The Schreier continuum and ends

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Abstract

In this paper we introduce the notion of the Schreier continuum for minimal sets of foliated bundles and show how the Schreier continuum can be used to calculate the number of ends of leaves in certain exceptional minimal sets of foliations in such bundles. We find examples of exceptional minimal sets that have path connected components with a variable number ends.

1 Introduction

Ghys shows in his seminal work [17] that the leaves of almost all (with respect to an appropriate harmonic measure) points of a compact foliated space have either 0, 1, 2 or a Cantor set of ends. Inspired by this result, Cantwell and Conlon [9] obtained similar results in the topological setting. They show that for a complete foliated space with a totally recurrent leaf (in particular, for compact minimal sets), the leaves of all points of a residual subset either have no ends, one end, two ends or a Cantor set of ends. Recall (see, e.g., [7]) that a space X is a *foliated space* if it admits a foliated atlas with transversals in some locally compact, second countable metrisable space Z. In the classical case of a foliated manifold, the transversals are in $\mathbb{R}^q$, where q is the codimension of the foliation.

In this paper we analyse the structure of ends of leaves in a transverse foliation $\mathcal{F}$ in a foliated bundle $p : E \to B$ over a closed manifold B. A leaf L of such a foliation is a covering space of B, and so there is a natural way of associating the

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structure of ends of the leaf with a coset space of the fundamental group of B (see, e.g., Scott [30] and references therein). However, in many cases this coset space is impossible to analyse directly. In the setting of foliated spaces, when the number of ends may vary from leaf to leaf, the task of determining all possible behaviours of ends of leaves purely algebraically becomes overwhelmingly difficult.

In this paper we shall introduce the notion of the *Schreier continuum* which allows one to find the ends of leaves within a minimal set geometrically. The Schreier continuum associated to a minimal set can be thought of as a global holonomy graph for the minimal set. Its path components in the leaf topology give holonomy graphs for the corresponding leaves of the original foliation (see Candel and Conlon [8] for a discussion of holonomy graphs), but the global topology of the entire continuum proves crucial to our analysis and is not captured by the ensemble of all the individual graphs.

Our construction works especially well when the minimal set intersects a fibre of the bundle in a totally disconnected space. In this case we can construct an inverse limit representation of the Schreier continuum which allows us to obtain information about ends of leaves. We carry out this procedure for some particular examples of solenoids and limit sets of suspensions over surfaces.

Solenoids as considered here are natural generalisations of the well-known one-dimensional solenoids and have the structure of a foliated bundle over a closed manifold with a Cantor set fiber. One-dimensional solenoids are important in the study of flows and have been identified as minimal sets of smooth flows [4, 18, 19, 22, 32] and are known to occur generically in Hamiltonian flows on closed manifolds [24]. The more general solenoids introduced in [25] provide an important class of continua, i.e. compact connected topological spaces, that have been the object of great interest in the study of continua [15, 27, 29]. It was shown in [14] that the equicontinuous minimal sets of foliations having the local structure of a disk in a leaf times a Cantor set are solenoids, and in [13] examples are provided of certain classes of solenoids embedded as minimal sets of smooth foliations of manifolds. Thus, solenoids are an important class of limit sets for foliations and an understanding of their end structure is vital to understanding their asymptotic behaviour, and the Schreier continuum provides us with the key tool to do so.

In particular, we consider the solenoid constructed by Schori [29]. The Schori solenoid is a non-homogeneous space which fibres over a genus 2 surface. Using the method of the Schreier continuum we show that in the Schori solenoid the number of ends varies from leaf to leaf. More precisely, there are three types of leaves: a single leaf with 4 ends, an uncountable infinity of leaves with 2 ends and an uncountable infinity of leaves with 1 end. Our method allows us to compute directly that the set of points whose leaves have 2 ends is meagre, i.e. topologically 'almost all' leaves in the solenoid have 1 end. We show how the Schori example can be modified so as to obtain for any $n \in \mathbb{N}$ a solenoid with a single $2n$-ended path-connected component where all other leaves have either 1 or 2 ends, with the set of points with leaves with 1 end being residual.

We show that our construction also works for the case when the minimal foliated space has a totally disconnected fibre and a Markov structure. All leaves in such a minimal set have a Cantor set of ends, which agrees with the results of

Cantwell and Conlon [10, 11] for Markov minimal sets of codimension one foliations.

In detail, our paper is organised as follows. In Section 2 we introduce the notion of the Schreier continuum and describe the inverse limit representation which can be associated to the Schreier continuum in certain cases. Section 3 explains in detail how to construct such a representation for a generalised solenoid. Section 4) contains computations of the end structures for some specific solenoids. In Section 5 we describe the construction of the inverse limit representation for Markov minimal set of foliations of bundles with 1-dimensional fibre. Section 6 indicates possible directions of further work.

2 The Schreier continuum for minimal sets of foliated bundles

Here we consider foliated bundles over a connected manifold B. That is, we have a foliated space E, a fibre bundle projection $p : E \to B$, the bundle admits local trivialisations that coincide with the charts in a foliation atlas for $\mathcal{F}$, and the mappings induced by p on the leaves of $\mathcal{F}$ in the leaf topology are covering maps of B (see, e.g., Candel and Conlon [7, Example 2.1.5]). Notice that here we do not require that the fibres of p be manifolds. In particular, if E' is any saturated ($\mathcal{F}$ invariant) subset of E, then the restriction of p to E' is also a foliated bundle.

For the sake of notation, fix a basepoint $b \in B$, and let F be the fibre $p^{-1}(b)$. One of the important features of a foliated bundle is that it admits a *total holonomy* homomorphism

$$h : \pi_1(B, b) \to \mathrm{Homeo}(F)$$

determined by the lifting of loops in B based at b to paths contained in the leaves of $\mathcal{F}$. This allows one to translate many questions about $\mathcal{F}$ into questions involving subgroups of $\mathrm{Homeo}(F)$. Associated to each point $x \in F$ is the subgroup of $\pi_1(B, b)$ corresponding to the stabiliser of x for the induced action on F, which we will refer to as the *kernel* of x and denote $\mathcal{K}_x$, a vital component of our analysis. While the action on F can be realised as a right or left action and depends on how we represent the group operation in $\pi_1(B, b)$, the group $\mathcal{K}_x$ itself is independent of those choices. It is important to realise that the kernel can vary from point to point.

In his classic work Hopf [21] showed that a non-compact regular covering space of a compact polyhedron has either one (e.g. the plane), two (e.g. the line) or a Cantor set of ends (e.g. the universal cover of the figure eight). When the foliated bundle is principal (that is, when the bundle automorphisms act transitively on the fibres of p), then all the leaves of $\mathcal{F}$ are regular coverings of B and so Hopf's theorem applies. Moreover, in this case the bundle automorphisms yield a homeomorphism between any two leaves and so all leaves have homeomorphic end spaces, either 0 (compact), 1, 2 or a Cantor set. However, as we shall see, there are many interesting cases of non principal foliated bundles where the end structure can vary from leaf to leaf.

Remark 2.1 In the more general setting of foliated space which do not necessarily have a bundle structure, examples of foliated spaces with variable number of ends can be constructed using the method due to Kenyon [23]. Thus in [5] one finds an example of a minimal foliated space with foliation by Riemann surfaces where a single leaf has 4 ends and other leaves have at most 2 ends. Since there is a leaf with the number of ends greater than 2, the results by Cantwell and Conlon [9] and Blanc [1] implies that such a foliated space must have a residual set of points with leaves with 1-end. Based on the Kenyon method, Blanc [2] also constructs a minimal foliated space with a residual set of planar 1-ended leaves which contains an uncountable infinity of leaves with all possible end structures.

2.1 Ends of a leaf in a foliated bundle

To motivate the construction of the Schreier continuum we first make some observations about the structure of leaves of $\mathcal{F}$. For this, recall (see, e.g. [20]) that the universal cover of B is the set of homotopy classes

$$\widetilde{B} = \{[\delta] \mid \delta : ([0,1],0) \to (B,b)\}$$

of paths δ starting at our chosen base point b which is given the topology with the basis of open sets of form

$$U_{[\delta]} = \{[\delta \cdot \eta] \mid \eta \text{ is a path in } U \text{ starting at } \delta(1)\},$$

where U is a contractible open set in B and $[\delta] \in \widetilde{B}$. With $\widetilde{b}$ denoting the homotopy class of the constant path at b, the covering projection is defined by

$$f : (\widetilde{B}, \widetilde{b}) \to (B,b) : \ [\delta] \mapsto \delta(1).$$

Given a point $x \in F$ and a leaf L_x of $\mathcal{F}$ through x, denote by

$$p_{L_x} = p|_{L_x} : (L_x, x) \to (B,b)$$

the induced covering. Then f lifts to a uniquely determined covering map

$$\tilde{f} : (\widetilde{B}, \widetilde{b}) \to (L_x, x)$$

which satisfies $p_{L_x} \circ \tilde{f} = f$. The fundamental group $\pi_1(B,b)$ acts on the left on $\widetilde{B}$ by the concatenation of paths

$$[\gamma] \cdot [\delta] = [\gamma \cdot \delta], \text{ where } [\gamma] \in \pi_1(B,b),$$

and $\tilde{f}([\delta]) = \tilde{f}([\delta'])$ if and only if $[\delta'] = [\gamma] \cdot [\delta]$ for some $[\gamma] \in \mathcal{K}_x$. Thus we are led to the following observation.

Proposition 2.1 *For each $x \in F$ the leaf L_x is homeomorphic to the orbit space*

$$\widetilde{B}/\mathcal{K}_x$$

where the action of $\mathcal{K}_x$ is the left action induced as a subgroup of $\pi_1(B,b)$.

Then a direct application of the generalisation of Hopf's theorem on regular coverings to irregular coverings (see, e.g., Scott [30, Lemma 1.2]) yields the following result.

Theorem 2.1 *For each $x \in F$ the leaf L_x has the same number of ends as the pair of groups $(\pi_1(B,b), \mathcal{K}_x)$.*

The ends of the pair of groups $(\pi_1(B,b), \mathcal{K}_x)$ are defined in terms of group cohomology, but since we will only be calculating the ends geometrically, we refer the reader to [30] for precise definitions. In practice it can be very difficult calculate the ends of pairs of groups algebraically. The number of ends can be realised geometrically as the number of ends of the associated Schreier diagram, whose definition we recall here for completeness.

Recall (see, e.g. [3]) that given a group G generated by $S = \{s_1, \ldots, s_n\}$, the *Cayley graph* Γ is the graph whose vertices are the elements of G and for which there is an edge labelled by $s_i \in S$ joining g to g' if and only if $g' = g\, s_i$. Then G acts on the left of Γ in a natural way. Given a subgroup C of G the *Schreier diagram* of (G, C) is the orbit space Γ/C of the left action of the subgroup C on Γ. While the actual construction of this graph depends on the choice of generators S, the number of ends of the graph is independent of this choice (see, e.g. [3]). In fact, the number of ends of any such Schreier diagram is the same as the number of ends of the pair (G, C) (see, e.g., Scott [30, Lemma 1.1]).

2.2 The Schreier space of a foliated bundle

For the construction of the *Schreier space* of a foliated bundle, we restrict ourselves to the case where the base manifold B is closed. Then we fix a finite set of generators $S = \{[\gamma_1], \ldots, [\gamma_n]\}$ of $\pi_1(B,b)$ that are represented by loops γ_i in B that intersect pairwise only at the base point b. If the dimension of B is 2 we can choose the generators to be given in the standard way by the boundary edges of a polygon whose quotient is homeomorphic to B, and when the dimension of B is greater than 2 it is not hard to see how to construct these loops given any finite set of generators and local charts.

Definition 2.1 *For a given foliated bundle (F, E, B, p) with a closed manifold B as base and generators $S = \{[\gamma_1], \ldots, [\gamma_n]\}$ of $\pi_1(B,b)$ as above, the* Schreier space *$\mathcal{S}$ is the subspace of E formed by taking the union of all lifts of the paths $\{\gamma_1, \ldots, \gamma_n\}$ to paths in E starting at points of F.*

Then if Γ_0 denotes the union of all the images of the loops γ_i in B, we see that the Cayley graph Γ of the fundamental group $\pi_1(B,b)$ is a regular covering

$$p_\Gamma : (\Gamma, e) \to (\Gamma_0, b),$$

where we choose the vertex e corresponding to the identity element of $\pi_1(B,b)$ as the base point of Γ and the covering projection maps all vertices in Γ to b and each edge to the corresponding loop. In turn, for the leaf L_x of a given $x \in F$,

using the leaf topology of L_x for the subspace $L_x \cap \mathcal{S}$ we see that the restriction of p_{L_x} to $L_x \cap \mathcal{S}$, which we denote by the same symbol, is a covering map

$$p_{L_x} : (L_x \cap \mathcal{S}, x) \to (\Gamma_0, b).$$

Since the elements in $\pi_1(\Gamma, e)$ correspond to relations on the elements of the generating set S in $\pi_1(B, b)$ we see that the covering map p_Γ lifts, as before, to a uniquely determined covering map

$$q : (\Gamma, e) \to (L_x \cap \mathcal{S}, x)$$

such that $q \circ p_{L_x} = p_\Gamma$, and $L_x \cap \mathcal{S}$ is homeomorphic to the orbit space $\Gamma/\mathcal{K}_x$ and so has the same number of ends. Theorem 2.1 then yields the following theorem.

Theorem 2.2 *For each $x \in F$ the leaf L_x has the same number of ends as $L_x \cap \mathcal{S}$.*

As we have seen, $L_x \cap \mathcal{S}$ in the leaf topology is simply the Schreier graph of the pair $(\pi_1(B, b), \mathcal{K}_x)$. While this could be derived fairly easily leaf by leaf by consideration of the holonomy graph, we shall see that the global topology of $\mathcal{S}$ allows one to deduce more about the structure of the ends than a leaf by leaf analysis.

2.3 The Schreier continuum of a foliated bundle

We shall refer to the sets of the form $L \cap \mathcal{S}$ for a leaf L of $\mathcal{F}$ as *leaves* of $\mathcal{S}$ and a set formed by a union of leaves will be called *saturated* as with foliations. A compact saturated subset of $\mathcal{S}$ is *minimal* if it is the closure of each of its leaves. If the sequence $\{x_i\}_{i=1}^{\infty}$ converges to x in $\mathcal{S}$, then the corresponding plaques in a foliation chart of x converge to the plaque of x. Hence, since the leaves of $\mathcal{S}$ are formed by lifting paths in B, a set is saturated in $\mathcal{S}$ if and only if the corresponding set is saturated in E and a set in $\mathcal{S}$ is minimal if and only if the corresponding set is a compact minimal subset of E. Since the leaves of $\mathcal{S}$ are connected, minimal sets are compact and connected; that is, they are continua. Thus, we refer to a minimal subcontinuum Λ of $\mathcal{S}$ as the *Schreier continuum* of the corresponding minimal set of E. If the foliation of the bundle (F, E, B, p) is minimal, the notions of the Schreier space and the Schreier continuum are interchangeable.

2.4 Schreier continua with Cantor set fibre

After product bundles, one of the most tractable classes of foliated bundles are those for which the fibres are totally disconnected. Hereafter we only consider a foliated bundle (F, E, B, p) where B is a closed manifold, E is a compact topological space, the fibre F is a totally disconnected topological space and the foliation $\mathcal{F}$ is minimal. For certain classes of such foliated bundles one can construct an inverse limit representation of the associated Schreier space $\mathcal{S}$ which leads to a global understanding of $\mathcal{S}$ and the space of ends for the leaves of E. This is possible if the set of generators S of the fundamental group $\pi_1(B, b)$ can be chosen compatible with such a representation in the following sense.

By assumption the fibre F is a totally disconnected compact perfect topological space and therefore is a Cantor set. There is a standard representation of F (see, e.g., [33]) as the inverse limit

$$F = \varprojlim \big\{ J^{(k)}, f_{k-1}^k, \mathbb{N}_0 \big\},$$

where $J^{(k)}$ is the indexing set of a cover $\mathcal{U}^k$ of F by disjoint clopen sets such that for each $k \in \mathbb{N}_0$ and each $j \in J^{(k)}$

$$\operatorname{diam} U_j^{(k)} < 1/k$$

and $\mathcal{U}^{(k)}$ refines $\mathcal{U}^{(k-1)}$ for each $k > 0$ with a refining mapping $f_{k-1}^k : J^{(k)} \to J^{(k-1)}$. The set of generators S for $\pi_1(B, b)$ is compatible with the inverse limit representation of F if for each $g \in S$ the corresponding homeomorphism $h(g) \in \mathrm{Homeo}(F)$ induces a well-defined mapping on the indexing sets $J^{(k)}$ which commutes with the refining maps f_{k-1}^k. In more detail, for each $U_j^{(k)}$ there is $i_j \in J^{(k)}$ such that

$$h(g)\big(U_j^{(k)}\big) \subset U_{i_j}^{(k)},$$

and $i_{f_{k-1}^k(j)} = f_{k-1}^k(i_j)$.

If S is such set of generators, then to each $J^{(k)}$ one associates a graph Λ_k whose vertices are the elements of $J^{(k)}$ and where there is an edge labelled by s joining two vertices if and only some lift of the loop $s \in S$ joins points from the sets corresponding to the vertices. The mappings f_{k-1}^k extend to the mappings $\nu_{k-1}^k : \Lambda_k \to \Lambda_{k-1}$ by sending an edge $e \in \Lambda_k$ to the edge in Λ_{k-1} labelled by the same letter and joining images of the endpoints of e in Λ_{k-1} under f_{k-1}^k. Then the associated Schreier continuum is given by

$$\Lambda = \varprojlim \{\Lambda_k,\, \nu_{k-1}^k,\, k \in \mathbb{N}_0\}.$$

Such an inverse limit representation can be constructed for generalised (weak) solenoids and exceptional Markov minimal sets of foliated bundles with a one dimensional fibre, as we show in the subsequent sections.

3 Schreier continua for solenoids

A general class of examples of foliated bundles with a Cantor set fibre can be found in (weak) solenoids. Here, a solenoid X_∞ will refer to the inverse limit

$$X_\infty = \varprojlim \{X_k, f_{k-1}^k, \mathbb{N}_0\} \tag{3.1}$$

of an inverse sequence of closed manifolds X_k where each bonding map $f_{k-1}^k : X_k \to X_{k-1}$ is a covering map of index greater than one. The projection $f_0 : X_\infty \to X_0$ is a fibre bundle projection with a profinite structure group (see Fokkink

and Oversteegen [15, Theorem 33]). When a solenoid is homogeneous (i.e., when the homeomorphism group acts transitively on the solenoid), the solenoid can be represented as the inverse limit of an inverse sequence where each covering map f_0^k is regular [15], and in this case the associated bundle is principal (see McCord [25, Theorem 5.6]). Thus, a solenoid can be considered a natural generalisation of covering spaces of closed manifolds where the fibre is no longer discrete but instead totally disconnected, and where regular covering spaces correspond to homogeneous solenoids. Locally, a solenoid is homeomorphic to $D \times$ Cantor Set, where D is a Euclidean disk of the same the dimension as the base manifold X_0. We shall see that a solenoid has the structure of a foliated space with transverse space a Cantor set.

3.1 Solenoids as minimal foliated bundles

Let $x_0 \in X_0$ be a point, denote $f_0^{-1}(x_0)$ by F and let $f_k : X_\infty \to X_k$ be the projection to X_k. As in Section 2 let

$$\widetilde{X}_0 = \{[\gamma] \mid \gamma : ([0,1], 0) \to (X_0, x_0)\}$$

be the universal covering of (X_0, x_0) with projection p. For a given point $x_\infty \in F$ we now construct a natural map

$$p_{x_\infty} : (\widetilde{X}_0, \widetilde{x}_0) \to (X_\infty, x_\infty)$$

whose image is the path component of x_∞ as follows. Let $[\delta] \in \widetilde{X}$. Since for each $k \in \mathbb{N}_0$ the composition

$$f_0^k = f_{k-1}^k \circ \cdots \circ f_0^1 : (X_k, x_k) \to (X_0, x_0)$$

is a covering map, then the path $\delta : ([0,1], 0) \to (X_0, x_0)$ lifts uniquely to a path $\delta_k : ([0,1], 0) \to (X_k, x_k)$ and the endpoint $\delta_k(1)$ is determined only by the homotopy class $[\delta]$. Thus, the function

$$p_{x_\infty}([\delta]) = (\delta_k(1))_{k \in \mathbb{N}_0}$$

is well-defined. As for each $k \in \mathbb{N}_0$ the projection of p_{x_∞} to (X_k, x_k) is a continuous universal cover, p_{x_∞} is continuous. As $p = p_0 \circ p_{x_\infty}$ is a covering map, it is not hard to show that p_{x_∞} is a fibration which then has a path component as its image (see, e.g., Clark [12, Theorem 2.3]). Thus, the solenoid is foliated by its path components and (F, X_∞, X_0, p_0) is a foliated bundle. And since each path component is dense, it is a minimal foliated bundle.

3.2 Schreier continua for solenoids

Recall from Section 2 that the fundamental group $\pi_1(X_0, x_0)$ acts on the left of $\widetilde{X}_0$ by the concatenation of paths

$$[\gamma] \cdot [\delta] = [\gamma \cdot \delta], \text{ where } [\gamma] \in \pi_1(B, b).$$

Denote by G_k the image of the homomorphism

$$(f_0^k)_\# : \pi_1(X_k, x_k) \to \pi_1(X_0, x_0),$$

then the kernel at x_∞, as defined in the previous section, is given by

$$\mathcal{K}_{x_\infty} = \bigcap_{k \in \mathbb{N}_0} G_k.$$

Thus, the Schreier continuum associated to X_∞ and some fixed finite set of generators S of $\pi_1(X_0, x_0)$ is given by

$$\Lambda = \varprojlim\{\Lambda_k,\, \nu_{k-1}^k,\, k \in \mathbb{N}_0\}.$$

where Λ_k is the Schreier diagram for the pair $(\pi_1(X_0, x_0), G_k)$. Alternatively, consider Λ_0 to be the image of the Schreier space $\mathcal{S}$ under the covering projection f_0, and each Λ_k as the lift of Λ_0 to X_k by the covering map f_0^k.

Example 3.1 In the case that the solenoid is a principal bundle and all path-connected components are homeomorphic, the end structure can typically be found directly by analysing the kernels $\mathcal{K}_x$ at points $x \in F$, and all possible end structures do occur. For example, let $X_k = \mathbb{S}^1$ and each f_{k-1}^k be a 2-fold covering map. The inverse limit

$$\Sigma_2 = \varprojlim\{\mathbb{S}^1, f_{k-1}^k, \mathbb{N}_0\}$$

is usually called the dyadic solenoid. The universal cover of each leaf is $\mathbb{R}$ and for each $x \in \Sigma_2$ the kernel $\mathcal{K}_x$ is trivial. Therefore, each leaf in Σ_2 is homeomorphic to $\mathbb{R}$ and so has 2 ends. Generalising this example to 2 dimensions one obtains a solenoid $\Sigma_2 \times \Sigma_2$ which fibres over a 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$. The universal cover of each leaf is $\mathbb{R}^2$, and for each $x \in \Sigma_2 \times \Sigma_2$ the kernel $\mathcal{K}_x$ is trivial and so each leaf has 1 end. It is known that a free group on 2 generators F_2 can be realised as the fundamental group of a 3-manifold, and F_2 is geometrically residually finite (see Scott [31] for general results about residually finite groups). Thus, choosing an appropriate decreasing chain of subgroups G_i of finite index with $\cap_i G_i = \emptyset$ with each G_i normal in F_2, one can construct a solenoid in which each leaf has a Cantor set of ends.

Example 3.2 More interesting examples of ends occur when the solenoid does not have the structure of the principal bundle and the structure of the set of ends vary from leaf to leaf. One such example is given by Rogers and Tollefson [27]. Their solenoid is the inverse limit of 2-fold coverings of the Klein bottle K_i by itself

$$K_\infty = \varprojlim\{K_i, f_{i-1}^i, \mathbb{N}_0\}.$$

The solenoid K_∞ is double covered by the solenoid $\mathbb{S}^1 \times \Sigma_2$ with non-trivial covering transformation of the covering map

$$p_\infty : \mathbb{S}^1 \times \Sigma_2 \to K_\infty$$

represented by the involution $\beta \times \alpha_\infty : \mathbb{S}^1 \times \Sigma_2 \to \mathbb{S}^1 \times \Sigma_2$, where β is the rotation in the circle by π and α_∞ is the map induced by the reflections $\alpha_i : \mathbb{S}^1 \to \mathbb{S}^1$. The involution $\beta \times \alpha_\infty$ has a unique fixed point x, so it fixes a single path-connected component of $\mathbb{S}^1 \times \Sigma_2$ and is a homeomorphism when restricted to any other path-connected component. Thus every path-component in K_∞ other than the exceptional one is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and has 2 ends. The exceptional path-component is homeomorphic to the quotient space

$$\mathbb{S}^1 \times \mathbb{R} / \big\{(t, s) \sim (-t, \beta(s))\big\},$$

it is non-orientable and has 1 end.

4 Examples of solenoidal minimal sets

We now consider an example where a direct algebraic technique is not sufficient. We construct the Schreier continuum of the Schori example, i.e. the solenoid

$$X_{\text{Sch}} = \varprojlim \{X_k, f_k^{k+1}, \mathbb{N}_0\},$$

where X_0 is a genus 2 surface, and for every $k \in \mathbb{N}_0$ the bonding map $f_k^{k+1} : X_{k+1} \to X_k$ is a 3-to-1 non-regular covering projection which is described in detail in Section 4.1. This solenoid was introduced in [29] as an example of a non-homogeneous weak solenoid. The aim of this section is to prove the following theorem.

Theorem 4.1 *The minimal foliation of the Schori solenoid has a residual set of points whose leaves have* 1 *ends, an uncountable family of leaves with* 2 *ends and a single leaf with* 4 *ends.*

The proof is detailed in Section 4.2. In Section 4.3 we show how to modify Schori's solenoid in such a way that the exceptional leaf has any given even number of ends. In Section 4.4 we describe the Schreier continuum for the Rogers-Tollefson example (see also Example 3.2).

4.1 The Schori solenoid: the construction

We recall briefly the construction of the Schori solenoid, for details one is referred to Schori [29].

The construction is by induction. Let X_0 be a genus 2 surface. For $n > 0$ let an n-handle be a 2-torus with n mutually disjoint open disks taken out, then X_0 is a union of two 1-handles H_0 and F_0 which intersect along their boundaries. Let C_0 and D_0 be simple closed curves in H_0 and F_0 respectively (see Figure 1, a)). For $k > 0$ let X_k be a genus $m_k = 3^k + 1$ surface with two 2^k-handles H_k and F_k distinguished, and let two simple closed curves C_k and D_k be chosen in H_k and F_k respectively. Then the next component in the sequence is obtained in the following manner: take out the curves C_k and D_k from X_k and pull the cut handles apart by an appropriate homeomorphism to obtain a cut surface $\widehat{X}_k$.

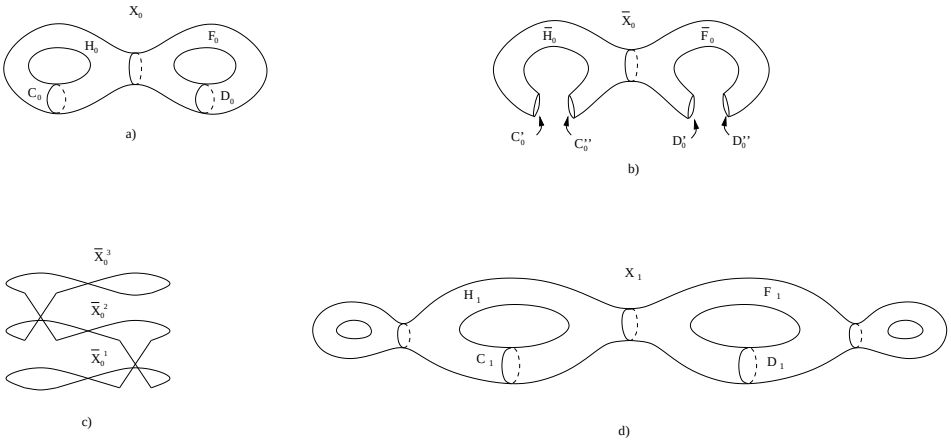


Figure 1: Construction of the Schori example: a) Choice of the handles H_0 and F_0 and closed curves C_0 and D_0 in X_0, b) The cut surface $\bar{X}_0$, c) Identifications between $\bar{X}_0^i$, $i = 1, 2, 3$. Each $\bar{X}_0^i$ is represented by a cut copy of figure 8, and identifications are depicted with straight lines, d) The surface X_1 and the choice of the handles H_1 and F_1 and closed curves C_1 and D_1.

Let $\bar{X}_k = \mathrm{Cl}(\widehat{X}_k)$, and denote by C_k' and C_k'' (resp. D_k' and D_k'') the boundary circles in H_k (resp. F_k) (see Figure 1, b)). Consider three copies $\bar{X}_k^1$, $\bar{X}_k^2$ and $\bar{X}_k^3$ of $\bar{X}_k$, so that for $i = 1, 2, 3$ $\bar{H}_k^i$ and $\bar{F}_k^i$ are the cut handles in $\bar{X}_k^i$, and $C_k^{i\,\prime}$ and $C_k^{i\,\prime\prime}$ (resp. $D_k^{i\,\prime}$ and $D_k^{i\,\prime\prime}$) are the boundary circles in H_k^i (resp. F_k^i). Then in H_k^i-handles identify $C_k^{1\,\prime}$ with $C_k^{1\,\prime\prime}$, $C_k^{2\,\prime}$ with $C_k^{3\,\prime\prime}$, and $C_k^{3\,\prime}$ with $C_k^{2\,\prime\prime}$, and in the F_k^i-handles identify $D_k^{1\,\prime}$ with $D_k^{2\,\prime\prime}$, $D_k^{2\,\prime}$ with $D_k^{1\,\prime\prime}$, and $D_k^{3\,\prime}$ with $D_k^{3\,\prime\prime}$ (see Figure 1, c)). Denote the identification space by X_{k+1}, and let H_{k+1} (resp. F_{k+1}) be the image of $H_k^2 \bigsqcup H_k^3$ (resp. $F_k^1 \bigsqcup F_k^2$) under the identification map (see Figure 1, d)). Define the mapping $f_k^{k+1} : X_{k+1} \to X_k$ by sending a point $(x, i) \in \bar{X}_k^i$ to $x \in X_k$. The obtained map is a 3-to-1 covering projection, and set

$$X_{\mathrm{Sch}} = \varprojlim\{X_k, f_k^{k+1}, \mathbb{N}_0\}.$$

4.2 The Schreier continuum in the Schori example

Consider the Schori solenoid X_{Sch}, and let $x_0 \in H_0 \cap F_0$. For each $k > 0$ there is a unique point $x_k \in X_k$ such that $x_k \in H_k \cap F_k$ and

$$f_0^k(x_k) = f_{k-1}^k \circ f_{k-2}^{k-1} \circ \cdots \circ f_0^1(x_k) = x_0,$$

so $(x_k)_{k=0}^{\infty} \in X_{\mathrm{Sch}}$. In the sequel we omit the subscript and superscript in the notation for a point in X_{Sch} writing (x_k) instead of $(x_k)_{k=0}^{\infty}$. Denote by $f_k : X_{\mathrm{Sch}} \to X_k$ the projection, and by F the fibre $f_0^{-1}(x_0)$ of the bundle $p_0 : X_{\mathrm{Sch}} \to X_0$ at x_0.

4.2.1 The group chain and the Schreier continuum

Let $G_0 = \pi_1(X_0, x_0)$, and for $k > 0$ denote $G_k = f_{0*}^k \pi_1(X_k, x_k)$. As it was shown in Section 3, the Schreier continuum for a pointed space $(X_{\mathrm{Sch}}, (x_k))$ is constructed as an inverse sequence of Schreier diagrams Λ'_k of pairs of groups (G_0, G_k). So the first step in the construction is to compute the group chain $G \supset G_1 \supset G_2 \supset \dots$

Choose loops a, b, α, β representing the generators of the fundamental group $\pi_1(X_0, x_0)$ in the standard way (see Section 2.2) with a relation

$$\mathrm{rel}_0 = [a, \alpha][b, \beta] = a\alpha a^{-1}\alpha^{-1} b\beta b^{-1}\beta^{-1},$$

where concatenation denotes the usual multiplication of paths. Calculation of G_k by an inductive procedure gives the following. For $k = 0$, distinguish the following subsets of generators of G_0,

$$S_{0ab} = \emptyset, \qquad S_{0ba} = \emptyset \tag{4.2a}$$

$$S_{0a} = \{a, \alpha\}, \qquad S_{0b} = \{b, \beta\}. \tag{4.2b}$$

For $k \geq 1$ define

$$\begin{aligned} S_{kab} &= S_{(k-1)ab} \cup a^{2^{k-1}} S_{(k-1)ab} a^{-2^{k-1}} \cup a^{2^{k-1}} S_{(k-1)b} a^{-2^{k-1}}, \\ S_{kba} &= S_{(k-1)ba} \cup b^{2^{k-1}} S_{(k-1)ba} b^{-2^{k-1}} \cup b^{2^{k-1}} S_{(k-1)a} b^{-2^{k-1}}, \\ S_{ka} &= \{a^{2^k}, \alpha\} \cup S_{kab}, \\ S_{kb} &= \{b^{2^k}, \beta\} \cup S_{kba}. \end{aligned} \tag{4.2c}$$

Then for $k \geq 0$ we have

$$G_k = \langle a^{2^k}, \alpha, b^{2^k}, \beta, S_{kab}, S_{kba} \mid \mathrm{rel}_k = \mathrm{id}, \mathrm{rel}_0 = \mathrm{id} \rangle,$$

where rel_k is the corresponding relation for the m_k-genus surface. The subgroup G_k has index 3^k in G_0.

Let $\bar{S}_{0a}$ and $\bar{S}_{0b}$ be sets of formal inverses of elements in $\bar{S}_{0a}$ and $\bar{S}_{0b}$ respectively, and denote $S = S_{0a} \cup S_{0b} \cup \bar{S}_{0a} \cup \bar{S}_{0b}$. For each $k \geq 0$ let Λ'_k be the Schreier diagram of the coset space G_0/G_k constructed with respect to S as in Section 3, and $\Lambda' = \lim\{\Lambda'_k, f_k^{k+1}, \mathbb{N}_0\}$ be the corresponding Schreier continuum. Let $v = (v_k)$ be a vertex of Λ', and denote by L_v its path-connected component.

4.2.2 The simplified Schreier continuum

We call an edge labelled by λ a λ-edge and write e_λ. Then $s(e_\lambda)$ is the starting vertex of the edge, and $t(e_\lambda)$ is the ending vertex. If $s(e_\Lambda) = t(e_\lambda) = v$, e_λ is called a λ-loop based at v. If it is important to keep track the starting point of an edge, we write e_λ^v. A path in Λ' is a continuous map $\gamma : [0, 1] \to \Lambda'$, so its image is contained in a path-connected component of Λ'. We allow paths to start and end in the interior of edges. A subgraph L' of Λ' is called path-connected, if for any $v, w \in L'$ there is an edge-path $\gamma \subset L'$ which starts at v and ends at w.

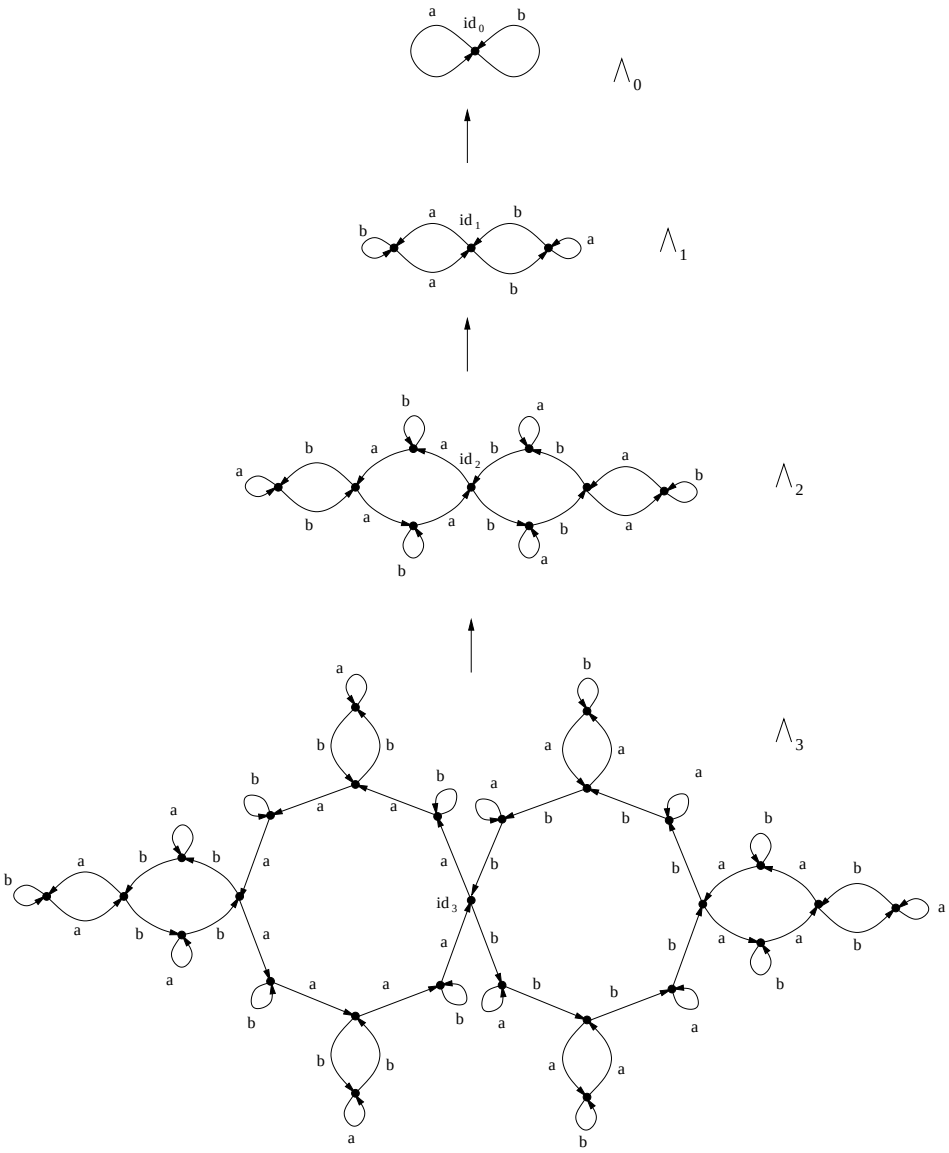


Figure 2: Graphs Λ_i for $i = 0, 1, 2, 3$.

We notice that any vertex $w = (w_k) \in \Lambda'$ is a base point of an α-, α^{-1}-, β- and β^{-1}-loops, since by construction any α-, α^{-1}-, β- and β^{-1}-loop based at $w_k \in \Lambda'_k$, $k \in \mathbb{N}_0$ lifts to a loop in Λ'_{k+1} at w_{k+1} labelled by the same letter. Given an exhaustion of a path-connected component L' of w by compact sets, eventually every such loop would be contained in a compact sets and will not make any contribution to the number of ends of L. This motivates the following lemma.

Lemma 4.1 *Let $S' = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}\}$, for $k \in \mathbb{N}_0$ let*

$$\Lambda_k = \Lambda'_k \setminus \bigcup_{w_k \in V(\Lambda'_k)} \Big(\bigcup_{\lambda \in S'} \text{int } e_\lambda^{w_k} \Big),$$

and $r_k : \Lambda'_k \to \Lambda_k$ be a retraction which maps each λ-loop $e_\lambda^{w_n}$, where $\lambda \in S'$, to its base point w_n and is the identity on the rest of Λ'_k. Denote by $i_k : \Lambda_k \to \Lambda'_k$ the inclusion, and define a continuous map $\nu_{k-1}^k : \Lambda_k \to \Lambda_{k-1}$ by

$$\nu_{k-1}^k(x) = r_{k-1} \circ {\nu'}_{k-1}^k \circ i_k(x). \tag{4.3}$$

Then $\{\Lambda_k, \nu_{k-1}^k, \mathbb{N}_0\}$ is an inverse sequence of topological spaces, and there is an induced map

$$r_\infty : \Lambda' \to \Lambda = \varprojlim \{\Lambda_k, \nu_k^{k+1}, \mathbb{N}_0\}, \tag{4.4}$$

which is a bijection between the sets of path-connected components of Λ' and Λ. Moreover, for any vertex $v \in \Lambda'$ the path-connected components L'_v and $r(L'_v)$ have the same number of ends.

Proof. The mappings (4.3) satisfy $\nu_i^k = \nu_i^j \circ \nu_j^k$ for any $i < j < k$, and Λ_k are compact, so the inverse limit Λ exists and is non-empty [26]. The retractions r_k satisfy $r_k \circ {\nu'}_k^{k+1} = \nu_k^{k+1} \circ r_{k+1}$ and are surjections onto compact spaces Λ_k, so they induce a continuous surjection (4.4) on the inverse limit spaces. On the other hand, the inclusions i_k induce a continuous mapping $i_\infty : \Lambda \to \Lambda'$ such that $r_\infty \circ i_\infty = \text{id}_\Lambda$. It follows that (4.4) is a bijection between path-connected components of Λ' and Λ, and $r_\infty(L'_v) = L_v$.

Let d' be a complete length metric on the leaves of Λ' such that the restrictions of ν'_k to leaves is a local isometry. In particular, each edge in Λ' has length 1 in associated to the metric length structure. Denote by d the induced length metric on the subspace Λ of Λ'. Denote by $B'(w, s)$ and $B(w, s)$ compact balls of radius s about a vertex w with respect to d' and d respectively, and consider the exhaustion $\{B'(v, z + \frac{1}{2})\}_{z \in \mathbb{N}}$ of L'_v by closed balls.

Notice that if $\lambda \in S'$ and e_λ^w is a loop, then for any $x \in \text{int } e_\lambda^w$ we have $d'(w, x) \leq \frac{1}{2}$. Therefore, for any $z \in \mathbb{N}$ the intersection $B'(v, z + \frac{1}{2}) \cap e_\lambda^w$ is either empty or contains the edge e_λ^w. Therefore, there is a bijection between the sets of path-connected components of the complements $L'_v \setminus B'(v, z + \frac{1}{2})$ and $L_v \setminus r_\infty(B'(v, z + \frac{1}{2}))$. The sequence

$$\Big\{ r_\infty(B'(v, z + \frac{1}{2}) \Big\}_{z \in \mathbb{N}} = \Big\{ B(v, z + \frac{1}{2}) \Big\}_{z \in \mathbb{N}}$$

is an exhaustion of L_v, and it follows that L'_v and L_v have the same number of ends.

□

In the case of the Schori example Λ_0 is a figure 8, and each Λ_k is the Schreier graph of a coset space F_2/G'_k, where F_2 is a free group on two generators and $G'_k \subset F_2$ is a subgroup of finite index (see Figure 2) with the following set of generators: denote by $\bar{S}_{0a} = \{a\}$, $\bar{S}_{0b} = \{b\}$, for $k > 1$ define S_{kab} and S_{kba} as in (4.2c), and

$$S_{ka} = \{a^{2^k}\} \cup S_{kab}, \qquad\qquad S_{kb} = \{b^{2^k}\} \cup S_{kba}.$$

Then

$$G'_k = \langle a^{2^k}, b^{2^k}, S_{kab}, S_{kba} \rangle.$$

4.2.3 Ends of special path-connected components in the Schori example

As in Lemma 4.1, we consider the sequence of topological spaces Λ_k, $k \in \mathbb{N}_0$, with length structure induced from Λ'_k and the corresponding length metric d_k. Since Λ_k is compact, d_k is bounded. In fact, we can obtain a more precise estimate, which will be of use later.

For $k \in \mathbb{N}_0$ we denote by id_k the coset $[\mathrm{id}] \in F_2/G'_k$, and by a^m (resp. b^m), $m = 1, \dots, 2^k - 1$, the coset $[a^m]$ (resp. $[b^m]$). Denote by $\mathcal{A}_k$ the connected component of $\Lambda_k - \{\mathrm{id}_k\}$, containing vertices $a^1, \dots, a^{2^k-1}$, and by $\mathcal{B}_k$ the other connected component. Then $\mathcal{A}_k \cup \{\mathrm{id}_k\}$ and $\mathcal{B}_k \cup \{\mathrm{id}_k\}$ are compact connected subspaces of Λ_k.

Lemma 4.2 *The restriction of the length metric d_k to $\mathcal{A}_k \cup \{\mathrm{id}_k\}$ or to $\mathcal{B}_k \cup \{\mathrm{id}_k\}$ has the least upper bound $N_k = 2^k - \frac{1}{2}$, and for any two positive integers $m, \ell < 2^k$*

$$d_k(a^\ell, a^m) = \min\{|m - \ell|, 2^k - |m - \ell|\},$$

and the same holds for $d_k(b^\ell, b^m)$.

Corollary 4.1 *For any positive integers $m, \ell < 2^{k-1}$*

$$d_k(a^\ell, a^m) = |m - \ell|.$$

Denote by id the point $(\mathrm{id}_k) \in \Lambda$.

Corollary 4.2 *If m, ℓ are positive integers, then for $(a^m), (a^\ell) \in L_{\mathrm{id}}$*

$$d\big((a^m), (a^\ell)\big) = |m - \ell|,$$

and there is a corresponding geodesic in L_{id} consisting only of a- or only of a^{-1}-edges.

In the following proposition we will also use a notation a^{-m} for a coset $[a^{-m}] \in F_2/G'_k$, where $m \in \mathbb{N}_0$, so that vertices in F_2/G'_k now have two names since $[a^\ell] = [a^{-m}]$ with possibly $\ell \neq m$. We will see that $(a^{-m}) \neq (a^\ell)$ in L_{id} for any $m, \ell > 0$ in the proof of Proposition 4.1.

Proposition 4.1 *The pathwise connected component L_{id} of Λ has four ends.*

Proof. The idea is to choose a cofinal sequence of compact subsets in L_{id} so that their complements are easy to handle. Such a sequence is a sequence of compact balls $B(\mathrm{id}, N_k)$ where $N_k = 2^k - \frac{1}{2}$ (see Lemma 4.2).

We say that a subgraph Γ_A is associated to a subset $A \subset V(L_{\mathrm{id}})$ of the set of vertices of L_{id} if Γ_A contains A and if Γ_A contains all edges $e_\lambda \in L_{\mathrm{id}}$ such that $s(e_\lambda), t(e_\lambda) \in A$. We claim that for each $k \in \mathbb{N}_0$ and for each subset of $V(L_{\mathrm{id}})$ given by

$$\begin{aligned} A_k^+ &= \{(a^m) \mid m > N_k\}, & A_k^- &= \{(a^{-m}) \mid m > N_k\}, \\ B_k^+ &= \{(b^m) \mid m > N_k\}, & B_k^- &= \{(b^{-m}) \mid m > N_k\} \end{aligned} \tag{4.5}$$

the associated graph lies in a distinct unbounded connected component of the complement $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$, and each $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$ has at most four unbounded connected components.

Clearly $\Gamma_{A_k^+}$ is path-connected. Since $B(\mathrm{id}, N_k)$ contains an interior point of an edge e_λ if and only if it contains at least one of its vertices,

$$\Gamma_{A_k^+} \cap B(\mathrm{id}, N_k) = \emptyset$$

and so $\Gamma_{A_k^+}$ is contained in a path-connected component of $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$. By Corollary 4.2 geodesics between vertices in A are contained in $\Gamma_{A_k^+}$ and they can be of arbitrary large length, so $\Gamma_{A_k^+}$ is contained in an unbounded path-connected component of the complement. By a similar argument each of the sets A_k^-, B_k^+, B_k^- is contained in an unbounded connected component of the complement $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$.

We show that $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ lie in different path-connected components of $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$. Assume to the contrary that there is a path γ between (a^m) and (a^{-n}) entirely contained in $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$. Choose $j \in \mathbb{N}_0$ so that for each $i \geq j$ the path γ is contained in the fundamental domain of the covering map $\nu_i : L_{\mathrm{id}} \to \Lambda_i$, then its projection $\gamma_i = \nu_i \circ \gamma$ is contained in $\Lambda_i \backslash B_i(\mathrm{id}_i, N_k)$, where $B_i(\mathrm{id}_i, N_k)$ is a closed ball about id_i of radius N_k with respect to the length metric d_i on Λ_i. Then

$$\ell(\gamma) \geq \ell_i(\gamma_i) \geq 2^i - (m + n),$$

where $\ell(\gamma)$ and $\ell_i(\gamma_i)$ are the lengths of paths in the length structures associated to d and d_i respectively. Since i can be arbitrary large, $\ell(\gamma)$ cannot be finite. It follows that $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ (and, by a similar argument, $\Gamma_{B_k^+}$ and $\Gamma_{B_k^-}$) are in different path connected components of $L_{\mathrm{id}} \backslash B(\mathrm{id}, N_k)$.

Next, suppose $\Gamma_{A_k^+}$ and $\Gamma_{B_k^+}$ (resp. $\Gamma_{B_k^-}$) are in the same path-connected component of $L_{\rm id}\backslash B({\rm id}, N_k)$ and choose $j \in \mathbb{N}$ so that a path γ between $(a^m) \in A_k^+$ and $b^n \in B_k^+$ (resp. $b^{-n} \in B_k^-$) is contained in the fundamental domain of the covering map $\nu_j : L_{\rm id} \to \Lambda_j$. Then $\gamma_j = \nu_j \circ \gamma$ is contained in a path-connected component of $\Lambda_j\backslash B_j({\rm id}_j, N_k)$, which implies that either $a^m, b^n \in \mathcal{A}_j$ or $a^m, b^n \in \mathcal{B}_j$ (resp. $a^m, b^{-n} \in \mathcal{A}_j$ or $a^m, b^{-n} \in \mathcal{B}_j$), a contradiction. Therefore, $\Gamma_{A_k^+}$ and $\Gamma_{B_k^+}$ (resp. $\Gamma_{B_k^-}$) are in different path-connected components of $L_{\rm id}\backslash B({\rm id}, N_k)$. Repeat the same argument for $\Gamma_{A_k^-}$. Conclude that $L_{\rm id}\backslash B({\rm id}, N_k)$ has at least four unbounded path-connected components.

We show that $L_{\rm id}\backslash B({\rm id}, N_k)$ has at most 4 path-connected components. Note that each $B({\rm id}, N_k)$ is precisely the fundamental domain of the covering map $\nu_k : L_{\rm id} \to \Lambda_k$. Then for $i > k$ the coboundary of the set of vertices $V(B_i({\rm id}_i, N_k))$ consists of exactly 4 edges. Let $v = (v_n) \in L_{\rm id}\backslash B({\rm id}, N_k)$, and γ be a path in $L_{\rm id}$ between id and v. Choose an integer $j > k$ such that $i > j$ implies that the path γ is contained in the fundamental domain of the covering map $\nu_i : L_{\rm id} \to \Lambda_i$, and so projects faithfully on Λ_i. Since ${\rm id}_i = \nu_i({\rm id}) \in B_i({\rm id}_i, N_k)$ and $v_i \in \Lambda_i\backslash B_i({\rm id}_i, N_k)$, the projection $\gamma_i = \nu_i \circ \gamma$ contains an edge of the coboundary of $V(B_i({\rm id}_i, N_k))$. It follows v is in a path-connected component containing one of the sets in (4.5).

□

Let $q_0 = {\rm id}_0$, $q_1 = a$ and for $k \in \mathbb{N}_0$, $k > 1$ define

$$q_{2k} = b^{2^k} q_{2k-1}, \qquad\qquad q_{2k+1} = a^{2^k+1} q_{2k}. \tag{4.6}$$

Then $\nu_{k-1}^k(q_k) = q_{k-1}$, and $q = (q_k)$ is a vertex in Λ.

Lemma 4.3 *A path-connected component L_q of Λ has 1 end.*

Proof. We claim that for each $k > 0$ the set $U_k = L_q\backslash B(q, N_k - \frac{1}{2})$ is path-connected. First notice that for any $i > k$ we have

$$\nu_i\Big(B\big(q, N_k - \tfrac{1}{2}\big)\Big) = B_i\big(q_i, N_k - \tfrac{1}{2}\big),$$

and if $d(q, v) > N_k - \frac{1}{2}$, then $\nu_i(v) \in B_i\big(q_i, N_k - \frac{1}{2}\big)$ for at most a finite number of $i \in \mathbb{N}_0$. Denote by $\mathcal{Q}_i$ and $\mathcal{P}_i$ path-connected components of $\Lambda_i\backslash\{{\rm id}_i\}$ so that $q_i \in \mathcal{Q}_i$, and notice that if $i > k$ then $\nu_i\Big(B\big(q, N_k - \frac{1}{2}\big)\Big) \subset \mathcal{Q}_i \cup \{{\rm id}_i\}$. By construction of Λ_i and the choice of the radius of the balls (see Lemma 4.2) the complement $\big(\mathcal{Q}_i \cup \{{\rm id}_i\}\big)\backslash\nu_i\Big(B\big(q, N_k - \frac{1}{2}\big)\Big)$ is path-connected.

Let $s = (s_n)$ and $t = (t_n)$ be points in U_k, and γ and δ be paths joining q with s and t respectively. Choose $j > k$ so that

$$\max\{\ell(\gamma), \ell(\delta)\} < N_j - \frac{1}{2},$$

and for any $i > j$ we have $s_i, t_i \in \Lambda_i\backslash B_i\big(q_i, N_k - \frac{1}{2}\big)$. Then for all $i \geq j$ the paths $\gamma_i = \nu_i \circ \gamma$ and $\delta_i = \nu_i \circ \delta$ are contained in $\mathcal{Q}_i \cup \{{\rm id}_i\}$. Choose a path c_j contained in $\mathcal{Q}_j \cup \{{\rm id}_j\}$ joining s_j and t_j. Then for any $i > j$ the lift c_i of c_j with an endpoint s_i has t_i as another endpoint. So U_k is path-connected and L_q has one end.

□

Let $q'_0 = \mathrm{id}_0$, $q'_1 = b$ and define for $k \in \mathbb{N}_0$, $k > 1$

$$q'_{2k} = a^{2^k} q'_{2k-1}, \qquad q'_{2k+1} = b^{2^k+1} q'_{2k}, \tag{4.7}$$

then $\nu_{k-1}^k(q'_k) = q'_{k-1}$, and $q' = (q'_k)$ is a vertex in Λ. By the argument similar to that of Lemma 4.3 the path-connected component $L_{q'}$ has one end.

4.2.4 Other path-connected components in the Schori example

The remaining path-connected components of Λ can be divided into two groups: those containing a so-called 'dyadic' point, and those containing a so-called 'flip-flopping' point. First we recall and introduce some notation.

As in Lemma 4.2.3, denote by $\mathcal{A}_k$ the path-connected component of $\Lambda_k \backslash \{\mathrm{id}_k\}$ containing cosets $a^i = [a^i] \in G/G_k$, and by $\mathcal{B}_k$ the other one. The vertex id_k has three preimages under $\nu_k^{k+1} : \Lambda_{k+1} \to \Lambda_k$, those being the vertices $\mathrm{id}_{k+1}, a^{2^k}, b^{2^k}$. Then $\mathcal{A}_{k+1} \backslash \{a^{2^k}\}$ consists of three path-connected components: the component $\mathcal{A}_{k+1}^+$ containing vertices a^i for $i < 2^k$, the component $\mathcal{A}_{k+1}^-$ containing vertices a^i for $i > 2^k$, and the remaining component $\mathcal{TA}_{k+1}$. Similarly, $\mathcal{B}_{k+1} \backslash \{b^{2^k}\}$ consists of path-connected components $\mathcal{B}_{k+1}^+$, $\mathcal{B}_{k+1}^-$ and $\mathcal{TB}_{k+1}$.

Let $v = (v_k) \in \Lambda$ be a vertex, and assume that $v \notin L_{\mathrm{id}}$. If for some k we have $v_k \in \mathcal{A}_k$ (resp. $v_k \in \mathcal{B}_k$), then either $v_{k+1} \in \mathcal{A}_{k+1}^+ \cup \mathcal{A}_{k+1}^-$ (resp. $v_{k+1} \in \mathcal{B}_{k+1}^+ \cup \mathcal{B}_{k+1}^-$) or $v_{k+1} \in \mathcal{TB}_{k+1}$ (resp. $v_{k+1} \in \mathcal{TA}_{k+1}$), and then the following situations are possible:

1. $v_k \in \mathcal{A}_k^+ \cup \mathcal{A}_k^- \cup \mathcal{B}_k^+ \cup \mathcal{B}_k^-$ for at most a finite number of k's, then either $v_k \in L_q$ or $v_k \in L_{q'}$.

2. $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$ for at most a finite number of indices, then v is called a *dyadic* point. For example, let $j \in \mathbb{N}_0$ be odd, choose $0 \le \ell_j < 2^j$ and define for $2k > j$

$$\ell_{2k} = \ell_{2k-1} + 2^{2k-1}, \qquad \ell_{2k+1} = \ell_{2k}.$$

   Then (a^{ℓ_n}) is a dyadic point.

3. there is a cofinal subset $I \in \mathbb{N}_0$ such that for any $k \in I'$ we have $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$, and the subset $\mathbb{N}_0 \backslash I$ is also cofinal. In this case v is called a *flip-flopping* point.

Lemma 4.4 *Let $v \in \Lambda$ be a dyadic point. Then L_v has 2 ends.*

Proof. Suppose that for $k > n$ we have $v_k \in \mathcal{A}_k$ (resp. $v_k \in \mathcal{B}_k$). Then the distance

$$d_k(v_k, \{a^1, \dots, a^{2^k-1}\}) = \inf\{d_k(v_k, a^\ell) \mid 1 \le \ell \le 2^k - 1\}$$

is constant for all $k > n$. Therefore, every dyadic point is in the path-connected component of a point $\tilde{a} = (a^{\ell_k})$, where $0 < \ell_k < 2^k$.

We construct an increasing sequence $K_0 \subset K_1 \subset K_2 \subset \dots$ of compact subsets in L_v such that $\tilde{a} \in K_0$ and for $i > 0$ $L_v \backslash K_i$ has precisely 2 unbounded path-connected components. Set $K_0 = B(\tilde{a}, N_0)$ and suppose K_{i-1} is given. Then there exists $j_i > 0$ such that $K_{i-1} \subset B(\tilde{a}, N_{j_i})$. Construct K_i as follows. Lemma 4.2 together with a simple computation show that $d_k(a_k^\ell, \mathrm{id}_k)$ increases with k, so there exists $n > 0$ such that for all $k \geq n$ the projection $\nu_k\big(B(\tilde{a}, N_{j_i})\big) \subset \mathcal{A}_k$. We can also assume that $2^i < 2^{n-2}$. Denote $\pm m = \ell_n \pm (2^{j_i} - 1)$. The coboundary of $V\big(B_n(a^{\ell_n}, N_{j_i})\big)$ contains the set of edges $W_n = \{e_a^m, e_{a^{-1}}^m, e_a^{-m}, e_{a^{-1}}^{-m}\} \subset \Lambda_n$, and a possibly empty set $E_n = \{e_{\lambda_1}, \dots, e_{\lambda_{s_n}}\}$. There is a finite set of vertices $V_{E_n} = \{w_1^n, \dots, w_m^n\} \in \Lambda_n$ such that a geodesic δ joining a^{ℓ_n} to w_t contains an edge from E_n. The graph $K' = B_n(a^{\ell_n}, N_{j_i}) \cup \Gamma_{V_{E_n}} \cup E_n$ is pathwise connected. For any $k > n$ there is a unique preimage $\tilde{K}' \subset \Lambda_k$ of K' containing a^{ℓ_k}, and its coboundary is precisely the set $W_k = \{e_a^m, e_{a^{-1}}^m, e_a^{-m}, e_{a^{-1}}^{-m}\} \subset \Lambda_k$. The inverse limit of $\tilde{K}'$ is a subset $K_i \subset L_v$ such that $K_i \ni \tilde{a}$.

Denote by $\bar{a}_k^m$ a vertex in Λ_k corresponding to the coset $[a^m a^{\ell_k}]$, and define

$$A_i^+ = \{\bar{a}^m = (\bar{a}_k^m) \mid m > N_{j_i}\}, \qquad A_i^- = \{\bar{a}^{-m} = (\bar{a}_k^{-m}) \mid m > N_{j_i}\}. \tag{4.8}$$

The graphs $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ are clearly path-connected. To see that a path-connected component of $L_v \backslash K_i$ containing such graph is unbounded we show that $\bar{a}^m = \bar{a}^s$ if and only if $m = s$, which implies that $\Gamma_{A_k^\pm}$ contains geodesics of arbitrary length. Indeed, choose $n \in \mathbb{N}_0$ such that $\max\{m, s\} < 2^n$, then for all $k > n$ we have $\bar{a}_k^s = \bar{a}_k^t$ if and only if $s = t$. Next, consider the points $\bar{a}^{2^{j_i}} \in \mathcal{A}_i^+$, $a^{-2^{j_i}} \in \mathcal{A}_i^-$, and assume there exists a path γ contained in $L_v \backslash K_i$ joining $\bar{a}^{2^{j_i}}$ and $\bar{a}^{-2^{j_i}}$. Choose $n > 0$ so big that for $k > n$ the projection $\nu_k(K_i) \subset \mathcal{A}_k$ and $2^{j_i} < 2^{n-2}$. The projection $\gamma_k = \nu_k \circ \gamma$ is a path joining $\bar{a}_k^{2^{j_i}}$ and $\bar{a}_k^{-2^{j_i}}$ and so must contain a sequence of a-edges. Assuming $a_k^{2^{j_i}}$ to be a starting point of γ_k, the first edge traversed by γ_k must be e_b, $e_{b^{-1}}$ or e_a. In the two former cases γ contains a loop based at $\bar{a}^{2^{j_i}}$, so we can as well assume that the first edge traversed by γ_k is e_a. By a similar argument γ_k must contain all a- or all a^{-1}-edges in Λ_k such that $e_\lambda \cap \nu_k(K_i) = \emptyset$. Then we have the following estimate on the length $\ell(\gamma)$ and $\ell_k(\gamma_k)$

$$\ell(\gamma) \geq \ell_k(\gamma_k) \geq 2^k - (2^{j_i} + 1),$$

which implies that γ has unbounded length. Therefore, $\Gamma_{A_k^+}$ and $\Gamma_{A_k^-}$ lie in different path-connected components of $L_v \backslash K_i$. We show that $L_v \backslash K_i$ has precisely two path-connected components. Assume $v \in L_v \backslash K_i$ and let δ be a path joining v with $\tilde{a}$. Then δ_k contains an edge of W_k, and it follows that v is in a path-connected component of A_i^+ or A_i^-. So there is at most 2 path-connected components.

□

Lemma 4.5 *Let $v \in \Lambda$ be a flip-flopping point. Then L_v has 1 end.*

Proof. We construct a sequence of compact sets $K_0 \subset K_1 \subset K_2 \dots$ such that $K_0 \ni v$ and $L_v \backslash K_i$ has one unbounded path-connected component. Set $K_0 = B(v, N_0)$, and suppose K_{i-1} is given. Then there exists $j_i > 0$ such that $K_{i-1} \subset B(v, N_{j_i})$. Since $d_k(a^{2^k}, \mathrm{id}_k)$ and $d_k(b^{2^k}, \mathrm{id}_k)$ increase with k, there is an $n > 0$ such that for any $k \geq n$ such that $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$ we have $B_k(v_k, N_{j_i}) \subset \mathcal{TA}_k \cup \mathcal{TB}_k$. Then $\nu_k\big(B(v, N_{j_i})\big) = B_k(v_k, N_{j_i})$. Denote by U_n the path-connected component of $\Lambda_n \backslash B_n(v_n, N_{j_i})$ containing id_n, and let $\mathcal{U}_n$ be a possibly empty set of remaining path-connected components. If $\bar{U}_n \in \mathcal{U}_n$, then $\bar{U}_n \subset \mathcal{TA}_n \cup \mathcal{TB}_n$, and for any $k \geq n$ the preimage $\bar{U}_k \ni v_k$ is a path-connected component of $\Lambda_k \backslash B_k(v_k, N_{j_i})$. Let $W_k = B_k(v_k, N_{j_i}) \cup \mathcal{U}_k$, then for each $k \geq n$ the complement $\Lambda_k \backslash W_k$ is path-connected. Set

$$K_i = \varprojlim \{W_k, \nu_k^{k+1}, k \geq n\}.$$

Let $s = (s_k), t = (t_k) \in L_v \backslash K_i$, and γ and δ be paths joining s and t with v respectively. Let $I \subset \mathbb{N}_0$ be a cofinal subsequence of indices such that if $k \in I$ then $v_k \in \mathcal{TA}_k \cup \mathcal{TB}_k$. Let $M = \max\{\ell(\gamma), \ell(\delta)\}$ and choose $n \in I$ such that $B_n(v_n, M) \subset \mathcal{TA}_n \cup \mathcal{TB}_n$. Then $\gamma_n = \nu_n \circ \gamma$ and $\delta_n = \nu_n \circ \delta$ are contained in $\mathcal{TA}_n \cup \mathcal{TB}_n$, and so either $s_n, t_n \in \mathcal{TA}_n \cup \{a^{2^n}\}$ or $s_n, t_n \in \mathcal{TB}_n \cup \{b^{2^n}\}$. Then there exists a path c_n contained in one of these sets, which joins s_n with t_n. From the construction of Λ_k it follows that for any $k \in I$ such that $k > n$ the lift c_k of c_n such that one of its endpoints is s_k has t_k as another endpoint. So there is a path c in $L_v \backslash K_i$ joining s and t. It follows that L_v has one end.

□

4.2.5 The number of leaves with one or two ends

It was shown in Blanc [1] that if a minimal compact foliated metric space space has a residual set of points whose leaves have 2 ends, then any leaf in $\mathcal{F}$ has one or two ends. In the Schori example the leaf L_{id} has four ends, therefore, the minimal foliation of X_{Sch} must have a residual set of points with leaves with one end. That is indeed the case, as Proposition 4.2 shows.

Denote by $\mathcal{AB} \subset V(\Lambda)$ the subset of vertices which lie in 4- or 2-ended path-connected components of Λ.

Proposition 4.2 *The set $\mathcal{AB} \subset V(\Lambda)$ of vertices which belong to 4- and 2-ended path-connected components of Λ is meagre.*

Proof. Consider the subset

$$A_0 = \big\{(a^{\ell_k}) \in V(\Lambda) \;\big|\; a^{\ell_{k-1}} = a^{\ell_k} \mod 2^{k-1}\big\}.$$

All vertices in A_0 lie either in L_{id} or in a 2-ended path-connected component of Λ, the first alternative occurring if (a^{ℓ_k}) is eventually constant. Denote also

$$A_m = \Big\{(v_k) \in V(\Lambda) \;\Big|\; \min_{(u_k) \in A_0} d\big((v_k), (u_k)\big) \leq m\Big\},$$

where d denotes metric on the leaves of Λ. We are going to show that for each $m \geq 0$ the set A_m is nowhere dense. For that it is enough to show that A_m is closed, i.e. $\mathrm{Cl}(A_m) = A_m$, since A_m has empty interior by minimality of 1-ended path-connected components.

Lemma 4.6 *The subset $A_m \subset V(\Lambda)$ is closed.*

Proof. First let $m = 0$. Let $\big\{(a^{\ell_k})_s\big\}$ be a sequence of elements in A_0 converging to a point $(\bar{a}_k) \in \mathrm{Cl}(A_0)$. Then for each $k \in \mathbb{N}$ there exists N_k such that for all $s > N_k$ and all $i < k$ we have

$$a_s^{\ell_i} = \bar{a}_i.$$

It follows that each $\bar{a}_k \in \nu_k(A_0)$ and so $(\bar{a}_k) \in A_0$.

Let $m > 0$, and $\big\{(v_k)_s\big\} \in A_m$ be a sequence converging to $(\bar{v}_k) \in \mathrm{Cl}(A_m)$. Then

$$\min_{(u_k)\in A_0} d\big((\bar{v}_k), (u_k)\big) \leq m$$

since otherwise there exists an open ball around $\bar{v}_k$, for example, of radius $\frac{1}{2}$, which does not contain points of $\big\{(v_k)_s\big\}$. Thus $(\bar{v}_k) \in A_m$.

□

Similarly, denote

$$B_0 = \big\{(b^{\ell_k}) \in V(\Lambda) \;\big|\; b^{\ell_{k-1}} = b^{\ell_k} \mod 2^{k-1}\big\}$$

and for $m > 0$ set

$$B_m = \Big\{(v_k) \in V(\Lambda) \;\Big|\; \min_{(u_k)\in B_0} d\big((v_k), (u_k)\big) \leq m\Big\}.$$

Then for $m \geq 0$ the set B_m is nowhere dense, and $A_m \cup B_m$ is also nowhere dense. Any 2-ended path-connected component intersects $A_0 \cup B_0$, so

$$\mathcal{AB} = \bigcup_{m=0}^{\infty} \Big(A_m \cup B_m\Big),$$

and $\mathcal{AB}$ is a meagre set.

□

Proposition 4.2 implies that the set of vertices in Λ which lie in 1-ended path-connected components is residual, so the generic leaf in X_{Sch} has 1 end [9].

Remark 4.1 The number of path-connected components of Λ with 2 ends is uncountable. Consider the dyadic solenoid

$$\Sigma = \varprojlim\{\mathbb{S}^1, f_{k-1}^k, \mathbb{N}_0\},$$

where $f^k_{k-1} : \mathbb{S}^1 \to \mathbb{S}^1$ is a 2-fold covering. The foliation of Σ by path-connected components has an uncountable infinity of leaves. Recall from Section 4.2.3 that there is a subset of leaves with 2 ends such that each leaf contains a dyadic point (a^{ℓ_k}), and notice that to $(a^{\ell_k}) \in X_{\mathrm{Sch}}$ one can associate a unique point in Σ. Using an argument on the existence of paths as in Section 4.2.3 one sees that if two points in Σ represent different path-connected components, then the corresponding points $(a^{\ell_i}) \in X_{\mathrm{Sch}}$ represent different path-connected components of X_{Sch}. Therefore, X_{Sch} has an uncountable infinity of leaves with 2 ends.

4.3 Generalised Schori example: solenoid with a $2n$-ended path-connected component

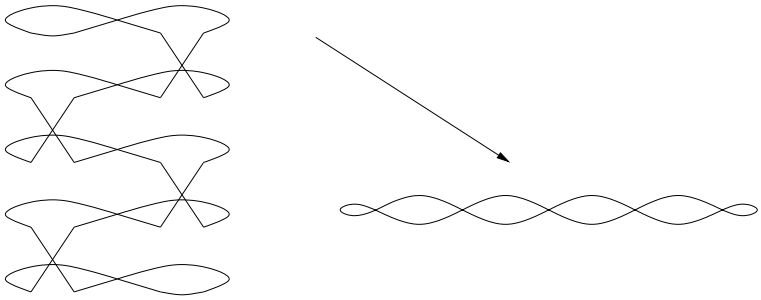
The Schori example can be generalised to obtain a solenoid with a $2n$-ended path component, for any $n \in \mathbb{N}$. The construction is based on that of the original Schori example.

4.3.1 The generalised Schori example with a $4n$-ended path-connected component

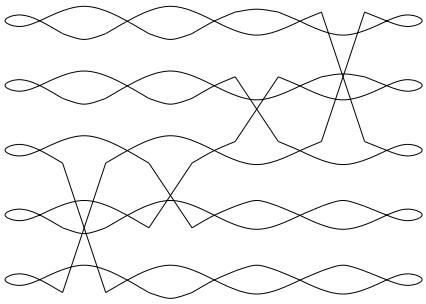
Let X_0 be a genus 2 surface with two n-handles H_1 and H_2 distinguished, let C_1 and C_2 be closed meridional curves in the interior of H_1 and H_2 respectively. The handles H_1 and H_2 intersect along their boundary curves. Cut the handles along C_1 and C_2, pull them apart by an appropriate homeomorphisms to obtain the cut surface $\widehat{X}_0$. Set $\bar{X}_0 = \mathrm{Cl}(\widehat{\mathrm{X}}_0)$and consider copies $\bar{X}_0^{(i)}$, $i = 1, \dots, 2n+1$ of $\bar{X}_0$ with handles $\bar{H}_1^{(i)}$ and $\bar{H}_2^{(i)}$ and pairs of boundary circles $C_1^{(i)\prime}$ and $C_1^{(i)\prime\prime}$, and $C_2^{(i)\prime}$ and $C_2^{(i)\prime\prime}$ in each handle. Make identifications so that for $i = 1, \dots, n$ $C_1^{(2i-1)\prime}$ is identified with $C_1^{(2i)\prime\prime}$, $C_1^{(2i-1)\prime\prime}$ is with $C_1^{(2i)\prime}$, $C_2^{(2i)\prime}$ with $C_2^{(2i+1)\prime\prime}$, $C_2^{(2i)\prime\prime}$ with $C_2^{(2i+1)\prime}$ (see Figure 3 for $n = 2$), and other cuts are identified trivially, i.e. $C_t^{(s)\prime}$ is identified with $C_t^{(s)\prime\prime}$. Denote the obtained genus $2n+2$ surface by X_1. Distinguish $2n$ handles $H_1, \dots, H_{2n}$ (those produced as a result of non-trivial identifications). Define $f^1_0 : X_1 \to X_0$ as in the Schori example.

Let X_k be a genus $\big((2n+1)^k - 2kn\big)$-surface, let $2n$ 2^k-handles $H_1, \dots, H_{2n}$ in X_k be distinguished, and let $C_1, \dots, C_{2n}$ be simple closed curves in the handles. Following the procedure described in the first step obtain $\bar{X}_k^{(i)}$, $i = 1, \dots, 2n+1$ cut surfaces, so that in each of them the cut handles $\bar{H}_j^{(i)}$, $j = 1, \dots, 2n$, are distinguished, and $C_j^{(i)\prime}$ and $C_j^{(i)\prime\prime}$ are boundary curves contained in $\bar{H}_j^{(i)}$. Then make identifications as follows (see Figure 4 for $n = 2$):

1. for $1 \le s \le n$ identify $C_s^{(s)\prime}$ with $C_s^{(n+1)\prime\prime}$, and $C_s^{(s)\prime\prime}$ with $C_s^{(n+1)\prime}$,
2. for $n+2 \le s \le n+1$ identify $C_{s-1}^{(s)\prime}$ with $C_{s-1}^{(n+1)\prime\prime}$, and $C_{s-1}^{(s)\prime\prime}$ with $C_{s-1}^{(n+1)\prime}$,
3. identify other handles trivially.



a)



b)

Figure 3: Schreier diagrams for the weak solenoid with an 8-ended path component: a) the 5-fold covering space Λ_1 of the figure 8, b) the 5-fold covering space Λ_2 of Λ_1

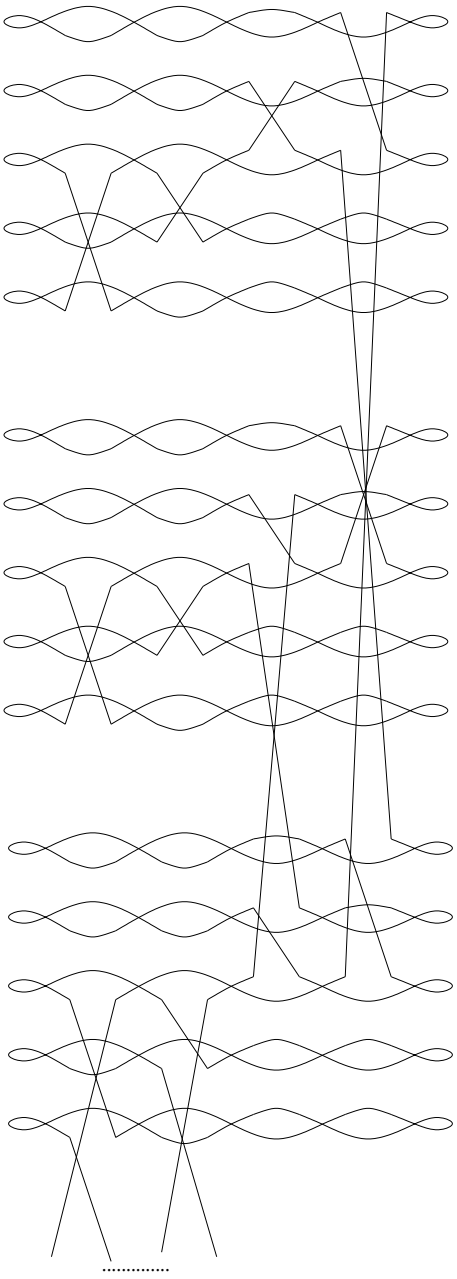


Figure 4: Schreier diagrams for the weak solenoid with an 8-ended path component: the 5-fold covering space Λ_3 of Λ_2, three first layers.

Denote the resulting identification space X_{k+1}, and define $f_k^{k+1} : X_{k+1} \to X_k$ as in the Schori example. Obtain X^{4n}_{Sch} as the inverse limit of the sequence of covering spaces f^k_{k-1}.

Choose a point $x_0 \in H_1 \cap H_2$. For each $k > 0$ there is a unique point x_k such that $f_0^k(x_k) = f^k_{k-1} \circ \cdots \circ f^1_0(x_k) = x_0$ and $x_0 \in H_n \cap H_{n+1}$, where H_i are handles in X_k. Denote $x = (x_k)$. Using the method of the Schreier continuum one obtains that the path-connected component L_x has $4n$ ends, topologically almost all leaves in X^{4n}_{Sch} have 1 end and there is an uncountable infinity of path-connected components with 2 ends.

4.3.2 The generalised Schori example with a $4n+2$-ended path component

Such an example is obtained by a slight alteration of the one in Section 4.3.1: at the first step one identifies $2n$ cut copies of a genus 2 surface X_0 along $2n-1$ cuts to obtain a genus $2n$ surface X_1. On the k-th step one considers a genus $\big((2n)^k - k(2n-1)\big)$-surface X_k with $(2n-1)$ handles, which are identified as in Section 4.3.1, only there is one less identification of the second type. The resulting space X_{k+1} is a covering space of X_0.

So one obtains a weak solenoid X^{4n+2}_{Sch} with one $4n+2$-ended path component, which also has path-connected components with 1 and 2 ends. By the same argument as in Section 4.3.1 the generic path-connected component has 1 end, and there is an uncountable infinity of path-connected components with 2 ends.

4.4 Rogers-Tollefson example revisited

The example introduced by Rogers and Tollefson [27] (see also Example 3.2) is that of a non-homogeneous solenoid

$$K_\infty = \varprojlim\{K_i, f^i_{i-1}, \mathbb{N}_0\},$$

where K_i is the Klein bottle and f^i_{i-1} is a 2-fold covering map constructed as follows (see Fokkink and Oversteegen [15]).

Represent the 2-torus as the quotient space $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and define

$$\bar{f} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : (x, y) \mapsto (x, 2y), \tag{4.9}$$

a 2-fold covering map of the torus by itself. The Klein bottle is the quotient of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ under the map $i : (x, y) \mapsto (x + \frac{1}{2}, -y)$, and the covering projection (4.9) induces a 2-fold covering map $f : K \to K$. Define the curves b and a in K by $b(t) = (\frac{1}{2}t, 0)$ and $a(t) = (0, t)$ for $t \in [0, 1]$. Then the homotopy classes $a = [a]$ and $b = [b]$ generate the fundamental group $\pi_1(K, \{0\})$ with a single relation $bab^{-1} = a^{-1}$. The induced map of the fundamental group is

$$f_\sharp : \pi_1(K, \{0\}) \to \pi_1(K, \{0\}) : (b, a) \mapsto (b, a^2),$$

with the corresponding decreasing chain of groups $G_k = \langle b, a^{2k}\rangle$. Although f is a regular covering map, the composition of f with itself is not regular. The kernel

of the group chain $\{G_k\}$ depends on the point, more precisely, for $\mathrm{id} = (\{0\})$ we have $\mathcal{K}_{\mathrm{id}} = \langle b \rangle$ and for any other point $x \in f_0^{-1}(\{0\})$ which does not belong to the path-connected component of id we have $K_x = \langle b^2 \rangle$. The Schreier diagrams for the pairs (G_0, G_k) with $k = 0, 1, 2, 3$ are shown in Figure 5.

Using the argument on the lengths of paths similar to the ones in Section 4.2 we obtain that the path-connected component of the point $\mathrm{id} \in K_\infty$ has 1 end, and any other path-connected component in K_∞ has 2 ends.

5 Schreier continua for Markov minimal sets

Another class of systems for which there exists a 'nice' inverse limit representation of the Schreier continuum is that of foliated circle bundles with codimension 1 foliations and Markov minimal sets, i.e. exceptional minimal sets with a holonomy pseudogroup generated, in a certain sense, by a subshift of finite type (see Cantwell and Conlon [10, 11]). We describe the inverse limit representation and give an example of computation of ends of leaves in a Markov minimal set using the Schreier continuum.

5.1 Foliated bundles with Markov minimal sets

Let B be a closed manifold, let $b \in B$ be a point and let

$$h : \pi_1(B, b) \to \mathrm{Homeo}(\mathbb{S}^1) \tag{5.10}$$

be a representation of the fundamental group of B onto the group of orientation-preserving circle homeomorphisms. Denote by $E = \widetilde{B} \times_{\pi_1(B,b)} \mathbb{S}^1$, where $\widetilde{B}$ is the universal cover of B, the suspension of the representation (5.10) (see, e.g. [6]). The projection $\pi : E \to B$ induced by the covering projection of the universal cover, is a foliated bundle with fibre $\mathbb{S}^1$ and foliation $\mathcal{F}$ induced by the product foliation on $\widetilde{B} \times \mathbb{S}^1$.

Fix an embedding $\rho : \mathbb{S}^1 \to E$ of the fibre of π such that $F = \rho(\mathbb{S}^1)$ is a fibre, and let $T \subset \mathbb{S}^1$ be a global transversal of $\mathcal{F}$, i.e. an open relatively compact subset such that each leaf of $\mathcal{F}$ meets $\rho(T)$ at least once. Assume that the foliation $\mathcal{F}$ has an exceptional minimal set X, i.e. $C = X \cap \rho(T)$ is a closed, totally disconnected and perfect subset of the compact fibre F, and is, therefore, a Cantor set [6]. Denote by Γ the holonomy pseudogroup of $\mathcal{F}$ on T, and by Γ_C the pseudogroup on C induced by Γ. We further assume that X is a Markov minimal set.

Recall (see Cantwell and Conlon [10, 11]) that a Markov minimal set is an exceptional minimal set of a codimension 1 foliation with a holonomy pseudogroup generated, in a certain sense, by a subshift of finite type. In more detail, if $m \geq 2$ is an integer and if $\mathcal{K}$ is the Cantor set $\{1, 2, \ldots, m\}^{\mathbb{N}}$ with typical element $\imath = (i_1, i_2, \ldots)$, then a *subshift of finite type* is a map $\sigma : \mathcal{K}_P \to \mathcal{K}_P$ on the subspace

$$\mathcal{K}_P = \{\imath \in \mathcal{K} \mid p_{i_j i_{j+1}} = 1, 1 \leq j \leq \infty\} \subset \mathcal{K}$$

associated to an $m \times m$ matrix $P = (p_{ij})$ with 0 and 1 entries, which is given by

$$\sigma(i_1, i_2, i_3 \ldots) = (i_2, i_3 \ldots).$$

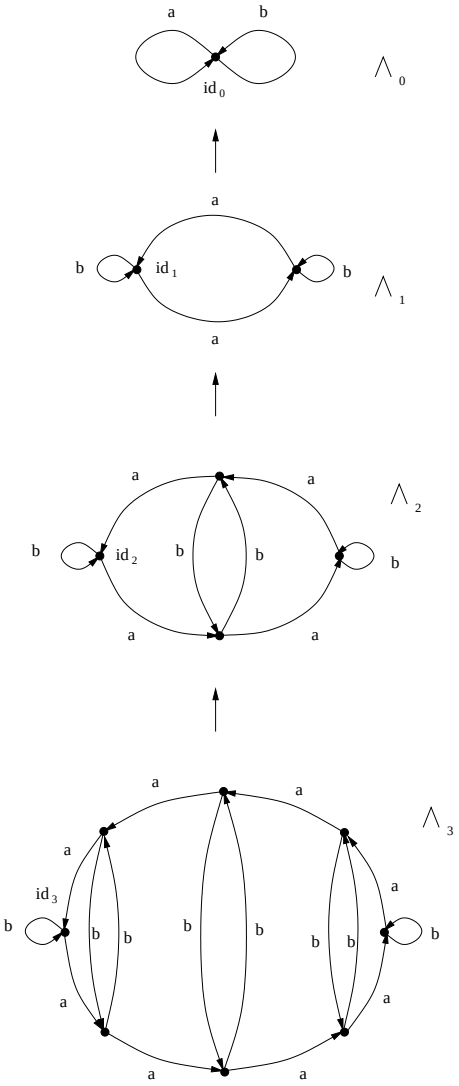


Figure 5: Schreier diagrams for the Rogers-Tollefson solenoid for $i = 0, 1, 2, 3$

A minimal set X is called *Markov*, and $C = X \cap \rho(T)$ is called a *Markov* Γ*-minimal set* if it is semi-conjugated to a subshift of finite type. More precisely, C is Markov Γ-minimal if there exist a homeomorphism $\tau : C \to C$ and a continuous surjection $\eta : C \to \mathcal{K}_P$ such that the pseudogroup Γ_C is generated by local restrictions of τ, for any $x \in C$ the preimage $\eta^{-1}(\eta(x))$ is either a single point or a pair of endpoints of a gap of C, and

$$\eta \circ \tau = \sigma \circ \eta.$$

A property of Markov minimal sets which we use to construct the inverse limit representation of the Schreier continuum is that one can choose generators of the pseudogroup on $\rho(T)$ in such a way that their ranges are disjoint. More precisely, one can find an open neighbourhood $W \subset \rho(T)$ of C and an embedding $\varphi : W \to \rho(T)$ equipped with the following data. There is a finitely generated pseudogroup $\Gamma' = \langle h_1, \ldots, h_m \rangle$ on $\rho(T)$ with Γ'-minimal set C' such that $C' = \varphi(C)$ and

$$\varphi \circ \Gamma_C \circ \varphi^{-1} = \Gamma'|_{C'},$$

where Γ' has the following properties. The ranges of h_j are disjoint, and there is a set of closed non-degenerate subintervals $\{I_1, \ldots, I_m\}$ of $\rho(T)$ with $I_j \subset R(h_j)$, $1 \leq j \leq m$, such that $I_j \cap D(h_i) \neq \emptyset$ if and only if $I_j \subseteq h_i^{-1}(I_i)$. The minimal set C' of such a system is a subset of

$$Z_0 = \mathrm{Bd}\Big(\bigcap_{n=1}^{\infty} \big(\bigcup_{|w_n|=n} I_{w_n} \big) \Big),$$

where I_{w_n} with $w_n = h_{i_1} \circ \cdots \circ h_{i_n}$ denotes the image $w_n \circ h_{i_n}^{-1}(I_{i_n})$. To such a system one can associate a matrix $P' = (p_{ij})$ where $p_{ij} = 1$ if and only if $I_j \subset D(h_j)$ and $p_{ij} = 0$ otherwise. Since the ranges of the generators are disjoint, there is a well-defined map

$$T : \bigcup_{i=1}^{m} R(h_i) \to \bigcup_{i=1}^{m} D(h_i)$$

defined by $T(x) = h_i^{-1}(x)$ if $x \in R(h_i)$. The restriction $\tau' = T|_{Z_0}$ is conjugated to the subshift of finite type associated to the matrix P', and if the foliation $\mathcal{F}$ of E is at least C^2, then the data above can be chosen so that the conjugating map is a homeomorphism. The system $S = (\{I_1, \ldots, I_m\}, \{h_1, \ldots, h_m\}, P')$ is called the *Markov system* and Γ' is called the Markov pseudogroup.

5.2 Ends of leaves in Markov minimal sets

As in Section 5.1 let Γ be a pseudogroup of a foliation $\mathcal{F}$ of a foliated bundle $\pi : E \to B$ with Markov minimal set X associated to a transversal $T \subset \mathbb{S}^1$, and let $C = X \cap \rho(T)$. We notice that here T can be only a subset of $\mathbb{S}^1$, and the pseudogroup Γ is the restriction of a subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ to T (see also Sacksteder's example in Section 5.3). We will also denote by Γ the induced pseudogroup on $\rho(T)$.

As it was indicated in Section 5.1 we can assume that Γ is Markov with the associated Markov system $S = (\{I_1, \dots, I_m\}, \{h_1, \dots, h_m\}, P)$ such that the conjugacy map η to the subshift of finite type is a homeomorphism. Let $x \in C$ be a point and obtain a directed graph $|\Gamma(x)|$ of the Γ-orbit of x as follows.

Denote by c_i, $i = 1, \dots, m$, the number of entries $p_{\lambda i} = 1$ in the matrix P, $\lambda = 1, \dots, m$, i.e. the number of intervals I_λ such that $I_\lambda \subset D(h_i)$. The set of vertices of $|\Gamma(x)|$ is a subset of points $y \in C$ such that there exists a word $w_n = h_{i_1} \circ \dots \circ h_{i_n}$ such that $w_n(x) = y$. If v_1 and v_2 are vertices of $|\Gamma(x)|$, then v_1 and v_2 are joined by an edge labelled by h_i if and only if $h_i(v_1) = v_2$. Since ranges of h_i, $i = 1, \dots, m$, are disjoint, each vertex is an ending vertex of exactly one edge. If v is an ending vertex of an edge labelled by h_i, then v is the starting vertex of c_i edges, and h_λ is an edge starting at v if and only if $p_{\lambda i} = 1$. So if e is an edge, then $\tau(t(e)) = s(e)$.

Remark 5.1 Here we label edges by symbols corresponding to the generators of a subgroup of Homeo($\mathbb{S}^1$) rather than by symbols denoting elements of the fundamental group of B. We also drop the edges corresponding to the generators in the kernel of the homomorphism h, so we obtain the *simplified* Schreier continuum.

The graph $|\Gamma(x)|$ has the same number of ends as the leaf L_x of X passing through $x \in C$ [10]. It was computed in [10] that $|\Gamma(x)|$ has a Cantor set of ends. The same result can be obtained with the help of the Schreier continuum.

5.2.1 The Schreier continuum for Markov minimal sets

If Γ is C^2 Markov, there is a natural way to associate to the Γ-minimal Markov set $C \subset F$ the Schreier continuum. As before, $\Gamma_C = \Gamma | C$.

Theorem 5.1 *Let Γ be a C^2 Markov pseudogroup on an open set $W \subset \rho(T)$ with the associated Markov system $S = (\{I_1, \dots, I_m\}, \{h_1, \dots, h_m\}, P)$, $m \geq 2$, and let $C \subset T$ be a Γ-invariant Markov minimal set. Then there exist an inverse sequence $\{J^{(k)}, f^k_{k-1}, \mathbb{N}\}$ of finite discrete spaces $J^{(k)}$ and a family of continuous mappings $\theta_k : C \to J^{(k)}$ such that $f^k_{k-1} \circ \theta_k = \theta_{k-1}$ and the induced mapping*

$$\theta_\infty : C \to J^\infty = \varprojlim \{J^{(k)}, f^k_{k-1}, \mathbb{N}\} \tag{5.11}$$

is a homeomorphism. Moreover, there is an induced action of Γ_C on $J^{(k)}$ and J^∞ such that the projections $f_k : J^\infty \to J^{(k)}$ are Γ_C-equivariant.

Proof. Let $X_i = h_i^{-1}(I_i)$ and denote $I_{i_1 i_2 \dots i_k} = h_{i_1} \circ h_{i_2} \circ \dots \circ h_{i_k}(X_{i_k})$. Let $V_j \subset W$ be an open subset such that $V_j \supset X_j = h_j^{-1}(I_j)$, $V_i \cap V_j = \emptyset$ if $i \neq j$, and $V_j \subset \mathrm{dom}(X_j)$. For $k \in \mathbb{N}$ let

$$J^{(k)} = \{(j_1, \dots, j_k) \mid p_{j_\ell j_{\ell-1}} = 1,\ \ell = 1, \dots, k-1\}$$

be the set of admissible k-tuples, then clopen sets

$$U^{(k)}_{j_1 \dots j_k} = C \cap I_{j_1 \dots j_k} = C \cap (h_{j_1} \circ \dots \circ h_{j_k}(V_{j_k}))$$

are disjoint, and $\mathscr{U}^{(k)}$ covers C. Define $f_k^{k+1} : J^{(k+1)} \to J^{(k)}$ by

$$f_k^{k+1}(j_1, \ldots, j_{k+1}) = (j_1, \ldots, j_k),$$

and $\theta_k : C \to J^{(k)}$ by saying that

$$\theta_k(x) = (j_1, \ldots, j_k) \qquad \text{if and only if} \qquad x \in U_{j_1 \ldots j_k}.$$

Both f_k^{k+1} and θ_k are continuous with respect to the discrete topology on $J^{(k)}$. The inverse limit space

$$J^\infty = \varprojlim\{J^{(k)}, f_{k-1}^k, \mathbb{N}\} \subset \prod_{k \in \mathbb{N}} J^{(k)}$$

is naturally identified with K_P, and the induced mapping

$$\theta^\infty : C \to J^\infty = \varprojlim\{J^{(k)}, f_{k-1}^k, \mathbb{N}\}$$

is a homeomorphism. Define the map $\mu : \{1, \ldots, m\}^n \to \{1, \ldots, m\}^{n-1}$, $n \geq 2$, by

$$\mu(i_1, i_2, \ldots, i_n) = (i_1, \ldots, i_{n-1}),$$

and the action of $w_n = h_{i_1} \circ \cdots \circ h_{i_n} \in \Gamma_C$ on $J^{(k)}$ is by

$$w_n \cdot (j_1, \ldots, j_k) = \mu^n(i_1, i_2, \ldots, i_n, j_1 \ldots, j_k). \tag{5.12}$$

Then θ_k and f_k^{k+1} are Γ-equivariant. It follows that θ^∞ is equivariant with respect to the Γ_C-action on J^∞ given by

$$w_n \cdot (t_1, t_2 \ldots) = (w_n \cdot t_1, w_n \cdot t_2, \ldots),$$

and the projections $f_k : J^\infty \to J^{(k)}$ are also Γ_C-equivariant.

□

We now construct the Schreier continuum. To each $J^{(k)}$ we associate a graph Λ_k in the following manner. Vertices of Λ_k are elements of $J^{(k)}$, and a vertex $v = (j_1, \ldots, j_k)$ is the starting point of an edge labelled by h_i if and only if $p_{ij_1} = 1$. The ending point of such an edge is a vertex $v' = \mu(i, j_1, \ldots, j_k) = (i, j_1, \ldots, j_{k-1})$. Put a length structure on Λ_k so that the length of each edge is 1, and let ρ_k be the associated complete length metric. Extending f_k^{k+1} to the edges of the graph one obtains continuous mappings $\nu_k^{k+1} : \Lambda_{k+1} \to \Lambda_k$. Let

$$\Lambda = \varprojlim\{\Lambda_k, \nu_k^{k+1}, \mathbb{N}\}$$

and put a metric ρ_∞ on Λ in such a way that if L_v is a path-connected component of a point $v = (v_i) \in \Lambda$, then the restriction $\nu_k|_{L_v} : L_v \to \Lambda_1$ of the projection $\nu_k : \Lambda \to \Lambda_1$ is a local isometry with respect to the complete length metric ρ on L_v associated to the length structure on L_v induced by ρ_∞. Then, if v is a vertex of Λ and $x \in C$ is such that $\theta_\infty(x) = v$, the path-connected component L_v is isometric to the graph $|\Gamma(x)|$ of the Γ-orbit of x, and so has the same number of ends.

5.2.2 Ends of leaves in Markov minimal sets

We show that L_v has a Cantor set of ends. Recall that no vertex of a graph $|\Gamma(x)|$ of the Γ-orbit of $x \in C$ is an ending point of two distinct edges. Since for $v = \theta_\infty(x)$ the graphs $|\Gamma(x)|$ and L_v are isometric, no vertex of the latter is an ending point of two distinct edges. This property comes down to the following property of the graphs in the inverse sequence.

Lemma 5.1 *Let Λ_k be a graph in the inverse sequence associated to a Markov minimal system $S = (\{I_1, \ldots, I_m\}, \{h_1, \ldots, h_m\}, P)$, $m \geq 2$, and let e_1 and e_2 be edges with the same ending point but distinct starting points. Let e_1' and e_2' be edges in Λ_{k+1} such that $\nu_k^{k+1}(e_1') = e_1$ and $\nu_k^{k+1}(e_2') = e_2$. Then e_1' and e_2' have distinct endpoints.*

Proof. Assume the edge e_1 is marked by h_α and the edge e_2 is marked by h_β. Let $s(e_1) = (i_1, \ldots, i_k)$ and $s(e_2) = (j_1, \ldots, j_k)$, then $t(e_1) = t(e_2)$ reads

$$\mu(\alpha, i_1, \ldots, i_{k-1}, i_k) = \mu(\beta, j_1, \ldots, j_{k-1}, j_k),$$

which implies $\alpha = \beta$ and $i_\ell = j_\ell$ for $\ell = 1, \ldots, k-1$. By a similar argument $t(e_1') = t(e_2')$ implies $i_k = j_k$, which contradicts the assumption that $s(e_1)$ and $s(e_2)$ are distinct.

□

Let γ be a path in L_v consisting of edges $e_1 e_2 \ldots e_n$ and so of the length n. We say that γ has a reduced form if it does not contain concatenated pairs of the form $e\bar{e}$ or $\bar{e}e$, where $\bar{e}$ corresponds to traversing the edge e in the direction opposite to its orientation.

Lemma 5.2 *Let $x \in C$, and L_v be a path connected component of $v \in \Lambda$. Let $B(v, n) \subset L_v$ be a closed ball of radius n about v with respect to the length metric ρ on L_v. For each vertex $w \in B(v, n+1) \backslash B(v, n)$ let e_w be an edge which has w as a vertex and which has another vertex inside $B(v, n)$. Define*

$$W_n^+ = \{w \in B(v, n+1) \backslash B(v, n) \mid t(e_w) = w\},$$
$$W_n^- = \{w \in B(v, n+1) \backslash B(v, n) \mid s(e_w) = w\},$$

and assume card $Y_n^+ >$ card Y_n^-. *Then the number of distinct path-connected components of $\Lambda_x \backslash B(x, n)$ is at least* card $Y_n^+ -$ card Y_n^-.

Proof. Let $w, w' \in W_n^+ \cup W_n^-$, and suppose there is a path γ in $L_v \backslash B(v, n)$ with $\gamma(0) = w$ and $\gamma(1) = w'$. We can assume that γ is in a reduced form. Since no vertex in L_v can be an ending vertex of a two distinct edges, there is only three possibilities: both $w, w' \in W_n^-$, $w \in W_n^+$ and $w' \in W_n^-$, $w \in W_n^-$ and $w' \in W_n^+$. The statement of the lemma follows.

□

By Lemma 5.2 if W_n^- is an empty set, then each point of W_n^+ represents a distinct path connected component of $L_v \backslash B(v,n)$.

A path γ is called an edge-loop if $\gamma(0) = \gamma(1)$.

Corollary 5.1 *A path connected component of Λ contains at most one edge-loop.*

Remark 5.2 The statement of Corollary 5.1 concerns the simplified Schreier continuum. If a path component of the Schreier continuum does not contain any edge-loops the corresponding leaf of the foliation $\mathcal{F}$ might have more complicated geometry. For example, in Sacksteder's example in Section 5.3 every leaf is a surface of infinite genus.

We are now in a position to compute the number of ends of path components in the Schreier continuum of a Markov minimal set.

Theorem 5.2 *Every path connected component in the Schreier continuum Λ associated to a Markov minimal set has a Cantor set of ends.*

Proof. Assume first that a path connected component L_v contains an edge-loop γ. Let $n \in \mathbb{N}$ be large enough so that $B(v,n)$ contains all vertices of edges in γ in its interior. Then for all $m \geq n$ the set W_m^- is empty. Since C is perfect, no end is isolated.

Assume that L_v contains no edge-loops. By an inductive argument, one can see that for any $n \geq 1$ the set W_n^- contains precisely one vertex. It follows that no end is isolated.

□

5.3 Sacksteder's example [28, 6]

Represent $\mathbb{S}^1$ as the unit interval in $\mathbb{R}$ with endpoints identified, and let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be given by

$$f(z) = z + 1/3 \mod 1,$$

and define $g : \mathbb{S}^1 \to \mathbb{S}^1$ so that

$$g(z) = \begin{cases} \frac{z}{3}, & 0 \leq z \leq \frac{1}{2} \\ 3z - \frac{5}{3}, & \frac{2}{3} \leq z \leq \frac{5}{6} \end{cases},$$

and g is defined elsewhere in $[0,1]$ in such a way that g is smooth, $g(1) = 1$, and g has a smooth inverse on $[0,1]$. Let B be an orientable genus 2 surface and a_1, b_1, a_2, b_2 be the generators of the fundamental group $\pi_1(B, b_0)$, $b_0 \in B$, subject to a relation $[a_1, b_1][a_2, b_2] = \mathrm{id}$. The choice of the base point $b_0 \in B$ fixes the identification of $\pi_1(B, b_0)$ with the group G of covering transformation of $p : \mathbb{H}_2 \to B$, where $\mathbb{H}_2$ is the hyperbolic plane. Define the homomorphism $h : G \to \mathrm{Diff}^\infty(\mathbb{S}^1)$ by

$$h(a_1) = f, \qquad h(a_2) = g, \qquad h(b_1) = h(b_2) = \mathrm{id},$$

and let $\pi : \mathbb{H}_2 \times_G \mathbb{S}^1 \to B$ be the suspension of h. The total holonomy group $G_{\mathcal{F}}$ on $\mathbb{S}^1$ is a subgroup of $\mathrm{Diff}^\infty(\mathbb{S}^1)$ generated by f and g. The $G_{\mathcal{F}}$-minimal subset $\mathbb{S}^1$ is a Cantor set

$$C = \bigcap_{n=1}^{\infty} C_n,$$

where

$$C_0 = \left[0, \tfrac{1}{6}\right] \cup \left[\tfrac{1}{3}, \tfrac{1}{2}\right] \cup \left[\tfrac{2}{3}, \tfrac{5}{6}\right],$$

and for each union of disjoint intervals C_n the set C_{n+1} is obtained by removing the middle third from each of the intervals, as in the construction of the ternary Cantor set (see [28, 6] for details). It follows that the foliation $\mathcal{F}$ has an exceptional minimal set X.

To see that X is Markov, notice that every relatively compact open neighbourhood $W \subset \mathbb{S}^1$ of $\left[0, \frac{1}{6}\right]$ is a global transversal of $\mathcal{F}$. Choose W so that $W \cap \left[\frac{1}{3}, \frac{1}{2}\right] = \emptyset$ and $W \cap \left[\frac{2}{3}, \frac{5}{6}\right] = \emptyset$. The restricted holonomy pseudogroup is generated by homeomorphisms $h_0 = g|_W$ and $h_1 = g \circ f|_W$ which have disjoint ranges. Let

$$I_0 = h_0\Big(\Big[0, \tfrac{1}{6}\Big]\Big), \qquad\qquad I_1 = h_1\Big(\Big[0, \tfrac{1}{6}\Big]\Big),$$

and let P be the 2×2 matrix with unit entries. Then $S = (\{I_0, I_1\}, \{h_0, h_1\}, P)$ is a Markov system. Figure 6 presents first components in the inverse sequence of graphs associated to the Schreier continuum.

6 Conclusions

For a foliated bundle $p : E \to B$ with foliation $\mathcal{F}$ we have introduced the notion of the *Schreier continuum* Λ which can be thought of as the union of holonomy graphs of leaves of the minimal set of $\mathcal{F}$ with suitable topology, and have shown that in the case when the minimal set of $\mathcal{F}$ is transversely a Cantor set, a great deal can be said about asymptotic properties of leaves in the minimal set by means of the study of the associated inverse limit representation of Λ. We have carried out the construction of the inverse limit representation of Λ together with the corresponding analysis in the case when $E = X$ is a generalised solenoid or X is a Markov minimal set of a codimension 1 foliation.

In particular, using the method of the Schreier continuum we have shown that in the non-homogeneous solenoid of Schori [29] the number of ends varies from leaf to leaf, more precisely, there are three types of leaves: a single leaf with 4 ends, an uncountable infinity of leaves with 2 ends and an uncountable infinity of leaves with 1 end. We confirmed by direct computation that the set of points in the solenoid corresponding to leaves with 1 ends is residual and, although the set

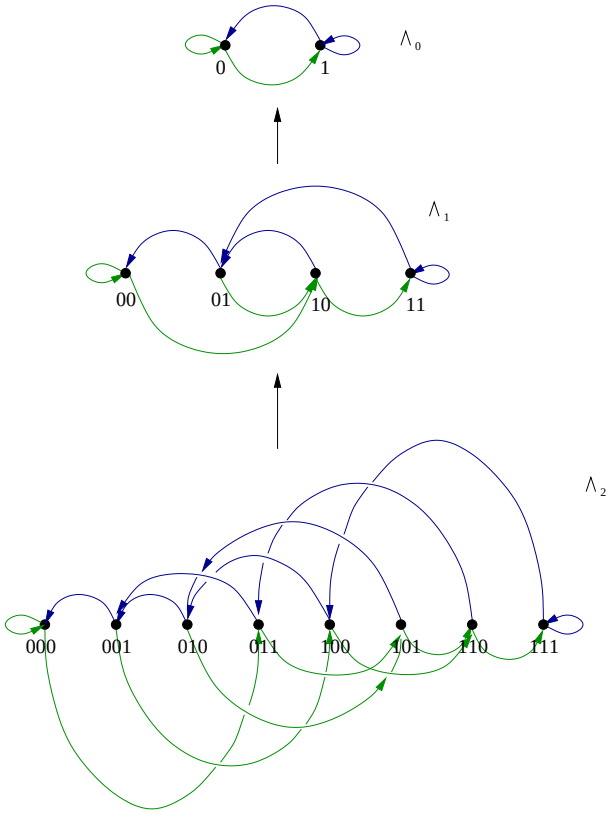


Figure 6: The graphs Λ_i for a Markov system on 2 generators, $i = 0, 1, 2$.

of points with leaves with 2 ends is meagre, there is still an uncountable number of such leaves.

We defined the notion of the Schreier continuum for foliated bundles, and in a future paper we intend to show how to extend this notion to a wider class of foliated spaces.

Before the introduction of the Schreier continuum there was no systematic way of finding the ends of path components of solenoids. We have seen how the Schreier continuum can be used to analyse quite complicated examples, and so it is natural to ask the following question.

Question 1 *What types of end structures can a solenoid have?*

Blanc [1] proves that if a residual set of points in a foliated space E has leaves with 2 ends, then almost every leaf in E (with respect to a harmonic measure) has 2 ends. Blanc also announces an example showing that this need not be the case if E has a residual set of points with leaves with 1 end. It would be of interest to determine whether this can occur in solenoids.

Question 2 *For the structure of ends in a solenoid, does topologically 'almost all' imply measure-theoretically 'almost all' ?*

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