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The Tate Thomason Conjecture

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Abstract

We prove the Tate Thomason conjecture using $\mathcal{K}$ where $\mathcal{K}$ is the complex topology spectrum. (see the introduction below). Fundamental to our proof is Theorem 2.2. Much of this work is related to the paper [TH89]

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Introduction:

By a spectrum we mean the following: A spectrum X is a collection of simplicial sets X_n for $n \geq 0$ together with morphisms of simplicial sets $\sigma_n : \Sigma X_n \mapsto Y_n$. A morphism of spectra $f : X \mapsto Y$ is a collection of morphisms $f_n : X_n \mapsto Y_n$ of simplicial sets that commute with the structure maps σ_n, ie $\sigma_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$. We are going to consider the stable homotopy category $\mathcal{S}$ derived from the Bousfield-Friedlander model category of simplicial spectra [BF78]. The topological K spectrum is written $\mathcal{K}$ and its localization is written L most of the time. The spectra we consider are K(X) where X is a scheme over a finite field F_q with $q = p^s$ where $p \neq l$, and K(X) is Quillen's Algebraic K theory spectra related to X. See [Jar] page 148-150. Our aim is to prove the Tate Thomason conjecture ([TH89]): Let X_∞ be $X \times_{Spec(F_q)} Spec(\bar{F}_q)$, where X is a smooth projective variety over $Spec(F_q)$, and $\bar{F}_q$ is the algebraic closure of F_q, then the homotopy group $\pi_{-1}(LK(X_\infty))$ is reduced. This statement implies the Tate Conjecture, (See Remark 0.2 below and also see [TH89] page 390 diagram 21). The proof is based on theorem 2.2 from section 2.

0. The Tate Conjecture

For the projective smooth variety X_∞ over the finite field F_q, consider the i-th Chow Group $CH^i(X_\infty)$ generated by cycles of codimension i on X_∞ modulo rational equivalence. If l is a prime invertible in F_q we can define a Q_l-linear cycle morphism from the i-th Chow group into the $2i$-th l-adic etale cohomology with i-th Tate twist coefficient:

$$\gamma^i_{Q_l} : CH^i(X_\infty) \otimes Q_l \mapsto H^{2i}(X_\infty, Q_l(i))$$

Consider F_q as a finite subfield of $\bar{F}_q$. Let $H^{2i}(X_\infty, Q_l(i))^{Gal(\bar{F}_q/Fq)}$ be the subspace of $H^{2i}(X_\infty, Q_l(i))$ of finite order under the action of $Gal(\bar{F}_q/F_q)$, then the Tate Conjecture states:

0.1 **Tate Conjecture**: *The image of the cycle class morphism $\gamma^i_{Q_l}$ in the cohomology group $H^{2i}(X_\infty, Q_l(i))$ is exactly $H^{2i}(X_\infty, Q_l(i))^{Gal(\bar{F}_q/F_q)}$*

0.2 *Thomason's reformulation of the Tate Conjecture:*

Take the smooth projective variety X_∞. Let $X_n = X \otimes F_{q^n}$ where X is a smooth projective variety over F_q as mentioned in the introduction. Let $\mathcal{K}$ be the complex topology spectrum. Thomason proves in [Th89] the following theorem:

Theorem 0.1 (Tate Thomason's Conjecture): *The Tate Conjecture is equivalent to the finiteness statement that for all $n \in N$, See [TH89]*

$$Hom(Q/Z_{(l)}, \pi_{-1}(L_{\mathcal{K}}K(X_n))) = 0$$

Remark 0.1: observe that in page 390 diagram 21 Thomason states that the arrow $f \otimes Q$ should be 0 for the Tate conjecture to be true. Hence we only need to prove that

$$Hom(Q/Z_{(l)}, \pi_{-1}(L_{\mathcal{K}}K(X_\infty))) = 0$$

Remark 0.2: Though the above Hom being 0 is sufficient to prove the Tate Conjecture, to prove the equivalence with the Tate Conjecture we need the Hom with the X_n [TH89]

Definition 0.1: *A group G which verifies that $Hom(Q/Z_{(l)}, G) = 0$ is said to be l reduced.*

We will simplify all over this work the terminology by saying that a group is reduced when we really mean that it is l-reduced.

The main lemmas stated by Thomason in [Th89] to prove the Tate Conjecture are

Lemma 0.1: *Let X be a smooth projective variety over F_q with $q = p^n$ and p prime. If l is a prime number different from p, and if $K^{Top}(X)$ is in Thomason's notation the topological K-theory spectrum, and $(...)\widehat{}$ is the l-adic completion of a spectrum, then*

$$L_{\mathcal{K}}K(X)\widehat{} \cong K^{Top}(X)\widehat{}$$

The proof of this lemma follows from Thomason's descent theorem proved in [TH85, 4.1]which relates algebraic K- theory to topological K-theory. See also ([TH89], page 388, equation (14) and references therein)

Remark 0.3: It also follows from [TH89] page 388 equation (13) that

$$K^{Top}/l^{\nu}(X) \cong L_{\mathcal{K}}(X)/l^{\nu}$$

See also the descent problem in ([M96], section 4)

Lemma 0.2:(See [TH89],the lemma of page 387)

The image of $colim K_0^{Top}(X_n)\widehat{} \otimes Q$ in $K_0^{Top}(X_\infty)\widehat{} \otimes Q$ consists precisely of those elements of finite orbit under $Gal(\bar{F}_q/F_q)$.

Remark 0.4:(See [TH89], page 387 square diagram 10) which is equal to

$$colim(K_0(X_n) \otimes Q_l) = K_0(X_\infty) \otimes Q_l \to colim K_0^{Top}(X_n)\widehat{} \otimes Q) \downarrow K_0^{Top}(X_\infty)\widehat{} \otimes Q$$

Where the equality means an isomorphism and in light of this, lemma 0.2, and the well established fact that $K_0(X_\infty) \otimes Q_l$ is isomorphic to $\bigoplus_{i=1}^{d} CH^i(X_\infty) \otimes Q_l$ with d the dimension of X, and that $K_0^{Top}(X_\infty)\widehat{} \otimes Q$ is isomorphic to $\bigoplus_{i=1}^{d} H_{et}^{2i}(X_\infty, Q_l(i))$,(see [TH89, page 390)the above composed map is exactly the cycle map $\gamma(X_\infty)_{Q_l}$ and the Tate Conjecture is equivalent to the conjecture that $K_0(X_\infty) \otimes Q_l$ and $colim(K_0^{Top}(X_n)\widehat{} \otimes Q)$ have the same image in $K_0^{Top}(X_\infty)\widehat{} \otimes Q$.

Lemma 0.3: is diagram 21 on page 390

Remark 0.5: To our knowledge, Thomason never said anything about the l-reducibility of the homotopy group $\pi_{-1}L_{\mathcal{K}}K(X_\infty))$

1. The Moore Spectrum and an important exact sequence

Remark 1.1: Given l^ν, with $\nu \in N$,and $M(Z/l^\nu)$ the Moore spectrum of the ring Z/l^ν,and if X is a smooth variety over a field k where $k = F_{q^n}$ or $k = \bar{F}_q$ with $q = p^n$ and L is the localization functor at the complex K-Theory spectrum $\mathcal{K}$, then

$LK(X)/l^\nu = LK(X) \wedge M(Z/l^\nu) = K(X) \wedge L\Sigma^\infty S^o \wedge M(Z_/l^\nu) = K(X)/l^\nu \wedge L\Sigma^\infty S^o = L(K(X)/l^\nu)$. These equalities follow from the smash property for L.

Remark 1.2: $\pi_*(L(K(X)/l^\nu))$ is l^ν-torsion.

The claim follows from the exact sequence

$$0 \mapsto \pi_*(LK(X) \otimes Z/l^\nu \mapsto \pi_*(LK(X) \wedge M(Z/l^\nu)) \mapsto Tor^1(\pi_{*-1}(LK(X)), Z/l^\nu) \mapsto 0$$

which splits (See [TH85] Appendix A, (6))

2. **Main Theorem.**

Theorem 2.2: *The Tate Module of the abelian group $\pi_m(LK(X_\infty))$ is trivial for every integer m*

As we have mentioned above, we have by [TH85] Appendix 6) that the exact sequence below splits

$$0 \mapsto \pi_m(LK(X_\infty)) \otimes Z/l^\nu \mapsto \pi_m(LK(X_\infty)) \wedge M(Z/l^\nu) \mapsto Tor^1(\pi_{m-1}(LK(X_\infty)), Z/l^\nu) \mapsto 0$$

Taking inverse limit over ν on the above exact sequence, we get an exact sequence since lim is left exact, and moreover exact in the case of the above exact sequence since the left extreme of that sequence is a surjective inverse system, and therefore Mittag-Leffler. Then, if we prove that

$$lim\pi_m(LK(X_\infty)) \otimes Z/l^\nu = (\pi_m(LK(X_\infty)))^l \cong lim\pi_m(LK(X_\infty)) \wedge M(Z/l^\nu)$$

where $(\pi_m(LK(X_\infty)))^l$ stands for l-adic completion of that abelian group, with l prime to p the characteristic of the base field of our smooth projective variety, being the lim exact here, we conclude that the Tate Module of $\pi_m(LK(X_\infty))$ is trivial since it is equal to (taking inverse limit over ν for fixed l)

$$limTor^1(\pi_{m-1}(LK(X_\infty)), Z/l^\nu)$$

Again,taking inverse limit with respect to the parameter ν with l fixed,

$$limTor^1(\pi_{m-1}LK(X_\infty)), Z/l^\nu)) = M = 0$$

.

where $M = \lim Tor^1(\pi_{m-1}LK(X_\infty), Z/l^\nu)$ is equal to

$$\Pi_{i=1}^\infty \{g_i = l^i - torsion - element \in \pi_{m-1}(LK(X_\infty)) : lg_{i+1} = g_i\}$$

by definition of the inverse limit and since $Tor^1(-, Z/l^\nu)$ are the l^ν -torsion elements of $\pi_{m-1}LK(X_\infty)$
See [Ro79] 8.4 page 223 and also the fact that $Tor^1_Z(A, B) = Tor^1_Z(B, A)$ on page 222).

Now, consider the image of a map f in $Hom(Q/Z_{(l)}, \pi_{m-1}(LK(X_\infty))$

Since $Q/Z_{(l)}$ is an l quasicycle divisible group (see [LF], if D is the image of f then D is an increasing union of groups D_n, of order l^n, where each D_n is generated by one generator g_n wich is l^n-torsion, and also $lg_n = g_{n-1}$, if the Tate module M trivial, this sequence of g_n has to be trivial and therefore $f = 0$

Therefore we now know that details about the Tate Module M of the homotopy groups $\pi_{m-1}LK(X_\infty)$ are of fundamental importance. The above arguments show that the Tate module M is isomorphic to $Hom(Q/Z_{(l)}, \pi_{m-1}(LK(X_\infty))$ by taking the sequence of generators of the increasing sequence of cyclic subgroups of $Q/Z_{(l)}$ to the sequence of it's images which belong to M. By [TH89] pp 389 (15) we have an exact sequence

$$0 \mapsto Ext(Q/Z_{(l)}, \pi_m(LK(X_\infty))) \mapsto \pi_m((LK(X_\infty))^l) \mapsto Hom(Q/Z_{(l)}, \pi_{m-1}(LK(X_\infty)) \mapsto 0$$

We can compare that exact sequence with the above exact sequence after taking lim and using the fact that $lim\pi_m(LK(X_\infty)) \wedge M(Z/l^\nu) \cong \pi_m((LK(X_\infty))^l)$ by [TH89] (6) pp 387 and (13) pp 388

$$0 \mapsto (\pi_m(LK(X_\infty)))^l \mapsto \pi_m((LK(X_\infty))^l) \mapsto M = \lim Tor^1(\pi_{m-1}LK(X_\infty), Z/l^\nu) \mapsto 0$$

We have the same middle term in both exact sequences and moreover we also proved that the extreme right terms are isomorphic. Therefore by the snake lema, the extreme left terms must also be isomorphic. By [TH89] pp 389 (16) we have another exact sequence see [TH89] pp 389 (16)

$$0 \mapsto lim^1 Hom(Z/l^k Z, \pi_m((LK(X_\infty)))) \mapsto Ext(Q/Z_(l), \pi_m(LK(X_\infty))) \mapsto lim \pi_m(LK(X_\infty)) \otimes Z/l^\nu \mapsto 0$$

Since we have just shown that Ext is isomorphic to the right extreme of the above exact sequence, therefore by [TH89] pp 389 below (17) that means that the l-torsion subgroup of $\pi_m((LK(X_\infty)))$ is finite for any integer m. But if f is in $Hom(Q/Z_{(l)}, \pi_{m-1}(LK(X_\infty)))$ it's image is l-torsion, and hence must be finite, but that is not possible since $Q/Z_{(l)}$ is divisible f cannot have a finite image. Hence $f = 0$ and we conclude that the Tate module M is trivial as stated in theorem 2.2, and moreover we can state the following theorem:

Theorem 2.3: $Hom(Q/Z_{(l)}, \pi_{m-1}(LK(X_\infty)) = 0$ *for all* m

Remark:2.4 : this theorem 2.3 shows what we conjectured above in order to research the Tate module M, that homotopy groups of the l-adic completion of the $LK(X_\infty)$ spectrum are isomorphic to the l-adic completion of the homotopy groups of those spectra, which at first sight is not trivial.

Corollary 2.5: $\pi_{-1}(LK(X_\infty))$ *is reduced*

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valmont8ar@hotmail.com