Mathematical model of Lame Problem for Simplified Elastic Theory applied to Controlled-Clearance Pressure Balances.

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Abstract: This paper is based on the original work of the master degree thesis [1] and also represents a revision of the models and the correlations with the analytical solutions given by other authors in the previous publications from 2003 to 2007. All the publications from 2003 to 2007 about simplified analytical methods applied to controlled-clearance pressure balances are not original re-elaborations (with some errors) of the thesis[1].
The analysis described in this paper starts with the mathematical model of thick-walled cylinder based on the solution of the Lame Equations applied to Mechanical theory of elastic equilibrium [5] for the formulation of the so called *Simplified Elastic Theory* that represents an analytical approach used in the study of the pressure balances. This analysis is well known as *Lame problem*. The solution of the Lame problem is used to determine the pressure distortion coefficient λ of controlled-clearance pressure balances. The analysis in this paper includes the case of pressure balances with compound cylinder used for high pressures measurements and completes the literature of the simplified analytical methods applied to the pressures balances. Two new formulas are developed by the author (eqs. 25-31) to determine directly the jacket pressure distortion coefficient n_j of controlled-clearance pressure balances in the case of single material cylinder and of compound cylinder (two materials).

1. Introduction

As mentioned in the abstract, the previous publications about simplified elastic theory are in the following:

- [2] is the first publication based on the thesis [1]: in this work are reported (extracted from [1]) the analytical demonstrations and the two formulas obtained by equations 23) and 28) for the evaluation of the pressure distortion coefficient of controlled clearance pressure balances with cylinder made of a single and two materials respectively.
- also in [3] there are the two equations mentioned above and extracted from [1] : the section 5.3 (from page 94 to page 100) titled "The simplified method based on Lame equation", containing the two formulas (eqs. 23 and 28) given in [1] to evaluate the pressure distortion coefficient of controlled clearance pressure balances.

As described by Dadson et al. [4], one of the most important developments of the recent history of the precise measurement of high pressures, is the introduction of the 'controlled-clearance system to the construction of the pressure balances assemblies and its application in the field of high pressures.

One of the most important problem to improve the performances of the pressure balances especially in the field of high pressure measurements is the accurate calculation of the pressure distortion coefficient λ.

In this regard, the approaches are classified in theoretical and experimental; a theoretical method described in this paper is *the simplified elastic theory*; for this method the literature gives two formulas based on the simplified elastic theory [6](Johnson and Newall 1953; Tsiklis 1968) for evaluating the coefficient λ and are applicable only in the case of piston-cylinder assemblies with simply and reentrant geometry. As demonstrated in [1] the same analysis and the same hypotheses used for simply and reentrant geometry are made to determine the formulas of the pressure distortion coefficients for controlled-clearance pressures balances.

The analysis of this paper is summarized in the following steps:

I) differential formulation of the *Lame Problem*: mathematical model of *Lame Theory* applied to the elastic theory (Mechanical theory of elastic equilibrium) described in section 2;

II) model chosen: most important to schematize the assembly with geometrical, material and loads conditions; a wrong model as used in [2,3] gives erroneous formulas for λ (section 3-4);

III) boundary conditions: for the solution of *Lame Problem* to determine the radial displacements of the particular piston-cylinder assembly analysed (section 3.1-4.1);

IV) evaluation of the pressure distortion coefficient for controlled-clearance assemblies on the basis of the hypotheses given in [6] and described in the section 3.2-4.2.

2. Lame Problem

Among the theoretical methods, the simplified elastic theory is an analytical approach to evaluate the pressure distortion coefficient by the solution of a system of differential equations named *Lame equations*. The results of this analysis lead to the determination of the stresses and distortions for thick walled-cylinder. The method is formally known as *Lame Problem* and the differential formulation of this problem is based on the analysis of the equilibrium of an elementary volume [5] as shown in fig.1.

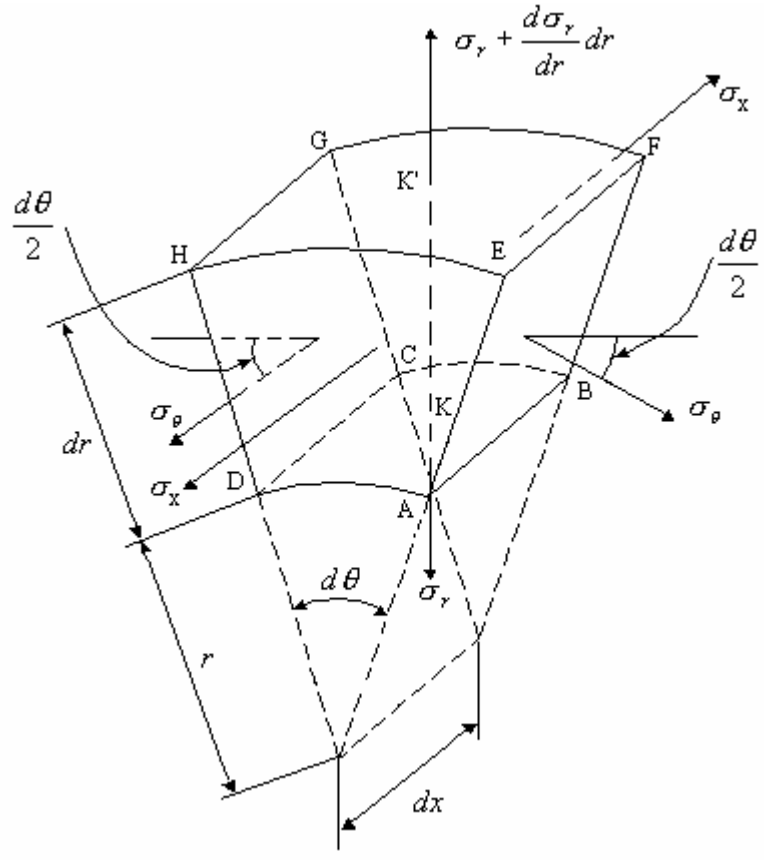


Fig. 1 The state of stress in the axialsymmetric cylindrical structures according to the simplified elastic theory (extracted from [1]).

With cylindrical coordinates *x, y, r*, the inner radius of the elementary volume is *r* and the external radius is *r+dr* with *dr* assumed as infinitesimal. The volume is transversely delimited by the surfaces *ADHE* , *BCGF*, by the surfaces *ABFE* , *CDHG* on two planes rotated of an infinitesimal angle $d\vartheta$ and by the surfaces *ABCD*, *EFGH* defined by the inner radius *r* and outer radius *r+dr* respectively.

With the following notation:

- σ_θ: circumferential stress;
- σ_r: radial stress applied on the surface *ABCD*;

- $\sigma_r + \frac{d\sigma_r}{dr} dr$: radial stress applied on the surface *EFGH*;
- $r \cdot d\vartheta \cdot dx$ and $(r + dr) \cdot d\vartheta \cdot dx$: internal surface *ABCD* and the external surface *EFGH* of the elementary volume respectively;
- $dr \cdot dx$: surface *ABEF* and *CDGH*
- $\sigma_r \cdot r \cdot d\vartheta \cdot dx$: force applied to the internal surface *ABCD* of the elementary volume;
- $\left(\sigma_r + \frac{d\sigma_r}{dr} dr\right) \cdot (r + dr) \cdot d\vartheta \cdot dx$: force applied to external surface *EFGH* of the elementary volume;
- $\sigma_\theta \cdot sen\frac{d\theta}{2} \cong \sigma_\theta \cdot \frac{d\theta}{2}$: vertical component (parallel to the *r* direction KK') of the circumferential stress;
- $\sigma_\theta \cdot \frac{d\theta}{2} \cdot drdx$: vertical component of the force acting on the surfaces *ABFE* and *CDHG*.
- *E* and ν: Young's Modulus and Poisson's ratio respectively;

the formulation of the *Lame Problem* is based on the following hypotheses:

i) axial-symmetry: there is no displacement in θ direction but only displacements *u* in *r* direction;

ii) radial displacements *u* only function of the coordinate *r*:

$$u(r, x, \theta) = u(r)$$

iii) shear stresses on the elementary volume must be zero: $\tau_{xr} = \tau_{x\theta} = \tau_{r\theta} = 0$

iv) by axial-symmetry and constant thickness, the radial and circumferential stresses are function only of the coordinate *r*:

$$\begin{cases} \sigma_r(r, x, \theta) = \sigma_r(r) \\ \sigma_\theta(r, x, \theta) = \sigma_\theta(r) \end{cases}$$

v) longitudinal stresses σ_x acting on transversal sections independent from the coordinates *x*, *r*:

$$\frac{\partial \sigma_x}{\partial r} = \frac{\partial \sigma_x}{\partial x} = 0$$

vi) axial symmetrical loads: because nothing varies in the θ direction,

vii) the problem is solved by linear elastic approach.

The radial force equilibrium yields:

$$\left(\sigma_r + \frac{d\sigma_r}{dr}dr\right)\cdot(r+dr)d\theta\cdot dx - \sigma_r\cdot rd\theta\cdot dx - 2\sigma_\theta dxdr\sin\left(\frac{d\theta}{2}\right)\cong$$

$$\cong\left(\sigma_r + \frac{d\sigma_r}{dr}dr\right)\cdot(r+dr)d\theta\cdot dx - \sigma_r\cdot rd\theta\cdot dx - 2\sigma_\theta dxdr\frac{d\theta}{2} =$$

$$= \sigma_r rd\theta\cdot dx + \sigma_r dr\cdot d\theta\cdot dx + \frac{d\sigma_r}{dr}dr^2 d\theta\cdot dx + \frac{d\sigma_r}{dr}rdrd\theta\cdot dx - \sigma_r rd\theta\cdot dx - \sigma_\theta drd\theta dx =$$

$$= \sigma_r dr\cdot d\theta\cdot dx + \frac{d\sigma_r}{dr}dr^2 d\theta\cdot dx + \frac{d\sigma_r}{dr}rdrd\theta\cdot dx - \sigma_\theta drd\theta dx = 0$$

ignoring highest infinitesimal order term $\frac{d\sigma_r}{dr}dr^2 d\theta\cdot dx$ and with opportune simplifications, we get:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad 1)$$

In order to solve the problem there is one equation with two variables σ_r, σ_θ. Applying Strain-Displacement Equations we can determine the second equation:

$$\begin{aligned}\varepsilon_r &= \frac{du}{dr}\\ \varepsilon_\theta &= \frac{2\pi\cdot(r+u)-2\pi r}{2\pi r} = \frac{u}{r}\end{aligned} \qquad 2)$$

From Constitutive Equations given from Hooke's generalized law we get:

$$\begin{cases}\varepsilon_r = \frac{1}{E}\cdot[\sigma_r - \nu\cdot(\sigma_x + \sigma_\theta)]\\ \varepsilon_\theta = \frac{1}{E}\cdot[\sigma_\theta - \nu\cdot(\sigma_r + \sigma_x)]\end{cases} \qquad 3)$$

from the first equation of 3):

$$\sigma_r = E\cdot\varepsilon_r + \nu\cdot(\sigma_x + \sigma_\theta) \qquad 4)$$

Substituting 4) in second equation of 3):

$$\varepsilon_\theta = \frac{1}{E}\cdot[\sigma_\theta - \nu\cdot E\varepsilon_r - \nu^2\cdot\sigma_x - \nu^2\cdot\sigma_\theta - \nu\cdot\sigma_x] = \frac{1}{E}\cdot[\sigma_\theta\cdot(1-\nu^2) - \nu\sigma_x\cdot(1+\nu) - \nu E\cdot\varepsilon_r]$$

$$E\varepsilon_\theta = \sigma_\theta\cdot(1-\nu^2) - \nu\sigma_x\cdot(1+\nu) - \nu E\cdot\varepsilon_r \quad \Rightarrow \quad E\varepsilon_\theta + \nu\sigma_x\cdot(1+\nu) + \nu E\cdot\varepsilon_r = \sigma_\theta\cdot(1-\nu^2)$$

$$\sigma_\theta = \frac{1}{1-\nu^2}\cdot[E\varepsilon_\theta + \nu\sigma_x\cdot(1+\nu) + \nu E\varepsilon_r]$$

and finally:

$$\sigma_\theta = \frac{E}{1-\nu^2}\cdot\left(\varepsilon_\theta + \nu\varepsilon_r\right) + \frac{\nu\sigma_x}{1-\nu} \qquad 5)$$

Substituting eq. 5) in eq. 4):

$$\sigma_r = E\varepsilon_r + \nu\sigma_x + \frac{\nu E}{1-\nu^2}\cdot\left(\varepsilon_\theta + \nu\varepsilon_r\right) + \frac{\nu^2\sigma_x}{1-\nu} = \frac{E\varepsilon_r\left(1-\nu^2\right) + \nu E\cdot\left(\varepsilon_\theta + \nu\varepsilon_r\right)}{1-\nu^2} + \frac{\nu\sigma_x(1-\nu) + \nu^2\sigma_x}{1-\nu} =$$

$$= \frac{E\varepsilon_r - E\varepsilon_r\nu^2 + \nu E\varepsilon_\theta + E\varepsilon_r\nu^2}{1-\nu^2} + \frac{\nu\sigma_x - \nu^2\sigma_x + \nu^2\sigma_x}{1-\nu}$$

and finally:

$$\sigma_r = \frac{E}{1-\nu^2}\cdot\left(\varepsilon_r + \nu\varepsilon_\theta\right) + \frac{\nu\sigma_x}{1-\nu} \qquad 6)$$

Substituting Strain-Displacement Equations 2) in eq. 5 and eq. 6) we get the following stress-displacement equations:

$$\begin{cases} \sigma_\theta = \dfrac{E}{1-\nu^2}\cdot\left(\dfrac{u}{r} + \nu\dfrac{du}{dr}\right) + \dfrac{\nu\sigma_x}{1-\nu} \\ \sigma_r = \dfrac{E}{1-\nu^2}\cdot\left(\dfrac{du}{dr} + \nu\dfrac{u}{r}\right) + \dfrac{\nu\sigma_x}{1-\nu} \end{cases} \qquad 7)$$

Substituting eqs. 7) in eq. 1):

$$\frac{E}{1-\nu^2}\cdot\frac{d}{dr}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) + \frac{E}{1-\nu^2}\cdot\frac{1}{r}\left(\frac{du}{dr} + \nu\frac{u}{r} - \frac{u}{r} - \nu\frac{du}{dr}\right) = 0$$

$$\left(\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu\frac{u}{r^2}\right) + \left(\frac{1}{r}\frac{du}{dr} + \nu\frac{u}{r^2} - \frac{u}{r^2} - \frac{\nu}{r}\frac{du}{dr}\right) = 0$$

$$\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu\frac{u}{r^2} + \frac{1}{r}\frac{du}{dr} + \nu\frac{u}{r^2} - \frac{u}{r^2} - \frac{\nu}{r}\frac{du}{dr} = 0$$

opportune simplifications lead to differential equation :

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

and in a compact form:

$$\frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) = 0 \qquad 8)$$

Eq. 8) can be written also as shown:

$$\frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) = \frac{d}{dr}\left[\frac{1}{r}\left(r\cdot\frac{du}{dr} + u\right)\right] = \frac{d}{dr}\left[\frac{1}{r}\left(r\cdot\frac{du}{dr} + \frac{dr}{dr}\cdot u\right)\right] = 0$$

and finally we get:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(u\cdot r)\right] = 0 \qquad 9)$$

that is a compact form of eq. 8). The integration of eq. 9) yields:

$$\frac{1}{r}\frac{d}{dr}(u \cdot r) = C_0$$

$$u \cdot r = C_0 \cdot \frac{r^2}{2} + C_2$$

$$u = C_0 \cdot \frac{r}{2} + \frac{C_2}{r}$$

and assuming: $\frac{C_0}{2} = C_1$, the equations above give the radial displacement as a function of radius:

$$u = C_1 \cdot r + \frac{C_2}{r} \quad \text{defined for } r \neq 0 \qquad 10)$$

Substituting eq. 10) in stress-displacement equations 7) we can determine the circumferential and the radial stresses at any point through the thickness of the wall. With geometrical and loads conditions, opportune boundary conditions are necessary to determine the integration constants C_1 and C_2.

3. Controlled clearance pressure balances with a single material cylinder

For controlled clearance pressure balance the solution of *Lame problem* is based on the following model for the assembly (fig.3):

I. thick-walled cylinder with open ends of inner radius r_c and outer radius R_c subjected to the internal pressure p and external jacket pressure p_j (fig.2). Where p is the pressure distribution in the clearance.

II. no end-loading on cylinder: $\sigma_x = 0$ [6]

III. piston subjected to longitudinal stress $\sigma_x = \cos t = -P$ and external pressure p. Where P is the pressure measurement applied on the piston base.

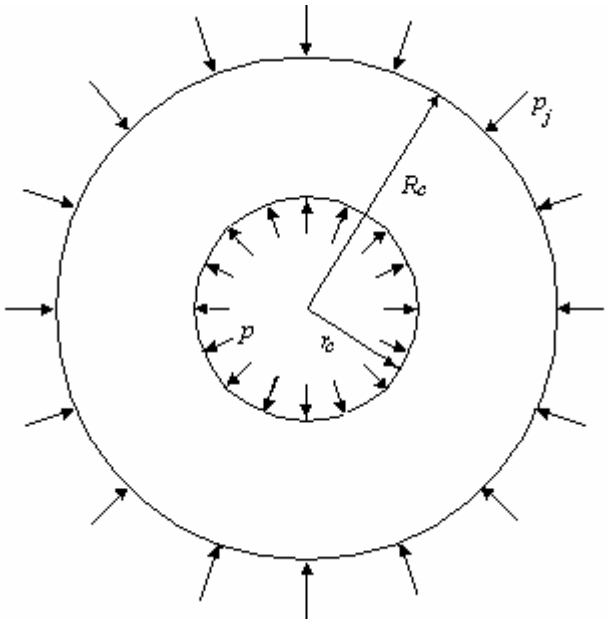


Fig.2 Transversal section of the cylinder of a controlled-clearance pressure balance.

From the eq. 10) we can obtain the parametric strains:

$$\begin{aligned}\frac{du}{dr} &= C_1 - \frac{C_2}{r^2} = \varepsilon_r \\ \frac{u}{r} &= C_1 + \frac{C_2}{r^2} = \varepsilon_\theta\end{aligned} \qquad 11)$$

Substituting eq. 11) in stress-displacement equations 7) yields:

$$\begin{cases}\sigma_r(r) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 - \dfrac{C_2}{r^2} + \nu_c \cdot \left(C_1 + \dfrac{C_2}{r^2} \right) \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c} \\ \sigma_\theta(r) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 + \dfrac{C_2}{r^2} + \nu_c \cdot \left(C_1 - \dfrac{C_2}{r^2} \right) \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c}\end{cases}$$

by opportune simplifications we get the following equations:

$$\begin{cases}\sigma_r(r) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 \cdot (1+\nu_c) - C_2 \cdot (1-\nu_c) \dfrac{1}{r^2} \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c} \\ \sigma_\theta(r) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 \cdot (1+\nu_c) + C_2 \cdot (1-\nu_c) \dfrac{1}{r^2} \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c}\end{cases} \qquad 12)$$

and for the analysis of the cylinder: $r_c \le r \le R_c$.

Where E_c and ν_c are the Young modulus and Poisson ratio of the cylinder respectively.

3.1 Boundary conditions

The following boundary conditions are necessary to determine the integration constants C_1 and C_2:

$$\begin{cases}\sigma_r(R_c) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 \cdot (1+\nu_c) - C_2 \cdot (1-\nu_c) \dfrac{1}{R_c^{\,2}} \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c} = -p_j \\ \sigma_r(r_c) = \dfrac{E_c}{1-\nu_c^{\,2}} \cdot \left[C_1 \cdot (1+\nu_c) - C_2 \cdot (1-\nu_c) \dfrac{1}{r_c^{\,2}} \right] + \dfrac{\nu_c \sigma_x}{1-\nu_c} = -p\end{cases} \qquad 13)$$

The solution of the system above is carried out starting from following difference:

$$\sigma_r(R_c) - \sigma_r(r_c) = \frac{E_c}{1-\nu_c^{\,2}} \cdot \left[-C_2 \cdot (1-\nu_c) \frac{1}{R_c^{\,2}} + C_2 \cdot (1-\nu_c) \frac{1}{r_c^{\,2}} \right] = -p_j + p$$

$$-C_2 \cdot (1-\nu_c) \frac{1}{R_c^{\,2}} + C_2 \cdot (1-\nu_c) \frac{1}{r_c^{\,2}} = (p - p_j) \cdot \left(\frac{1-\nu_c^{\,2}}{E_c} \right)$$

$$C_2 \cdot \left[\frac{\left(R_c^{\,2} - r_c^{\,2} \right) \cdot (1-\nu_c)}{r_c^{\,2} \cdot R_c^{\,2}} \right] = (p - p_j) \cdot \left(\frac{1-\nu_c^{\,2}}{E_c} \right)$$

and we can obtain the first integration constant:

$$C_2 = \left(p - p_j\right) \cdot \left(\frac{1-\nu_c{}^2}{E_c}\right) \cdot \left[\frac{r_c{}^2 \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right) \cdot \left(1-\nu_c\right)}\right] \qquad 14)$$

Substituting eq. 14) in the second eq. in 13) gives:

$$\frac{E_c}{1-\nu_c{}^2} \cdot \left[C_1 \cdot \left(1+\nu_c\right) - \frac{\left(p - p_j\right) \cdot \left(1-\nu_c{}^2\right) \cdot r_c{}^2 \cdot R_c{}^2}{E_c \cdot \left(R_c{}^2 - r_c{}^2\right) \cdot \left(1-\nu_c\right)} \cdot \left(1-\nu_c\right) \frac{1}{r_c{}^2}\right] = -p - \frac{\nu_c \sigma_x}{1-\nu_c}$$

$$\frac{E_c}{1-\nu_c{}^2} \cdot \left[C_1 \cdot \left(1+\nu_c\right) - \frac{\left(p - p_j\right) \cdot \left(1-\nu_c{}^2\right) \cdot R_c{}^2}{E_c \cdot \left(R_c{}^2 - r_c{}^2\right)}\right] = -p - \frac{\nu_c \sigma_x}{1-\nu_c}$$

$$\frac{C_1 \cdot \left(1+\nu_c\right) \cdot E_c}{1-\nu_c{}^2} - \frac{\left(p - p_j\right) \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)} = -p - \frac{\nu_c \sigma_x}{1-\nu_c}$$

$$\frac{C_1 \cdot E_c}{1-\nu_c} = \frac{\left(p - p_j\right) \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)} - p - \frac{\nu_c \sigma_x}{1-\nu_c}$$

$$C_1 = \left[\frac{\left(p - p_j\right) \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)}\right] \cdot \frac{1-\nu_c}{E_c} - p \cdot \left(\frac{1-\nu_c}{E_c}\right) - \frac{\nu_c \sigma_x}{1-\nu_c} \cdot \left(\frac{1-\nu_c}{E_c}\right) = \left[\frac{\left(p - p_j\right) \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)} - p\right] \cdot \frac{1-\nu_c}{E_c} - \frac{\nu_c \sigma_x}{E_c}$$

$$C_1 = \left[\frac{\left(p - p_j\right) \cdot R_c{}^2 - p \cdot \left(R_c{}^2 - r_c{}^2\right)}{\left(R_c{}^2 - r_c{}^2\right)}\right] \cdot \frac{1-\nu_c}{E_c} - \frac{\nu_c \sigma_x}{E_c} = \left[\frac{p \cdot R_c{}^2 - p_j \cdot R_c{}^2 - p \cdot R_c{}^2 + p \cdot r_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)}\right] \cdot \frac{1-\nu_c}{E_c} - \frac{\nu_c \sigma_x}{E_c}$$

and the other integration constant is:

$$C_1 = \left[\frac{p \cdot r_c{}^2 - p_j \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)}\right] \cdot \frac{1-\nu_c}{E_c} - \frac{\nu_c \sigma_x}{E_c} \qquad 15)$$

Substituting eqs. 14) and 15) in eq. 10):

$$U(r) = \left[\frac{p \cdot r_c{}^2 - p_j \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right)} \cdot \frac{1-\nu_c}{E_c} - \frac{\nu_c \sigma_x}{E_c}\right] r + \left(\frac{p - p_j}{r}\right) \cdot \left(\frac{1-\nu_c{}^2}{E_c}\right) \cdot \left[\frac{r_c{}^2 \cdot R_c{}^2}{\left(R_c{}^2 - r_c{}^2\right) \cdot \left(1-\nu_c\right)}\right]$$

we can obtain the radial displacement of the cylinder as a function of radius:

$$U(r) = \frac{1-\nu_c}{E_c}\left(\frac{p r_c^2 - p_j R_c^2}{R_c^2 - r_c^2}\right) r + \frac{1+\nu_c}{E_c}\left(\frac{r_c^2 R_c^2}{r}\right) \frac{p - p_j}{R_c^2 - r_c^2} - \frac{\nu_c}{E_c} \sigma_x r \qquad 16)$$

The hypothesis of no end-loading on cylinder ($\sigma_x = 0$) yields:

$$U(r) = \frac{1-\nu_c}{E_c}\left(\frac{p r_c^2 - p_j R_c^2}{R_c^2 - r_c^2}\right) r + \frac{1+\nu_c}{E_c}\left(\frac{r_c^2 R_c^2}{r}\right) \frac{p - p_j}{R_c^2 - r_c^2} \qquad 17)$$

the eq. 17) is used to determine the piston cylinder radial displacements on the internal surface:

$$U(r_c) = \frac{1-\nu_c}{E_c}\left(\frac{pr_c^2 - p_j R_c^2}{R_c^2 - r_c^2}\right) r_c + \frac{1+\nu_c}{E_c}\left(\frac{r_c^2 R_c^2}{r_c}\right)\frac{p - p_j}{R_c^2 - r_c^2} =$$

$$= \frac{r_c}{E_c}\left[\frac{(1-\nu_c)\cdot\left(pr_c^2 - p_j R_c^2\right) + (1+\nu_c)\cdot(p - p_j)R_c^2}{R_c^2 - r_c^2}\right] =$$

$$\frac{r_c}{E_c}\left[\frac{pr_c^2 - p\nu_c r_c^2 - p_j R_c^2 + p_j \nu_c R_c^2 + pR_c^2 - p_j R_c^2 + p\nu_c R_c^2 - p_j \nu_c R_c^2}{R_c^2 - r_c^2}\right]$$

$$= \frac{r_c}{E_c}\left[\frac{p\left(R_c^2 + r_c^2\right) - 2p_j R_c^2 + \nu_c p\left(R_c^2 - r_c^2\right)}{R_c^2 - r_c^2}\right]$$

and finally:

$$U(r_c) = \frac{r_c p}{E_c}\left[\frac{R_c^2 + r_c^2 - 2\left(\frac{p_j}{p}\right)R_c^2}{R_c^2 - r_c^2} + \nu_c\right] \qquad 18)$$

Applying the eq. 16) to the piston and for the hypothesis III of the model (page 7) we get:

$$u(r) = -\left(\frac{1-\nu_p}{E_p}\right) p \cdot r - \frac{\nu_c}{E_c}\sigma_x r$$

and for $\sigma_x = -P$ we get the piston radial displacements as a function of radius:

$$u(r) = -\left(\frac{1-\nu_p}{E_p}\right) p \cdot r + \left(\frac{\nu_p \cdot P}{E_p}\right) r \qquad 19)$$

The eq. 19) is used to determine the piston radial displacement for $r = r_p$:

$$u(r_p) = \frac{r_p}{E_p}\left[(\nu_p - 1)p + \nu_p P\right] \qquad 20)$$

Where E_p and ν_p are the Young's Modulus and Poisson ratio of the piston respectively.

3.2 Pressure distortion coefficient

The pressure distortion coefficient of a controlled-clearance pressure balance is calculated on the basis of the solution of the *Lame Problem* for this model given by the equations 18) and 20). As mentioned above the calculation is performed with the same hypotheses formulated [6] for the case of simple configuration of the assembly:

i) constant gap between the piston and the cylinder;

ii) the dimensions of the clearance are negligible; the radius of the piston is equal to the internal radius of the cylinder: $r_p \cong r_c$ and $h_0 \cong 0$; where h_0 is the undistorted gap between piston and cylinder;

iii) linear pressure profile in the clearance along the engagement piston-cylinder length L: the analysis is made assuming an average constant pressure profile with pressure value equal to *0,5 P*.

According to the hypothesis ii) page 4):

$$\frac{dU}{dx} = \frac{du}{dx} = 0 \;\Rightarrow \begin{cases} u(0) = u(L) \\ U(0) = U(L) \end{cases}$$

Remembering the pressure distortion coefficient given in [6]:

$$\lambda = \frac{1}{1+\frac{h_0}{r_p}} \cdot \left[\frac{u(0)+U(0)}{P \cdot r_p} + \frac{1}{P^2 \cdot r_p} \int_0^l p(x) \cdot \left(\frac{dU}{dx} + \frac{du}{dx} \right) dx \right]$$

applying the hypothesis ii) we get:

$$\begin{cases} \lambda = \dfrac{1}{1+\frac{h_0}{r_p}} \cdot \left(\dfrac{u(0)+U(0)}{P \cdot r_p} \right) \cong \dfrac{u(0)+U(0)}{P \cdot r_c} \\ h_0 \cong 0; \quad r_c \cong r_p \end{cases} \qquad 21)$$

substituting $p = \frac{P}{2}$ in eqs. 18) and 20), the radial displacements of the cylinder and the piston are:

$$\begin{cases} U(0) = U(l) = \dfrac{r_c P}{2E_c} \left[\dfrac{\left(R_c^{\,2} + r_c^{\,2}\right) - \frac{4p_j}{P}}{R_c^{\,2} - r_c^{\,2}} + \nu_c \right] = \dfrac{r_c P}{2E_c} \left[\dfrac{r_c^2 + R_c^{\,2} - 4t \cdot R_c^2}{R_c^{\,2} - r_c^{\,2}} + \nu_c \right] \\ u(0) = u(l) = \dfrac{r_p}{E_p} \left[\nu_p \dfrac{P}{2} - \dfrac{P}{2} + \nu_p P \right] \cong \dfrac{r_c P}{2E_p} \left(3\nu_p - 1 \right) \end{cases} \qquad 22)$$

where $t = \frac{p_j}{P}$. Substituting eq. 22) in th eq. 21) we get:

$$\lambda = \frac{1}{P \cdot r_c} \cdot \frac{r_c P}{2E_p} \left(3\nu_p - 1 \right) + \frac{1}{P \cdot r_c} \cdot \frac{r_c P}{2E_c} \left(\frac{r_c^2 + R_c^{\,2} - 4t \cdot R_c^2}{R_c^{\,2} - r_c^{\,2}} + \nu_c \right)$$

and with opportune simplifications the pressure distortion coefficient for controlled-clearance pressure balance is the following formula given in [1](Saragosa Silvano, 2001):

$$\lambda = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_c}\left(\frac{R_c^{\,2} + r_c^2 - 4t \cdot R_c^2}{R_c^{\,2} - r_c^{\,2}} + \nu_c\right) \qquad 23)$$

As a particular case of eq. 23) we assume $p_j = 0 \Rightarrow t = 0$ and is carried out the pressure distortion coefficient for pressure balance in free-deformation mode:

$$\lambda = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_c}\left(\frac{R_c^{\,2} + r_c^2}{R_c^{\,2} - r_c^{\,2}} + \nu_c\right) \qquad 24)$$

the equation above is the formula of the pressure distortion coefficient for a simple configuration pressure balance derived by Johnson e Newall 1953; Tsiklis, 1968 as mentioned in [6].

The eq. 23) for the controlled-clearance pressure balance is a general case of the equation 24) given in the literature.

In the original formula represented by eq. 23) *t* was successively changed by other authors [2][3] in $\chi = \frac{p_j}{\bar{p}}$ with $\bar{p} = \frac{P}{2}$ and is carried out the formula (Buonanno et al., 2003):

$$\lambda = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_c}\left(\frac{R_c^{\,2} + r_c^2 - 2\chi \cdot R_c^2}{R_c^{\,2} - r_c^{\,2}} + \nu_c\right)$$

but this is the same formula given in [1] with different notations and gives the same numerical values.

From eqs. 23) and 24) is possible to derive the equation of jacket pressure distortion coefficient n_j for this controlled-clearance configuration; the standard method [7] [8] is based on the evaluation of λ_{FD} and λ_{CC} with eq. 23) and eq. 24). Then the coefficient n_j is given from the eq. $\lambda_{FD} - \lambda_{CC} = \frac{n_j}{s}$ (with $s = \frac{1}{t} = \frac{P}{p_j}$). From eqs. 23) and 24):

$$\lambda_{FD} - \lambda_{CC} = \frac{n_j}{s} = \frac{1}{2E_c}\left(\frac{4t \cdot R_c^2}{R_c^{\,2} - r_c^{\,2}}\right)$$

where *CC* and *FD* means controlled clearance and free deformation respectively. With the introduction of the ratio $\beta_c = \frac{r_c}{R_c}$ we get:

$$\lambda_{FD} - \lambda_{CC} = \frac{1}{E_c}\left(\frac{2t}{1 - \beta_c^{\,2}}\right) = \frac{n_j}{s}$$

$$n_j = \frac{s}{E_c}\left(\frac{2t}{1 - \beta_c^{\,2}}\right)$$

and finally a new equation (Saragosa Silvano, 2009):

$$n_j = \frac{2}{E_c\left(1 - \beta_c^{\,2}\right)} \qquad 25)$$

the equation 25) is used to determine directly the jacket pressure distortion coefficient. With equation 23) is carried out λ_{FD}:

$$\lambda_{FD} = \lambda_{CC} + \frac{n_j}{s} \qquad 26)$$

4. Controlled clearance pressure balances with cylinder compound

The technology based on axial symmetric structures with cylinder compound (two materials) is used for the construction of the pressure balances assemblies used in high pressures measurements.

The analysis of this configuration is based on the same hypotheses made for the case of single material cylinder (pag. 7):

I) thick-walled cylinder 1 with open ends of inner radius r_c and outer radius r_m subjected to the internal pressure p and external pressure p_m (fig.4a). Where p is the pressure distribution in the clearance between the piston and the cylinder. Thick-walled cylinder 2 with open ends of inner radius r_m and outer radius R_c subjected to the internal pressure p_m and external jacket pressure p_j (fig.4b).

II) no end-loading on cylinder compound: $\sigma_x = 0$

III) piston subjected to longitudinal stress $\sigma_x = \cos t = -P$ and external pressure p distribution.

Assuming the following notation in this section (fig.3):

- R_c: radius of the external surface of cylinder 2;
- r_c: radius of the internal surface of cylinder 1;
- p: pressure applied on the internal surface of cylinder 1: pressure distribution in the clearance between piston and cylinder.
- p_j: pressure applied on the external surface of cylinder 2: jacket pressure.
- r_m: radius of the external surface of cylinder 1 and of the internal surface of cylinder 2;
- E_1, ν_1: Young modulus and Poisson ratio for the material of the internal cylinder respectively;
- E_2, ν_2: Young modulus and Poisson ratio for the material of the external cylinder respectively;
- $U_1(r)$: radial elastic displacements of the internal cylinder.
- $U_2(r)$: radial elastic displacements of the external cylinder.
- p_m: interfacial pressure;
- P: pressure applied to the piston base: pressure measurement.

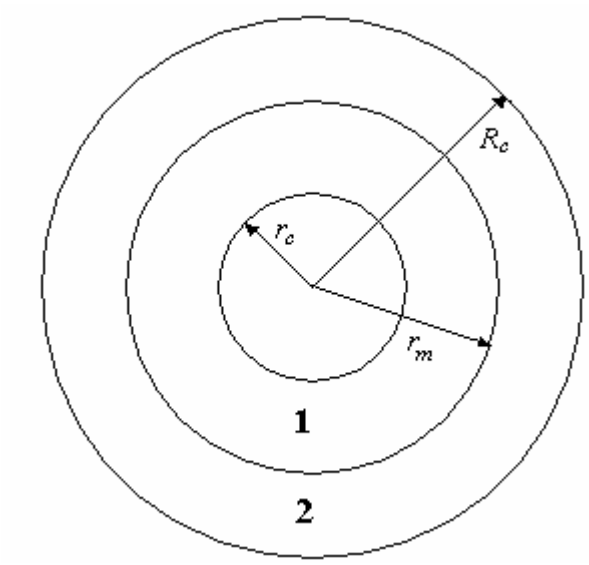


Fig. 3 Transversal section of the cylinder compound for a controlled-clearance pressure balance.

Even for the controlled-clearance pressure balances with cylinder compound, the calculation of the pressure distortion coefficient λ is based on the solution of the *Lame Problem*. In this theoretical analysis the literature gives two models:

- MODEL I (Saragosa, 2001).

Model without the external cylinder (fig.4a): the effect of the cylinder 2 is replaced by the pressure p_m acting on external surface of the internal cylinder (cylinder 1) and the assembly can be schematized as a controlled-clearance assembly with cylinder made of single material and jacket pressure p_m. The pressure distortion coefficient can be evaluated by the equation 23).

- MODEL II (Buonanno *et al*., 2003).

Without the internal cylinder 1 fig. 4b): obtained "removing the internal layer 1 an replacing it with the contact pressure, p_{med}, that it applies on the layer 2" as reported in [2] and in pag.99 of [3]. This is a wrong model for the solution of the *Lame Problem* because:

a) this is not a model of a pressure balance;
b) model II was erroneously correlated to the equation 28) by the authors of the previous publications [2,3]: wrong model for right formula; actually the analysis based on the model II gives another equation (completely different from eq.28) for the pressure distortion coefficient of the pressure balance with cylinder compound; the piston is not present and in this case in the eq. 21) the ratio $\frac{(3\nu_p - 1)}{2E_p}$ is not present.

   This model is valid only for the calculation of interfacial pressure p_m.

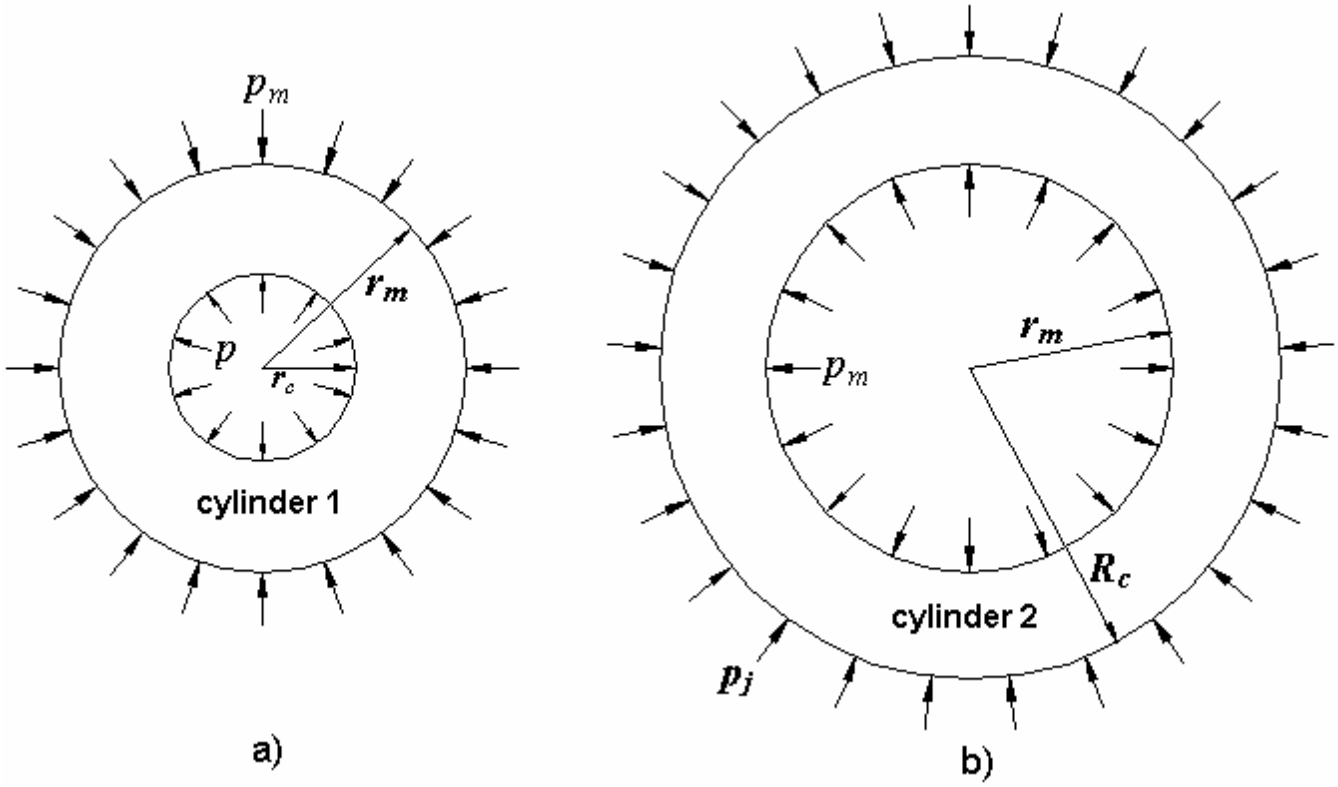


Fig. 4 Transversal sections of the cylinder for two hypothetical models.

On the basis of the above considerations this analysis will be performed using the model I.

4.1 Boundary conditions

In the case of controlled-clearance pressure balances with cylinder made of two material, the first step is the evaluation of the interfacial pressure p_m for $r = r_m$. In this case, the boundary conditions for internal cylinder is:

$$U_1(r_m) = \frac{1-\nu_1}{E_1}\left(\frac{pr_c^2 - p_m r_m^2}{r_m^2 - r_c^2}\right) r_m + \frac{1+\nu_1}{E_1}\left(\frac{r_c^2 r_m^2}{r_m}\right)\frac{p - p_m}{r_m^2 - r_c^2} =$$

$$= \frac{r_m}{E_1}\left(\frac{pr_c^2 - p_m r_m^2 - p\nu_1 r_c^2 + p_m \nu_1 r_m^2 + pr_c^2 - p_m r_c^2 + pr_c^2\nu_1 - p_m r_c^2 \nu_1}{E_1(r_m^{\,2} - r_c^2)}\right) =$$

$$= \frac{r_m}{E_1}\left[\frac{p_m r_m^2(\nu_1 - 1) - p_m r_c^2(1+\nu_1) + 2pr_c^2}{(r_m^{\,2} - r_c^2)}\right]$$

and for the external cylinder 2 is:

$$U_2(r_m)=\frac{r_m}{E_2}\left[\frac{(1-\nu_2)\cdot\left(p_m r_m^2-p_j {R_c}^2\right)+R_c^2(1+\nu_2)\cdot(p_m-p_j)}{R_c^2-r_m^2}\right]=$$

$$=\frac{r_m}{E_2}\left(\frac{p_m r_m^2-p_j R_c^2-\nu_2 p_m r_m^2+p_j R_c^2\nu_2+p_m R_c^2-p_j R_c^2+p_m R_c^2\nu_2-p_j R_c^2\nu_2}{(R_c^2-r_m^2)}\right)=$$

$$=\frac{r_m}{E_2}\left[\frac{p_m r_m^2(1-\nu_2)+p_m R_c^2(1+\nu_2)-2p_j R_c^2}{(R_c^2-r_m^2)}\right]$$

for $r=r_m$ the radial elastic displacements of the points of the internal cylinder are equal to radial elastic displacements of the points of the external cylinder:

$$U_1(r_m)=\frac{r_m}{E_1}\left[\frac{p_m r_m^2(\nu_1-1)-p_m r_c^2(1+\nu_1)+2pr_c^2}{({r_m}^2-r_c^2)}\right]=U_2(r_m)=\frac{r_m}{E_2}\left[\frac{p_m r_m^2(1-\nu_2)+p_m R_c^2(1+\nu_2)-2p_j R_c^2}{(R_c^2-r_m^2)}\right]$$

$$\frac{p_m\left[r_m^2(\nu_1-1)-r_c^2(1+\nu_1)\right]+2pr_c^2}{E_1(r_m^2-r_c^2)}=\frac{p_m\left[r_m^2(1-\nu_2)+R_c^2(1+\nu_2)\right]-2p_j R_c^2}{E_2(R_c^2-r_m^2)}$$

$$\frac{p_m\left[(\nu_1-1)-(1+\nu_1)\cdot\beta_1^2\right]+2p\beta_1^2}{E_1(1-\beta_1^2)}=\frac{p_m\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right]-2p_j}{E_2(1-\beta_2^2)}$$

with the introduction of the ratios $\beta_1^2=\frac{r_c^2}{r_m^2}$, $\beta_2^2=\frac{r_m^2}{R_c^2}$ and $n=\frac{p_j}{p}$:

$$\frac{2p\beta_1^2}{E_1(1-\beta_1^2)}+\frac{2p_j}{E_2(1-\beta_2^2)}=2p\left[\frac{\beta_1^2}{E_1(1-\beta_1^2)}+\frac{n}{E_2(1-\beta_2^2)}\right]=$$
$$=\frac{p_m\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right]}{E_2(1-\beta_2^2)}-\frac{p_m\left[(\nu_1-1)-(1+\nu_1)\cdot\beta_1^2\right]}{E_1(1-\beta_1^2)} \qquad 27)$$

and with:

$$A=\frac{[(\nu_1-1)-(1+\nu_1)\cdot\beta_1^2]}{E_1(1-\beta_1^2)};\quad B=\frac{[\beta_2^2(1-\nu_2)+(1+\nu_2)]}{E_2(1-\beta_2^2)};\quad C=\frac{[\beta_1^2\cdot E_2(1-\beta_2^2)+nE_1(1-\beta_1^2)]}{E_1E_2(1-\beta_1^2)\cdot(1-\beta_2^2)}$$

$$2pC=p_m(B-A)$$

the interfacial pressure is:

$$p_m=\frac{2pC}{B-A}$$

4.2 Pressure distortion coefficient

Substituting in the eq. 23) p_j with p_m $\left(t=\frac{p_m}{P}\right)$ and $p=\frac{P}{2}$ (hypothesis iii page 11) in the equation of the the interfacial pressure, the pressure distortion coefficient is given by the following equation:

$$\lambda = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_1}\left(\frac{r_m{}^2 + r_c^2 - 4\frac{p_m r_m^2}{P}}{r_m{}^2 - r_c{}^2} + \nu_1 \right) = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_1}\left(\frac{r_m{}^2 + r_c^2 - 4\frac{2\frac{P}{2}Cr_m^2}{P(B-A)}}{r_m{}^2 - r_c{}^2} + \nu_1 \right)$$

and finally the pressure distortion coefficient for a controlled-clearance pressure balance with cylinder compound is (Saragosa Silvano., 2001)[1]:

$$\lambda = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_1}\left(\frac{r_m{}^2 + r_c^2 - 4\frac{Cr_m^2}{(B-A)}}{r_m{}^2 - r_c{}^2} + \nu_1 \right) \qquad 28)$$

this equation was successively reported in [2] and [3] and used in [8](page 94). Starting from eq. 26:

$$2p\left[\frac{\beta_1^2}{E_1(1-\beta_1^2)} + \frac{n}{E_2(1-\beta_2^2)} \right] = \frac{p_m\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right]}{E_2(1-\beta_2^2)} + \frac{p_m\left[(1-\nu_1)+(1+\nu_1)\cdot\beta_1^2\right]}{E_1(1-\beta_1^2)} =$$

$$= p_m\left[\frac{\beta_2^2(1-\nu_2)+(1+\nu_2)}{E_2(1-\beta_2^2)} + \frac{(1-\nu_1)+(1+\nu_1)\cdot\beta_1^2}{E_1(1-\beta_1^2)} \right]$$

and by assuming:

$$a = \frac{\beta_1^2}{E_1(1-\beta_1^2)} + \frac{n}{E_2(1-\beta_2^2)}$$

$$b = \frac{\beta_2^2(1-\nu_2)+(1+\nu_2)}{E_2(1-\beta_2^2)} + \frac{\beta_1^2(1+\nu_1)+(1-\nu_1)}{E_1(1-\beta_1^2)}$$

is carried out the interfacial pressure:

$$p_m = \frac{2pa}{b}$$

For $p = \frac{P}{2}$:

$$p_m = \frac{2pa}{b} = \frac{2\cdot\frac{P}{2}\cdot a}{b} = \frac{P\cdot a}{b}$$

and finally with $t = \frac{p_m}{P} = \frac{P\cdot a}{bP} = \frac{a}{b}$:

$$\lambda_{CC} = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_1}\left(\frac{r_m{}^2 + r_c^2 - \frac{4ar_m^2}{b}}{r_m{}^2 - r_c{}^2} + \nu_1 \right) \qquad 29)$$

this is the same formula (more compact) given above in eq. 28). The case of free deformation mode derives from the equation 29) assuming $p_j = 0 \Rightarrow n = 0$:

$$f = \frac{\beta_1^2}{E_1(1-\beta_1^2)}$$

$$b = \frac{\beta_2^2(1-\nu_2)+(1+\nu_2)}{E_2(1-\beta_2^2)} + \frac{\cdot\,\beta_1^2(1+\nu_1)+(1-\nu_1)}{E_1(1-\beta_1^2)}$$

the pressure distortion coefficient for a pressure balance in free deformation mode with cylinder compound is:

$$\lambda_{FD} = \frac{(3\nu_p - 1)}{2E_p} + \frac{1}{2E_1}\left(\frac{r_m{}^2 + r_c^2 - \frac{4f \cdot r_m^2}{b}}{r_m{}^2 - r_c{}^2} + \nu_1 \right) \qquad 30)$$

From eqs. 28) and 30) is possible to derive the equation of jacket pressure distortion coefficient n_j for the controlled-clearance assemblies with cylinder compound:

$$\lambda_{FD} - \lambda_{CC} = \frac{1}{2E_1}\left[\frac{\frac{4nr_m^2}{E_2(1-\beta_2^2)}}{b(r_m{}^2 - r_c{}^2)} \right] = \frac{2n}{E_1E_2b(1-\beta_1^2)(1-\beta_2^2)} =$$

$$= \frac{2n}{E_1(1-\beta_1^2)\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right] + E_2(1-\beta_2^2)\left[\beta_1^2(1+\nu_1)+(1-\nu_1)\right]}$$

and finally for $p = \frac{P}{2}$:

$$\lambda_{FD} - \lambda_{CC} = \frac{2 \cdot \frac{2p_j}{P}}{E_1(1-\beta_1^2)\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right] + E_2(1-\beta_2^2)\left[\beta_1^2(1+\nu_1)+(1-\nu_1)\right]}$$

for $s = \frac{P}{p_j}$ we get the second new formula (Saragosa S. 2009):

$$n_j = s(\lambda_{FD} - \lambda_{CC}) = \frac{4}{E_1(1-\beta_1^2)\left[\beta_2^2(1-\nu_2)+(1+\nu_2)\right] + E_2(1-\beta_2^2)\left[\beta_1^2(1+\nu_1)+(1-\nu_1)\right]} \qquad 31)$$

5. Comparisons of calculations

Calculations made using the simplified analytical method are compared with experimental measurements and with results of Finite Elements analysis Methods (FEM) made at Physikalisch-Technische Bundesanstalt, Braunschweig, Germany (PTB), Bureau National de Métrologie, Laboratoire National d'Essais, Paris, France (BNM-LNE) for pressure balance BNM-LNE 200 Mpa and Istituto di Metrologia "G. Colonnetti", Consiglio Nazionale delle Ricerche, Turin, Italy (IMGC-CNR now INRIM), PTB, BNM-LNE and Ulusal Metroloji Enstitüsü, Gebze/Kocaeli, Turkey (UME) for pressure balance PTB 1GPa DH 7594.

The controlled-clearance pressure balance BNM-LNE 200 MPa (fig.5) characterized in Euromet project 256 with the two assemblies unit #4 and unit #5 [7] is described in table1.

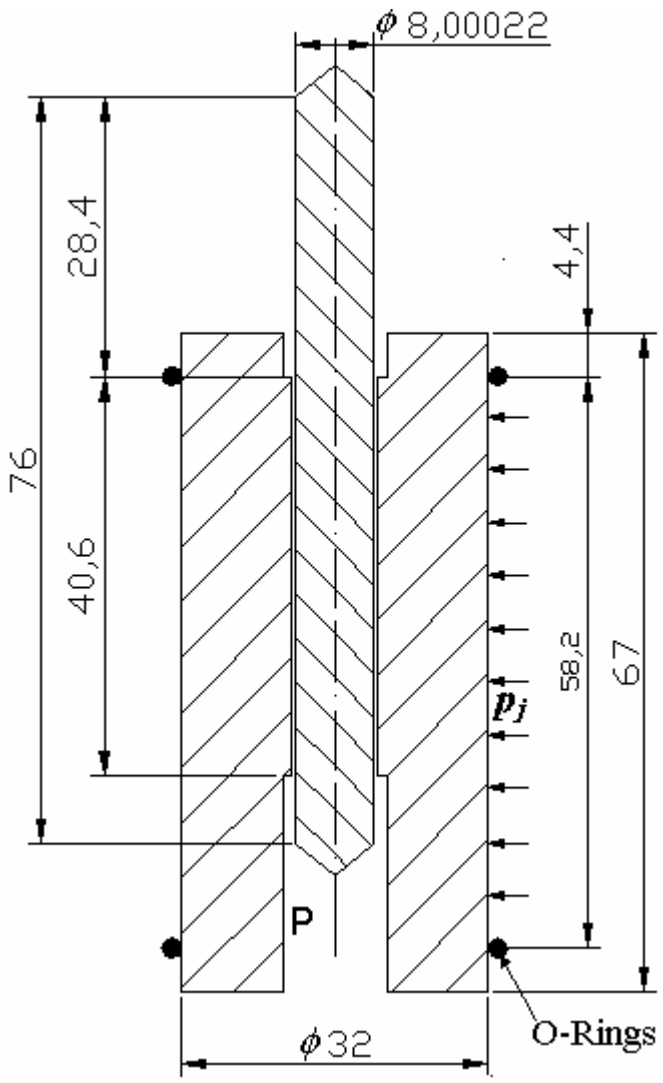


Fig.5 BNM-LNE 200 Mpa assembly unit #5 (made with Autocad with dimensional data extracted from [7]).

Table 1. Main characteristics of BNM-LNE 200 Mpa assembly unit #5; those for unit #4 are given in parentheses (extracted from [7]).

Property	**Value**
Geometry	Simple/Controlled clearance
Pressure scale (MPa)	From 6 to 200
Piston radius (mm)	4,00011 (3,99995)
Cylinder radius r_c (mm)	4,00036 (4,00052)
Cylinder radius R_c (mm)	16
Undistorted clearance(μm)	0,25 (0,57)
Material (piston and cylinder)	Tungsten carbide
Young modulus (MPa)	630000
Poisson ratio	0,218
Clearance length (mm)	40,6
p_j/P (controlled-clearance mode)	1/4

Assuming $t = \frac{p_j}{P} = \frac{1}{4}$ in the case of controlled clearance mode the pressure distortion coefficient for this assembly is carried out applying eq. 23): $\lambda_{cc} = -0{,}0487 \ \ p.p.m.\ MPa^{-1}$

In table 2 is reported [7] a comparison of calculated distortion coefficients for both units #4 and #5.

With $\beta_c = \frac{4}{16} = 0{,}25$, from eq. 25 we get the jacket pressure distortion coefficient:

$$n_j = \frac{2}{630000 \cdot \left(1 - 0{,}25^2\right)} = 3{,}39\ p.p.m.\ MPa^{-1}$$

Table 2 Comparisons of calculations (values in ** as new results of simple theory, all the others extracted from [7]).

P(MPa)	method	λ (*p.p.m.* Mpa^{-1})	n_j (*p.p.m.* Mpa^{-1})
BNM-LNE 200 MPa unit #5, controlled-clearance mode			
	Experimental	-0,02	3,56 (120 MPa) 3,52 (200 MPa)
	Lame	**-0,0487****	**3,39****
5	PTB	0,011	
40	PTB	0,097	
120	PTB	0,154	
	NPL	0,153	
200	PTB	0,166	
	NPL	0,165	
	PTB*	-0,043	
BNM-LNE 200 MPa unit #4, controlled-clearance mode			
	Experimental	-0,14	3,80 (120-200 MPa)
	Lame	-0,0487	3,39
120	PTB	0,093	
	NPL	0,083	
200	PTB	0,117	
	NPL	0,111	
	PTB*	-0,050	

*Whole cylinder line loaded

As shown in table 3 results of FEM analysis are larger than the experimental ones (<0) and than the coefficient determined by the simplified theory (<0). Only in the case of FEM analysis performed with appropriate boundary conditions the results are nearest the coefficient determined according to the simplified theory. For BNM-LNE 200 MPa simplified analytical method gave an indication to improve the FEM simulations.

The controlled-clearance pressure balance PTB 1GPa DH-7594 used in the Physikalisch-Technische Bundesanstalt (fig.6) was characterized in the Euromet project 463 and is described in table 3. For the FEM simulations four model were chosen [8]. One of these models (model 2) is analysed in this section to compare the calculations: model with an ideal undistorted gap profile between piston and cylinder assuming the elastic constants reported in table 3. This model is the nearest to the model chosen for the simplified elastic theory.

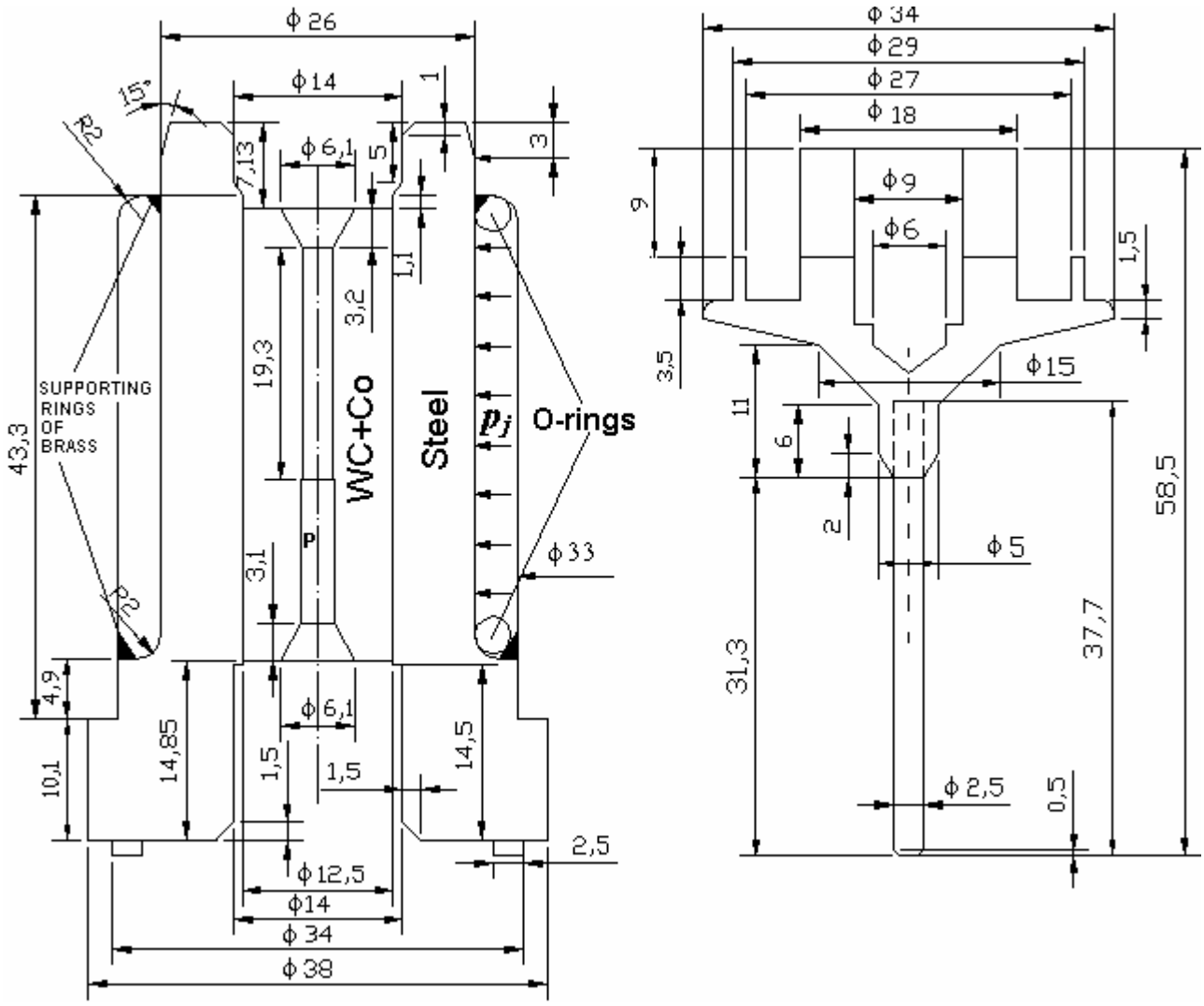


Fig.6 Pressure balance PTB 1GPa DH-7594 (made with Autocad with dimensional data extracted from [1]).

Table 3. Main characteristics of PTB 1GPa DH-7594 (extracted from [1]).

Property	**Value**
Geometry	Simple/Controlled clearance
Pressure scale (MPa)	Up to1000 (Controlled clearance mode)
Piston radius (mm)	1,24899
Cylinder radius r_c (mm)	1,24931
Cylinder radius r_m (mm)	6,25
Cylinder radius R_c (mm)	13
Undistorted clearance(μm)	0,32
Material piston and inner cylinder	Tungsten carbide+Co
Young modulus piston (MPa)	543000
Young modulus inner cylinder (MPa)	630000
Young modulus steel cylinder (MPa)	200000
Poisson ratio (steel)	0,29
Poisson ratio piston (carbide)	0,238
Poisson ratio cylinder (carbide)	0,22
Axial clearance length (mm)	19,3
p_i/P (controlled-clearance mode)	1/10

With $\frac{p_j}{P} = \frac{1}{10}$ [8] in the case of controlled clearance mode the pressure distortion coefficient for this assembly is carried out applying eq. 28 or eq. 29:

$$\lambda_{cc} = 0{,}354\ p.p.m.\ MPa^{-1}$$

The jacket pressure distortion coefficient is given by eq.31:

$$n_j = 3{,}973\, p.p.m.\ MPa^{-1}$$

In the case of free deformation mode, the eq. 30 or eq. 26 with $s = \frac{P}{p_j} = 10$ yields:

$$\lambda_{FD} = 0{,}751\, p.p.m.\ MPa^{-1}$$

Comparison with the other methods is reported in table 4.

Table 4. results of simple theory, FEM analysis (ideal undistorted gap) and experimental ones (extracted from [8]).

P(MPa)	method	λ_{CC} ($p.p.m.$ Mpa^{-1})	λ_{FD} ($p.p.m.$ Mpa^{-1})	n_j ($p.p.m.$ Mpa^{-1})
PTB 1GPa DH-7594				
	Experimental	0,437±0,012	0,776±0,011	3,39±0,17
	Lame	**0,354***	**0,751***	**3,973***
100	PTB	0,371	0,761	
250	PTB	0,365	0,761	
400	PTB	0,360	0,761	
	IMGC/Cassino	0,357	0,752	
	BNM-LNE	0,335	0,729	
	UME	0,370	0,776	
600	PTB	0,356	0,762	
800	PTB	0,354		
1000	PTB	0,353		
	IMGC/Cassino	0,325		
	BNM-LNE	0,332		
	UME	0,356		

* values obtained using respectively eqs. 28) 30) and 31) that agree with results reported in [8]

For the model used the application of simplified elastic theory gave results in good agreement with those of FEM analysis.

Of course in the case of controlled-clearance FEM simulations made for the model with real undistorted clearance gave mean pressure distortion coefficient values in good agreement with the experimental ones.

6. Conclusions

From the comparisons described above for two different configurations of controlled-clearance pressure balances we can obtain:

- comparison of the results of FEM simulations and simplified elastic theory in the case of regular undistorted gap and cylinder compound reveals a good agreement as reported also from the characterizations of other controlled-clearance pressure balances in Euromet Project 463.
- simple method in same cases gives indicative values of pressure distortion coefficients to optimize FEM simulation starting from pressure balance model with ideal cylindrical undistorted clearance profile [7].

The limits of the simplified elastic theory regardless FEM analysis are:

- pressure profile in the clearance between piston and cylinder is not linear as experimentally demonstrated by Welch et al. (1984) and by FEM simulations [6] [7] [8];
- the analysis based on the simplified elastic theory is made without taking account of the density and viscosity of liquid media in the clearance between piston and cylinder[6] but the iterative FEM analysis comprises the mechanical theory of elastic equilibrium and integration of the Navier-Stokes differential equations in the case of laminar flow;
- evaluation of the pressure distortion coefficient with simplified elastic theory is made assuming a constant clearance otherwise a real irregular undistorted gap profile. From recent researches [8] dimensions of the clearance were identified as a main uncertainty source for the evaluation of the pressure distortion coefficient. In fact FEM analysis performed by introducing in the model the real undistorted clearance showed a remarkable difference with the regular undistorted gap for calculations of the pressure distortion coefficient [8].

Actually FEM analysis is the most diffused theoretical method to characterize pressure balances. The results of FEM simulations can be used to design assemblies with good performances. But the simplified elastic theory represents a method to validate FEM calculations especially in the case of deviation of FEM calculations from experimental pressure distortion coefficients [7].

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