arXiv:1007.1199v2 [math.RT] 15 Jul 2010

REPRESENTATION OF NELSON ALGEBRAS BY ROUGH SETS DETERMINED BY QUASIORDERS

JOUNI JÄRVINEN AND SÁNDOR RADELECZKI

Abstract. In this paper, we show that every quasiorder R induces a Nelson algebra $\mathbb{RS}$ such that the underlying rough set lattice RS is algebraic. We note that $\mathbb{RS}$ is a three-valued Łukasiewicz algebra if and only if R is an equivalence. Our main result proves that if $\mathbb{A}$ is a Nelson algebra defined on an algebraic lattice, then there exists a set U and a quasiorder R on U such that $\mathbb{A} \cong \mathbb{RS}$.

1. Introduction

Nelson algebras, also called $\mathcal{N}$-lattices or quasi-pseudo-Boolean algebras, were introduced by H. Rasiowa as algebraic counterparts of the constructive logic with strong negation by D. Nelson and A. A. Markov (see [21]). They can be considered also as a generalisation of Boolean algebras. It is well known that any Boolean algebra defined on an algebraic lattice is isomorphic to the powerset algebra $\wp(U)$ of some set U. In this paper, we prove an analogous result for Nelson algebras with algebraic underlying lattices and algebras of rough sets determined by quasiorders.

Rough sets are introduced by Z. Pawlak in [19]. In rough set theory it is assumed that our knowledge about a universe of discourse U is given in terms of a binary relation reflecting the distinguishability or indistinguishability of the elements of U. Originally, Pawlak assumed that this binary relation is an equivalence, but in the literature numerous studies can be found in which approximations are determined also by other types of relations.

If R is a given binary relation on U, then for any subset $X \subseteq U$, the *lower approximation* of X is defined as

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}$$

and the *upper approximation of* X is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid x\,R\,y\}$. The *rough set of* X is the pair $\mathcal{A}(X) = (X^{\blacktriangledown}, X^{\blacktriangle})$ and the set of all *rough sets* is

$$RS = \{\mathcal{A}(X) \mid X \subseteq U\}.$$

The set RS may be canonically ordered by the coordinatewise order: $\mathcal{A}(X) \leq \mathcal{A}(Y)$ holds in RS if $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ and $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$.

The structure of RS is well studied in the case when R is an equivalence; see [5, 7, 8, 11, 17, 18, 20]. In particular, J. Pomykała and J. A. Pomykała showed in [20] that RS is a Stone lattice. Later this result was improved

2010 *Mathematics Subject Classification.* Primary 06B15; Secondary 06D30, 08A05, 68T37, 06D10.

by S. D. Comer [5] by showing that RS is a regular double Stone lattice. In [8], M. Gehrke and E. Walker proved that RS is isomorphic to $\mathbf{2}^I \times \mathbf{3}^J$, where I is the set of singleton R-classes and J is the set of non-singleton equivalence classes of R. Additionally, RS forms a three-valued Łukasiewicz algebra, as shown by P. Pagliani [17]. If R is reflexive and symmetric or just transitive, then RS is not necessarily even a semilattice. If R is symmetric and transitive, then the structure of RS is as in case of equivalences [10].

In [12], we proved that any RS determined by a quasiorder R is a completely distributive lattice isomorphic to a complete ring of sets, and we described its completely join-irreducible elements. We also showed that $\mathbb{RS} = (RS, \cup, \cap, c, (\emptyset, \emptyset), (U, U))$ is a De Morgan algebra, where the operation c is defined by $c\colon \mathcal{A}(X) \mapsto \mathcal{A}(U \setminus X)$. In this paper, we prove that $\mathbb{RS}$ is in fact a Nelson algebra defined on an algebraic lattice. The main objective of this work is to prove the following representation theorem.

Theorem 1.1. *Let $\mathbb{A} = (A, \vee, \wedge, c, 0, 1)$ be a Nelson algebra defined on an algebraic lattice. Then, there exists a set U and a quasiorder R on U such that $\mathbb{A} \cong \mathbb{RS}$.*

As a corollary we can also show that if $\mathbb{A}$ is a semisimple Nelson algebra with an underlying algebraic lattice, then there exists a set U and an equivalence R on U such that $\mathbb{A} \cong \mathbb{RS}$.

The paper is structured as follows. In the next section we recall some notions and facts related to De Morgan, Kleene, Nelson, and Heyting algebras. Section 3 summarises some more or less known properties of completely join-irreducible elements of completely distributive Kleene algebras, which will be used in the proofs of our main results. In Section 4, we prove that rough set lattices induced by quasiorders determine Nelson algebras. We also show that $\mathbb{RS}$ is a three-valued Łukasiewicz algebra only in case R is an equivalence. Section 5 contains the proof of Theorem 1.1 and some of its consequences.

2. Preliminaries

Systematic treatments of De Morgan and Kleene algebras can be found in [1, 21]. A *De Morgan algebra* $\mathbb{A} = (A, \vee, \wedge, c, 0, 1)$ is an algebra of type $(2, 2, 1, 0, 0)$ such that A is a bounded distributive lattice with a least element 0 and a greatest element 1, and c is a unary operation that satisfies for all $x, y \in A$,

$$c(c(x)) = x;$$
$$x \leq y \text{ if and only if } c(x) \geq c(y).$$

This definition implies that c is an isomorphism between the lattice A and its dual A^∂. Hence, it satisfies the equations:

$$\begin{aligned} c(x \vee y) &= c(x) \wedge c(y); \\ c(x \wedge y) &= c(x) \vee c(y). \end{aligned}$$

An element x of a complete lattice L is *completely join-irreducible* if for every subset S of L, $x = \bigvee S$ implies that $x \in S$. The set of completely join-irreducible elements of L is denoted by $\mathcal{J}(L)$ — or simply by $\mathcal{J}$ when there is no danger of confusion. For any x, let $J(x) = \{j \in \mathcal{J} \mid j \leq x\}$.

A complete lattice L is *completely distributive* if for any doubly indexed family of elements $\{x_{i,j}\}_{i\in I,\, j\in J}$ of L, we have

$$\bigwedge_{i\in I}\Big(\bigvee_{j\in J} x_{i,j}\Big) = \bigvee_{f\colon I\to J}\Big(\bigwedge_{i\in I} x_{i,f(i)}\Big),$$

that is, any meet of joins may be converted into the join of all possible elements obtained by taking the meet over $i \in I$ of elements $x_{i,k}$, where k depends on i.

We say that the De Morgan algebra $\mathbb{A}$ is *completely distributive*, if its underlying lattice A is completely distributive. In such a case, we may define for any $j \in \mathcal{J}$ the element

$$j^* = \bigwedge\{x \in A \mid x \not\leq c(j)\}.$$

It is well known that $j^* \in \mathcal{J}$ (see e.g. [14]). The next lemma for a finite $\mathbb{A}$ was proved in [14], and it is contained implicitly in [3].

Lemma 2.1. *If $\mathbb{A}$ is a completely distributive De Morgan algebra, then for all $i, j \in \mathcal{J}$:*
(a) $j^* \not\leq c(j)$;
(b) $i \leq j$ *implies* $i^* \geq j^*$;
(c) $j^{**} = j$.

Notice that statements (b) and (c) of Lemma 2.1 mean that the map $j \mapsto j^*$ is an order-isomorphism between the ordered set $\mathcal{J}$ and its dual $\mathcal{J}^\partial$.

A complete lattice L is said to be *algebraic* if any element $x \in L$ is the join of a set of compact elements of L (see e.g. [9]). A *complete ring of sets* is a family of sets $\mathcal{F}$ such that $\bigcup\mathcal{H}$ and $\bigcap\mathcal{H}$ belong to $\mathcal{F}$ for any $\mathcal{H} \subseteq \mathcal{F}$.

In the next remark we give some conditions under which a lattice is isomorphic to a complete ring of sets (cf. [6]).

Remark 2.2. Let L be a lattice. The following are equivalent:
(a) L is isomorphic to a complete ring of sets;
(b) L is algebraic and completely distributive;
(c) L is distributive and doubly algebraic (i.e. both L and L^∂ are algebraic);
(d) L is algebraic, distributive and every element of L is a join of completely join-irreducible elements of L.

If $\mathbb{A}$ is a De Morgan algebra whose underlying lattice is algebraic, then A is doubly algebraic, since A is self-dual. Thus, A has all equivalent properties of Remark 2.2. The next connection between the maps $c\colon A \to A$ and ${}^*\colon \mathcal{J} \to \mathcal{J}$ was proved in [14] for finite algebras and in the completely distributive case it is contained implicitly in [22] (see also [23]).

Lemma 2.3. *If $\mathbb{A}$ is a De Morgan algebra defined on an algebraic lattice, then for all $x \in A$, $c(x) = \bigvee\{j \in \mathcal{J} \mid j^* \not\leq x\}$.*

Let L and K be two completely distributive lattices such that any element of them is a join of completely join-irreducible elements, and assume that $\varphi\colon \mathcal{J}(L) \to \mathcal{J}(K)$ is an order-isomorphism of ordered sets. G. Birkhoff [2] proved that in this case the map $\Phi\colon L \to K$,

$$\Phi(x) = \bigvee \varphi(J(x))$$

is a lattice-isomorphism. One may extend this result to De Morgan algebras.

Corollary 2.4. *Let $\mathbb{L} = (L, \vee, \wedge, c, 0, 1)$ and $\mathbb{K} = (K, \vee, \wedge, c, 0, 1)$ be two De Morgan algebras defined on algebraic lattices. If $\varphi\colon \mathcal{J}(L) \to \mathcal{J}(K)$ is an order-isomorphism such that*

$$\varphi(j^*) = \varphi(j)^*$$

for all $j \in \mathcal{J}(L)$, then Φ is an isomorphism between the algebras $\mathbb{L}$ and $\mathbb{K}$.

Proof. Since the map $\Phi\colon L \to K$ is a lattice-isomorphism that maps $J(x)$ onto $J(\Phi(x))$, the set $\{j \in \mathcal{J}(L) \mid j \not\leq x\}$ is mapped by Φ onto the set $\{k \in \mathcal{J}(K) \mid k \not\leq \Phi(x)\}$. By using this fact, we prove that

$$\Phi(c(x)) = c(\Phi(x)).$$

Indeed, since $j \in \mathcal{J}(L)$, $\Phi(j) = k$ implies $\Phi(j^*) = \varphi(j^*) = \varphi(j)^* = \Phi(j)^* = k^*$, the set $\{j^* \mid j \in \mathcal{J}(L) \text{ and } j \not\leq x\}$ is mapped by Φ onto $\{k^* \mid k \in \mathcal{J}(K) \text{ and } k \not\leq \Phi(x)\}$. By Lemma 2.3, $c(x) = \bigvee\{j \in \mathcal{J}(L) \mid j^* \not\leq x\} = \bigvee\{j^* \mid j \in \mathcal{J}(L) \text{ and } j \not\leq x\}$; recall that $j = j^{**}$. Similarly, we get $c(\Phi(x)) = \bigvee\{k^* \mid k \in \mathcal{J}(K) \text{ and } k \not\leq \Phi(x)\}$. Hence, we have $\Phi(c(x)) = \bigvee\{\Phi(j^*) \mid j \in \mathcal{J}(L) \text{ and } j^* \not\leq x\} = \bigvee\{k^* \mid k \in \mathcal{J}(K) \text{ and } k \not\leq \Phi(x)\} = c(\Phi(x))$. □

A De Morgan algebra $\mathbb{A}$ is a *Kleene algebra* if for all $x, y \in A$,

$$x \wedge c(x) \leq y \vee c(y).$$

We define for a Kleene algebra $\mathbb{A}$ two sets:

$$A^+ = \{x \vee c(x) \mid x \in A\} \quad \text{and} \quad A^- = \{x \wedge c(x) \mid x \in A\}.$$

The proof of the following lemma is straightforward and is omitted.

Lemma 2.5. *Let $\mathbb{A}$ be a Kleene algebra. Then,*

(a) $c(A^+) = A^-$ *and* $c(A^-) = A^+$;
(b) $a \leq c(b)$ *for all* $a, b \in A^-$;
(c) $c(a) \leq b$ *for all* $a, b \in A^+$;
(d) $a \in A^-$ *iff* $a \leq c(a)$;
(e) $a \in A^+$ *iff* $c(a) \leq a$.

A nonempty subset I of a lattice L is an *ideal*, if for every $x, y \in I$ also $x \vee y \in I$, and $x \in I$ and $y \leq x$ imply $y \in I$. The ideal I is a *prime ideal* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$. *Filters* and *prime filters* of a lattice are defined dually. Clearly, if $F \subseteq L$ is a prime filter, then $L \setminus F$ is a prime ideal. If L is a distributive lattice and j is a join-irreducible element of it, then $[j)$ is a prime filter. Additionally, for any Kleene algebra $\mathbb{A}$, A^- is an ideal and A^+ is a filter of A, as noted in e.g. [13].

Let $\mathbb{A}$ be a Kleene algebra defined on a complete lattice A. We denote

$$\alpha = \bigvee A^- \quad \text{and} \quad \beta = \bigwedge A^+.$$

Lemma 2.5 implies easily the following result.

Corollary 2.6. *If $\mathbb{A}$ is a Kleene algebra defined on a complete lattice, then $A^- = (\alpha]$, $A^+ = [\beta)$, and $c(\alpha) = \beta$.*

A *Heyting algebra* L is a bounded lattice such that for all $a, b \in L$, there is a greatest element x of L such that

$$a \wedge x \leq b.$$

This element is the *relative pseudocomplement* of a with respect to b, and is denoted $a \Rightarrow b$. It is well known that any completely distributive lattice L is a Heyting algebra such that the relative pseudocomplement is defined as

$$a \Rightarrow b = \bigvee \{x \in L \mid a \wedge x \leq b\}.$$

The following definitions can be found in [3], for instance. A *quasi-Nelson* algebra is a Kleene algebra $\mathbb{A}$ such that for each pair a and b of its elements, the *weak relative pseudocomplement*

$$a \to b = a \Rightarrow (c(a) \vee b)$$

exists. Clearly, any Kleene algebra $\mathbb{A}$ defined on an algebraic lattice A is a quasi-Nelson algebra, because A is a Heyting algebra.

A *Nelson algebra* is a quasi-Nelson algebra satisfying the equation

$$(a \wedge b) \to c = a \to (b \to c).$$

3. Completely Distributive Kleene and Nelson Algebras

In this section, we present some more or less known properties that are used in the proofs of our main results. First, we show that in completely distributive Kleene algebras, the set of completely join-irreducible elements $\mathcal{J}$ can be divided into three disjoint sets in terms of the map $^*\colon \mathcal{J} \to \mathcal{J}$.

For any set A, let X' denote the set-theoretical complement $A \setminus X$ of any subset $X \subseteq A$. Let $\mathbb{A}$ be a De Morgan algebra. We denote the set of its prime filters by $\mathcal{F}_p$. A map $g\colon \mathcal{F}_p \to \mathcal{F}_p$ is defined by setting

$$g(P) = c(P)'$$

for all $P \in \mathcal{F}_p$. Since $c(P)$ is a prime ideal of the lattice A, $c(P)'$ is a prime filter of A. It is easy to verify (see e.g. [3]) that for all $P \in \mathcal{F}_p$,

$$g(g(P)) = P$$

and for any $P, Q \in \mathcal{F}_p$, we have

$$P \subseteq Q \text{ implies } g(Q) \subseteq g(P).$$

If $\mathbb{A}$ is a completely distributive De Morgan algebra, the map g is related to the map $^*\colon \mathcal{J} \to \mathcal{J}$, because for every $j \in \mathcal{J}$,

$$g([j)) = \{c(x) \mid j \leq x\}' = \{x \mid x \leq c(j)\}' = \{x \mid x \not\leq c(j)\} = [j^*).$$

Let $\mathbb{A}$ be a Kleene algebra. It is known (see e.g. [3]) that in this case for any prime filter P of A, we have either $g(P) \subseteq P$ or $P \subseteq g(P)$. In [3], the following two sets are defined:

$$\mathcal{F}_p^+ = \{P \in \mathcal{F}_p \mid P \subseteq g(P)\};$$
$$\mathcal{F}_p^- = \{P \in \mathcal{F}_p \mid g(P) \subseteq P\}.$$

Then, $\mathcal{F}_p = \mathcal{F}_p^+ \cup \mathcal{F}_p^-$.

Remark 3.1. From the above, it follows that for any $j \in \mathcal{J}$ in a completely distributive Kleene algebra $\mathbb{A}$, j and j^* are comparable. Indeed, for any $j \in \mathcal{J}$, either $[j) \in \mathcal{F}_p^+$ or $[j) \in \mathcal{F}_p^-$ holds. If $[j) \in \mathcal{F}_p^+$, then $[j) \subseteq g([j)) = [j^*)$ implies $j^* \leq j$, and for $[j) \in \mathcal{F}_p^-$, $[j^*) = g([j)) \subseteq [j)$ gives $j \leq j^*$.

Let $\mathbb{A}$ be a completely distributive Kleene algebra. We may now define three disjoint sets:

$$\mathcal{J}^- = \{j \in \mathcal{J} \mid j < j^*\};$$
$$\mathcal{J}^* = \{j \in \mathcal{J} \mid j = j^*\};$$
$$\mathcal{J}^+ = \{j \in \mathcal{J} \mid j > j^*\}.$$

Then clearly,

$$[j) \in \mathcal{F}_p^+ \iff j \in \mathcal{J}^+ \cup \mathcal{J}^*,$$
$$[j) \in \mathcal{F}_p^- \iff j \in \mathcal{J}^- \cup \mathcal{J}^*,$$

and in view of Remark 3.1, we have $\mathcal{J} = \mathcal{J}^- \cup \mathcal{J}^* \cup \mathcal{J}^+$. The next lemma contains some simple known facts, but it is proved to make our proofs consistent.

Lemma 3.2. *If $\mathbb{A}$ is a completely distributive Kleene algebra, then for all $j \in \mathcal{J}$:*
(a) *If $j \notin A^-$, then $j^* \leq j$;*
(b) *$\mathcal{J}^- = \mathcal{J} \cap A^-$;*
(c) *$j \in \mathcal{J}^- \iff j^* \in \mathcal{J}^+$;*

Proof. (a) If $j \notin A^-$, then $j \not\leq c(j)$ by Lemma 2.5(d) and so $j^* \leq j$.

(b) If $j \in \mathcal{J}^-$, then $j \notin A^-$ is not possible by (a), and we obtain $\mathcal{J}^- \subseteq \mathcal{J} \cap A^-$. Conversely, if $j \in \mathcal{J} \cap A^-$, then $j \leq c(j)$ and hence $j^* \leq j$ is not possible. Thus, we get $j < j^*$, that is, $j \in \mathcal{J}^-$. This proves $\mathcal{J}^- = \mathcal{J} \cap A^-$.

(c) If $j \in \mathcal{J}^-$, then $j^* > j = j^{**}$ and $j^* \in \mathcal{J}^+$. The other direction is proved analogously. □

Following A. Monteiro [14], a Kleene algebra $\mathbb{A}$ is said to have the *interpolation property*, if for any $P, Q \in \mathcal{F}_p^+$ such that $P \subseteq g(Q)$, there exists a prime filter F of A fulfilling the conditions $P \subseteq F \subseteq g(P)$ and $Q \subseteq F \subseteq g(Q)$ (see also [16]).

Theorem 3.3. [3, Theorem 3.5] *A quasi-Nelson algebra is a Nelson algebra if and only if it has the interpolation property.*

Lemma 3.4. [3, Lemma 2.2] *A Kleene algebra $\mathbb{A}$ has the interpolation property if and only if for given $P, Q \in \mathcal{F}_p^+$ such that $P \subseteq g(Q)$,*

$$a \wedge b \not\leq c(a) \vee c(b)$$

for all $a \in P$ and $b \in Q$.

We note that in case of a finite Kleene algebra, the interpolation property is equivalent to condition:

(M) For any p and q in $\mathcal{J}$ such that $p^*, q^* \leq p, q$, there is k in $\mathcal{J}$ such that

$$p^*, q^* \leq k \leq p, q$$

(see [14]). Moreover, the equivalence of these two conditions in case of Kleene algebras with an algebraic underlying lattice implicitly follows from

[22, Theorem 2]. Since the approach and the terminology of [22] is notably different from ours, to avoid recalling several notions from there and to make our proofs self-consistent, below we present a direct proof.

Proposition 3.5. *If $\mathbb{A}$ is a Kleene algebra defined on an algebraic lattice, then $\mathbb{A}$ has the interpolation property if and only if condition* (M) *is satisfied.*

Proof. ($\Rightarrow$) Suppose that $\mathbb{A}$ has the interpolation property and that for some $p, q \in \mathcal{J}$, $p^*, q^* \leq p, q$ is satisfied. Then, for the prime filters $P = [p)$ and $Q = [q)$, we have $g(P) = [p^*)$, $g(Q) = [q^*)$, and $P, Q \subseteq g(P), g(Q)$. It is clear that $P, Q \in \mathcal{F}_p^+$, $p \in P$, and $q \in Q$. Therefore, by Lemma 3.4,

$$(\dagger) \qquad p \wedge q \not\leq c(p) \vee c(q).$$

Because A is an algebraic lattice and $\mathbb{A}$ is a De Morgan algebra, each element x of A is the join of completely join-irreducible elements below x. Then ($\dagger$) implies that there exists a completely join-irreducible element $k \in \mathcal{J}$ such that $k \leq p \wedge q$, but $k \not\leq c(p) \vee c(q)$. Now clearly $k \leq p, q$ and $k \not\leq c(p), c(q)$, that is, $p^*, q^* \leq k$ and condition (M) holds.

($\Leftarrow$) Assume that condition (M) is satisfied, but $\mathbb{A}$ does not have the interpolation property. Then, by Lemma 3.4, there exists two prime filters P and Q satisfying $P \subseteq g(P)$, $Q \subseteq g(Q)$, and $P \subseteq g(Q)$, as well as elements $a \in P$ and $b \in Q$ such that

$$a \wedge b \leq c(a) \vee c(b).$$

First, we show that for any $F \in \mathcal{F}_p^+$ and $x \in F$, the element

$$x^u = \bigvee \{j \mid j \in J(x) \setminus \mathcal{J}^-\}$$

belongs to F. For that, we also define the element

$$x^d = \bigvee \{j \mid j \in J(x) \cap \mathcal{J}^-\}.$$

Clearly, $x = x^u \vee x^d$. Since $x \in F$ and F is a prime filter, we have that $x^u \in F$ or $x^d \in F$. Now $x^d \in A^-$, because $\mathcal{J}^- \subseteq A^-$ and A^- is a complete sublattice of A by Corollary 2.6. Thus, $x^d \leq c(x^d)$ by Lemma 2.5(d). If $x^d \in F$, then $c(x^d) \in F$ contradicting $c(x^d) \in c(F) = g(F)' \subseteq F'$. Therefore, we must have $x^d \notin F$ and $x^u \in F$.

Secondly, we prove that there exist $p \in J(a) \setminus \mathcal{J}^-$ and $q \in J(b) \setminus \mathcal{J}^-$ such that $p \not\leq c(q)$. If we assume that for all $p \in J(a) \setminus \mathcal{J}^-$ and $q \in J(b) \setminus \mathcal{J}^-$, $p \leq c(q)$ holds, then

$$\begin{aligned} a^u &= \bigvee \{p \mid p \in J(a) \setminus \mathcal{J}^-\} \\ &\leq \bigwedge \{c(q) \mid q \in J(b) \setminus \mathcal{J}^-\} \\ &= c\big(\bigvee \{q \mid q \in J(b) \setminus \mathcal{J}^-\}\big) \\ &= c(b^u). \end{aligned}$$

As P is a filter and $a^u \in P$, we obtain $c(b^u) \in P$, which contradicts $c(b^u) \in c(Q) = g(Q)' \subseteq P'$. Thus, $p \not\leq c(q)$ for some $p \in J(a) \setminus \mathcal{J}^-$ and $q \in J(b) \setminus \mathcal{J}^-$.

Finally, we have $p^* \leq p$ and $q^* \leq q$, and $p \not\leq c(q)$ implies $q^* \leq p$ by the definition of q^*. From this, by Lemma 2.1, we also get $p^* \leq q^{**} = q$. By our original assumption, there exists an element $k \in \mathcal{J}$ such that

$$p^*, q^* \leq k \leq p, q.$$

Notice that this also directly implies $p^*, q^* \leq k^* \leq p, q$.

The elements $a \in P$ and $b \in Q$ satisfy $a \wedge b \leq c(a) \vee c(b) = c(a \wedge b)$. Because $p \leq a$ and $q \leq b$,

$$k, k^* \leq p \wedge q \leq a \wedge b \leq c(a \wedge b) \leq c(k), c(k^*).$$

This then means that both k and k^* are in $\mathcal{J}^-$. But this is impossible by Lemma 3.2(c). Therefore, the interpolation property must hold. □

4. Nelson Algebras of Rough Sets Determined by Quasiorders

In [12], we proved that if U is a non-empty set and R is a quasiorder on U, then RS is a complete sublattice of $\wp(U) \times \wp(U)$, where $\wp(U)$ denotes the family of all subsets of U. Since $\wp(U) \times \wp(U)$ is an algebraic, completely distributive lattice, this implies that RS is also an algebraic completely distributive lattice. Thus, RS has the properties listed in Remark 2.2 and

$$\bigwedge_{i \in I} \mathcal{A}(X_i) = \Big(\bigcap_{i \in I} X_i^{\blacktriangledown}, \bigcap_{i \in I} X_i^{\blacktriangle} \Big) \quad \text{and} \quad \bigvee_{i \in I} \mathcal{A}(X_i) = \Big(\bigcup_{i \in I} X_i^{\blacktriangledown}, \bigcup_{i \in I} X_i^{\blacktriangle} \Big)$$

for all $\{\mathcal{A}(X_i) \mid i \in I\} \subseteq RS$. It is easy to observe that $(\emptyset, \emptyset)$ is the least and (U, U) is the greatest element of RS. We also showed that the set of completely join-irreducible elements of RS is

$$(\star) \qquad \mathcal{J} = \{(\emptyset, \{x\}^{\blacktriangle}) \mid \ |R(x)| \geq 2\} \cup \{(R(x), R(x)^{\blacktriangle}) \mid x \in U\},$$

and that every element can be represented as a join of elements in $\mathcal{J}$.

In addition, we proved that the map

$$c \colon RS \to RS,\ \mathcal{A}(X) \mapsto \mathcal{A}(X')$$

is a De Morgan complement, and therefore

$$\mathbb{RS} = (RS, \vee, \wedge, c, (\emptyset, \emptyset), (U, U))$$

is a De Morgan algebra. Note that

$$\mathcal{A}(X') = ((X')^{\blacktriangledown}, (X')^{\blacktriangle}) = ((X^{\blacktriangle})', (X^{\blacktriangledown})').$$

Additionally, $\{x\}^{\blacktriangle} = \{y \in U \mid y \, R \, x\} = R^{-1}(x)$ for all $x \in U$. Because R is reflexive, $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$, and transitivity means $X^{\blacktriangle\blacktriangle} \subseteq X^{\blacktriangle}$ and $X^{\blacktriangledown} \subseteq X^{\blacktriangledown\blacktriangledown}$ for all $X \subseteq U$. In fact, ${}^{\blacktriangle} \colon \wp(U) \to \wp(U)$ is a topological closure operator, and ${}^{\blacktriangledown} \colon \wp(U) \to \wp(U)$ is a topological interior operator; see [11].

Lemma 4.1. *If R is a quasiorder on a non-empty set U, then $\mathbb{RS}$ is a quasi-Nelson algebra.*

Proof. Since RS is a completely distributive lattice, we have to only show that $\mathbb{RS}$ is a Kleene algebra. Let $x = (X^{\blacktriangledown}, X^{\blacktriangle})$ and $y = (Y^{\blacktriangledown}, Y^{\blacktriangle})$. Then,

$$x \wedge c(x) = (X^{\blacktriangledown} \cap (X^{\blacktriangle})', X^{\blacktriangle} \cap (X^{\blacktriangledown})') = (\emptyset, X^{\blacktriangle} \setminus X^{\blacktriangledown});$$
$$y \vee c(y) = (Y^{\blacktriangledown} \cup (Y^{\blacktriangle})', Y^{\blacktriangle} \cup (Y^{\blacktriangledown})') = ((Y^{\blacktriangle} \setminus Y^{\blacktriangledown})', U).$$

Hence, the condition $x \wedge c(x) \leq y \vee c(y)$ is satisfied. □

Our next lemma presents some properties of the completely join-irreducible elements of RS. The set $\mathcal{J}$ is defined as in $(\star)$.

Lemma 4.2. *If R is a quasiorder on a non-empty set U, then the following assertions hold:*

(a) $\mathcal{J}^- = \{(\emptyset, \{x\}^{\blacktriangle}) \mid \ |R(x)| \geq 2\}$;
(b) $(\emptyset, \{x\}^{\blacktriangle})^* = (R(x), R(x)^{\blacktriangle})$ *for all* $(\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{J}^-$;
(c) $\mathcal{J}^+ = \{(R(x), R(x)^{\blacktriangle}) \mid \ |R(x)| \geq 2\}$;
(d) $\mathcal{J}^* = \{(\{x\}, \{x\}^{\blacktriangle}) \mid \ R(x) = \{x\}\}$.

Proof. (a) Let $j = (X^{\blacktriangledown}, X^{\blacktriangle}) \in \mathcal{J}^-$. Then, $j \leq c(j)$ and $j = j \wedge c(j) = (\emptyset, X^{\blacktriangle} \setminus X^{\blacktriangledown})$. This implies $j = (\emptyset, \{x\}^{\blacktriangle})$ for some x such that $|R(x)| \geq 2$. Conversely, if $j = (\emptyset, \{x\}^{\blacktriangle})$, then $c(j) = (U \setminus \{x\}^{\blacktriangle}, U)$ implying $j \leq c(j)$ and $j \in \mathcal{J}^-$.

(b) Let $x \in U$ be such that $|R(x)| \geq 2$. Then, $j = (\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{J}^-$ and by Lemma 3.2(c), $j^* \in \mathcal{J}^+$ and therefore we must have $j^* = (R(y), R(y)^{\blacktriangle})$ for some $y \in U$. Because j^* is the least element not included in $c(j) = (U \setminus \{x\}^{\blacktriangle}, U)$, we have $R(y) \nsubseteq U \setminus \{x\}^{\blacktriangle}$, that is, $R(y) \cap \{x\}^{\blacktriangle} \neq \emptyset$. Thus, there exists $z \in U$ such that $y\,R\,z$ and $z\,R\,x$ implying $y\,R\,x$. This gives $R(x) \subseteq R(y)$ and $R(x)^{\blacktriangle} \subseteq R(y)^{\blacktriangle}$. Hence, $(R(x), R(x)^{\blacktriangle}) \leq (R(y), R(y)^{\blacktriangle}) = j^*$. As $(R(x), R(x)^{\blacktriangle}) \nleq c(j)$, we get $(R(x), R(x)^{\blacktriangle}) = j^*$.

(c) is now obvious by (a), (b), and Lemma 3.2.

(d) If $j = j^*$, then by $(\star)$, (a), and (c) we must have $j = (R(x), R(x)^{\blacktriangle}) = (\{x\}, \{x\}^{\blacktriangle})$ for some $x \in U$ such that $R(x) = \{x\}$.

Conversely, suppose $j = (\{x\}, \{x\}^{\blacktriangle})$ for some x such that $R(x) = \{x\}$. Then, $j \notin \mathcal{J}^-$ and, by Lemma 3.2(a), either $j^* \in \mathcal{J}^-$ or $j^* \in \mathcal{J}^*$. Clearly, $j^* \in \mathcal{J}^-$ is not possible, since $R(x) = \{x\}$. □

Proposition 4.3. *If R is a quasiorder on a non-empty set U, then $\mathbb{RS}$ is a Nelson algebra such that the underlying lattice RS is algebraic.*

Proof. By Theorem 3.3 and Proposition 3.5, we have to show only that the quasi-Nelson algebra $\mathbb{RS}$ has property (M). Let $p, q \in \mathcal{J}$, where $\mathcal{J}$ is defined as in $(\star)$, and suppose that

$$p^*, q^* \leq p, q.$$

We will show that in this case there exists an element $k \in \mathcal{J}$ such that

$$p^*, q^* \leq k \leq p, q.$$

We may exclude the cases $p = p^*$ and $q = q^*$, because they imply directly $k = p$ or $k = q$.

Now, $p^*, q^* < p, q$ implies $p, q \in \mathcal{J}^+$. Hence, $p = (R(x), R(x)^{\blacktriangle})$ and $q = (R(y), R(y)^{\blacktriangle})$ for some elements $x, y \in U$ such that $|R(x)|, |R(y)| \geq 2$. By Lemma 4.2(b), $p^* = (\emptyset, \{x\}^{\blacktriangle})$ and $q^* = (\emptyset, \{y\}^{\blacktriangle})$. Then, $p^* \leq q$ gives $x \in \{x\}^{\blacktriangle} \subseteq R(y)^{\blacktriangle}$. Hence, there exists an element $z \in U$ such that $y\,R\,z$ and $x\,R\,z$. We have to consider two cases: (i) $|R(z)| \geq 2$ and (ii) $R(z) = \{z\}$.

(i) Assume that $|R(z)| \geq 2$. Then clearly, $k = (\emptyset, \{z\}^{\blacktriangle})$ is a completely join-irreducible element. Observe that $x, y \in \{z\}^{\blacktriangle}$ implies $\{x\}^{\blacktriangle}, \{y\}^{\blacktriangle} \subseteq \{z\}^{\blacktriangle}$, whence we obtain $p^* \leq k$ and $q^* \leq k$. Since $z \in R(x)$ and $z \in R(y)$, we get also $\{z\}^{\blacktriangle} \subseteq R(x)^{\blacktriangle}$ and $\{z\}^{\blacktriangle} \subseteq R(y)^{\blacktriangle}$ implying $k \leq p$ and $k \leq q$.

(ii) Suppose $R(z) = \{z\}$. Then $k = (\{z\}, \{z\}^{\blacktriangle})$ is a completely join-irreducible element. Because $x\,R\,z$ and $y\,R\,z$, we obtain

$$\begin{aligned} p^* = (\emptyset, \{x\}^{\blacktriangle}) &\leq (\{z\}, \{z\}^{\blacktriangle}) = k, \\ q^* = (\emptyset, \{y\}^{\blacktriangle}) &\leq (\{z\}, \{z\}^{\blacktriangle}) = k, \\ k = (\{z\}, \{z\}^{\blacktriangle}) &\leq (R(x), R(x)^{\blacktriangle}) = p, \quad \text{and} \\ k = (\{z\}, \{z\}^{\blacktriangle}) &\leq (R(y), R(y)^{\blacktriangle}) = q. \end{aligned}$$

Hence, $p^*, q^* \leq k \leq p, q$ is satisfied in both cases (i) and (ii). □

Example 4.4. For any binary relation R on U, a set C is called a *connected component* of R, if C is an equivalence class of the smallest equivalence relation containing R. In [12], we presented a decomposition theorem stating that for any left-total relation,

$$RS \cong \prod_{C \in \mathfrak{Co}} RS(C),$$

where $\mathfrak{Co}$ is the set of connected components of R and $RS(C)$ is the set of rough sets on the component C determined by the restriction of R to the set C. Note that R is said to be *left-total* if for any x, there exists y such that $x\,R\,y$.

For any equivalence, the connected components are just equivalence classes. If an equivalence class C consists of a single element, say a, then $RS(C) = \{(\emptyset, \emptyset), (\{a\}, \{a\})$, and $RS(C) = \{(\emptyset, \emptyset), (\emptyset, C), (C, C)\}$ in case $|C| \geq 2$. This then means that RS is isomorphic to the direct product of chains of two and three elements.

In case of quasiorders, the height of components cannot be limited. Let us consider the following simple case. Assume that $U = \{1, 2, \ldots, n\}$ is a set of n consecutive natural numbers and consider its usual order $\leq$. For any $X \subseteq U$, $X^{\blacktriangle} = \{1, \ldots, i\}$, where i is the maximal element of X and $X^{\blacktriangledown} = \{j, \ldots, n\}$, where j is the least j such that $\{j, \ldots, n\} \subseteq X$. All elements of U belong to the same component and RS has the members:

$$\begin{aligned} \mathcal{A}(\{1, \ldots, n\}) &= (\{1, \ldots, n\}, U) \\ &\vdots \\ \mathcal{A}(\{n-1, n\}) &= (\{n-1, n\}, U) \\ \mathcal{A}(\{n\}) &= (\{n\}, U) \\ \mathcal{A}(\{n-1\}) &= (\emptyset, \{1, \ldots, n-1\}) \\ &\vdots \\ \mathcal{A}(\{1\}) &= (\emptyset, \{1\}) \\ \mathcal{A}(\emptyset) &= (\emptyset, \emptyset) \end{aligned}$$

Thus, RS forms a chain of $2n$ elements.

We conclude this section by considering three-valued Łukasiewicz algebras. It is known that three-valued Łukasiewicz algebras coincide with the semisimple Nelson algebras — see [4], where further references can be found.

Following A. Monteiro [15], we can define a *three-valued Łukasiewicz algebra* as an algebra $(A, \vee, \wedge, c, \Delta, 1)$ such that A is a distributive lattice and the following equations are satisfied:

(Ł1) $c(c(x)) = x$
(Ł2) $c(x \wedge y) = c(x) \vee c(y)$
(Ł3) $c(x) \vee \Delta(x) = 1$
(Ł4) $x \wedge c(x) = c(x) \wedge \Delta(x)$
(Ł5) $\Delta(x \wedge y) = \Delta(x) \wedge \Delta(y)$.

Proposition 4.5. *If R is a quasiorder, then the rough set lattice $\mathbb{RS}$ is a three-valued Łukasiewicz algebra if and only if R is an equivalence.*

Proof. If R is an equivalence, then $\mathbb{RS}$ is a three-valued Łukasiewicz algebra such that $\Delta(\mathcal{A}(X)) = (X^{\blacktriangle}, X^{\blacktriangle})$ as shown by P. Pagliani [17, 18].

Conversely, suppose by contradiction that $\mathbb{RS}$ is a three-valued Łukasiewicz algebra, but the quasiorder R is not symmetric. Then, there exist $x, y \in U$ such that $(x, y) \in R$, but $(y, x) \notin R$. Now $|R(x)| \geq 2$, $j = (\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{J}^-$, and $j \leq c(j)$. For j, there exists the element $\Delta(j)$ satisfying (Ł4), that is, $c(j) \wedge \Delta(j) = j \wedge c(j) = j$. From this we obtain

$$(U \setminus \{x\}^{\blacktriangle}, U) \wedge \Delta(j) = (\emptyset, \{x\}^{\blacktriangle}).$$

This means that $\Delta(j) = (Y^{\blacktriangledown}, Y^{\blacktriangle})$, where $Y^{\blacktriangle} = \{x\}^{\blacktriangle}$ and $Y^{\blacktriangledown} \subseteq \{x\}^{\blacktriangle}$. Assume that $Y^{\blacktriangledown} \neq \emptyset$. Then, there exists $z \in Y^{\blacktriangledown} \subseteq \{x\}^{\blacktriangle}$. So, $z\,R\,x$, and now $x\,R\,y$ implies $z\,R\,y$, that is, $y \in R(z)$. Since $z \in Y^{\blacktriangledown}$, we have $y \in R(z) \subseteq Y \subseteq Y^{\blacktriangle} = \{x\}^{\blacktriangle}$. But $y \in \{x\}^{\blacktriangle}$ is not possible, since $(y, x) \notin R$. So, we must have $Y^{\blacktriangledown} = \emptyset$ and $\Delta(j) = (\emptyset, \{x\}^{\blacktriangle}) = j$. Then,

$$c(j) \vee \Delta(j) = c(j) \vee j = c(j) = (U \setminus \{x\}^{\blacktriangle}, U) \neq (U, U).$$

This contradicts (Ł3) and therefore $\mathbb{RS}$ is not a three-valued Łukasiewicz algebra. □

5. Proof of the Representation Theorem

Next, we present the proof of Theorem 1.1. Let $\mathbb{A}$ be a Nelson algebra such that its underlying lattice A is algebraic. We denote by $\mathcal{J}$ the set of completely join-irreducible elements of A. We define a mapping $\rho\colon \mathcal{J} \to \mathcal{J}^- \cup \mathcal{J}^*$. For all $j \in \mathcal{J}$, let

$$\rho(j) = \begin{cases} j & \text{if } j \in \mathcal{J}^- \cup \mathcal{J}^*, \\ j^* & \text{otherwise.} \end{cases}$$

Notice that $\rho(\rho(j)) = \rho(j)$ and $\rho(j) = \rho(j^*)$ for all $j \in \mathcal{J}$.

Proof of Theorem 1.1. Let us set $U = \mathcal{J}$ and define a binary relation R on U by

$$x\,R\,y \iff \rho(x) \leq \rho(y).$$

Then, R is a quasiorder on U. Observe that $x\,R\,x^*$ and $x^*\,R\,x$ for all $x \in U$. Thus, $R(x) = R(x^*)$ and $\{x\}^{\blacktriangle} = \{x^*\}^{\blacktriangle}$ for every $x \in U$. If $x \in \mathcal{J}^- \cup \mathcal{J}^+$, then $|R(x)| \geq 2$, because $x \neq x^*$. If $x \in \mathcal{J}^*$, then $R(x) = \{x\}$. Namely, if $x \in \mathcal{J}^*$ and $x\,R\,y$, then $x = \rho(x) \leq \rho(y)$. Now $\rho(y)^* \leq x^* = x \leq \rho(y)$ implies $\rho(y) \in \mathcal{J}^*$ since $\rho(y) \in \mathcal{J}^+$ is not possible. So, $\rho(y) = \rho(y)^*$, which implies $y \in \mathcal{J}^*$ by the definition of ρ. Now, $x \leq y$ and $y \leq x$ yield $y = x$.

The algebra $\mathbb{RS}$ of rough sets determined by the relation R is a Nelson algebra such that the underlying lattice RS is an algebraic lattice by Proposition 4.3. Let $\mathcal{J}(RS)$ denote the sets of completely join-irreducible elements of RS. We show first that $\mathcal{J}$ and $\mathcal{J}(RS)$ are order-isomorphic.

Let us define a mapping $\varphi\colon \mathcal{J} \to \mathcal{J}(RS)$ by setting

$$\varphi(x) = \begin{cases} (\emptyset, \{x\}^{\blacktriangle}) & \text{if } x \in \mathcal{J}^-; \\ (R(x), R(x)^{\blacktriangle}) & \text{otherwise.} \end{cases}$$

The map φ is well-defined. Namely, if $x \in \mathcal{J}^-$, then $|R(x)| \geq 2$ and $\varphi(x) = (\emptyset, \{x\}^{\blacktriangle}) \in \mathcal{J}(RS)^-$. If $x \in \mathcal{J}^+$, then also $|R(x)| \geq 2$, and $\varphi(x) = (R(x), R(x)^{\blacktriangle}) \in \mathcal{J}(RS)^+$. For any $x \in \mathcal{J}^*$, $R(x) = \{x\}$ gives $\varphi(x) = (R(x), R(x)^{\blacktriangle}) = (\{x\}, \{x\}^{\blacktriangle}) \in \mathcal{J}(RS)^*$ (cf. Lemma 4.2).

We show that φ is an order-embedding. The proof is divided to four cases:

(i) $x \in \mathcal{J}^-$ and $y \in \mathcal{J}^-$
(ii) $x \in \mathcal{J}^-$ and $y \notin \mathcal{J}^-$
(iii) $x \notin \mathcal{J}^-$ and $y \notin \mathcal{J}^-$
(iv) $x \notin \mathcal{J}^-$ and $y \in \mathcal{J}^-$

(i) Let $x, y \in \mathcal{J}^-$. Then, $\rho(x) = x$ and $\rho(y) = y$. Now $x \leq y$ implies $\rho(x) \leq \rho(y)$ and $x\,R\,y$. So, $\{x\}^{\blacktriangle} \subseteq \{y\}^{\blacktriangle}$ and $\varphi(x) = (\emptyset, \{x\}^{\blacktriangle}) \leq (\emptyset, \{y\}^{\blacktriangle})) = \varphi(y)$. Conversely, if $\varphi(x) \leq \varphi(y)$, then $x \in \{x\}^{\blacktriangle} \subseteq \{y\}^{\blacktriangle}$ and $x\,R\,y$. Thus, $x = \rho(x) \leq \rho(y) = y$.

(ii) Let $x \in \mathcal{J}^-$ and $y \notin \mathcal{J}^-$. Then, $\rho(x) = x$ and $\rho(y) = y^*$. If $x \leq y$, then $y^* \leq x^*$ and we have $x, y^* \leq x^*$ and $x, y^* \leq y^{**} = y$. Thus, $x^{**}, y^* \leq x^*, y$, which implies by Proposition 3.5 that there exists $z \in \mathcal{J}$ such that $x, y^* \leq z, z^* \leq x^*, y$. This then gives $\rho(x), \rho(y) \leq \rho(z)$ and we obtain $x\,R\,z$ and $y\,R\,z$. Therefore, $z \in R(y)$, $x \in R(y)^{\blacktriangle}$, $\{x\}^{\blacktriangle} \subseteq R(y)^{\blacktriangle\blacktriangle} = R(y)^{\blacktriangle}$, and $\varphi(x) = (\emptyset, \{x\}^{\blacktriangle}) \leq (R(y), R(y)^{\blacktriangle}) = \varphi(y)$.

On the other hand, if $\varphi(x) \leq \varphi(y)$, then $x \in \{x\}^{\blacktriangle} \subseteq R(y)^{\blacktriangle}$ and so there exists z such that $x\,R\,z$ and $y\,R\,z$. Thus, $x = \rho(x) \leq \rho(z)$ and $y^* = \rho(y) \leq \rho(z)$. If $\rho(z) = z$, then $z \in \mathcal{J}^- \cup \mathcal{J}^*$ and $z \leq z^*$. We have $x \leq z$ and $y^* \leq z \leq z^*$ implying $x \leq z \leq y$. If $\rho(z) = z^* \neq z$, then $z \in \mathcal{J}^+$ and $z^* < z$. Then $x \leq z^* < z$ and $y^* \leq z^*$ giving $x < z \leq y$.

(iii) Now $x, y \notin \mathcal{J}^-$, $\rho(x) = x^*$, and $\rho(y) = y^*$. If $x \leq y$, then $\rho(y) = y^* \leq x^* = \rho(x)$ and $y\,R\,x$. Therefore, $R(x) \subseteq R(y)$, and $\varphi(x) = (R(x), R(x)^{\blacktriangle}) \leq (R(y), R(y)^{\blacktriangle}) = \varphi(y)$. Conversely, if $\varphi(x) \leq \varphi(y)$, then $R(x) \subseteq R(y)$. Since $x \in R(x)$, we have $y\,R\,x$. Thus, $y^* = \rho(y) \leq \rho(x) = x^*$ and $x \leq y$.

(iv) If $x \notin \mathcal{J}^-$ and $y \in \mathcal{J}^-$, then $x \leq y$ is impossible, because $\mathcal{J}^-$ is a down-set. Similarly, $\varphi(x) = (R(x), R(x)^{\blacktriangle}) \leq (\emptyset, \{y\}^{\blacktriangle}) = \varphi(y)$ is not possible, because $R(x) \neq \emptyset$.

We have shown that $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ in all cases. Hence, $\varphi\colon \mathcal{J} \to \mathcal{J}(RS)$ is an order-embedding. Next we prove that φ is onto $\mathcal{J}(RS)$. Let $j \in \mathcal{J}(RS)$. We consider three disjoint cases.

(i) Assume $j = (\emptyset, \{x\}^{\blacktriangle})$ for some $x \in U = \mathcal{J}$ such that $|R(x)| \geq 2$. If $x \in \mathcal{J}^-$, then $\varphi(x) = (\emptyset, \{x\}^{\blacktriangle}) = j$. If $x \in \mathcal{J}^+$, then $\varphi(x) = (R(x), R(x)^{\blacktriangle})$ and $\varphi(x^*) = (\emptyset, \{x^*\}^{\blacktriangle}) = (\emptyset, \{x\}^{\blacktriangle}) = j$. Case $x \in \mathcal{J}^*$ may be excluded, since it means $R(x) = \{x\}$.

(ii) If $j = (\{x\}, \{x\}^{\blacktriangle})$ for some $x \in U$ such that $R(x) = \{x\}$, then $x\,R\,x^*$ and $x^*R\,x$ imply $x = x^*$. Thus, $x \in \mathcal{J}^*$, and we infer $\varphi(x) = j$.

(iii) Suppose that $j = (R(x), R(x)^{\blacktriangle})$ for some $x \in U$ such that $|R(x)| \geq 2$. Similarly as in case (i), if $x \in \mathcal{J}^+$, then $\varphi(x) = j$ and if $x \in \mathcal{J}^-$, then $\varphi(x^*) = j$. Case $x \in \mathcal{J}^*$ is not possible.

Thus, φ is an order-isomorphism of the ordered set $\mathcal{J}$ onto the ordered set $\mathcal{J}(RS)$. Because A and RS are isomorphic as lattices, they are isomorphic as Heyting algebras also. To show the isomorphism of the algebras $\mathbb{A}$ and $\mathbb{RS}$, by Corollary 2.4 it is sufficient to prove that $\varphi(j^*) = \varphi(j)^*$ for all $j \in \mathcal{J}$.

We consider three cases:

(i) If $x \in \mathcal{J}^-$, then $x^* \in \mathcal{J}^+$ and

$$\varphi(x^*) = (R(x^*), (R(x^*)^{\blacktriangle}) = (R(x), R(x)^{\blacktriangle}) = (\emptyset, \{x\}^{\blacktriangle})^* = \varphi(x)^*.$$

(ii) If $x \in \mathcal{J}^*$, then $x = x^*$ and

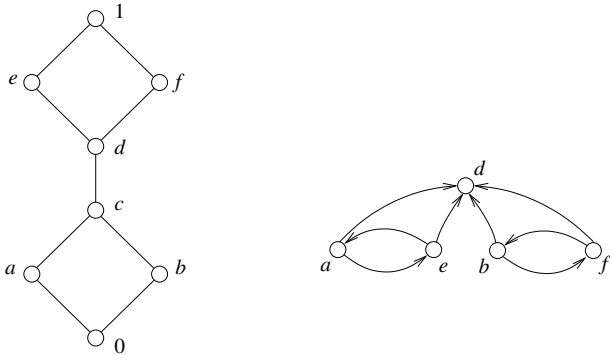
$$\varphi(x^*) = (\{x^*\}, \{x^*\}^{\blacktriangle}) = (\{x\}, \{x\}^{\blacktriangle}) = \varphi(x) = \varphi(x)^*.$$

(iii) If $x \in \mathcal{J}^+$, then $x^* \in \mathcal{J}^-$ and

$$\varphi(x^*) = (\emptyset, \{x^*\}^{\blacktriangle}) = (\emptyset, \{x\}^{\blacktriangle}) = (R(x), R(x)^{\blacktriangle})^* = \varphi(x)^*.$$

Thus, $\mathbb{A}$ and $\mathbb{RS}$ are isomorphic Nelson algebras. □

Example 5.1. Let us consider the Nelson algebra $\mathbb{A}$ presented in Figure 5.1(a). The operation c is defined by $c(0) = 1$, $c(a) = f$, $c(b) = e$, and $c(c) = d$. The set of completely join-irreducible elements $\mathcal{J}$ is $\{a, b, d, e, f\}$. It is easy to observe that $a^* = e$, $b^* = f$, and $d^* = d$. Hence, $\mathcal{J}^- = \{a, b\}$, $\mathcal{J}^* = \{d\}$, and $\mathcal{J}^+ = \{e, f\}$. Let us consider the construction of the relation R presented in the proof of Theorem 1.1.



(a) (b)

FIGURE 1.

The map ρ is defined as $\rho(a) = \rho(e) = a$, $\rho(d) = d$, and $\rho(b) = \rho(f) = b$. The relation R on $U = \mathcal{J}$ is then defined by $x\,R\,y$ iff $\rho(x) \leq \rho(y)$ and is depicted in Figure 5.1(b). Note that since R is reflexive, there should be an arrow from each circle to the circle itself, but such loops are omitted. Now $RS = \{(\emptyset, \emptyset), (\emptyset, \{a, e\}), (\emptyset, \{b, f\}), (\emptyset, \{a, b, e, f\}), (\{d\}, U), (\{a, d, e\}, U), (\{b, d, f\}, U), (U, U)\}$ and clearly $\mathbb{RS} \cong \mathbb{A}$.

For rough sets lattices determined by equivalences, there exists the following representation theorem: for every regular double Stone algebra A, there exists a set U and an equivalence R on U such that A is isomorphic to a subalgebra of RS [7, 8]. By applying Theorem 1.1 and Proposition 4.5, we can prove the following result for rough sets determined by equivalences.

Corollary 5.2. *Let $\mathbb{A}$ be a semisimple Nelson algebra defined on an algebraic lattice. Then, there exists a set U and an equivalence R on U such that $\mathbb{A} \cong \mathbb{RS}$.*

Proof. Suppose that $\mathbb{A}$ is a semisimple Nelson algebra. Then, by Theorem 1.1, there exists a set U and a quasiorder R on U such that $\mathbb{A}$ and $\mathbb{RS}$ are isomorphic Nelson algebras. Because A is a semisimple Nelson algebra, then $\mathbb{A}$ and $\mathbb{RS}$ are mutually isomorphic three-valued Łukasiewicz algebras. This implies by Proposition 4.5 that R must be an equivalence. □

Let us denote by $R_{\mathcal{J}}$ the quasiorder on $\mathcal{J}$ constructed in the proof of Theorem 1.1. Let us note that the set $\mathcal{J}$ and the relation $R_{\mathcal{J}}$ are not necessary minimal in the sense that there may exist a set U of smaller cardinality than $\mathcal{J}$ and a quasiorder R on U determining the same Nelson algebra. For instance, in case of Example 5.1, the same algebra can be obtained by the relation $R = \delta_U \cup \{(a, c), (b, c)\}$ on the three-element set $U = \{a, b, c\}$, where δ_U denotes the identity relation of U. However, $R_{\mathcal{J}}$ has the property that any equivalence class of $R_{\mathcal{J}} \cap {R_{\mathcal{J}}}^{-1}$ is of the form $\{j, j^*\}$, where $j \in \mathcal{J}$. Therefore, any rough set algebra $\mathbb{RS}$ determined by a quasiorder can be generated also by the quasiorder $R_{\mathcal{J}}$ having the property that equivalence classes of $R_{\mathcal{J}} \cap {R_{\mathcal{J}}}^{-1}$ have at most two elements. In case of three-valued Łukasiewicz algebras this then means that each equivalence class of $R_{\mathcal{J}}$ has at most two elements. Hence, the construction is minimal in the above sense.

Remark 5.3. Notice that the Nelson algebra $\mathbb{A}$ is isomorphic to a Boolean algebra whenever the quasiorder $R_{\mathcal{J}}$ of our construction is a partial order; in fact, for our construction, the following statements are equivalent:

(a) $R_{\mathcal{J}}$ is a partial order;
(b) $R_{\mathcal{J}} = \delta_{\mathcal{J}}$;
(c) $\mathbb{RS} \cong (\wp(\mathcal{J}), \cup, \cap, ', \emptyset, \mathcal{J})$;
(d) $\mathbb{A}$ is a Boolean algebra.

Namely, if $R_{\mathcal{J}}$ is a partial order, then our construction gives $j = j^*$ for all $j \in \mathcal{J}$. So, by Lemma 2.1, $i \leq j$ implies $i = j$ for all $i, j \in \mathcal{J}$. Thus, $(\mathcal{J}, \leq)$ is an antichain and $R_{\mathcal{J}} = \delta_{\mathcal{J}}$. As (b)⇒(a) is obvious, we get (a)⇔(b).

If $R_{\mathcal{J}} = \delta_{\mathcal{J}}$, then $RS = \{(X, X) \mid X \subseteq \mathcal{J}\}$ and $c(X, X) = (\mathcal{J} \setminus X, \mathcal{J} \setminus X)$. Therefore, we have $\mathbb{RS} \cong (\wp(\mathcal{J}), \cup, \cap, ', \emptyset, \mathcal{J})$. Thus, (b)⇒(c), and (c)⇒(d) is clear by Theorem 1.1. If $\mathbb{A}$ is defined on an algebraic Boolean lattice, then this lattice is atomistic and $\mathcal{J}$ coincides with set of its atoms. This means that $(\mathcal{J}, \leq)$ is an antichain and $R_{\mathcal{J}} = \delta_{\mathcal{J}}$ by our construction. So, (d)⇒(b).

References

[1] Raymond Balbes and Philip Dwinger, *Distributive lattices*, University of Missouri Press, Columbia, Missouri, 1974.

[2] Garrett Birkhoff, *Lattice theory*, 3rd ed., Colloquim publications, vol. XXV, American Mathematical Society (AMS), Providence, Rhode Island, 1995.
[3] Roberto Cignoli, *The class of Kleene algebras satisfying an interpolation property and Nelson algebras*, Algebra Universalis (1986), 262–292.
[4] ———, *The algebras of Łukasiewicz many-valued logic: A historical overview*, Lecture Notes in Computer Science **4460** (2007), 69–83.
[5] Stephen D. Comer, *An algebraic approach to the approximation of information*, Fundamenta Informaticae **14** (1991), 492–502.
[6] B. A. Davey and H. A. Priestley, *Introduction to lattices and order*, 2nd ed., Cambridge University Press, 2002.
[7] Stéphane P. Demri and Ewa S. Orłowska, *Incomplete information: Structure, inference, complexity*, Monographs in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, Heidelberg, 2002.
[8] Mai Gehrke and Elbert Walker, *On the structure of rough sets*, Bulletin of Polish Academy of Sciences. Mathematics **40** (1992), 235–245.
[9] George Grätzer, *General lattice theory*, 2nd ed., Birkhäuser, Basel, 1998.
[10] Jouni Järvinen, *The ordered set of rough sets*, Lecture Notes in Computer Science **3066** (2004), 49–58.
[11] ———, *Lattice theory for rough sets*, Transactions on Rough Sets **VI** (2007), 400–498.
[12] Jouni Järvinen, Sándor Radeleczki, and Laura Veres, *Rough sets determined by quasiorders*, Order **26** (2009), 337–355.
[13] Kalle Kaarli and Alden F. Pixley, *Polynomial completeness in algebraic systems*, Chapman & Hall/CRC, Boca Raton, 2000.
[14] Antonio Monteiro, *Construction des algébres de Nelson finies*, Bulletin de l'Academie Polonaise des Sciences **11** (1963), 359–362.
[15] ———, *Sur la définition des algèbres de Łukasiewicz trivalentes*, Bulletin Mathématique de la Société Scientifique Mathématique Physique R. P. Roumanie **7** (1963), 3–12.
[16] ———, *Les éléments réguliers d'un $\mathcal{N}$-lattice*, Anais da Academia Brasileira de Ciências **52** (1980), 653–656.
[17] Piero Pagliani, *Rough set systems and logico-algebraic structures*, Incomplete Information: Rough Set Analysis (Ewa Orłowska, ed.), Physica-Verlag, Heidelberg, 1997, pp. 109–190.
[18] Piero Pagliani and Mihir Chakraborty, *A geometry of approximation. Rough set theory: Logic, algebra and topology of conceptual patterns*, Springer, 2008.
[19] Zdzisław Pawlak, *Rough sets*, International Journal of Computer and Information Sciences **11** (1982), 341–356.
[20] Jacek Pomykała and Janusz A. Pomykała, *The Stone algebra of rough sets*, Bulletin of Polish Academy of Sciences. Mathematics **36** (1988), 495–512.
[21] Helena Rasiowa, *An algebraic approach to non-classical logics*, Studies in Logic and the Foundations of Mathematics, no. 78, North-Holland, 1974.
[22] Dimiter Vakarelov, *Notes on N-lattices and constructive logic with strong negation*, Studia Logica **36** (1977), 109–125.
[23] Jules Varlet, *Fixed points in finite de Morgan algebras*, Discrete Mathematics **53** (1985), 265–280.

Jouni Järvinen: Department of Information Technology, FI-20014 University of Turku, Finland
E-mail address: Jouni.Jarvinen@utu.fi

Sándor Radeleczki: Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary
E-mail address: matradi@uni-miskolc.hu