arXiv:1007.1112v1 [math.PR] 7 Jul 2010

ASYMPTOTIC SUPPORT THEOREM FOR PLANAR ISOTROPIC BROWNIAN FLOWS

By Moritz Biskamp

Technische Universität Berlin

It has been shown by various authors that the diameter of a given non-trivial bounded connected set $\mathcal{X}$ grows linearly in time under the action of an isotropic Brownian flow (IBF), which has a non-negative top-Lyapunov exponent. In case of a planar IBF with a positive top-Lyapunov exponent the precise deterministic linear growth rate K of the diameter is known to exist. In this paper we will extend this result to an asymptotic support theorem for the time-scaled trajectories of a planar IBF φ, which has a positive top-Lyapunov exponent, starting in a non-trivial compact connected set $\mathcal{X} \subseteq \mathbf{R}^2$, i.e. we will show convergence in probability of the set of time-scaled trajectories in the Hausdorff distance to the set of Lipschitz continuous functions on $[0,1]$ starting in 0 with Lipschitz constant K.

1. Introduction. Isotropic Brownian flows (IBFs) are a fairly natural class of stochastic flows and have been first introduced by K. Itô [8] and A. M. Yaglom [16]. For this class of stochastic flows the image of a single point is a Brownian motion, and the covariance tensor between two different Brownian motions is an isotropic function of their positions. IBFs, and in particular their local structure, have been extensively studied in the 1980's by [11] and [3] among others.

The study of the global behaviour of stochastic flows was stimulated by Carmona's conjecture [4, Section 5.2.] that the diameter of the image of a compact set could expand linearly in time but not faster. For stochastic flows this conjecture was proved by Cranston, Scheutzow, and Steinsaltz [6] and improved by Lisei and Scheutzow [12] as well as by Scheutzow [13]. Even more surprising than this upper bound is maybe the existence of points that move with linear speed, although each individual point as a diffusion grows on average like the square-root of the time. This lower bound was proved first for IBFs, which have a strictly positive top-Lyapunov exponent, by Cranston, Scheutzow, and Steinsaltz [5] and under more general conditions by Scheutzow and Steinsaltz [14]. Nevertheless, upper and lower bounds for the linear growth turn out to be far from each other in some examples. In case

AMS 2000 subject classifications: Primary 60G17, 37C10; secondary 60G15, 37H10

Keywords and phrases: Stochastic flows, isotropic Brownian flows, asymptotic expansion, asymptotic support theorem

of planar periodic stochastic flows (stochastic flows on the torus) Dolgopyat, Kaloshin, and Koralov [7] used a new approach based on the so-called *stable norm*, to identify the precise deterministic linear growth rate of such flows. By this approach van Bargen [15] identified the precise deterministic growth rate for planar IBFs, which have a strictly positive top-Lyapunov exponent.

Not only the linear growth rate has been analyzed in the last years but also the behaviour of the individual trajectories of stochastic flows. Scheutzow and Steinsaltz [14] investigated so-called *ball-chasing* properties of the flow, which is the existence of a trajectory that follows a given Lipschitz path in a logarithmic neighbourhood [14, Theorem 4.2], where the Lipschitz constant is basically the lower bound of linear growth mentioned in the previous paragraph.

Here we are looking at the individual trajectories of a planar IBF, or to be more precise at the linear time-scaled versions. Getting a better understanding of these trajectories yields a deeper understanding of the expansion of non-trivial bounded connected sets under the action of an IBF. In this paper we will show convergence in probability of the set of time-scaled trajectories in the Hausdorff distance to the set of Lipschitz continuous functions starting in 0 with Lipschitz constant K, which is the deterministic growth rate for a planar IBF mentioned above. Roughly speaking we will show the following: On the one hand we show that for any given Lipschitz function with Lipschitz constant K starting in 0 there exists a trajectory that approximates this Lipschitz function; this represents the upper bound for the expansion of the initial set. On the other hand, for any time-scaled trajectory there exists a Lipschitz function with Lipschitz constant K starting in 0 such that this function is close to the time-scaled trajectory; which corresponds to the lower bound for the expansion of the initial set. As far as the author knows such a complete characterization of the asymptotic behaviour of the trajectories of stochastic flows is a novelty in the present context and hence yields a new and deeper understanding of the expansion of non-trivial bounded connected sets under the action of IBFs.

The paper is organized as follows: In Section 2.1 we first introduce the notion of stochastic flows, and in particular of IBFs and some of their main properties used within this paper. After stating the main theorem in Section 3 we first introduce the notion of *stable norm* in Section 4. The proof of the main theorem is divided into the proof of the upper bound (Section 5.1) and the lower bound (Section 5.2).

2. Preliminaries.

2.1. *Isotropic Brownian Flows.* We provide a short introduction to isotropic Brownian flows (IBF) following mainly [2].

A *stochastic flow of homeomorphisms* on $\mathbf{R}^d$ is a family of random homeomorphisms $\{\varphi_{s,t} : s,t \in \mathbf{R}_+\}$ of $\mathbf{R}^d$, which almost surely satisfies the flow property, i.e. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $s,t,u \in \mathbf{R}_+$, and $\varphi_{t,t} = \mathrm{id}|_{\mathbf{R}^d}$ for all $t \in \mathbf{R}_+$ and is jointly continuous, i.e. $(s,t,x) \mapsto \varphi_{s,t}(x)$ is continuous. The flow is called a *Brownian flow* if the increments $\varphi_{s,t}$ on disjoint intervals are independent and time homogeneous.

Due to [10, Theorem 4.2.8], under suitable regularity conditions, Brownian flows of homeomorphisms can be realized as solutions of Kunita-type stochastic differential equations

$$\varphi_{s,t}(x) = x + \int_s^t M(\mathrm{d}u, \varphi_{s,u}(x)) + \int_s^t v(\varphi_{s,u}(x))\mathrm{d}u, \qquad s \le t,$$

where $v : \mathbf{R}^d \to \mathbf{R}^d$ is a vector field and $M : \mathbf{R}_+ \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is a mean-zero Gaussian martingale field on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. M is called the generating *Brownian field* and its distribution is determined by the covariances

$$\mathbf{E}\left[\langle M(t,x),\xi\rangle \langle M(s,y),\eta\rangle\right] = (s \wedge t)\langle b(x,y)\xi,\eta\rangle, \qquad \xi,\eta \in \mathbf{R}^d,$$

where $b : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^{d\times d}$ is a covariance tensor. The distribution of the flow $\{\varphi_{s,t} : s,t \in \mathbf{R}_+\}$ is determined by the functions $b(x,y)$ and $v(x)$. If there is no risk of ambiguity we will abbreviate $\varphi_{0,t}(x)$ by $\varphi_t(x)$. Due to the independent increments and the flow property, a Brownian flow satisfies according to [10, Theorem 4.2.1] a Markov property in the following sense: Let $\mathcal{F}_{s,t}$ be the least sub σ-algebra of $\mathcal{F}$ containing all null sets and $\bigcap_{\varepsilon>0}\{\varphi_{u,r} : s-\varepsilon \le u, r \le t+\varepsilon\}$. Then for any $\{\mathcal{F}_{s,t} : t \in [s,\infty)\}$-stopping-time τ we have

$$\mathcal{L}[\varphi_{\tau,r}(\varphi_{s,\tau}(\cdot)) : r \ge \tau | \mathcal{F}_{s,\tau}] = \mathcal{L}_{s,\tau}[\varphi_{s,s+r-\tau}(\cdot) : r \ge \tau]. \tag{1}$$

An *isotropic Brownian flow* on $\mathbf{R}^d$ is a Brownian flow of homeomorphisms of $\mathbf{R}^d$, where the distribution of each $\varphi_{s,t}$ is invariant under rigid transformations of $\mathbf{R}^d$. The invariance in distribution of $\varphi_{s,t}$ under rigid motions implies the invariance in distribution of the generating Brownian field $M(t,x)$, in this case $M(t,x)$ is said to be an *isotropic Brownian field*. The invariance under translations implies that $b(x,y) = b(x-y,0) \equiv b(x-y)$ and then the invariance under rotations and reflections implies that

$$b(x) = O^T b(Ox) O, \tag{2}$$

for all orthogonal matrices O on $\mathbf{R}^d$. Moreover we have $v(x) \equiv 0$. In this paper we will assume $b \in C^\infty$, since we will use results of [15], where smoothness of b has to be assumed. In this case $\varphi_{s,t}(\cdot) \in C^\infty(\mathbf{R}^d)$ are diffeomorphims. Furthermore, the isotropy property (2) implies that $b(0) = c\,\mathrm{id}|_{\mathbf{R}^d}$ for some constant $c > 0$. At the cost of rescaling time by a constant factor we can and will assume that $b(0) = \mathrm{id}|_{\mathbf{R}^d}$. In order to avoid the trivial case where the flow consists of translations, we assume also that $b(x) \not\equiv \mathrm{id}|_{\mathbf{R}^d}$. Since the properties of the flow we are interested in do not depend on rigid translations of the space by a Brownian motion added to the generated IBF, we can and will assume that $\lim_{|x|\to\infty} b(x) = 0$.

According to [16, Section 4] (and as described in [3]) a covariance tensor with the above properties can be written in the form

$$b_{ij}(x) = \begin{cases} (B_L(|x|) - B_N(|x|))\frac{x_i x_j}{|x|^2} + \delta_{ij} B_N(|x|) & \text{if } x \neq 0, \\ \delta_{ij} & \text{if } x = 0, \end{cases}$$

for $i, j = 1, \ldots, d$, where B_L and B_N are the so-called longitudinal and transverse (normal) covariance functions defined by

$$B_L(r) := b_{ii}(re_i), \qquad B_N(r) := b_{ii}(re_j),$$

for $r \geq 0$ and $i \neq j$, where e_i denotes the i^{th} unit vector in $\mathbf{R}^d$. For future reference define

$$\beta_L := -B_L''(0) > 0, \qquad \beta_N := -B_N''(0) > 0,$$

to be the negative second right-hand derivative of the longitudinal and respectively transverse covariance function. In Lemma A.1 we will give an estimate on the longitudinal and transverse respectively covariance function in terms of β_L and β_N respectively.

Lyapunov exponents can be defined for dynamical systems and characterise the exponential rate of separation of infinitesimally close trajectories. Baxendale and Harris [3] have shown under the assumptions mentioned above, that IBFs have Lyapunov exponents, which satisfy

$$\mu_i = \frac{1}{2}\left((d-i)\beta_N - i\beta_L\right), \qquad i = 1, \ldots, d.$$

The top-Lyapunov exponent μ_1, and more precisely its sign, crucially affects the asymptotic behaviour of the flow. As shown in [5] and [14] respectively a non-negative top Lyapunov exponent $\mu_1 \geq 0$ implies that any non-trivial bounded set (a set is said to be *non-trivial* if it is connected and contains

more than one point) does not contract to a single point under the action of the flow. On the other hand, if $\mu_1 < 0$ then according to [14] there is a positive probability that a small set contracts to a single point and hence our result cannot be true. By this remark and since we would like to use results from [15] we always will assume a strictly positive top-Lyapunov exponent. But we conjecture that the results in [15] and hence our main result is also true for $\mu_1 = 0$. For more details on Lyapunov exponents for random dynamical systems we refer to [1].

If the flow φ is restricted to $\{(s,t) \in \mathbf{R}_+ \times \mathbf{R}_+ : s \leq t\}$ it is called the *forward flow*, whereas if restricted to $\{(s,t) \in \mathbf{R}_+ \times \mathbf{R}_+ : s \geq t\}$ it is called the *backward flow*. In Kunita [10, Theorem 4.2.10] the generating Brownian field of the backward Brownian flow has been calculated. If the flow is isotropic it turns out that it is in fact equal to the generating Brownian field of the forward Brownian flow (see [3, (3.7)]). This implies that for fixed $T > 0$ we have

$$\mathcal{L}[\varphi_{s,t}(\cdot) : 0 \leq s \leq t \leq T] = \mathcal{L}[\varphi_{T-s,T-t}(\cdot) : 0 \leq s \leq t \leq T], \tag{3}$$

the so-called *time reversal* property of IBFs.

2.2. *Time-scaled Trajectories.* Let $\mathcal{X} \subseteq \mathbf{R}^d$ be compact and denote the *set of time-scaled trajectories of the flow starting in* $\mathcal{X}$ up to some time $T > 0$ by

$$F_T(\mathcal{X}, \omega) := \bigcup_{x \in \mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T} \varphi_{0,tT}(x, \omega) \right\}$$

for $\omega \in \Omega$. Since $\mathcal{X}$ is compact and $(x,t) \mapsto \varphi_{0,t}(x)$ is continuous we have that $F_T(\mathcal{X})$ is a compact subset of the continuous functions on $[0,1]$ with respect to the supremum norm $\|\cdot\|_\infty$. Further denote by $\mathrm{Lip}_0(K)$ the set of Lipschitz continuous functions f on $[0,1]$ with $f(0) = 0$ and Lipschitz constant K, which is as well a compact set with respect to $\|\cdot\|_\infty$. The Hausdorff distance between two non-empty compact sets A and B of a metric space is defined by

$$d_H(A,B) := \max \left\{ \sup_{x \in A} d(x,B) \,;\, \sup_{y \in B} d(y,A) \right\},$$

where d denotes the metric. Since $F_T(\mathcal{X})$ and $\mathrm{Lip}_0(K)$ are compact subsets of $C([0,1], \|\cdot\|_\infty)$, the function

$$(T, \omega) \mapsto d_H(F_T(\mathcal{X}, \omega), \mathrm{Lip}_0(K))$$

is well defined.

3. Main Theorem. From here on we will consider the case of planar IBFs, i.e. the dimension of the space will be $d = 2$. Given a planar IBF φ, which has a strictly positive top-Lyapunov exponent, our main result is: For any non-trivial compact connected set $\mathcal{X} \subseteq \mathbf{R}^2$ we have convergence in probability of $d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K))$ to 0 for $T \to \infty$, i.e. the following theorem.

THEOREM 3.1. *Let φ be a planar IBF, which has a strictly positive top-Lyapunov exponent. Then there exists a deterministic constant $K > 0$ such that for any $\varepsilon > 0$ and any non-trivial compact and connected set $\mathcal{X} \subseteq \mathbf{R}^2$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K)) > \varepsilon\right) = 0,$$

where d_H denotes the Hausdorff distance, $F_T(\mathcal{X})$ the set of time-scaled trajectories (see Section 2.2) and $\mathrm{Lip}_0(K)$ the set of Lipschitz continuous functions on $[0,1]$ starting in 0 with Lipschitz constant K.

The theorem will be proved in Section 5.

4. Stable Norm. The concept of stable norm presented in this section traces back to Dolgopyat, Kaloshin, and Koralov [7], where they considered planar periodic stochastic flows.

Denote by $B_r(w)$ the closed ball in $\mathbf{R}^2$ of radius r around $w \in \mathbf{R}^2$. For any $R \geq 1$ let $\mathcal{C}_R$ be the set of all connected compact large subsets of $\mathbf{R}^2$ fully contained in $B_{2R}(0)$, where a set is called *large* if its diameter is greater or equal than 1. For any $v \in \mathbf{R}^2$, $\mathcal{X} \subseteq \mathbf{R}^2$ and $s \geq 0$ define the stopping time

$$\tau^R(\mathcal{X}, v, s) := \inf\left\{t \geq 0 : \varphi_{s,s+t}(\mathcal{X}) \cap B_R(v) \neq \emptyset\,;\, \mathrm{diam}(\varphi_{s,s+t}(\mathcal{X})) \geq 1\right\},$$

which is the first time when starting at time s the initial set $\mathcal{X}$ under the action of the flow hits an R-neighbourhood of v as a large set. For $s = 0$ we will abbreviate in the following $\tau^R(\mathcal{X}, v, 0)$ by $\tau^R(\mathcal{X}, v)$. By temporal homogeneity of the flow the laws of $\tau^R(\mathcal{X}, v, s)$ and $\tau^R(\mathcal{X}, v)$ coincide. If only the distribution matters, we will use $\tau^R(\mathcal{X}, v)$. Then it is known from [15] that for $v \in \mathbf{R}^2$ the following limit (uniformly in $\mathcal{X} \in \mathcal{C}_R$) exists

$$\|v\|^R := \lim_{t\to\infty} \frac{1}{t} \sup_{\gamma\in\mathcal{C}_R} \mathbf{E}\left[\tau^R(\gamma, vt)\right] = \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left[\tau^R(\mathcal{X}, vt)\right].$$

This limit is called *stable norm* of v. Further it is known that $\|\cdot\|^R$ does not depend on the precise choice of $R \geq 1$ and it is indeed a norm on $\mathbf{R}^2$, see [15,

Section 3.2.2]. Hence for the sequel fix some arbitrary $R \geq 1$. If we denote the closed unit ball in $\mathbf{R}^2$ with respect to $\|\cdot\|^R$ by $\mathcal{B}$ then it has been shown by van Bargen [15, Theorem 2.1] that for any $\varepsilon > 0$ and any non-trivial bounded connected $\mathcal{X} \subseteq \mathbf{R}^2$

$$\lim_{T\to\infty} \mathbf{P}\left((1-\varepsilon)T\mathcal{B} \subseteq \bigcup_{x\in\mathcal{X}} \bigcup_{0\leq t\leq T} \varphi_t(x) \subseteq (1+\varepsilon)T\mathcal{B}\right) = 1. \tag{4}$$

For our purpose this immediately implies that for $\varepsilon > 0$ and $t \in (0,1]$ we have

$$\lim_{T\to\infty} \mathbf{P}\left(\varphi_{tT}(\mathcal{X}) \subseteq tT(1+\varepsilon)\mathcal{B}\right) = 1. \tag{5}$$

Since the flow is isotropic $\mathcal{B}$ is a ball in $\mathbf{R}^2$ with (Euclidean) radius K, i.e. $K = 1/\|e_1\|^R > 0$. This deterministic constant K is the Lipschitz constant in Theorem 3.1.

In the sequel we will need the following lemma from [15] on convergence in probability of the time-scaled hitting time to the stable norm.

LEMMA 4.1. *For any $\varepsilon > 0$ and $v \in \mathbf{R}^2$ we have*

$$\lim_{T\to\infty} \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\left|\frac{\tau^R(\gamma, Tv)}{T} - \|v\|^R\right| > \varepsilon\right) = 0,$$

in particular for any $m \in \mathbf{N}$ there exists a constant $c_m^{(1)}$ such that

$$\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, Tv) > \left(\|v\|^R + \varepsilon\right)T\right) \leq c_m^{(1)} T^{-m}.$$

PROOF. [15, Corollary 4.7] and [15, (3.27)]. □

The following lemma ensures that the diameter uniformly in $\gamma \in \mathcal{C}_R$ under the action of the flow stays large after $\sqrt{T}$ with high probability for T large.

LEMMA 4.2. *For any $m \in \mathbf{N}$ there exists a constant $c_m^{(2)}$ such that for T large*

$$\sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\inf_{s\geq\sqrt{T}} \operatorname{diam}(\varphi_s(\gamma)) < 1\right) \leq c_m^{(2)} T^{-m}.$$

PROOF. Following the ideas of [15, (3.15) and (3.16)] for any $m \in \mathbf{N}$ there exists some constant $\tilde{c}_m^{(2)}$ such that for sufficiently small $\delta > 0$ and $n \in \mathbf{N}$ large we have

$$\sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(S_n(\gamma)\right) := \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\inf_{\substack{s \in \mathbf{N} \\ s \geq \lfloor \sqrt{n} \rfloor}} \operatorname{diam}(\varphi_s(\gamma)) < \delta n \right) \leq \tilde{c}_m^{(2)} n^{-m}.$$

Similar to [13, Lemma 6] for $x, y \in \mathbf{R}^2$ there exists a Brownian motion W such we have almost surely

$$\inf_{0 \leq t \leq 1} \|\varphi_t(x) - \varphi_t(y)\| \geq \|x - y\| \exp\left(-\frac{\kappa}{2} + \sqrt{\kappa} \inf_{0 \leq t \leq 1} W_t \right),$$

where according to Lemma A.1 we have $\kappa := \max\{\beta_L; \beta_N\}$. For $\gamma \in \mathcal{C}_R$ and any integer $k \geq \lfloor \sqrt{T} \rfloor$ we choose on $S_{\lfloor T \rfloor}(\gamma)^c$ points $x^{(k)}, y^{(k)} \in \varphi_k(\gamma)$ such that $\left\| x^{(k)} - y^{(k)} \right\| = \delta k$. Hence we get for $m \in \mathbf{N}$ and k large enough

$$\begin{aligned}
&\sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\inf_{k \leq t \leq k+1} \operatorname{diam}(\varphi_t(\gamma)) < 1 \,\middle|\, S_{\lfloor T \rfloor}(\gamma)^c \right) \\
&\qquad \leq \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\inf_{0 \leq t \leq 1} \left\| \varphi_t(x^{(k)}) - \varphi_t(y^{(k)}) \right\| < 1 \,\middle|\, S_{\lfloor T \rfloor}(\gamma)^c \right) \\
&\qquad \leq \mathbf{P}\left(\delta k \exp\left(-\frac{\kappa}{2} + \sqrt{\kappa} \inf_{0 \leq t \leq 1} W_t \right) < 1 \right) \\
&\qquad \leq \frac{2}{\sqrt{2\pi}} (\delta k)^{1/2} \exp\left(-\frac{\log(\delta k)^2}{2\kappa} \right) \\
&\qquad \leq \frac{2}{\sqrt{2\pi}} (\delta k)^{1/2} \exp\left(-\frac{\log(\delta k^m)}{2\kappa} \right) = \frac{2}{\sqrt{2\pi}} \delta^{\frac{\kappa-1}{2\kappa}} k^{\frac{\kappa-m}{2\kappa}}.
\end{aligned}$$

Then there exists a constant $c_m^{(2)}$ such that for T large

$$\begin{aligned}
&\sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\inf_{s \geq \sqrt{T}} \operatorname{diam}(\varphi_s(\gamma)) < 1 \right) \\
&\qquad \leq \sum_{k \geq \lfloor \sqrt{T} \rfloor} \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(\inf_{k \leq t \leq k+1} \operatorname{diam}(\varphi_t(\gamma)) < 1 \,\middle|\, S_{\lfloor T \rfloor}(\gamma)^c \right) \\
&\qquad\qquad + \sup_{\gamma \in \mathcal{C}_R} \mathbf{P}\left(S_{\lfloor T \rfloor}(\gamma) \right) \\
&\qquad \leq c_m^{(2)} T^{-m},
\end{aligned}$$

which completes the proof. □

Remark. Observe that in the previous lemma uniform convergence in $\gamma \in \mathcal{C}_R$ is only achieved because the sets in $\mathcal{C}_R$ are large.

5. Proof of Theorem 3.1. As usual we consider a planar IBF φ, which has a strictly positive top-Lyapunov exponent. The upper bound (Section 5.1) and the lower bound (Section 5.2) of Theorem 3.1 will be proved for large sets, i.e. the initial set $\mathcal{X}$ is assumed to be in $\mathcal{C}_R$ for some arbitrary fixed $R \geq 1$. The generalization to non-trivial compact connected sets will be done in Section 5.3, which then completes the proof of Theorem 3.1.

5.1. *Upper Bound.* This section is devoted to the proof of the upper bound of Theorem 3.1, i.e. the following Theorem.

THEOREM 5.1. *For any $\varepsilon > 0$ and $\mathcal{X} \in \mathcal{C}_R$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{g\in F_T(\mathcal{X})} d\left(g, \mathrm{Lip}_0(K)\right) > \varepsilon\right) = 0,$$

where K is the Euclidean radius of the stable norm unit ball (see Section 4).

The proof of Theorem 5.1 is divided into several steps. The main idea is to show that the time-scaled trajectories behave like Lipschitz functions on some sufficiently small discrete grid (Lemma 5.2), and between two supporting points large growth of the initial set does not occur (Lemma 5.3). For the first estimate we have to control trajectories starting inside some linearly growing set, which extends the result of Lemma 4.1, where the initial set has a fixed diameter. The basic lemma to control this is the following.

LEMMA 5.1. *For all $\varepsilon > 0$, $v \in \mathbf{R}^2$ and $0 < \tilde{\varepsilon} \leq \frac{\varepsilon}{6\|e_1\|^R}$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\left|\frac{\tau^R(B_{\tilde{\varepsilon}T}(0), vT)}{T} - \|v\|^R\right| > \varepsilon\right) = 0.$$

PROOF. Since $B_R(0) \subset B_{\tilde{\varepsilon}T}(0)$ for T large we have because of Lemma 4.1

$$\begin{aligned}\mathbf{P}&\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) > \left(\|v\|^R + \varepsilon\right) T\right)\\ &\leq \mathbf{P}\left(\tau^R(B_R(0), vT) > \left(\|v\|^R + \varepsilon\right) T\right) \to 0.\end{aligned}$$

According to [15, Lemma 4.4] there exists a constant $\alpha > 0$ such that

$$\inf_{\gamma\in\mathcal{C}_R^*} \inf_{t\geq\alpha} \mathbf{P}\left(\varphi_t(\gamma) \cap \partial B_R(0) \neq \emptyset; \mathrm{diam}(\varphi_t(\gamma)) \geq 1\right) =: p_1 > 0, \tag{6}$$

where $\mathcal{C}_R^*$ denotes the set of all large connected subsets γ of $\mathbf{R}^2$ with $\gamma \cap \partial B_R(0) \neq \emptyset$. The estimate (6) basically tells that given some extra time α uniformly in $\gamma \in \mathcal{C}_R^*$ there is a positive probability that $\varphi_t(\gamma)$ will stay intersected with $\partial B_R(0)$. By spatial homogeneity, the time reversal property of IBFs (see (3)) and (6) we get

$$\begin{aligned}\mathbf{P}\left(\varphi_{t+\alpha}(B_R(0)) \cap B_{\tilde{\varepsilon}T}(vT) \neq \emptyset\right) &= \mathbf{P}\left(B_R(vT) \cap \varphi_{t+\alpha}(B_{\tilde{\varepsilon}T}(0)) \neq \emptyset\right)\\ &\geq \mathbf{P}\left(B_R(vT) \cap \varphi_{t+\alpha}(B_{\tilde{\varepsilon}T}(0)) \neq \emptyset \,\middle|\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right) \cdot\\ &\qquad \mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right)\\ &\geq p_1\, \mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right).\end{aligned}$$

According to Lemma 4.2 for any $m \in \mathbf{N}$ there exists a constant $c_m^{(2)}$ such that for $t \geq \sqrt{T}$ we have

$$\mathbf{P}\left(\operatorname{diam}\left(\varphi_{t+\alpha}(B_R(0))\right) < 1\right) \leq c_m^{(2)} T^{-m}.$$

Thus we get for $t \geq \sqrt{T}$

$$\mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq t\right) \leq \frac{1}{p_1} \mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0), vT) \leq t + \alpha\right) + \frac{c_m^{(2)}}{p_1} T^{-m}. \tag{7}$$

Further we have

$$\begin{aligned}\mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0), vT) \leq \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\right)&\\ \leq \mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0), vT) \leq \left(\|v\|^R - \frac{\varepsilon}{2}\right) T\right.&;\\ \left.\tau^R(B_R(0), vT) > \left(\|v\|^R - \frac{\varepsilon}{6}\right) T\right)&\\ + \mathbf{P}\left(\tau^R(B_R(0), vT) \leq \left(\|v\|^R - \frac{\varepsilon}{6}\right) T\right),&\end{aligned} \tag{8}$$

where the second term converges to 0 for $T \to \infty$ by Lemma 4.1. To estimate the first term consider an R-net on $\partial B_{\tilde{\varepsilon}T}(vT)$, i.e. there exists $N(\tilde{\varepsilon}T) \in \mathbf{N}$ and points $Tw_1, \ldots, Tw_{N(\tilde{\varepsilon}T)} \in \partial B_{\tilde{\varepsilon}T}(0)$ such that

$$\partial B_{\tilde{\varepsilon}T}(vT) \subseteq \bigcup_{i=1}^{N(\tilde{\varepsilon}T)} B_R((v + w_i)T),$$

where $N(\tilde{\varepsilon}T)$ grows at most polynomial in T for a fixed degree $\tilde{m} \in \mathbf{N}$. Thus

we get estimating the first term in (8) using isotropy of the flow

(9)

$$\begin{aligned}
&\mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0),vT)\le\left(\|v\|^R-\frac{\varepsilon}{2}\right)T\,;\,\tau^R(B_R(0),vT)>\left(\|v\|^R-\frac{\varepsilon}{6}\right)T\right)\\
&\quad\le\sum_{i=1}^{N(\tilde{\varepsilon}T)}\mathbf{P}\left(\tau^R(B_R(0),vT)>\left(\|v\|^R-\frac{\varepsilon}{6}\right)T\,\middle|\right.\\
&\qquad\qquad\left.\tau^R(B_R(0),(v+w_i)T)\le\left(\|v\|^R-\frac{\varepsilon}{2}\right)T\right)\\
&\quad\le\sum_{i=1}^{N(\tilde{\varepsilon}T)}\mathbf{P}\left(\tau^R(\varphi_{\tau^R(B_R(0),(v+w_i)T)}(B_R(0)),vT)>\frac{\varepsilon}{3}T\right)\\
&\quad\le N(\tilde{\varepsilon}T)\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R(\gamma,e_1\tilde{\varepsilon}T)>\frac{\varepsilon}{3}T\right)\\
&\quad\le N(\tilde{\varepsilon}T)\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R(\gamma,e_1\tilde{\varepsilon}T)>\left(\tilde{\varepsilon}\,\|e_1\|^R+\frac{\varepsilon}{6}\right)T\right).
\end{aligned}$$

This last probability converges according to Lemma 4.1 uniformly in $\gamma\in\mathcal{C}_R$ in $o(T^{-m})$ for $m>\tilde{m}$ to 0 as $T\to\infty$. Hence combining (7), (8) and (9) we get for $t=\left(\|v\|^R-\varepsilon\right)T$ and $T\ge\frac{2\alpha}{\varepsilon}$

$$\begin{aligned}
&\mathbf{P}\left(\tau^R(B_{\tilde{\varepsilon}T}(0),vT)\le\left(\|v\|^R-\varepsilon\right)T\right)\\
&\qquad\le\frac{1}{p_1}\mathbf{P}\left(\tau^{\tilde{\varepsilon}T}(B_R(0),vT)\le\left(\|v\|^R-\frac{\varepsilon}{2}\right)T\right)+\frac{c_m^{(2)}}{p_1}T^{-m}\\
&\qquad\to 0,
\end{aligned}$$

as $T\to\infty$, which completes the proof. □

Using Lemma 5.1 we will show that all time-scaled trajectories starting in a linearly growing set behave like a Lipschitz function for a given mesh size Δt.

LEMMA 5.2. *Let $\varepsilon\in(0,1)$ and $\Delta t\in(0,1)$. Then for $0<\tilde{\varepsilon}\le\frac{K(1+\varepsilon/2)\Delta t\varepsilon}{6(4+\varepsilon)}$ we have*

$$\lim_{T\to\infty}\mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)}\left|\frac{1}{T}x-\frac{1}{T}\varphi_{\Delta tT}(x)\right|\ge\Delta tK(1+\varepsilon)\right)=0.$$

PROOF. Since $|v| = K\|v\|^R$ and $\tilde{\varepsilon} \leq \Delta t K \frac{\varepsilon}{2}$ we have for some constant c^* specified below and T large

$$\begin{aligned}
&\mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} |x - \varphi_{\Delta tT}(x)| \geq \Delta t K(1+\varepsilon)T\right) \\
&\quad\leq \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} \|\varphi_{\Delta tT}(x)\|^R \geq \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right) \\
&\quad\leq \mathbf{P}\left(\exists\, x \in B_{\tilde{\varepsilon}T}(0) : \|\varphi_{\Delta tT}(x)\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right) \\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} \|\varphi_{\Delta tT}(x)\|^R > \Delta t\left(1+\frac{\varepsilon}{2}\right)T\right) \\
&\quad\leq \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \varphi_{\Delta tT}(B_{\tilde{\varepsilon}T}(0)) \cap B_R(vT) \neq \emptyset\right) \\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} |\varphi_{\Delta tT}(x)| > c^* \log(\Delta tT)\right) \\
&\quad\leq \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq \Delta tT\right) \\
&\qquad + \mathbf{P}\left(\operatorname{diam}(\varphi_{\Delta tT}(B_{\tilde{\varepsilon}T}(0))) < 1\right) \\
&\qquad + \mathbf{P}\left(\inf_{x\in B_{\tilde{\varepsilon}T}(0)} |\varphi_{\Delta tT}(x)| > c^* \log(\Delta tT)\right).
\end{aligned}$$

First observe that [14, Theorem 4.2] yields the existence of a constant c^* such that the probability that there exists some $x \in B_{\tilde{\varepsilon}T}(0)$, which remains in a logarithmic neighbourhood of the origin, i.e. $|\varphi_s(x)| \leq c^* \log s$ for all $s \geq \Delta tT$, converges to 1 for $T \to \infty$. Hence the third probability converges to 0 and because of Lemma 4.2 the second probability converges to 0 as well. Thus we get

(10)

$$\begin{aligned}
&\lim_{T\to\infty} \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)} |x - \varphi_{\Delta tT}(x)| \geq \Delta t K(1+\varepsilon)T\right) \\
&\quad\leq \lim_{T\to\infty} \mathbf{P}\left(\exists v \in \mathbf{R}^2 : \|v\|^R = \Delta t\left(1+\frac{\varepsilon}{2}\right) ;\, \tau^R(B_{\tilde{\varepsilon}T}(0), vT) \leq \Delta tT\right) \\
&\quad= \lim_{T\to\infty} \mathbf{P}\Bigg(\underbrace{\exists v \in \Delta t\partial\mathcal{B} : \tau^R\left(B_{\frac{\varepsilon}{1+\varepsilon/2}T}(0), vT\right) \leq \frac{\Delta t}{1+\frac{\varepsilon}{2}}T}_{:=S_1(T)}\Bigg),
\end{aligned}$$

where $\mathcal{B}$ denotes the unit ball with respect to the stable norm. Let now $\delta := \frac{\varepsilon\Delta t}{16\|e_1\|^R}$ and $v_1, \ldots, v_N$ a δ-net on $\Delta t\partial\mathcal{B}$. Because of Lemma 5.1 with

$\tilde{\eta} := \frac{\tilde{\varepsilon}}{(1+\varepsilon/2)} \leq \frac{K}{6}\frac{\Delta t\varepsilon}{4+\varepsilon}$ we have

$$\mathbf{P}\left(S_2(T)\right) := \mathbf{P}\left(\exists j : \tau^R\left(B_{\tilde{\eta}T}(0), v_jT\right) \leq \frac{\Delta t}{\left(1+\frac{\varepsilon}{4}\right)}T\right) \to 0. \tag{11}$$

Because of the isotropy of the flow we get

$$\begin{aligned}
&\mathbf{P}\left(S_2(T)^c | S_1(T)\right) \\
&\quad = \mathbf{P}\left(\forall j : \tau^R\left(B_{\tilde{\eta}T}(0), v_jT\right) > \frac{\Delta t}{\left(1+\frac{\varepsilon}{4}\right)}T \,\middle|\, S_1(T)\right) \\
&\quad \leq \mathbf{P}\left(\forall j : |v - v_j| \leq \delta\,;\, \tau^R\left(B_{\tilde{\eta}T}(0), v_jT\right) > \frac{\Delta t}{\left(1+\frac{\varepsilon}{4}\right)}T \,\middle|\, S_1(T)\right) \\
&\quad \leq \mathbf{P}\Bigg(\forall j : |v - v_j| \leq \delta\,;\, \tau^R(\varphi_{\tau^R(B_{\tilde{\eta}T}(0), vT)}(B_{\tilde{\eta}T}(0)), v_jT) > \\
&\qquad\qquad \left(\frac{1}{\left(1+\frac{\varepsilon}{4}\right)} - \frac{1}{\left(1+\frac{\varepsilon}{2}\right)}\right)\Delta tT \,\Bigg|\, S_1(T)\Bigg) \\
&\quad \leq \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, \delta e_1T) > \left(\frac{1}{\left(1+\frac{\varepsilon}{4}\right)} - \frac{1}{\left(1+\frac{\varepsilon}{2}\right)}\right)\Delta tT\right) \\
&\quad \leq \sup_{\gamma\in\mathcal{C}_R} \mathbf{P}\left(\tau^R(\gamma, \delta e_1T) > \left(\delta\left\|e_1\right\|^R + \frac{\varepsilon}{16}\Delta t\right)T\right),
\end{aligned} \tag{12}$$

which converges to 0 for $T \to \infty$ according to Lemma 4.1. Combining (11) and (12) now yields

$$\mathbf{P}\left(S_1(T)\right) \leq \mathbf{P}\left(S_2(T)^c | S_1(T)\right) + \mathbf{P}\left(S_2(T)\right) \to 0,$$

which completes the proof because of (10). □

The event that between two supporting points of the grid (chosen sufficiently small) the trajectories move not too much will be treated in the following Lemma. It is an application of the chaining techniques introduced by Scheutzow [13].

LEMMA 5.3. *For any bounded $\mathcal{X} \subseteq \mathbf{R}^2$, $a > 0$ and any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ with $\Delta t := \max_i |t_{i-1} - t_i| < \frac{a^2}{12\kappa}$ with $\kappa := \max\{\beta_L; \beta_N\}$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \sup_{t_i\leq t\leq t_{i+1}} \left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{tT}(x)\right| > a\right) = 0.$$

PROOF. Denote by $N(\mathcal{X},\delta)$ the minimal number of closed balls of radius $\delta>0$ needed to cover $\mathcal{X}$. Let $\mathcal{X}_j$, $j=1,\dots,N(\mathcal{X},e^{-3\kappa T})$ be compact sets of diameter at most $e^{-3\kappa T}$ which cover $\mathcal{X}$ and choose arbitrary points $x_j\in\mathcal{X}_j$. Then there exists a constant $L>0$ (depending only on $\mathcal{X}$) such that

$$N(\mathcal{X},e^{-3\kappa T})\le Le^{3\kappa T}.$$

We have

$$\mathbf{P}\left(\sup_{x\in\mathcal{X}}\max_i\sup_{t_i\le t\le t_{i+1}}\left|\frac{1}{T}\varphi_{t_iT}(x)-\frac{1}{T}\varphi_{tT}(x)\right|>a\right)\le P_1+P_2,$$

where

$$P_1:=Le^{3\kappa T}n\max_{i,j}\mathbf{P}\left(\sup_{t_i\le t\le t_{i+1}}|\varphi_{t_iT}(x_j)-\varphi_{tT}(x_j)|>Ta-1\right)$$

and

$$P_2:=Le^{3\kappa T}n\max_j\mathbf{P}\left(\sup_{0\le t\le 1}\operatorname{diam}(\varphi_{tT}(\mathcal{X}_j))>1\right).$$

Because of the temporal and spatial homogeneity of the flow and since the one point motion is Brownian we get by denoting a one-dimensional Brownian motion by W

$$\begin{aligned}P_1&\le 2Lne^{3\kappa T}\mathbf{P}\left(\sup_{0\le s\le\Delta tT}|W_s|>\frac{Ta-1}{\sqrt{2}}\right)\\&\le 8Lne^{3\kappa T}\frac{\sqrt{\Delta t}}{(a-1)\sqrt{2\pi T}}\exp\left(-\frac{a^2}{4\Delta t}T\right)\\&=8Ln\frac{\sqrt{\Delta t}}{(a-1)\sqrt{2\pi T}}\exp\left(\left(3\kappa-\frac{a^2}{4\Delta t}\right)T\right)\to 0\end{aligned}$$

for $T\to\infty$, see [9, Problem II.8.2]. On the other hand we use Theorem 1 of [13] to bound P_2. Because of Lemma A.1 the derivative of the quadratic variation of the difference $M(t,x)-M(t,y)$, where M is the generating isotropic Brownian field, satisfies the Lipschitz property with $\kappa=\max\{\beta_L;\beta_N\}>0$. Lemma 6 of [13] ensures that Theorem 1 can be applied with $\sigma^2=\kappa$ and $\Lambda=\frac{\kappa}{2}$. Hence there exists $\tilde{T}$ such that for $T\ge\tilde{T}$

$$\begin{aligned}P_2&\le Le^{3\kappa T}n\max_j\mathbf{P}\left(\sup_{x,y\in\mathcal{X}_j}\sup_{0\le s\le T}|\varphi_s(x)-\varphi_s(y)|>1\right)\\&\le Le^{3\kappa T}n\exp\left(-\left(\frac{1}{2\kappa}\left(3\kappa-\frac{\kappa}{2}\right)^2+\frac{\kappa}{16}\right)T\right)=Ln\exp\left(-\frac{\kappa}{16}T\right)\to 0,\end{aligned}$$

for $T\to\infty$, which completes the proof. □

The next Lemma tells that it is sufficient to analyze the Lipschitz behaviour of the time-scaled trajectories to get rid of the infimum over all Lipschitz functions.

LEMMA 5.4. *For any $\varepsilon > 0$, $\mathcal{X} \subseteq \mathbf{R}^2$ and any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ we have*

$$\left\{ \sup_{x\in\mathcal{X}} \inf_{f\in \mathrm{Lip}_0(K)} \max_i \left| \frac{1}{T}\varphi_{t_i T}(x) - f(t_i) \right| > \frac{\varepsilon}{3} \right\} \subseteq S_1 \cup S_2,$$

where

$$S_1 := \left\{ \sup_{x\in\mathcal{X}} \max_i \left| \frac{1}{T}\varphi_{t_i T}(x) - \frac{1}{T}\varphi_{t_{i+1} T}(x) \right| > (t_{i+1} - t_i)\left(K + \frac{\varepsilon}{3}\right) \right\}$$

and

$$S_2 := \left\{ \sup_{x\in\mathcal{X}} \max_i \left| \frac{1}{T}\varphi_{t_i T}(x) \right| > t_i \left(K + \frac{\varepsilon}{3}\right) \right\}.$$

PROOF. Let $x \in \mathcal{X}$. Then

$$\max_i \left| \frac{1}{T}\varphi_{t_i T}(x) - \frac{1}{T}\varphi_{t_{i+1} T}(x) \right| \le (t_{i+1} - t_i)\left(K + \frac{\varepsilon}{3}\right)$$

and

$$\max_i \left| \frac{1}{T}\varphi_{t_i T}(x) \right| \le t_i \left(K + \frac{\varepsilon}{3}\right)$$

imply that the function f_x defined by

$$f_x(0) = 0 \qquad \text{and} \qquad f_x(t_i) := \frac{1}{T}\varphi_{t_i T}(x) \frac{K}{\left(K + \frac{\varepsilon}{3}\right)}, \quad i \in \{1, \ldots, n\}$$

and linear interpolation for $t \in (t_i, t_{i+1})$ is Lipschitz continuous with Lipschitz constant K hence $f_x \in \mathrm{Lip}_0(K)$. Further for T large we have

$$\max_i \left| \frac{1}{T}\varphi_{t_i T}(x) - f_x(t_i) \right| \le \frac{\varepsilon}{3},$$

which completes the proof by taking complements and unifying over all $x \in \mathcal{X}$. □

Finally we provide the proof of Theorem 5.1.

PROOF OF THEOREM 5.1. For any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ with

$$\Delta t := \max_i \{t_{i+1} - t_i\} \leq \min\left\{\frac{\varepsilon}{3(K + \frac{\varepsilon}{3})}; \frac{\varepsilon^2}{108\kappa}\right\}$$

by triangle inequality and according to Lemma 5.4 we have

$$\begin{aligned}\mathbf{P}\left(\sup_{g \in F_T(\mathcal{X})} d\left(g, \mathrm{Lip}_0(K)\right) > \varepsilon\right) &= \mathbf{P}\left(\sup_{x \in \mathcal{X}} \inf_{f \in \mathrm{Lip}_0(K)} \left\| \frac{1}{T}\varphi_{\cdot T}(x) - f \right\|_\infty > \varepsilon\right) \\ &\leq P_1 + P_2 + P_3,\end{aligned}$$

where

$$P_1 := \mathbf{P}\left(\sup_{x \in \mathcal{X}} \max_i \left| \frac{1}{T}\varphi_{t_i T}(x) - \frac{1}{T}\varphi_{t_{i+1} T}(x) \right| > (t_{i+1} - t_i) K \left(1 + \frac{\varepsilon}{3}\right)\right)$$

and

$$P_2 := \mathbf{P}\left(\sup_{x \in \mathcal{X}} \max_i \left| \frac{1}{T}\varphi_{t_i T}(x) \right| > t_i K \left(1 + \frac{\varepsilon}{3}\right)\right)$$

and

$$P_3 := \mathbf{P}\left(\sup_{x \in \mathcal{X}} \max_i \sup_{t_i \leq t \leq t_{i+1}} \left| \frac{1}{T}\varphi_{t_i T}(x) - \frac{1}{T}\varphi_{tT}(x) \right| > \frac{\varepsilon}{3}\right).$$

According to Lemma 5.3 since $\Delta t \leq \frac{\varepsilon^2}{108\kappa}$ we immediately get $P_3 \to 0$. According to (5) we have

$$P_2 \leq \sum_{i=1}^n \mathbf{P}\left(\varphi_{t_i T}(\mathcal{X}) \not\subseteq t_i T \left(1 + \frac{\varepsilon}{3}\right) \mathcal{B}\right) \to 0,$$

where $\mathcal{B}$ denotes the unit ball with respect to the stable norm. For the convergence of P_1 it hence suffices to show that for all $i \in \{1, \ldots, n\}$

$$\mathbf{P}\left(\sup_{x \in \mathcal{X}} \left| \frac{1}{T}\varphi_{t_i T}(x) - \frac{1}{T}\varphi_{t_{i+1} T}(x) \right| > (t_{i+1} - t_i) K \left(1 + \frac{\varepsilon}{3}\right) \;\middle|\; \varphi_{t_i T}(\mathcal{X}) \subseteq t_i T (1 + \varepsilon) \mathcal{B}\right)$$

converges to 0 for $T \to \infty$. Let $\tilde{\varepsilon} \leq \frac{K(1+\varepsilon/6)\tilde{\Delta}t\varepsilon}{18(4+\varepsilon/3)}$, where $\tilde{\Delta}t := \min_i\{t_{i+1} - t_i\}$, then there exists for fixed $i \in \{1, \ldots, n\}$ an integer $N \in \mathbf{N}$ and $v_1, \ldots, v_N \in t_i(1+\varepsilon)\mathcal{B}$ such that

$$t_i T(1+\varepsilon)\mathcal{B} \subseteq \bigcup_{j=1}^{N} B_{\tilde{\varepsilon}T}(v_j T).$$

Hence we get using isotropy of the flow

$$\begin{aligned}
&\mathbf{P}\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{T}\varphi_{t_iT}(x) - \frac{1}{T}\varphi_{t_{i+1}T}(x)\right| > (t_{i+1}-t_i)K\left(1+\frac{\varepsilon}{3}\right)\ \middle|\right. \\
&\qquad\qquad \left.\varphi_{t_iT}(\mathcal{X}) \subseteq t_iT(1+\varepsilon)\mathcal{B}\right) \\
&\leq N \cdot \mathbf{P}\left(\sup_{x\in B_{\tilde{\varepsilon}T}(0)}\left|\frac{1}{T}x - \frac{1}{T}\varphi_{(t_{i+1}-t_i)T}(x)\right| > (t_{i+1}-t_i)K\left(1+\frac{\varepsilon}{3}\right)\right) \\
&\to 0
\end{aligned}$$

for $T \to \infty$ according to Lemma 5.2. Thus the assertion is proved.

□

5.2. *Lower Bound.* This section is devoted to the proof of the lower bound of Theorem 3.1, i.e. the following Theorem.

THEOREM 5.2. *For any $\varepsilon > 0$ and $\mathcal{X} \in \mathcal{C}_R$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{f\in \mathrm{Lip}_0(K)} d(f, F_T(\mathcal{X})) > \varepsilon\right) = 0,$$

where K is the Euclidean radius of the stable norm unit ball (see Section 4).

The proof of Theorem 5.2 is divided into several steps. Since the Lipschitz functions are compact with respect to the supremum norm the problem can be reduced to a finite set of Lipschitz functions (see proof of Theorem 5.2). The main idea is then to show that for any given Lipschitz function there exists a point in the initial set such that the image of this point under the action of the flow approximates the Lipschitz function on a discrete grid (Lemma 5.5). Further Lemma 5.3 tells that between two supporting points, if chosen sufficiently close, the trajectories move not too fast.

LEMMA 5.5. *For any $\varepsilon > 0$, $f \in \mathrm{Lip}_0(K-\varepsilon)$, $\mathcal{X} \in \mathcal{C}_R$ and any partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ we have*

$$\lim_{T\to\infty} \mathbf{P}\left(\inf_{x\in\mathcal{X}} \max_i \left|\frac{1}{T}\varphi_{t_iT}(x) - f(t_i)\right| \le \varepsilon\right) = 1.$$

PROOF. Consider the following sequence of random subsets of $\mathbf{R}^2$

$$\begin{aligned}\mathcal{X}_0^{(T)} &:= \mathcal{X},\\ \mathcal{X}_i^{(T)} &:= \varphi_{t_{i-1}T,t_iT}\left(\mathcal{X}_{i-1}^{(T)}\right) \cap B_{T^{2/3}}(Tf(t_i))\end{aligned}$$

for $i = 1,\ldots,n$, which is the part of $\varphi_{t_iT}(\mathcal{X})$ that has been close (in linear scaling) to $Tf(t_j)$ for all $0 \le j \le i$. Further define the set (abbreviating $\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right)$ by τ_i^R)

$$\gamma_i^{(T)} := \varphi_{t_{i-1}T,t_{i-1}T+\tau_i^R}\left(\mathcal{X}_{i-1}^{(T)}\right) \cap B_{2R}(Tf(t_i)),$$

for $i = 1,\ldots,n$, which is the part of $\mathcal{X}_{i-1}^{(T)}$ that is at first in a $2R$-neighbourhood of $Tf(t_i)$. Observe that $\mathcal{X}_{i-1}^{(T)} \neq \emptyset$ implies that τ_i^R is almost surely finite. To simplify notations we will denote the largest (with respect to the diameter) connected component of $\mathcal{X}_i^{(T)}$ and $\gamma_i^{(T)}$ respectively by the same symbol. Let $A_i^{(T)}$ be the event that $\mathcal{X}_{i-1}^{(T)}$ reaches an R-neighbourhood of $Tf(t_i)$ in time, i.e.

$$A_i^{(T)} := \left\{\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right) \le (t_i - t_{i-1})T\right\}$$

for $i = 1,\ldots,n$, and $B_i^{(T)}$ the event that there exists a point in the first intersection of $\mathcal{X}_{i-1}^{(T)}$ with an R-neighbourhood of $Tf(t_i)$ that stays close (in linear scaling) to $Tf(t_i)$ up to time t_iT and $\mathcal{X}_{i-1}^{(T)}$ is large at time t_i, i.e. on

$$\left\{\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right) \le (t_i - t_{i-1})T\right\}$$

that is (abbreviating $\tau^R\left(\mathcal{X}_{i-1}^{(T)}, Tf(t_i), Tt_{i-1}\right)$ by τ_i^R)

$$B_i^{(T)} := \left\{\inf_{x\in\gamma_{i-1}^{(T)}} \sup_{t_{i-1}T+\tau_i^R \le t \le t_iT} \left|\varphi_{t_{i-1}T+\tau_i^R,t}(x) - Tf(t_i)\right| \le T^{2/3}\,;\right.$$
$$\left.\operatorname{diam}\left(\varphi_{t_{i-1}T,t_iT}\left(\mathcal{X}_{i-1}^{(T)}\right)\right) \ge 1\right\}.$$

Hence we get by construction: if there exists $x \in \mathcal{X}$ such that $\varphi_\cdot(x)$ reaches successively the R-neighbourhoods of $Tf(t_i)$ for all $i \in \{1,\dots,n\}$ in time (before time t_iT) and is still close to these points at time t_iT then the time-scaled trajectory $\frac{1}{T}\varphi_{\cdot T}(x)$ starting in this particular x is close to the Lipschitz function f at the time t_i for all $i \in \{0,\dots,n\}$, i.e.

$$\begin{aligned}(13)\quad \mathbf{P}\left(\inf_{x\in\mathcal{X}}\max_i\left|\frac{1}{T}\varphi_{t_iT}(x)-f(t_i)\right|\le\varepsilon\right) &\ge \mathbf{P}\left(\bigcap_{i=1}^n A_i^{(T)}\cap\bigcap_{i=1}^n B_i^{(T)}\right)\\ &= \mathbf{P}\left(A_1^{(T)}\right)\mathbf{P}\left(B_1^{(T)}\,\middle|\,A_1^{(T)}\right)\cdots\mathbf{P}\left(B_n^{(T)}\,\middle|\,\bigcap_{i=1}^n A_i^{(T)}\cap\bigcap_{i=1}^{n-1}B_i^{(T)}\right).\end{aligned}$$

Observe that the conditional distribution $\mathcal{L}\left(\tau^R\left(\mathcal{X}_{i-1}^{(T)},Tf(t_i),Tt_{i-1}\right)\middle|\mathcal{X}_{i-1}^{(T)}\right)$ coincides with the conditional distribution $\mathcal{L}\left(\tau^R\left(\mathcal{X}_{i-1}^{(T)},Tf(t_i)\right)\middle|\mathcal{X}_{i-1}^{(T)}\right)$ for $i\in\{1,\dots,n\}$ and hence the results from Section 4 are applicable.

For any $k\in\{1,\dots,n\}$ because of the Markov property (1) of the flow we have

$$\begin{aligned}(14)\quad &\mathbf{P}\left(A_k^{(T)}\,\middle|\,\bigcap_{i=1}^{k-1}A_i^{(T)}\cap\bigcap_{i=1}^{k-1}B_i^{(T)}\right)\\ &\quad=\mathbf{P}\left(\tau^R\left(\mathcal{X}_{k-1}^{(T)},Tf(t_k),Tt_{k-1}\right)\le(t_k-t_{k-1})T\,\middle|\,\bigcap_{i=1}^{k-1}A_i^{(T)}\cap\bigcap_{i=1}^{k-1}B_i^{(T)}\right)\\ &\quad\ge\inf_{\gamma\in\mathcal{C}_R}\inf_{v\in B_1(0)}\mathbf{P}\left(\tau^R\left(\gamma,T(f(t_k)-f(t_{k-1}))+vT^{2/3}\right)\le(t_k-t_{k-1})T\right)\\ &\quad\ge 1-\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R\left(\gamma,T(f(t_k)-f(t_{k-1}))\right)>(t_k-t_{k-1})\frac{T}{1+\varepsilon/K}\right)\\ &\qquad-\sup_{\gamma\in\mathcal{C}_R}\sup_{v\in B_1(0)}\mathbf{P}\left(\tau^R(\gamma,vT^{2/3})>(t_k-t_{k-1})\frac{\varepsilon}{1+\varepsilon/K}T\right).\end{aligned}$$

Because of the isotropy of the flow the last probability reduces to

$$\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R(\gamma,e_1T^{2/3})>(t_k-t_{k-1})\frac{\varepsilon}{1+\varepsilon/K}T\right)\to 0,$$

and converges to 0 according to Lemma 4.1. Since $f\in\mathrm{Lip}_0(K-\varepsilon)$ and $|v|=K\|v\|^R$ we have $\|f(t_k)-f(t_{k-1})\|^R\le(t_k-t_{k-1})\left(1-\frac{\varepsilon}{K}\right)$, which

implies because of Lemma 4.1

$$\begin{aligned}
&\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R\left(\gamma,T(f(t_k)-f(t_{k-1}))\right)>(t_k-t_{k-1})\frac{T}{1+\varepsilon/K}\right)\\
&\quad\le\sup_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\tau^R\big(\gamma,T(f(t_k)-f(t_{k-1}))\big)>\right.\\
&\qquad\qquad\left.\|f(t_k)-f(t_{k-1})\|^R\,\frac{1}{1-(\varepsilon/K)^2}T\right)\\
&\quad\to 0,
\end{aligned}$$

and hence convergence to 0 of the first probability in (14). On the other hand we get for $k\in\{1,\dots,n\}$ by fixing some $\tilde{x}_{k-1}\in\gamma^{(T)}_{k-1}$ for T large (abbreviating $\tau^R\left(\mathcal{X}^{(T)}_{k-1},Tf(t_k),Tt_{k-1}\right)$ by τ^R_k)

$$\begin{aligned}
(15)\quad &\mathbf{P}\left(B^{(T)}_k\,\middle|\,\bigcap_{i=1}^{k}A^{(T)}_i\cap\bigcap_{i=1}^{k-1}B^{(T)}_i\right)\\
&\quad\ge\mathbf{P}\left(\sup_{t_{k-1}T+\tau^R_k\le t\le t_kT}\left|\varphi_{t_{k-1}T+\tau^R_k,t}(\tilde{x}_{k-1})-Tf(t_k)\right|\le T^{2/3}\,\middle|\right.\\
&\qquad\qquad\left.\bigcap_{i=1}^{k}A^{(T)}_i\cap\bigcap_{i=1}^{k-1}B^{(T)}_i\right)\\
&\qquad+\mathbf{P}\left(\operatorname{diam}\left(\varphi_{t_{k-1}T,t_kT}\left(\mathcal{X}^{(T)}_{k-1}\right)\right)\ge 1\,\middle|\,\bigcap_{i=1}^{k}A^{(T)}_i\cap\bigcap_{i=1}^{k-1}B^{(T)}_i\right)-1.
\end{aligned}$$

Since the one point motions are Brownian the first term can be estimated for some $\delta\in(0,1)$ via (denoting by $W=\left(W^{(1)},W^{(2)}\right)$ a 2-dimensional Brownian motion)

$$\begin{aligned}
(16)\quad &\mathbf{P}\left(\sup_{t_{k-1}+\tau^R_k\le t\le t_k}\left|\varphi_{t_{k-1}T+\tau^R_k,tT}(\tilde{x}_{k-1})-Tf(t_k)\right|\le T^{2/3}\,\middle|\right.\\
&\qquad\qquad\left.\bigcap_{i=1}^{k}A^{(T)}_i\cap\bigcap_{i=1}^{k-1}B^{(T)}_i\right)\\
&\quad\ge\mathbf{P}\left(\sup_{0\le t\le t_k-t_{k-1}}|W_{tT}|\le(1-\delta)T^{2/3}\right)\\
&\quad\ge 1-8\cdot\mathbf{P}\left(W^{(1)}_1>\frac{(1-\delta)}{\sqrt{2(t_k-t_{k-1})}}T^{1/6}\right)\\
&\quad\to 1,
\end{aligned}$$

see [9, Problem II.8.2]. Further we have because of Lemma 4.2

$$\begin{aligned}\mathbf{P}\left(\operatorname{diam}\left(\varphi_{t_{k-1}T,t_kT}\left(\mathcal{X}_{k-1}^{(T)}\right)\right)\geq 1\,\middle|\,\bigcap_{i=1}^{k}A_i^{(T)}\cap\bigcap_{i=1}^{k-1}B_i^{(T)}\right)\\ \geq \inf_{\gamma\in\mathcal{C}_R}\mathbf{P}\left(\operatorname{diam}\left(\varphi_{0,(t_k-t_{k-1})T}(\gamma)\right)\geq 1\right)\\ \to 1.\end{aligned}$$

This together with (16) yields convergence of (15) to 1. Combining (14) and (15) via (13) implies the assertion. □

Finally we provide the proof of Theorem 5.2.

Proof of Theorem 5.2. Because of compactness of the Lipschitz functions with respect to the supremum norm we can reduce the problem to a finite set of Lipschitz functions as follows. Since $\mathrm{Lip}_0(K-\frac{\varepsilon}{4})$ is compact with respect to $\|\cdot\|_\infty$ there exists some $N\in\mathbf{N}$ and $f_1,\dots,f_N\in\mathrm{Lip}_0(K-\frac{\varepsilon}{4})$ such that for any $g\in\mathrm{Lip}_0(K-\frac{\varepsilon}{4})$ there exists $j\in\{1,\dots,N\}$ with

$$\|g-f_j\|_\infty\leq\frac{\varepsilon}{4}.$$

If $f\in\mathrm{Lip}_0(K)$ then $\frac{K-\frac{\varepsilon}{4}}{K}f\in\mathrm{Lip}_0(K-\frac{\varepsilon}{4})$ and hence for any $f\in\mathrm{Lip}_0(K)$ because of $\|f\|_\infty\leq K$ there exists $j\in\{1,\dots,N\}$ such that

$$\|f-f_j\|_\infty\leq\left\|f-\frac{K-\frac{\varepsilon}{4}}{K}f\right\|_\infty+\left\|\frac{K-\frac{\varepsilon}{4}}{K}f-f_j\right\|_\infty\leq\frac{\varepsilon}{2}.$$

Thus we get

$$\begin{aligned}(17)\qquad &\mathbf{P}\left(\sup_{f\in\mathrm{Lip}_0(K)}\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f\right\|_\infty>\varepsilon\right)\\ &\quad=\mathbf{P}\left(\max_j\sup_{\substack{f\in\mathrm{Lip}_0(K)\\ |f-f_j|\leq\frac{\varepsilon}{2}}}\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f\right\|_\infty>\varepsilon\right)\\ &\quad\leq\sum_{j=1}^{N}\mathbf{P}\left(\inf_{x\in\mathcal{X}}\left\|\frac{1}{T}\varphi_{0,\cdot T}(x)-f_j\right\|_\infty>\frac{\varepsilon}{2}\right).\end{aligned}$$

Now choose a partition $0=t_0<t_1<\ldots<t_n=1$ of $[0,1]$ with $\Delta t:=\max_i\{t_{i+1}-t_i\}\leq\min\left\{\frac{\varepsilon^2}{768\kappa};\frac{\varepsilon}{8K}\right\}$, where $\kappa:=\max\{\beta_L;\beta_N\}$. Using the

triangle inequality we get for any $f \in \mathrm{Lip}_0(K - \frac{\varepsilon}{4})$ since

$$\max_i \sup_{t_i \le t \le t_{i+1}} |f(t_i) - f(t)| \le \left(K - \frac{\varepsilon}{4}\right) \Delta t \le \frac{\varepsilon}{8}$$

the estimate

$$\begin{aligned} (18) \quad & \mathbf{P}\left(\inf_{x\in\mathcal{X}} \left\| \frac{1}{T}\varphi_{0,\cdot T}(x) - f_i \right\|_\infty > \frac{\varepsilon}{2}\right) \\ & \le \mathbf{P}\left(\inf_{x\in\mathcal{X}} \max_i \left| \frac{1}{T}\varphi_{0,t_iT}(x) - f(t_i) \right| > \frac{\varepsilon}{4}\right) \\ & \quad + \mathbf{P}\left(\sup_{x\in\mathcal{X}} \max_i \sup_{t_i\le t\le t_{i+1}} \left| \frac{1}{T}\varphi_{0,t_iT}(x) - \frac{1}{T}\varphi_{0,tT}(x) \right| > \frac{\varepsilon}{8}\right). \end{aligned}$$

Because of Lemma 5.5 the first term in (18) converges to 0 for $T \to \infty$ and since $\Delta t \le \frac{\varepsilon^2}{768\kappa}$ Lemma 5.3 yields convergence of the second term to 0. Hence combining (17) and (18) proves the assertion.

□

5.3. *Proof of Theorem 3.1.*

Proof of Theorem 3.1. By definition of the Hausdorff distance it is sufficient to show

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{g\in F_T(\mathcal{X})} d\left(g, \mathrm{Lip}_0(K)\right) > \varepsilon\right) = 0 \tag{19}$$

and

$$\lim_{T\to\infty} \mathbf{P}\left(\sup_{f\in \mathrm{Lip}_0(K)} d(f, F_T(\mathcal{X})) > \varepsilon\right) = 0. \tag{20}$$

For $\mathcal{X} \in \mathcal{C}_R$ equation (19) is proved in Section 5.1, namely Theorem 5.1, whereas (20) is proved in Section 5.2, namely Theorem 5.2. For any non-trivial compact connected $\mathcal{X} \subseteq \mathbf{R}^2$ we need to construct a scaled flow on a diffusively scaled space such that the diameter of $\mathcal{X}$ becomes large and the results of Theorem 5.1 and Theorem 5.2 are applicable.

Let $r := \mathrm{diam}(\mathcal{X}) > 0$. Define the scaled space $\tilde{\mathbf{R}}^2 := \{\frac{x}{r} : x \in \mathbf{R}^2\}$ equipped with the usual Euclidean metric and consider the function

$$\tilde{\varphi} : \mathbf{R}_+ \times \mathbf{R}_+ \times \tilde{\mathbf{R}}^2 \times \Omega \to \tilde{\mathbf{R}}^2; \qquad \tilde{\varphi}_{s,t}(\tilde{x}, \omega) := \frac{1}{r}\varphi_{r^2 s, r^2 t}(r\tilde{x}, \omega).$$

Since φ is an IBF on $\mathbf{R}^2$ we have that $\tilde{\varphi}$ is also an IBF on $\tilde{\mathbf{R}}^2$ with generating isotropic Brownian field $\tilde{M}(t,\tilde{x}) = \frac{1}{r}M(r^2t, r\tilde{x})$ for $t \geq 0$, $\tilde{x} \in \tilde{\mathbf{R}}^2$ and covariance tensor $\tilde{b}(\tilde{x}) = b(r\tilde{x})$ for $\tilde{x} \in \tilde{\mathbf{R}}^2$ and thus has the same properties as φ, in particular the top-Lyapunov exponent of $\tilde{\varphi}$ is strictly positive. By construction of $\tilde{\mathbf{R}}^2$ the initial set $\frac{1}{r}\mathcal{X}$ has diameter 1 seen as a subset of $\tilde{\mathbf{R}}^2$. Denote the time-scaled trajectories of $\tilde{\varphi}$ by

$$\tilde{F}_T(\mathcal{X},\omega) := \bigcup_{x\in\frac{1}{r}\mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T}\tilde{\varphi}_{0,tT}(x,\omega) \right\}.$$

One can easily deduce from (4) using the definition of $\tilde{\varphi}$ that the Euclidean radius of the unit ball of the stable norm defined via $\tilde{\varphi}$ in $\tilde{\mathbf{R}}^2$ is $\tilde{K} = rK$. Thus it follows from (19) and (20) applied to $\tilde{\varphi}$ that

$$\lim_{T\to\infty} \mathbf{P}\left(d_H(\tilde{F}_T(\mathcal{X}), \mathrm{Lip}_0(\tilde{K})) > \varepsilon \right) = 0.$$

By definition of $\tilde{F}_T(\mathcal{X})$ one sees that this convergence also holds for the set $r^2\tilde{F}_{T/r^2}(\mathcal{X})$ and thus by definition of $\tilde{\varphi}$

$$F_T(\mathcal{X}) = r\tilde{F}_{T/r^2}(\mathcal{X}) \to \frac{1}{r}\,\mathrm{Lip}_0(\tilde{K}) = \mathrm{Lip}_0(K),$$

where convergence is meant in the Hausdorff distance in probability. This proves the assertion for any non-trivial compact connected set $\mathcal{X} \subseteq \mathbf{R}^2$. □

APPENDIX A: AN ESTIMATE ON THE COVARIANCE FUNCTION

One of the general assumptions for stochastic flows is a Lipschitz property of the derivative of the quadratic variation of the difference $M(t,x)-M(t,y)$, where M denotes the generating martingale field of the flow. In case of IBFs this property is achieved by an estimate of the second derivative of the covariance functions. The following proof is due to M. Scheutzow.

LEMMA A.1. *let φ be an IBF with generating isotropic Brownian field M. The function $\mathcal{A}(t,x,y) := \frac{\mathrm{d}}{\mathrm{d}t}\langle M(\cdot,x) - M(\cdot,y)\rangle_t$ satisfies for all $t \geq 0$, $x, y \in \mathbf{R}^2$*

$$\|\mathcal{A}(t,x,y)\| \leq \max\{\beta_L;\beta_N\}\,|x-y|^2,$$

where β_L and β_N are as in Section 2.1 and $\|\cdot\|$ denotes the spectral norm on $\mathbf{R}^{2\times 2}$.

PROOF. Observe that by definition of the covariance tensor we have

$$\mathcal{A}(t,x,y) = 2(b(0) - b(x-y)).$$

According to [15, Lemma 1.6] x is an eigenvector of $b(x)$ to the eigenvalue $B_L(|x|)$ and any vector $x^\perp \neq 0$ perpendicular to x is an eigenvector of $b(x)$ to the eigenvalue $B_N(|x|)$. Since the matrix $\mathcal{A}(t,x,y)$ is symmetric we have

$$\begin{aligned} \|\mathcal{A}(t,x,y)\| &= \|2(b(0) - b(x-y))\| \\ &= 2\max\{1 - B_L(|x-y|); 1 - B_N(|x-y|)\}. \end{aligned} \tag{21}$$

Now consider an $\mathbf{R}^2$-valued centered Gaussian process $U(x)$, $x \in \mathbf{R}^2$, with covariances $\mathbf{E}\left[U_i(x)U_j(y)\right] = b_{ij}(x-y)$ for $i,j \in \{1,2\}$. Then by stationarity and Schwartz' inequality we have for $r \geq 0$

$$\begin{aligned} B''(r) &= \lim_{h\to 0}\lim_{\delta\to 0} \mathbf{E}\left[\frac{U_1(he_1) - U_1(0)}{h}\frac{U_1(-(r+\delta)e_1) - U_1(-re_1)}{\delta}\right] \\ &= -\mathbf{E}\left[U_1'(re_1)U_1'(0)\right] \geq -\mathbf{E}\left[U_1'(0)^2\right] = B_L''(0). \end{aligned}$$

By Taylor expansion [3, Section 2] for each $r > 0$ there exists some $\theta \in (0,r)$ such that

$$B_L(r) = B_L(0) + \frac{1}{2}B_L''(\theta)r^2 \geq 1 + \frac{1}{2}B_L''(0)r^2 = 1 - \frac{\beta_L}{2}r^2.$$

The estimate on B_N follows in the same way, so from (21) we get

$$\|\mathcal{A}(t,x,y)\| \leq \max\{\beta_L;\beta_N\}\,|x-y|^2\,.$$

□

ACKNOWLEDGEMENT

The present research was supported by the International Research Training Group *Stochastic Models of Complex Processes* funded by the German Research Council (DFG). The author gratefully thanks Michael Scheutzow and Holger van Bargen from TU Berlin for fruitful discussions.

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Technische Universität Berlin
Institut für Mathematik, MA 7-4
Strasse des 17. Juni 136
10623 Berlin
Germany
E-mail: biskamp@math.tu-berlin.de