' are of the same type. Property 1 is proved.

Let us prove Property 2. Construct the reduction mapping (120). Property 1 implies that the covering (112) over the component $Z_{4,d\circ}$ is trivial. Namely, assume that the product of the full collection of the residues is imaginary. The value of this product remains fixed along each piecewise-linear path on $Z_{4,d\circ}$, therefore the value of the product determines an order of the preimages of a point on the covering space: after the order of preimages is changed the value of the product is changed to the opposite.

Therefore, a reduction of the structure mapping (117) into a mapping in the subspace $K(\mathbf{I}_c, 1) \subset K(\mathbf{D}, 1)$ is well-defined. Below, in Lemma 11 is proved that the considered reduction is well-defined into the smallest subspace $K(\mathbf{I}_d, 1) \subset K(\mathbf{I}_c, 1)$. The reduction (120) is well-defined.

Let us construct the reduction (118). Consider a point w on a maximal elementary stratum of the space $Z_{4,a\circ}$. Consider the collection of pairs of principle angle coordinates $\{(\check{x}_i, \check{y}_i)\}$ of the point $FF_{4\circ}(w) = [(x, y)]$, here and below we will use the denotations, which was introduced in the proof of the Property 1. Consider the collection $V = \{v_i\}$ of residues of pairs of principle coordinates ($r-1$ residues) and denote the product of the residues by $v(w)$. By Property 1 the value $v^{-1}(w)$ coincides with the residue of the pair of axillary coordinates of the point $[(x, y)]$. Let us define the following transformation θ_w of the collection of pairs of the angle coordinates:

$$\theta_w : \{(\check{x}_i, \check{y}_i)\} \mapsto \{(\check{x}_i, \mathbf{i}v(w)\check{y}_i)\}.$$

The collection of pairs $\theta_w(\{(\check{x}_i, \check{y}_i)\})$ in the right side of the formula determines the collection of residues ($(r-1)$ residues), the product of the residues is equal to $\mathbf{i}$. Let us call the transformation θ_w the standardization of the collection of pairs of the principle coordinates at the regular point w. Denote the collection $\theta_w(\{(\check{x}_i, \check{y}_i)\})$ by $\{\theta_w(\check{x}_i), \theta_w(\check{y}_i))\})$ and let us call this collection the standard collection of pairs of regular coordinates.

Assume w' is a point on a neighbor maximal stratum β' to the stratum β of the space $Z_{4,a\circ}$, the intersection of $Cl(\beta') \cap Cl(\beta)$ contains a stratum α of the deep 1. Let us consider the canonical transformation $\xi(\alpha)$ of the collection of pairs of principle coordinates at the point $FF_{4\circ}(w)$ into the collection of pairs of principle coordinates at the point $FF_{4\circ}(w')$. By this transformation a pair of disappeared coordinates at the point w is replaced by a pair of appeared coordinates at the point w' (up to a re-numeration). The considered pair of coordinates on α corresponds to the pair of axillary coordinates. The collection of pairs of principal coordinates, which is completed by the pair of axillary coordinates is the full collection of coordinates. The transformation $\xi(\alpha)$ of principal coordinates is defined by the formula: $\check{x}_{i_\beta} \mapsto \check{x}_{i_{\beta'}}$, $\check{y}_{i_\beta} \mapsto \check{y}_{i_{\beta'}}$.

Let us denote $\check{x}_{i_\beta}^{-1} \cdot \check{x}_{i_{\beta'}}$ by $\delta(x)$, and $\check{y}_{i_\beta}^{-1} \cdot \check{y}_{i_{\beta'}}$ by $\delta(y)$.

The formula of the transformation $\xi(\alpha)$ (up to a renumeration) is given by $\check{x}_{i_{\beta'}} = \delta(x)\check{x}_{i_\beta}$, $\check{y}_{i_{\beta'}} = \delta(y)\check{y}_{i_\beta}$. For a principal coordinate the transformation $\xi(\alpha)$ is the identity.

Let us define the transformation $\Xi(\alpha)$ of the standard collection of the pairs of principle coordinates by the formula:

$$\{(\theta(w)(\check{x}_i), \theta(w)(\check{y}_i))\} \mapsto \{(\delta(x)\theta(w)(\check{x}_i), \delta(x)\theta(w)(\check{y}_i))\}.$$

It is not difficult to see that the transformation $\Xi(\alpha)$ keeps the product of residues of principle coordinates. Therefore the collection of the coordinates in the right side of the formula corresponds to the standard collection of pairs of principle coordinates at the point w'.

The transformation $\Xi(\alpha)$ of the standard collection of pairs of all coordinates along the segment $[FF_{4\circ}(w), FF_{4\circ}(w')]$ on $\Sigma_\circ$ (see. (42)) is well-defined. The sequence of the transformations of coordinates along a closed piecewise-linear path, which starts at the point w, determines the transformation of the standard collection of all pairs of coordinates. This transformation determines the following homomorphism of the fundamental group:

$$\pi_1(Z_{4,a\circ}) \to K(\mathbf{H}, 1).$$

This homomorphism of the fundamental group determines the mapping

$$G'_{a\circ} : Z_{4,a\circ} \to K(\mathbf{H}, 1). \tag{122}$$

Let us prove that the image of the mapping (122) is inside the subspace $K(\mathbf{H}_a, 1) \subset K(\mathbf{H}, 1)$. In particular, because the group $\mathbf{H}_a$ is commutative (the group $\mathbf{H}_a$ is isomorphic to the group $\mathbb{Z}/4 \times \mathbb{Z}/2$), we get that the construction of the mapping (122) depends no of a choice of the marked point w.

Let us consider the mapping of the canonical 2-sheeted coverings

$$\bar{G}'_{a\circ} : \bar{Z}_{4,a\circ} \to K(\mathbf{H}_c, 1), \tag{123}$$

which is a 2-sheeted covering over the mapping (122). The 2-sheeted covering over the mapping $FF_{4\circ}$ admits a reduction mapping into the subspace $K(\mathbf{I}_d, 1) \subset K(\mathbf{I}_c, 1)$, see Lemma 11 below. Therefore, the mapping (123) admits a reduction into the subspace $K(\mathbf{H}_d, 1) \subset K(\mathbf{H}_c, 1)$. Therefore the mapping (122) admits a reduction into $K(\mathbf{H}_a, 1) \subset K(\mathbf{H}, 1)$, or into $K(\mathbf{H}_{b\times \dot{b}}, 1) \subset K(\mathbf{H}, 1)$. A straightforward calculation shows that the first case is realized.

Let us define the mapping (118) by the composition

$$Z_{4,a\circ} \to K(\mathbf{H}_a, 1) \to K(\mathbf{I}_a, 1).$$

The projection $K(\mathbf{H}_a, 1) \to K(\mathbf{I}_a, 1)$ could be defined by a two different ways. Let us define this mapping, such that the following condition is satisfied. Assume that a path $l \subset Z_{4,a\circ}$ admits an extension of the standard collection of the coordinates along itself, such that the standardization into an arbitrary regular point of l is the identity, This means that the product of residues of the pairs of principle coordinates at a regular point is equal to $\mathbf{i}$. Then

the mapping (118) into the space $K(\mathbf{I}_a, 1)$, is determined by the reduction mapping along l. (Assume that a path $l \subset Z_{4,a\circ}$ determines the canonical extension of the coordinate system, for which at an arbitrary regular point of the path the product of residues of all principal coordinates is equal to $-\mathbf{i}$. Then the mapping, which is constructed by the same formula, is conjugated to the reduction mapping (118).) The mapping (118) is well-defined. The definition of the mapping (119) is analogous. Property 2 is proved.

Properties 3 and 4 are evident, the proofs are omitted. Lemma 10 is proved.

Lemma 11. *1. The mapping $\eta_\circ \circ F'_{3\circ} : Z'_{3\circ} \to \Sigma_\circ \to K(\mathbf{D}, 1)$ admits a reduction by the mapping $G'_{3\circ} : Z'_{3\circ} \to K(\mathbf{I}_d, 1)$. into the subspace $K(\mathbf{I}_d, 1) \subset K(\mathbf{D}, 1)$.*

2. The mapping $G'_{3\circ}$ on the subpolyhedron $Z'_{3\circ} \subset Z_{3\circ}$ is extended to a mapping

$$G_{3\circ} : Z_{3\circ} \to K(\mathbf{I}_d, 1). \tag{124}$$

3. The mapping $G_{3\circ}$ is the 2-sheeted covering over the mapping $\hat{G}_{3\circ} : \hat{Z}_{3\circ} \to K(\mathbf{H}_d, 1)$, which corresponds to the 2-sheeted covering $K(\mathbf{I}_d, 1) \to K(\mathbf{H}_d, 1)$.

Proof of Lemma 11

Let us prove Property 1. Let $l : S^1 \subset Z_{3\circ}$ be an arbitrary PL-path, which is projected to the path $m = p(l) : S^1 \subset \bar{U}(Q_{diag})_\circ \subset \bar{\Sigma}_\circ$ by the mapping

$$p : ((x_1, x_2), y', x, y) \mapsto (x_1, x_2).$$

Let the path m be transversely intersecting the subspace of singular strata of the deep 2 in a finite number of singular points, which is denoted by $\{t_1, \ldots, t_j\}$.

On an elementary stratum of the deep 0 of the polyhedron $U(Q_{diag})_\circ$, which contains the point $pt = l(0)$, where $0 \in S^1$ is the marked point, let us fix a pair of angles coordinates $\{(\check{x}_1(0), \check{x}_2(0))\}$. Take the angle coordinate $\{\check{y}(0)\}$ of the point y and consider the triple $(\check{x}_1(0), \check{x}_2(0), \check{y}(0))$.

Extend in a natural way a coordinate system $(\check{x}_1(0), \check{x}_2(0), \check{y}(0))$ of the point pt along the path l, such that regular coordinates in a neighborhood of each critical point $\{t_1, \ldots, t_j\}$ are agree (the natural extension of coordinate system was defined by the formula (42)).

The transformation of the initial system of coordinates $(\check{x}_1(0), \check{x}_2(0), \check{y}'(0))$ in the system of coordinates $(\check{x}_1(2\pi), \check{x}_2(2\pi), \check{y}'(2\pi))$ is

well defined. This transformation is decomposed into the direct product of tree natural transformations $\eta_{x_1}(l) \in \mathbf{I}_d$, $\eta_{x_2}(l) \in \mathbf{I}_d$, $\eta_{y'}(l) \in \mathbf{I}_d$ for the first, second, and the third coordinate respectively.

Let us prove the following equations:

$$\eta_y(l) = \eta_{x_1}(l), \tag{125}$$

$$\eta_y(l) = \eta_{x_2}(l). \tag{126}$$

Let us prove the equation (125). Assume that the path l is contained inside the only maximal stratum of the space $Z'_{3\circ}$. Then the path m is contained in a regular stratum of the deep 1. In this case the equations (125), (126) are evident.

Denote by $\{t_1, \ldots, t_j\}$ the finite set of the critical points of the path l, in which critical point the path l crosses a stratum of the depth 2. In a infinitesimal small neighborhood of each critical point t_i there are two regular points of the path m, which are on strata of the deep 1. Denote these critical points by $\tau_{i,1}, \tau_{i,2}$. Without loss of the generality, the path m is defined such that in a neighborhood of each critical point t_i all coordinates $\{\check{x}_1\}$, included the critical coordinate, are fixed. The coordinates $\{\check{x}_2\}$, $\{\check{y'}\}$ can jump.

On the interval (t_{i-1}, t_i) the path l has the only axillary coordinate with the number $j([i-1, i])$ (about the definition of principal and axillary coordinate see Lemma 10, Property 1). On an infinitesimal small interval $(\tau_{i,1}, \tau_{i,2})$ the path l has two axillary coordinates with numbers $j([i-1, i])$ и $j([i, i+1])$.

Choose the path m, satisfied the following conditions, which are not restrictive:

— at the points $(\tau_{i,1}, \tau_{i,2})$ the collection of the principal coordinates are changed continuously;

— at the critical point $\tau_{i,1}$ the first axillary coordinate with the numbers $j([i-1, i])$, $j([i, i+1])$ is changed continuously; the second axillary coordinate with the number $j([i, i+1])$ could jump, the second axillary coordinate with the number $j([i-1, i])$ is continuous; the third axillary coordinate is continuous;

— at the critical point $\tau_{i,2}$ the first two axillary coordinates with the numbers $j([i-1, i])$, $j([i, i+1])$ are changed continuously; the second axillary coordinate with the number $j([i-1, i])$ could jump, the second axillary coordinate with the number $j([i, i+1])$ is continuous; the third axillary coordinate with the number $j([i, i+1])$ could jump by the multiplication on an element, which is denoted by

$$\theta(x_i) \in \mathbf{I}_a. \tag{127}$$

Let us prove that the value $\eta_y(l) \in \mathbf{I}_a$ is given by the following formula:

$$\eta_y(l) = \prod_i \theta(x_i). \tag{128}$$

For the path l let us consider a path l_0, which is also projected into m and which satisfies the equation $\theta(x_i) = 1$ for an arbitrary i. The homology class of the path $\eta_y(l)$ distinguishes from the homology class of the path $\eta_y(l_0)$ by the element $\prod_i \theta(x_i)$. The Formula (125) for the path l_0 is obvious.

For the path l the following equation is satisfies:

$$\prod_i \theta(x_i) = 1,$$

because the number r of coordinates satisfies the equation $r \equiv 0 \pmod 4$. The formula (125) is proved. Condition 1 is proved.

Condition 2 is following from the formula (126), which is proved analogously. The prove of Condition 3 is standardly and is omitted. Lemma 11 is proved.

Configuration space $YY_\circ$ of singularities

Let us define the space $Z_\circ$ as the cylinder of the 2-valued mapping $F_{4\circ} : Z_{4\circ} \to Z_{3\circ}$. Let us consider the following mapping $FF_\circ : Z_\circ \to \Sigma_\circ$, which is given by the formula (114). Define the following commutative diagram:

$$Y_\circ \subset \Sigma_\circ \xleftarrow{FF_\circ} Z_\circ, \tag{129}$$

where the subspace $Y_\circ \subset \Sigma_\circ$ was defined by the formula (69).

Let us define the polyhedron

$$YY_\circ \tag{130}$$

as the cylinder of the mapping $FF_\circ$ in Diagram (129), i.e. as a result of the gluing of the polyhedron $Y_\circ$ with the polyhedron $Z_\circ$ along the subspace $FF_\circ^{-1}(Y_\circ)$ by the mapping $FF_\circ|_{FF_\circ^{-1}(Y_\circ)}$.

Let us define the polyhedron $YY_{b\times\dot{b}\circ} \subset YY_\circ$, replacing in the formula (129) the polyhedron $Y_\circ$ to the subpolyhedron $Y_{b\times\dot{b}\circ}$, which is equipped with the attaching mapping

$$Y_{b\times\dot{b}\circ} \subset \Sigma_\circ \xleftarrow{F_\circ} Z_\circ, \tag{131}$$

where the polyhedron $Y_{b\times\dot{b}\circ}$ is defined as the covering space over the subpolyhedron (72) by the standard 2-sheeted covering

$$Y_{b\times\dot{b}\circ} \to \hat{Y}_{b\times\dot{b}\circ}.$$

Mappings $\eta_{YY_\circ} : YY_\circ \to K(\mathbf{D}, 1)$

Define the structure mapping $\eta_{YY_\circ} : YY_\circ \to K(\mathbf{D}, 1)$ by the formula:

$$\eta_\circ \circ F_\circ : YY_\circ \to \Sigma_\circ \to K(\mathbf{D}, 1), \tag{132}$$

where the mapping $F_\circ$ is defined as the result of the natural extension of the mapping (114) from the subpolyhedron $Z_\circ \subset YY_\circ$ on the polyhedron, and the mapping $\eta_\circ$ is defined by the formula (42). In the next subsection subspaces of the space $YY_\circ$, which are called the resolution space and which are the required spaces in Lemma 4 are well-defined.

Resolution spaces $R\Sigma_a$**,** $R\hat{K}_{b\times\dot{b}\circ}$ **and resolution mappings** $\phi_a : R\Sigma \to K(\mathbf{I}_a, 1)$**,** $\hat{\phi}_{b\times\dot{b}\circ} : R\hat{K}_{b\times\dot{b}\circ} \to K(\mathbf{H}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$

Let us construct a polyhedron $R\Sigma_a$. Consider the following subdiagram in Diagram (129)

$$Y_a \subset Y_\circ \subset \Sigma_\circ \overset{FF_\circ|_{Z_{4,a\circ}}}{\longleftarrow} Z_{4,a\circ}. \tag{133}$$

In this diagram the subspace $Y_a \subset Y_\circ$ is defined by the formula (73), and the mapping $FF_\circ|_{Z_{4,a\circ}}$ is considered as a partial (with the origin $FF_\circ^{-1}(Y_a)$). Define the space $R\Sigma_a$ as the cylinder of the partial multivalued mapping from Diagram (133) over the subspace. This is the required resolution space in Lemma 4.

Let us construct the polyhedron $RK_{b\times\dot{b}\circ}$. Consider the following subdiagram in Diagram (129)

$$Y_{b\times\dot{b}} \subset Y_\circ \subset \Sigma_\circ \overset{FF_\circ|_{Z_{4,b\times\dot{b}\circ}}}{\longleftarrow} Z_{4,b\times\dot{b}\circ}. \tag{134}$$

In this diagram the subspace $Y_{b\times\dot{b}} \subset Y_\circ$ is defined by the formula (72), and the mapping $FF_\circ|_{Z_{4,b\times\dot{b}\circ}}$ is considered as a partial (with the origin $FF_\circ^{-1}(Y_{b\times\dot{b}})$). Define the space $R\Sigma_{b\times\dot{b}}$ as the cylinder of the partial multivalued mapping from Diagram (134) over the subspace. The polyhedron $R\hat{K}_{b\times\dot{b}\circ}$ is defined as the base of the standard 2-sheeted covering $RK_{b\times\dot{b}\circ} \to R\hat{K}_{b\times\dot{b}\circ}$. This is the required resolution space in Lemma 4.

Let us define the structure mapping ϕ_a. Consider the restriction of the mapping (132) on the subspace $R\Sigma_a \subset YY_\circ$. Let us prove that this mapping admits a reduction by the mapping (20)

$$\phi_a : R\Sigma_a \to K(\mathbf{I}_a, 1).$$

Consider the mappings (118), (76). It is sufficiently to prove that these mappings satisfy the following equation:

$$\phi_{Y_a} \circ FF_{4\circ}|_{Z_{4,a\circ}} = G_{4,a\circ}, \tag{135}$$

where the mapping $G_{4,a\circ}$ is defined by the formula (118), and the mapping $FF_{4\circ}$ is defined by the formula (114). By the construction the following equation is satisfied: $\phi_{Y_a} \circ F_{3\circ} = G_{3\circ}$, where the mapping $G_{3\circ}$ is defined by the formula (124), and the mapping $F_{3\circ}$ is defined by the formula (115). The mapping $G_{4,a\circ}$ satisfies the following property. For an arbitrary homology class of peaswise-linear paths $l \subset Z_{4,a\circ}$, which are projected into the subspace $Y_a \subset \Sigma_\circ$, the formula (135) is satisfied. The condition of a general position of a path is not restrictive, and for an arbitrary path l the formula (135) is proved. The resolution mapping (20) is well-defined.

The structure mappings

$$\phi_{b\times\dot{b}} : R\Sigma_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}}, 1), \tag{136}$$

$$\hat{\phi}_{b\times\dot{b}} : R\hat{\Sigma}_{b\times\dot{b}} \to K(\mathbf{H}_{b\times\dot{b}}, 1) \tag{137}$$

are defined analogously.

Proof of Lemma 4

All resolution spaces and required structure mappings was defined below. Properties of the resolution spaces and the mappings are obviously. Lemma 4 is proved.

The final step of the proof of Lemma 9; the deformation of the mapping $i_2 \circ c'_1$ as a formal mapping

Denote the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^{n-7}$ by $proj$. Decompose the space $\mathbb{R}^n$ as the Cartesian product: $(\mathbb{R}^1)^3 \times \mathbb{C}^2 \times \mathbb{R}^{n-7} \cong \mathbb{R}^n$. Therefore, the fiber of $proj$, which is the direct sum of the standard Euclidean space of the dimension 4, which is equipped by the standard complex structure, and the standard Euclidean space of the dimension 3, which is orthogonal to $i_3(\mathbb{R}^{n-3})$.

Recall that by $proj_J$ the standard orthogonal projection of a neighborhood $U_{J,1}$ of the sphere $J \subset J \times \mathbb{R}^{n-k-7} \subset \mathbb{R}^{n-7}$ onto the sphere itself $i_1(J) \subset J \times \mathbb{R}^{n-k-7} \subset \mathbb{R}^{n-7}$ is well defined.

Recall that the deformation $i_1 \circ c \mapsto c'$ is defined as a vertical deformation along $proj_J$ inside the subspace $\mathbb{R}^{n-7}$. Consider the polyhedron (106) of boundary singularities of the mapping c'_1. Consider the inclusion $LL_\circ \subset$

$U(L)_\circ \setminus M'_\circ \subset N'_\circ$ of the polyhedron (106) into a deleted neighborhood of the subpolyhedron (104) in the polyhedron $N'_\circ$. Denote $U(L)_\circ \setminus M'_\circ$ by $U(LL)_\circ$.

The following mapping into the configuration space is well defined:

$$res : U(LL)_\circ \to Z_\circ. \tag{138}$$

Consider the following composition: $\eta_{Z_\circ} \circ res : U(LL)_\circ \to Z_\circ \to K(\mathbf{D}, 1)$, where the mapping $\eta_{Z_\circ} : Z_\circ \to K(\mathbf{D}, 1)$ is defined as the restriction of the mapping (132) on the subpolyhedron $Z_\circ \subset YY_\circ$. The mapping $\eta_{Z_\circ} \circ res$ is extended into the following mapping:

$$U(L_\circ) \to K(\mathbf{D}, 1), \tag{139}$$

because an each simplex of the subpolyhedron (106) in the corresponding simplex of the polyhedron $L'_\circ$ has the codimension $k - 7 \geq 3$.

The restriction of the mapping $\eta_{Z_\circ}$ on the subpolyhedron $Z_{4,a\circ} \cup Z_{4,b\times\dot{b}\circ} \cup Z_{4,d\circ}$ admits a reduction by the mappings (118), (119), (120). Therefore the mapping $\eta_{Z_\circ} \circ res$ also admits the corresponding reduction.

Consider the mapping $i_2 \circ c'$ and re-denote this mapping by $c : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-3}$ for short. Consider the associated with the mapping c the formal (equivariant) mapping $c^{(2)} : (\mathbb{RP}^{n-k})^2 \to \mathbb{R}^{n-5} \times \mathbb{R}^{n-5}$. The formal mapping $c^{(2)}$ is holonomic, in particular, this formal mapping is holonomic on the subpolyhedron $\bar{U}(L') \subset (\mathbb{RP}^{n-k})^2$.

Let us define a deformation $(c, c^{(2)}) \mapsto (c, d_1^{(2)})$ of the formal mappings, where $d_1^{(2)} : (\mathbb{RP}^{n-k})^2 \to (\mathbb{R}^{n-3})^2$, which is the identity on the subpolyhedron $\bar{U}(L')$.

For the formal mapping $d_1^{(2)}$ the following polyhedral of (formal) self-intersection are well defined:

$$N(d_1^{(2)})_{a,\circ} \subset N(d_1^{(2)})_\circ, \tag{140}$$

$$N(d_1^{(2)})_{b\times\dot{b}\circ} \subset N(d_1^{(2)})_\circ, \tag{141}$$

$$N(d_1^{(2)})_{d,\circ} \subset N(d_1^{(2)})_\circ. \tag{142}$$

(A definition of these polyhedra are analogous to the definition of polyhedra (27), (28), (29).)

The deformation τ is defined such that on the polyhedron of the formal self-intersection (141) the symmetry, which corresponds to the quadratic extension $\mathbf{I}_{b\times\dot{b}} \subset \mathbf{H}_{b\times\dot{b}}$, is kept. Therefore, the following regular 2-sheeted covering

$$N(d_1^{(2)})_{b\times\dot{b}\circ} \to \hat{N}(d_1^{(2)})_{b\times\dot{b}\circ} \tag{143}$$

is well-defined.

The deformation τ is defined such that the pair $(c, d_1^{(2)})$ has holonomic singularities. Recall that τ is fixed on $\bar{U}(L')$, in particular, the formal mapping $d_1^{(2)}$ is holonomic along the polyhedra (140), (141), (142) of the formal self-intersection of the formal mapping $d^{(2)}$.

Moreover, the deformation τ is defined such that the polyhedra (140), (141), (142) are subpolyhedra in the universal polyhedra (72), (73), and $Y_{d\circ}$. The construction of the deformation τ follows from Lemma (8), which is applied to the mapping outside $U(L)_\circ$, where this mapping is an immersion. The formal mapping d_1^2, which is satisfies the prescribed properties, is well-defined.

At the last step of the construction a deformation of the mapping $c \mapsto d$ into the required mapping, and a deformation of the formal mapping $d_1^{(2)} \mapsto d^{(2)}$ into the required formal mapping $d^{(2)} : (\mathbb{RP}^{n-k})^2 \to (\mathbb{R}^n)^2$, such that the pair $(d, d^{(2)}$ has holonomic singularities are well-defined.

This deformation is vertical with respect to the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^{n-3}$. Therefore all properties of the polyhedron of the formal self-intersection of the mapping $d_1^{(2)}$ are kept. Denote by $N(d^{(2)})_\circ$ the polyhedron of the formal self-intersection of the mapping $d^{(2)}$. Consider the subpolyhedron $M(d^{(2)})_\circ \subset N(d^{(2)})_\circ$, which is defined analogously to the polyhedron (105).

The formal mapping $d^{(2)}$ satisfies the following property: the image of the resolution mapping

$$rez(d^{(2)}) : (N(d^{(2)})_\circ \setminus M(d^{(2)})_\circ) \to YY_\circ \tag{144}$$

is inside the subpolyhedron $R\Sigma_a \cup RK_{b\times b\circ} \subset YY_\circ$.

For this construction is sufficiently to use the following statements:

1. The canonical 2-sheeted covering over the polyhedron $Z_{4,d\circ}$ is trivial, the restriction of this covering on a regular neighborhood of the boundary of the polyhedron $Z_{4,a\circ}$ is trivial. This property is proved in Lemma 10.

2. The canonical 2-sheeted covering over the polyhedron $Z_{3\circ}$ is trivial.

3. The canonical 2-sheeted covering over the subpolyhedron $Y_{d\circ} \subset Y_\circ \subset YY_\circ$, see (71), is trivial.

Using three vertical deformations, step by step, the deformation of the formal mapping $d_1^{(2)}$ into a formal mapping $d^{(2)}$, we get that the resolution mapping of the polyhedron $N(d^{(2)})_\circ$ of the formal self-intersection of the formal mapping $d^{(2)}$ satisfies Properties 1,2,3. Moreover, the polyhedron $N(d^{(2)})_\circ$ is decomposed into the following disjoin union:

$$N(d^{(2)})_\circ = N_a(d^{(2)}) \cup N_{b\times b\circ}(d^{(2)})$$

of 2 polyhedra. Moreover, the composition

$$(\phi_a \cup \phi_{b\times\dot{b}}) \circ rez(d^{(2)}) : N(d^{(2)})_\circ \to R\Sigma_a \cup RK_{b\times\dot{b}\circ} \to K(\mathbf{I}_a, 1) \cup K(\mathbf{I}_{b\times\dot{b}}, 1)$$

is well-defined and determines a reduction μ_a of the restriction of the structure mapping of the polyhedron $N(d^{(2)})_\circ$ to the component $N_a(d^{(2)})$. Lemma 9 is proved.

Proof of Lemma 2 B

Proof of Statement B of the lemma is analogous to the proof of Statement A of the lemma. Lemma 2 is proved.

7 Proof of Lemma 1, Lemma 3, Proposition 28 [A2], Proposition 31 [A2], and Lemma 35 [A2]

To prove Lemma 1 is sufficiently to repeat a part of Lemma 2 B., which is related with a subpolyhedron $RK_{b\times\dot{b}\circ}$ in the polyhedron of the self-intersection. We get the proof of Lemma 3, using re-denotations of the structure groups in the proof of the main result of [Akh1]. Lemmas 30, 32 from [A2] are proved analogously to 2. Lemma 35 from [A2] is proved analogously to Lemma 3. A detailed proof of the lemmas requires to make the paper greater.

Let us consider a sketch of the proof of [Proposition 30,A2]. Let us formulate Lemma 12, which is analogous to Lemma 9.

$\mathbf{E}_{b\times\dot{b}}$–structure of formal mappings with holonomic singularities

Consider the polyhedron $X_{b\times\dot{b}}\int_\chi S^1$, which is a skeleton of the Eilenberg-Mac Lane space $K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}), 1)$, correspondingly to [Formula (181), A2]. Consider the mapping $\mathrm{i}_{J_X\int_\chi S^1}\circ\varphi_{X_{b\times\dot{b}}} : X_{b\times\dot{b}}\int_\chi S^1 \to D^{n-1}\times S^1 \subset \mathbb{R}^n$, where the mapping $\varphi_{X_{b\times\dot{b}}}$ is defined by the [Formula (186), A2], and the mapping (embedding) $\mathrm{i}_{J_X\int_\chi S^1}$ is defined by the [Formula (190), A2]. We shall consider this mapping as a mapping with a holonomic singularity in the sense of [Definition 9, A2]. Denote this formal mapping by $(d_{\int,0}, d^{(2)}_{\int,0})$. Let us restrict this formal mapping $(d_{\int,0}, d^{(2)}_{\int,0})$ on the subpolyhedron $X_{b\times\dot{b}} \subset X_{b\times\dot{b}}\int_\chi S^1$, and denote this restriction by $(d_0, d^{(2)}_0)$.

Lemma 12. *There exists a C^0–small* PL*-deformation of the formal holonomic pair of mappings $(d_{\int,0}, d^{(2)}_{\int,0})$ to a pair of mappings $(d_\int, d^{(2)}_\int)$ with holonomic singularity, such that the polyhedron $N_{\int\circ}$ of formal self-intersection*

of the mapping $(d_{\int}, d_{\int}^{(2)})$ *is decomposed into the union of two subpolyhedra:*

$$N_{\int\circ} = N_{\int,\mathbf{E}_{b\times\dot{b}}} \cup N_{\int,[3]\circ}, \tag{145}$$

where $N_{\int,\mathbf{E}_{b\times\dot{b}}}$ *is closed.*

The restriction of the structure mapping $\zeta_\circ$ *on the subpolyhedron* $N_{\int,\mathbf{E}_{b\times\dot{b}}}$ *admits a reduction, which is given by the mapping* $\zeta_{b\times\dot{b}} : N_{\int,\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)$.

The polyhedron $N_{\int\circ}$ *contains a subpolyhedron* $N_\circ \subset N_{\int\circ}$*, which is decomposes into two components:*

$$N_\circ = N_{\mathbf{E}_{b\times\dot{b}}} \cup N_{[3]\circ},$$

where the components are defined as the corresponding components in the formula (145). The restriction of the structured map $\zeta_\circ$ *on the subpolyhedron* $N_{[3]\circ}$ *admits a reduction, which is given by the mapping*

$$\zeta_{b\times\dot{b}\times\mathbb{Z}/2\circ} : N_{[3]\circ} \to K((\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_\chi \mathbb{Z}, 1),$$

and which is satisfies the boundary condition, given by a mapping into the subspace $K(\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2, 1)$*. (In this formula the extension of the group* $\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2$ *(and analogous extensions below) are corresponding to the inclusion* $X_{b\times\dot{b}} \subset X_{b\times\dot{b}} \int_\chi S^1$*.)*

The mapping $\zeta_{b\times\dot{b}\times\mathbb{Z}/2\circ}$ *is a compressed by the canonical 2-sheeted covering* $N_{[3]\circ} \to \hat{N}_{[3]\circ}$*, and is a 2-sheeted covering mapping over the mapping*

$$\hat{\zeta}_{b\times\dot{b}\times\mathbb{Z}/2\circ} : \hat{N}_{[3]\circ} \to K((\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1),$$

which is satisfies the boundary condition, given by a mapping into the subspace $K((\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1)$*. In the previous formula the automorphism (involution)* $\hat{\chi} : \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2 \to \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2$ *is the identity on the subgroup* $\mathbf{E}_{b\times\dot{b}} \subset \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2$*, and is mapped the generator* $t \in \mathbb{Z}/2$ *into the element* $t_d t$*, where* t_d *is the generator of the subgroup* $\mathbf{I}_d \subset \mathbf{E}_{b\times\dot{b}}$*. Define the automorphism* $\chi : \mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2 \to \mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2$ *by the restriction of* $\hat{\chi}$ *on the subgroup.*

Let us formulated and proof a lemma, which is required to check [Formula (211), A2]. For an arbitrary pair of integers (s_1, s_2), $s_1 = 1 \pmod 2$, $s_2 = 1 \pmod 2$, $s = s_1 + s_2 = n - \frac{n-m_\sigma}{8}$, consider the homology class [(210), A2]. This homology class is defined as the image of the fundamental class of the manifold $X(s_1, s_2)$, which is naturally embedded into $X_{b\times\dot{b}}$.

Denote the restriction of $(d, d^{(2)})$ on $X(s_1, s_2)$ by $(d(s_1, s_2), d^{(2)}(s_1, s_2))$. Consider a polyhedron of the formal self-intersection of the mapping $(d(s_1, s_2), d^{(2)}(s_1, s_2))$, which is represented by a disjoin union of the two subpolyhedra. The canonical covering over the first polyhedron is a closed subpolyhedron into $\bar{N}_{\mathbf{E}_{b\times\dot{b}}}$, the canonical covering over the second polyhedron is the closure of an open subpolyhedron in $C\bar{N}_{[3]\circ}$, denote this closure by $C\overline{NX}(s_1, s_2)$.

Denote the fundamental class of the polyhedron $C\overline{NX}(s_1, s_2)$ by $[C\overline{NX}(s_1, s_2)] \in H_{n-\frac{n-m_\sigma}{4}}(K(\mathbf{I}_{b\times\dot{b}}, 1))$ (we have used the isomorphism [(42),A2]).

Let us prove the formulas [(211),A2] analogously to Lemma 2A, in which the formula (88) is proved (Condition 1 from [Lemma 26, A1]).

Proposition 13. *An arbitrary homology class* $[C\overline{NX}(s_1, s_2)]$ *is trivial.*

Proof of Proposition 13

Denote by $NX(s_1, s_2)_\circ$ an open polyhedron, which is the base of 2-sheeted covering space $\overline{NX}(s_1, s_2)_\circ$. The polyhedron $NX(s_1, s_2)_\circ$ is equipped with the structure mapping

$$\zeta(s_1, s_2)_\circ : NX(s_1, s_2)_\circ \to K((\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1),$$

and the regular neighborhood of the boundary is mapped by the considered structure mapping into the subspace

$$K(\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2, 1) \subset K((\mathbf{I}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1). \tag{146}$$

The manifold $X(s_1, s_2)$ is a 2-sheeted covering over the manifold $\hat{X}(s_1, s_2)$. Therefore, an open polyhedron, which is a base of the 2-sheeted covering with the covering space $NX(s_1, s_2)_\circ$ is well-defined. Let us denote this polyhedron by $\widehat{NX}(s_1, s_2)_\circ$. The polyhedron $\widehat{NX}(s_1, s_2)_\circ$ is equipped with a structure mapping

$$\hat{\zeta}_\circ : \widehat{NX}(s_1, s_2)_\circ \to K((\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1),$$

a regular neighborhood of the boundary is mapped by this mapping into the subspace

$$K(\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2, 1) \subset K((\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1). \tag{147}$$

Assume, that the image of the structure mapping $\zeta(s_1, s_2)_\circ$ is inside the subspace (146). Then the statement of the lemma is evident, because $C\overline{NX}(s_1, s_2)$ is a composition with a 2-sheeted covering over $CNX(s_1, s_2)$ (comp. with the initial step of the proof of Lemma 2A).

Let us consider a general case. We shall use the polyhedron $\widehat{NX}(s_1, s_2)_\circ$. The universal bundle over the space $K((\mathbf{E}_{b\times\dot{b}}\times\mathbb{Z}/2)\int_{\hat{\chi}}\mathbb{Z}, 1)$ is a 8-dimensional bundle. It is sufficiently to prove the formula for the cycle, which is defined as the intersection of the considered fundamental class with the Euler class of the pull-back of a suitable Whitney sum of the universal bundle. Denote the Euler class of the universal bundle by $\hat{\tau}_{b\times\dot{b}\times\mathbb{Z}/2,\int}$.

For an arbitrary pair of the positive integers (p_1, p_2), $p_1 = 1 \pmod 2$, $p_2 = 1 \pmod 2$, $p = p_1 + p_2 = \frac{n+6}{2}$, define the submanifold $XX(p_1, p_2)$ of the dimension p, $XX = \mathbb{R}\mathrm{P}^{p_1} \times \mathbb{R}\mathrm{P}^{p_2}$.

Define the embedding $XX(p_1, p_2) \subset X(s_1, s_2)$, as the Cartesian product of the coordinate embeddings $\mathbb{R}\mathrm{P}^{p_1} \subset \mathbb{R}\mathrm{P}^{s_1}$, $\mathbb{R}\mathrm{P}^{p_2} \subset \mathbb{R}\mathrm{P}^{s_2}$, which satisfies the restriction $s_1 - p_1 = s_2 - p_2 = \frac{s-p}{2}$. Define the formal mapping $(dd(p_1, p_2), dd^{(2)}(p_1, p_2))$ as the restriction of the formal mapping $(d(s_1, s_2), d^{(2)}(s_1, s_2))$ to the submanifold $XX(p_1, p_2)$. Denote by $NXX(p_1, p_2)_\circ$ an open polyhedron of the formal self-intersection of the mapping $(dd(p_1, p_2), dd(p_1, p_2)^{(2)})$. The following 6-dimensional subpoluhedron

$$NXX(p_1, p_2)_\circ \subset NX(s_1, s_2)_\circ$$

is well-defined, the fundamental class of this subpolyhedron is realized the homology Euler class of the bundle $\zeta_\circ^*(\tau_{b\times\dot{b}\times\mathbb{Z}/2,\int}^{\frac{s-p}{8}})$.

Let us prove that the homology class

$$[C\overline{NXX}(p_1, p_2)] \in H_6(\mathbf{I}_{b\times\dot{b}}, 1) \tag{148}$$

is trivial. We shall distinguishes the exceptional case, when $p_1 = 1$, or $p_2 = 1$. Consider non-exceptional case in which $p_1 \geq 3$, $p_2 \geq 3$. Let us prove that the homology class (148) is trivial.

The lens manifold $(\mathbb{R}\mathrm{P}^{p_1} \times \mathbb{R}\mathrm{P}^{p_2})/\mathbf{i}_{diag}$ is immersible into $\mathbb{R}^n$. Therefore the homology class of the boundary singularities of the polyhedron $\partial(\widehat{NXX}(p_1, p_2)_\circ)$ in the group $H_5(\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2, 1)$ is trivial. Let us omit below the marks $\circ$ and C in denotations.

Let us consider the 5-dimensional fundamental class $[\hat{p}^{-1}(pt)] \in H_5(\mathbf{E}_{b\times\dot{b}}\times \mathbb{Z}/2, 1)$ of the closed subpolyhedron $\hat{p}^{-1}(pt)$, where $\hat{p} : \widehat{NXX}(p_1, p_2) \to S^1$ is the projection, which is induced by the projection $p_{\mathbf{E}_{b\times\dot{b}}\times\mathbb{Z}/2,\int}$ of the universal space.

Assume that the homology class $[\hat{p}^{-1}(pt)]$ is trivial. Then, without loss of a generality, we may assume that the manifold $\hat{p}^{-1}(pt)$ is empty and the proof is reduced to the previous.

Assume that the homology class $[\hat{p}^{-1}(pt)]$ is non-trivial. Let us prove that the homology class $[\hat{p}^{-1}(pt)]^{!}$ is realized for a suitable mapping of a closed 6-dimensional manifold A, $\zeta_A : A \to (\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2) \int_{\hat{\chi}} \mathbb{Z}, 1)$, for which the homology class, defined analogously to (148), is trivial.

Let us decompose the fundamental class $[\hat{p}^{-1}(pt)]$ over the base of the group $Im(H_5(K(\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2, 1); \mathbb{Z}) \to H_5(K(\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2, 1))$. Consider the following epimorphisms:

$$\pi_b : \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2 \to \mathbf{E}_b \times \mathbb{Z}/2,$$

$$\pi_{\dot{b}} : \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2 \to \mathbf{E}_{\dot{b}} \times \mathbb{Z}/2.$$

Assume that the image of the homology class $[\hat{p}^{-1}(pt)]$ in the group $H_5(K(\mathbf{E}_b \times \mathbb{Z}/2, 1) \times K(\mathbf{E}_{\dot{b}} \times \mathbb{Z}/2, 1))$ by the homomorphism $(\pi_b \times \pi_{\dot{b}})_*$ is represented by the tensor product of a homology class of $H_2(K(\mathbf{E}_b \times \mathbb{Z}/2, 1)$ to a homology class of $H_3(K(\mathbf{E}_{\dot{b}} \times \mathbb{Z}/2, 1))$. The proof in the last cases is evident (or is is given after b is replaced by $\dot{b}$.)

The condition $\hat{\chi}_*([\hat{p}^{-1}(pt)]) = [\hat{p}^{-1}(pt)]$ is satisfied, because the boundary conditions on $\widehat{NXX}(p_1, p_2)_\circ$ determines the trivial homology class. Therefore, after the expansion of the element $\pi_{\dot{b},*}([\hat{p}^{-1}(pt)])$ over the standard base the generator of the factor $H_3(K(\mathbb{Z}/2, 1); \mathbb{Z})$ is not involved and $\pi_{\dot{b},*}([p^{-1}(pt)])$ is expressed by the generator of $H_3(K(\mathbf{E}_b, 1))$.

Analogous to the construction (98), without loss of a generality, we may assume that the homology class (148) is trivial. Therefore, without loss of a generality, we may assume, that $p^{-1}(pt) = \emptyset$, and we may repeat the previous proof as in the case, when the image of the structure mapping is inside the subspace (147).

Is sufficiently to prove that in the exceptional case the homology class (148) is trivial. Let us decomposes the homology class (148) over the standard base of the group $H_6(\mathbf{I}_{b\times\dot{b}}, 1)$. The generators of the group are $t_{3,b}t_{3,\dot{b}}$, $t_b t_{5,\dot{b}}$, $t_{5,b}t_{\dot{b}}$. In the exceptional case, evidently, that the generator $t_{3,b}t_{3,\dot{b}}$ is not involved. To prove that the last generators $t_b t_{5,\dot{b}}$, $t_{5,b}t_{\dot{b}}$ are not involved, let us intersect the 6-dimensional polyhedron $NXX(p_1, p_2)_\circ$ with 4-dimensional Euler class of the universal bundle, which is the bull-back by π_b, or by $\pi_{\dot{b}}$, correspondingly to the generators $t_b t_{5,\dot{b}}$, $t_{5,b}t_{\dot{b}}$. The proof is analogous to the previous proof, this proof is more simple, because the Euler class is represented by a 2-dimensional subpolyhedron in $NXX(p_1, p_2)_\circ$. Is sufficiently to consider the only generators of $H_1(K(\mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2, 1); \mathbb{Z}) = \mathbf{E}_{b\times\dot{b}} \times \mathbb{Z}/2$.

Lemma 13 is proved.

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Troitsk, IZMIRAN, 142190

pmakhmet@izmiran.ru