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CONCENTRATION OF EIGENFUNCTIONS ON 2-MANIFOLDS OUTSIDE CONVEX OBSTACLES

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Abstract. This paper concerns the concentration of Dirichlet and Neumann eigenfunctions of the Laplacian on a compact two-dimensional Riemannian manifold with strictly geodesically concave boundary. We link three inequalities which bound the concentration in different ways. We also prove one of these inequalities, which bounds the L^2 norms of the restrictions of eigenfunctions to broken geodesics.

1. Introduction

Let (M, g) be a compact two-dimenstional Riemannian manifold with smooth boundary. Assume that the boundary is strictly geodesically concave. This means that for any point x in ∂M, there is a geodesic in M which goes through x intersecting ∂M tangentially with exactly first order contact. Let e_j be Dirichlet or Neumann eigenfunctions of the Laplacian Δ_g forming an orthonormal basis of $L^2(M)$. Let $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ be the corresponding eigenvalues, normalized so that $-\Delta_g e_j = \lambda_j^2 e_j$. This paper concerns the concentration of the eigenfunctions e_j.

One way to measure the concentration of the eigenfunctions is by their L^p norms. For $p \geq 2$, the eigenfunctions satisfy

$$\|e_j\|_{L^p(M)} \lesssim \lambda_j^{\delta(p)} \tag{1.1}$$

This was proven for Dirichlet boundary conditions by Grieser [4] and for Neumann boundary conditions by the author [1]. We can interpret (1.1) as a way of bounding the concentration of the eigenfunctions. For $p > 2$, a natural problem is to determine when (1.1) is sharp, meaning

$$\limsup_{j\to\infty} \lambda_j^{-\delta(p)} \|e_j\|_{L^p(M)} > 0 \tag{1.2}$$

The main purpose of this paper is to give two conditions which are equivalent to (1.2) when $2 < p < 6$. Specifically, we will consider two other ways of measuring the concentration of eigenfunctions, and we will prove corresponding inequalities. We will then see that sharpness of these inequalities is equivalent to (1.2) when $2 < p < 6$.

Our second way of measuring the concentration of eigenfunctions is by the L^2 norms of their restrictions to broken geodesics. A broken geodesic is a curve in M which is geodesic away from the boundary and reflects off the boundary according to the reflection law for g. We bound this kind of concentration in the following theorem.

Theorem 1.1. *If γ is a broken geodesic in M of length $L \leq 1$, then for $\lambda_j \geq 1$,*

$$\|e_j\|_{L^2(\gamma)} \lesssim L^{\frac{1}{4}} \lambda_j^{\frac{1}{4}}$$

This extends a result of Burq-Gérard-Tzvetkov [3]. Their result dealt with compact two-dimensional Riemannian manifolds without boundary. Their work was motivated by Reznikov [7] who considered hyperbolic surfaces. Both suppressed the dependence on L in the right side. We want to make the dependence on L explicit for use in the proof of Proposition 1.6. Theorem 1.1 easily yields the following corollary.

Corollary 1.2. *If γ is a broken geodesic of unit length and $p \geq 2$, then for $\lambda_j \geq 1$,*

$$\|e_j\|_{L^2(\gamma)} \lesssim \lambda_j^{\frac{1}{2p}} \|e_j\|_{L^p(M)}$$

Bourgain [2] obtained this inequality for compact two-dimensional Riemannian manifolds without boundary. Theorem 1.1 yields Corollary 1.2 for the case $p = 2$. The case $p = \infty$ is trivial since the eigenfunctions are continuous. The other cases hold by interpolation.

We will link sharpness of Theorem 1.1 and sharpness of (1.1) for $2 < p < 6$. Let Π be the set of all unit length broken geodesics in M. We will show that for $2 < p < 6$, the inequality (1.2) is equivalent to

$$\limsup_{j\to\infty} \sup_{\gamma\in\Pi} \lambda_j^{-\frac{1}{4}} \|e_j\|_{L^2(\gamma)} > 0 \tag{1.3}$$

Our third way to measure the concentration of eigenfunctions is to take L^2 norms over certain neighborhoods of broken geodesics. For γ in Π, define the neighborhoods

$$\mathcal{N}_j(\gamma) = \left\{ x \in M : \inf_{y\in\gamma} d_g(x, y) < \lambda_j^{-\frac{1}{2}} \right\}$$

This will be a consequence of the following theorem, which is the main result of this paper.

Theorem 1.3. *Assume Λ is large and fix $\varepsilon > 0$. There is a constant C_ε such that for $\lambda_j \geq \Lambda$, the eigenfunctions e_j satisfy*

$$\|e_j\|^4_{L^4(M)} \leq C_\varepsilon \lambda_j^{\frac{1}{2}} \sup_{\gamma\in\Pi} \|e_j\|_{L^2(\mathcal{N}_j(\gamma))} + \varepsilon \lambda_j^{\frac{1}{2}} + C$$

This extends a result of Sogge [10], who considered compact two-dimensional Riemannian manifolds without boundary. Corollary 1.2 and Theorem 1.3 imply the following result.

Corollary 1.4. *Let e_{j_k} be a subsequence of eigenfunctions and let $2 < p < 6$. The following are equivalent:*

$$\limsup_{k\to\infty} \lambda_{j_k}^{-\delta(p)} \|e_{j_k}\|_{L^p(M)} > 0 \tag{1.4}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \|e_{j_k}\|_{L^2(\mathcal{N}_{j_k}(\gamma))} > 0 \tag{1.5}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \lambda_{j_k}^{-\frac{1}{4}} \|e_{j_k}\|_{L^2(\gamma)} > 0 \tag{1.6}$$

If (1.4) holds for some p in the range $2 < p < 6$, then it holds for all such p, by (1.1) and interpolation. So to prove Corollary 1.4, it suffices to consider the case $p = 4$. In this case, (1.4) implies (1.5) by Theorem 1.3. It is clear that (1.5) implies (1.6), and (1.6) implies (1.4) by Corollary 1.2.

A related problem is to determine when a subsequence e_{j_k} of eigenfunctions is quantum ergodic, meaning

$$\lim_{k\to\infty} \int_M (Ae_{j_k})\overline{e_{j_k}}\, dx = \int_{S^*M} \sigma_A \, dL$$

for every classical pseudodifferential operator A of order zero. Here dx is the Riemannian measure, S^*M is the unit cotangent bundle, σ_A is the principal symbol of A, and dL is the normalized Liouville measure. In particular, this implies that the probability measures $|e_{j_k}|^2\, dx$ converge weakly to the normalized Riemannian measure. In this case (1.5) cannot hold, so Corollary 1.4 implies the following.

Corollary 1.5. *Assume a subsequence e_{j_k} of eigenfunctions is quantum ergodic. Then*

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \lambda_{j_k}^{-\frac{1}{4}} \|e_{j_k}\|_{L^2(\gamma)} > 0 \tag{1.7}$$

and for $2 < p < 6$,

$$\limsup_{k\to\infty} \lambda_{j_k}^{-\delta(p)} \|e_{j_k}\|_{L^p(M)} > 0 \tag{1.8}$$

Zelditch-Zworski [12] proved that if the billiard flow is ergodic, then for Dirichlet boundary conditions there is a subsequence e_{j_k} of density one which is quantum ergodic. A subsequence is of density one when

$$\lim_{k\to\infty} \frac{k}{j_k} = 1$$

Their result demonstrates that the global dynamics of the billiard flow influence the concentration of eigenfunctions. Our last result also demonstrates this.

Proposition 1.6. *Fix a broken geodesic γ in M which has finite length and is not contained in a periodic broken geodesic. Then*

$$\limsup_{j\to\infty} \lambda_j^{-\frac{1}{4}} \|e_j\|_{L^2(\gamma)} = 0$$

That is, if Theorem 1.1 is sharp for a fixed broken geodesic, then it must be a segment of a periodic broken geodesic.

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2. Reductions

The beginning of the proofs of Theorem 1.1 and Theorem 1.3 are similar so we begin both in this section. We can assume that M is a subset of a compact two-dimensional Riemannian manifold (M_0, g). Let d_0 be the distance function on M_0 induced by g and let Δ_0 be the Laplacian on M_0. For the rest of this paper, we will assume $\lambda \geq 1$.

Fix a small $\delta > 0$, and choose a $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi(0) = 1$ and $\hat{\chi}$ supported on a closed interval contained strictly inside of $(\frac{1}{2}\delta, \delta)$. Define the translations

$\chi_\lambda(s) = \chi(s-\lambda)$. We will use the operators $\chi_\lambda(\sqrt{-\Delta_g})$ and $\chi_\lambda(\sqrt{-\Delta_0})$. Here $\sqrt{-\Delta_g}$ is defined with respect to the appropriate boundary conditions. Notice $\chi_{\lambda_j}(\sqrt{-\Delta_g})e_j = e_j$. To prove Theorem 1.1, it suffices to prove that

$$\|\chi_\lambda(\sqrt{-\Delta_g})f\|_{L^2(\gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M)} \tag{2.1}$$

Burq-Gérard-Tzvetkov [3] proved the following analogue. They suppressed the dependence on L in the right side, but it follows from their proof.

Theorem 2.1. *If γ is a smooth curve on M_0 of length $L \leq 1$, then*

$$\|\chi_\lambda(\sqrt{-\Delta_0})f\|_{L^2(\gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M_0)}$$

Let Π_0 be the set of all unit length geodesics in M_0. Fix $r \in (0,1)$. For $\gamma \in \Pi_0$, define the neighborhoods

$$\mathcal{T}_\lambda(\gamma) = \Big\{x \in M_0 : \inf_{y\in\gamma} d_0(x,y) < r\lambda^{-1/2}\Big\}$$

There is a constant Λ such that for any geodesic $\gamma \in \Pi_0$, there exists a fixed finite number of broken geodesics $\gamma_i \in \Pi$ such that $\mathcal{T}_\lambda(\gamma)\cap M \subset \bigcup \mathcal{B}_\lambda(\gamma_i)$ for $\lambda \geq \Lambda$. By (1.1), we know $\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{1/8}\|e_\lambda\|_{L^2(M)}$, so to prove Theorem 1.3 it suffices to show that

$$\begin{aligned}\int_M |\chi_\lambda(\sqrt{-\Delta_g})f(x)|^2|g(x)|^2\,dx \leq C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} +\ C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)}\end{aligned} \tag{2.2}$$

For $r = 1$, Sogge [10] proved the following analogue. Moreover, the same proof shows this holds for smaller values of r as well.

Theorem 2.2. *Fix $\varepsilon > 0$. There is a constant C_ε such that*

$$\begin{aligned}\int_{M_0} |\chi_\lambda(\sqrt{-\Delta_0})f(x)|^2|g(x)|^2\,dx \leq C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M_0)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M_0)} +\ C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M_0)}\end{aligned}$$

To prove inequalities (2.1) and (2.2), define projection operators Π_j on $L^2(M)$ by $\Pi_j f = \langle f, e_j\rangle e_j$. For $f \in L^2(M)$,

$$\begin{aligned}\chi_\lambda(\sqrt{-\Delta_g})f = \sum_{j=0}^\infty \chi_\lambda(\lambda_j)\Pi_j f = (2\pi)^{-1}\int \hat\chi(t)e^{-it\lambda}\sum_{j=0}^\infty e^{it\lambda_j}\Pi_j f\,dt \\ = (2\pi)^{-1}\int \hat\chi(t)e^{-it\lambda}e^{it\sqrt{-\Delta_g}}f\,dt\end{aligned} \tag{2.3}$$

Similarly, for $f \in L^2(M_0)$,

$$\chi_\lambda(\sqrt{-\Delta_0})f = (2\pi)^{-1}\int \hat\chi(t)e^{-it\lambda}e^{it\sqrt{-\Delta_0}}f\,dt \tag{2.4}$$

We will reduce the problem by following Smith-Sogge [9] to analyze the half-wave operator. Define the set

$$H_\delta = \Big\{x \in M : d(x,\partial M) \leq \delta\Big\}$$

and let E_δ be the complement of H_δ in M. If t is in supp $\hat{\chi}$, then

$$\left(e^{it\sqrt{-\Delta_g}}f\right)\Big|_{E_\delta} = \left(e^{it\sqrt{-\Delta_0}}f\right)\Big|_{E_\delta}$$

So (2.3) and (2.4) imply that

$$\left(\chi_\lambda(\sqrt{-\Delta_g})f\right)\Big|_{E_\delta} = \left(\chi_\lambda(\sqrt{-\Delta_0})f\right)\Big|_{E_\delta}$$

For a broken geodesic γ on M of length L, Theorem 2.1 yields

$$\|\chi_\lambda(\sqrt{-\Delta_g})f\|_{L^2(\gamma\cap E_\delta)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M)}$$

So to prove (2.1), it remains to prove

$$\|\chi_\lambda(\sqrt{-\Delta_g})f\|_{L^2(\gamma\cap H_\delta)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M)} \tag{2.5}$$

Similarly, Theorem 2.2 yields

$$\begin{aligned}\int_{E_\delta} |\chi_\lambda(\sqrt{-\Delta_g})f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} +\ C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)}\end{aligned}$$

So to prove (2.2), it remains to prove

$$\begin{aligned}\int_{H_\delta} |\chi_\lambda(\sqrt{-\Delta_g})f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M)}\|g\|^2_{L^4(M)} +\ C\|f\|^2_{L^2(M)}\|g\|^2_{L^2(M)}\end{aligned} \tag{2.6}$$

It is equivalent to show (2.5) and (2.6) with $\chi_\lambda(\sqrt{-\Delta_g})e^{it_0\sqrt{-\Delta_g}}f$ in place of $\chi_\lambda(\sqrt{-\Delta_g})f$ for some fixed t_0, because

$$\|e^{-it_0\sqrt{-\Delta_g}}f\|_{L^2(M)} = \|f\|_{L^2(M)}$$

Adapting (2.3) gives

$$\chi_\lambda(\sqrt{-\Delta_g})e^{it_0\sqrt{-\Delta_g}}f = (2\pi)^{-1}\int \hat{\chi}(t)e^{-it\lambda}e^{i(t+t_0)\sqrt{-\Delta_g}}f\,dt$$

For an operator A from M_0 to $\mathbb{R}\times M_0$, define associated operators

$$I_\lambda(A)f(x) = \int_{\mathbb{R}} \hat{\chi}(t)e^{-it\lambda}Af(t,x)\,dt$$

Here we can identify operators from M to $\mathbb{R}\times M$ with operators from M_0 to $\mathbb{R}\times M_0$ whose kernels are supported in $M\times(\mathbb{R}\times M)$. In particular, we then have $I_\lambda(E_g) = 2\pi\,\chi_\lambda(\sqrt{-\Delta_g})e^{it_0\sqrt{-\Delta_g}}$ where E_g is the operator given by

$$E_gf(t,x) = \left(e^{i(t+t_0)\sqrt{-\Delta_g}}f\right)(x)$$

We can rewrite (2.5) and (2.6), respectively, as

$$\|I_\lambda(E_g)f\|_{L^2(\gamma\cap H_\delta)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M_0)}$$

and

$$\begin{aligned}\int_{H_\delta} |I_\lambda(E_g)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M_0)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M)} +\ C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M)}\end{aligned}$$

The set of operators A such that $I_\lambda(A)$ satisfies

$$\|I_\lambda(A)f\|_{L^2(\gamma\cap H_\delta)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(M_0)} \tag{2.7}$$

and

$$\int_{H_\delta} |I_\lambda(A)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(M_0)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(M_0)}\|g\|^2_{L^4(M)} + C\|f\|^2_{L^2(M_0)}\|g\|^2_{L^2(M)} \tag{2.8}$$

is a complex vector space. This set includes any operator A whose kernel $K(t,x,y)$ is uniformly bounded over the region

$$\Big\{(t,x,y) : t\in \operatorname{supp}\hat\chi, x\in H_\delta, y\in M_0\Big\}$$

because then the kernel of $I_\lambda(A)$ is uniformly bounded, independent of λ. In this case the estimates (2.7) and (2.8) are trivial. In particular, this set contains all smoothing operators, by compactness.

Since ∂M is strictly geodesically concave, there is a $c_0>0$ such that if $t_0>0$ is small then any unit speed broken geodesic γ with $d(\gamma(0),\partial M)\le c_0t_0^2$ must satisfy

$$d(\gamma(t),\partial M)\ge c_0t_0^2$$

for $\frac12 t_0\le t\le 4t_0$. Now define Ω to be the set of points y in M such that there is a unit speed broken geodesic γ with $\gamma(0)=y$ and $d(\gamma(t_0+t),\partial M)\le 2\delta$ for some $t\in[-\delta,\delta]$. We assume that $2\delta<c_0t_0^2$ and $\delta<\frac12 t_0$, which implies $d(\omega,\partial M)\ge c_0t_0^2$.

If the kernel of E_g has a singularity at (t,x,y) then there is a broken geodesic of length $t+t_0$ with endpoints at x and y. So there is a smooth function α with support in Ω such that the kernel of the operator

$$f\to E_g(1-\alpha)f$$

is smooth over the region $\{(t,x,y) : t\in\operatorname{supp}\hat\chi, x\in H_\delta, y\in M_0\}$. This reduces the problem to only considering f with support in Ω.

Define an operator E_0 from M_0 to $\mathbb{R}\times M_0$ by

$$E_0f(t,x)=\Big(e^{i(t+t_0)\sqrt{-\Delta_0}}f\Big)(x)$$

Here we must treat the cases of Dirichlet and Neumann boundary conditions somewhat differently. Let $\mathcal{R}$ be an operator from M_0 to $\mathbb{R}\times\partial M$. For Dirichlet boundary conditions, let $\mathcal{R}$ be given by

$$\mathcal{R}f=(E_0f)\big|_{\mathbb{R}\times\partial M}$$

For Neumann boundary conditions, let $\mathcal{R}$ be given by

$$\mathcal{R}f=(\partial_\nu E_0f)\big|_{\mathbb{R}\times\partial M}$$

Here ∂_ν is the inward pointing normal derivative on ∂M.

Let $\Box_g=\partial_t^2-\Delta_g$ and $\Box_0=\partial_t^2-\Delta_0$. Let W be the forward solution operator of the appropriate boundary value problem for $\Box_g$, mapping data on $\mathbb{R}\times\partial M$ which vanish for $t\le -t_0$ to functions on $\mathbb{R}\times M$. For Dirichlet boundary conditions, the equation $u=Wh$ means u solves

$$\begin{cases} \Box_g u &= 0 \\ u &= 0 \quad \text{for } t\le -t_0 \\ u|_{\mathbb{R}\times\partial M} &= h \end{cases}$$

For Neumann boundary conditions, the equation $u = Wh$ means u solves

$$\begin{cases} \Box_g u & = 0 \\ u & = 0 \quad \text{for } t \leq -t_0 \\ (\partial_\nu u)|_{\mathbb{R}\times\partial M} & = h \end{cases}$$

Now over $[-t_0, t_0] \times M$, for f supported in ω,

$$E_g f = E_0 f - W\mathcal{R}_+ f$$

where $\mathcal{R}_+$ is $\mathcal{R}$ smoothly cutoff on the left to $t \in [-t_0, t_0]$. Since we are assuming that $\delta < \frac{1}{2}t_0$, we have $[\frac{1}{2}\delta, \delta] \subset (-t_0, t_0)$.

We can break up the cotangent bundle of $\mathbb{R} \times \partial M$ into three time-independent conic regions. These are the elliptic and hyperbolic regions where the Dirichlet problem is elliptic and hyperbolic, respectively, and the glancing region which is the region between them. We can break up the identity operator into a sum of time-independent conic pseudodifferential cutoffs as

$$I = \Pi_e + \Pi_h + \Pi_g$$

where Π_e and Π_h are essentially supported strictly inside the elliptic and hyperbolic regions, respectively, and Π_g is essentially supported in a small conic set about the glancing region. Then over $[-t_0, t_0] \times M$,

$$E_g f = E_0 f - W\Pi_e\mathcal{R}_+ f - W\Pi_h\mathcal{R}_+ f - W\Pi_g\mathcal{R}_+ f$$

The operator $I_\lambda(E_0)$ is equal to $\chi_\lambda(\sqrt{-\Delta_0}) \circ e^{it_0\sqrt{-\Delta_0}}$, so it satisfies (2.7) and (2.8) by Theorem 2.1 and 2.2.

The projection of any characteristic direction of $\Box_g$ onto $T^*(\mathbb{R}\times\partial M)$ is contained in the hyperbolic or glancing regions, so $W\Pi_e\mathcal{R}_+$ is smoothing. This implies that $I_\lambda(W\Pi_e\mathcal{R}_+)$ satisfies (2.7) and (2.8).

On the essential support of Π_h, we can solve the forward Dirichlet and Neumann problems for $\Box_g$ locally, modulo smoothing operators, on an open set in $\mathbb{R} \times M_0$ around $\mathbb{R} \times \partial M$. This gives a positive constant t_1 and an operator $\tilde{W}$ from $\mathbb{R} \times \partial M$ to $\mathbb{R} \times M_0$ such that $\Box_0 \tilde{W} v$ is smooth over $[-2t_1, 2t_1] \times M_0$ and $(W - \tilde{W})\Pi_h v$ is smooth over $\mathbb{R} \times M$ for any v supported by $t \in [-t_1, t_1]$,

We can assume $t_0 \leq t_1$ and define operators J_1 and J_2 by

$$J_1 f = \left(\tilde{W}\Pi_h\mathcal{R}_+ f\right)\Big|_{t=-t_0}$$

$$J_2 f = (-\Delta_0)^{-1/2}\left(\left(\partial_t\tilde{W}\Pi_h\mathcal{R}_+ f\right)\Big|_{t=-t_0}\right)$$

These are non-degenerate Fourier integral operators of order zero from M_0 to M_0.

Define operators C_0 and S_0 from M_0 to $\mathbb{R} \times M_0$ by

$$C_0 f(t,x) = \left(\cos\left((t+t_0)\sqrt{-\Delta_0}\right)f\right)(x)$$

and

$$S_0 f(t,x) = \left(\sin\left((t+t_0)\sqrt{-\Delta_0}\right)f\right)(x)$$

We can write $W\Pi_h\mathcal{R}_+ f$, modulo smoothing operators, as $C_0 J_1 f + S_0 J_2 f$. By the L^2 continuity of J_1 and J_2, it remains to show that $I_\lambda(C_0)$ and $I_\lambda(S_0)$ satisfy (2.7) and (2.8). This will complete the argument for the term $W\Pi_h\mathcal{R}_+ f$. Define an operator $\tilde{E}_0$ from M_0 to $\mathbb{R} \times M_0$ by

$$\tilde{E}_0 f(t,x) = \left(e^{-it\sqrt{-\Delta_0}}f\right)(x)$$

Since $I_\lambda(E_0)$ satisfies (2.7) and (2.8), it suffices, by Euler's formula, to show that the same is true for $I_\lambda(\tilde{E}_0 \circ e^{-it_0\sqrt{-\Delta_0}})$. It is equivalent to consider the operators $I_\lambda(\tilde{E}_0)$, because $e^{it_0\sqrt{-\Delta_0}}$ is unitary on $L^2(M_0)$.

If δ is small, we can apply the parametrix construction of Theorem 4.1.2 in Sogge [11]. Then over the region where $t \in \operatorname{supp} \hat{\chi}$, the operator $\tilde{E}_0$ is equal, modulo smoothing operators, to an operator Q, which is given in appropriately chosen coordinate charts by

$$Qf(x) = \iint e^{i[\varphi_0(x,y,\xi) - tp_0(y,\xi)]} q(t,x,y,\xi) f(y)\, d\xi dy$$

where φ_0 is smooth, p_0 is the principal symbol of $\sqrt{-\Delta_0}$, and q is a symbol of type $(1,0)$ and order zero. In such a coordinate chart, the kernel of $I_\lambda(Q)$ is

$$\iint \hat{\chi}(t) e^{i[\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda]} q(t,x,y,\xi)\, dt d\xi$$

Since $p_0(y,\xi) \sim |\xi|$ and $\lambda \geq 1$,

$$\Big|\frac{\partial}{\partial t}\Big(\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda\Big)\Big| = |p_0(y,\xi) + \lambda| \gtrsim 1 + |\xi|$$

An integration by parts argument shows that for any positive integer N,

$$\int \hat{\chi}(t) e^{i[\varphi_0(x,y,\xi) - tp_0(y,\xi) - t\lambda]} q(t,x,y,\xi)\, dt \lesssim (1+|\xi|)^{-N}$$

So the kernel of $I_\lambda(Q)$ is uniformly bounded, independent of λ. This implies that $I_\lambda(Q)$ satisfies (2.7) and (2.8). This completes the argument for the term $W\Pi_h\mathcal{R}_+ f$.

Now we break up Π_g into a finite sum of pseudodifferential cutoffs, each essentially supported in a suitably small conic neighborhood of a glancing ray. This breaks up $W\Pi_g\mathcal{R}_+ f$ into a finite sum and the Melrose-Taylor parametrix [6] can be applied to each term. We will use coordinates for M_0, chosen so that M is given by $x_2 > 0$. Then each term in this sum can be written, modulo smoothing operators, in the form $G \circ K$, where K is a non-degenerate Fourier integral operator of order zero from M to $\mathbb{R}^2$ and G is an operator from $\mathbb{R}^2$ to $\mathbb{R}^3$ with kernel

$$\int e^{i\theta(x,\xi) + it\xi_1 - y\cdot\xi}\Big(A_+\big(\zeta(x,\xi)\big)a(x,\xi) + A'_+\big(\zeta(x,\xi)\big)b(x,\xi)\Big)H\big(\zeta_0(\xi)\big)\, d\xi$$

The functions a and b are symbols of type $(1,0)$ and order $1/6$ and $-1/6$, respectively, and both are supported by x in a small ball about the origin and by ξ is in a small conic neighborhood of the ξ_1-axis. Let Ai be the Airy function. Then A_+ is given by $A_+(z) = Ai(e^{-\frac{2}{3}\pi i}z)$. The function H depends on the boundary conditions. For Dirichlet boundary conditions, it is given by

$$H(s) = \frac{Ai(s)}{A_+(s)}$$

For Neumann boundary conditions, it is instead given by

$$H(s) = \frac{Ai'(s)}{A'_+(s)}$$

The function ζ_0 is defined by $\zeta_0(\xi) = -\xi_1^{-1/3}\xi_2$, and the phases θ and ζ are real, smooth, and homogeneous in ξ of degree 1 and $2/3$, respectively, with

$$\zeta\big((x_1,0),\xi\big) = \zeta_0(\xi) \quad \text{and} \quad \frac{\partial\zeta}{\partial x_2}\big((x_1,0),\xi\big) < 0 \tag{2.9}$$

Let $\langle\ ,\ \rangle_x$ be the inner product given by g. In the region $\zeta(x,\xi)\le 0$, the functions θ and ζ satisfy

$$\begin{cases} \xi_1^2 - \langle d_x\theta, d_x\theta\rangle_x + \zeta\langle d_x\zeta, d_x\zeta\rangle_x = 0 \\ \langle d_x\theta, d_x\zeta\rangle_x = 0 \end{cases} \tag{2.10}$$

Also, θ and ζ satisfy these equations to infinite order at $x_2 = 0$ in the region $\zeta(x,\xi) > 0$.

For an operator A from $\mathbb{R}^2$ to $\mathbb{R}^3$, define associated operators

$$I_\lambda(A)f(x) = \int_{\mathbb{R}} \hat\chi(t)e^{-it\lambda}Af(t,x)\,dt$$

Fix a small $r > 0$ and define the set

$$S_r = \Big\{x\in\mathbb{R}^2 : |x|\le r, x_2\ge 0\Big\}$$

Consider the set of operators A with the following properties. First assume that for a broken geodesic γ in S_r of length $L\le 1$, and for f with fixed compact support,

$$\|I_\lambda(A)f\|_{L^2(\gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)} \tag{2.11}$$

Also assume that for any $\varepsilon > 0$, there is a constant C_ε such that for f with fixed compact support,

$$\begin{aligned}\int_{S_r} |I_\lambda(A)f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ +\ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} +\ C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)}\end{aligned} \tag{2.12}$$

By the L^2 continuity of K, it suffices to show that $I_\lambda(G)$ satisfies (2.11) and (2.12). The set of operators A such that $I_\lambda(A)$ satisfies (2.11) and (2.12) is a complex vector space. This set includes any operator A whose kernel $K(t,x,y)$ is uniformly bounded over compact subsets of

$$\Big\{(t,x,y) : t\in \operatorname{supp}\hat\chi, x\in S_r, y\in\mathbb{R}^2\Big\}$$

because then the kernel of $I_\lambda(A)$ is uniformly bounded, independent of λ, over compact subsets of $S_r\times\mathbb{R}^2$. In this case the estimates (2.11) and (2.12) are trivial. In particular, this applies when A is smoothing.

Let ρ be a smooth function with $\rho(s) = 0$ for $s\ge -1$ and $\rho(s) = 1$ for $s\le -2$. Following Zworski [13], we break up G into $G_m + G_d$, where the kernel of G_m is

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}\Big((\rho A_+)\big(\zeta(x,\xi)\big)a(x,\xi) + (\rho A_+)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)H\big(\zeta_0(\xi)\big)\,d\xi$$

and the kernel of G_d is

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}q(x,\xi)\,d\xi$$

Here $q(x,\xi)$ equals

$$\Big(\big((1-\rho)A_+\big)\big(\zeta(x,\xi)\big)a(x,\xi) + \big((1-\rho)A_+\big)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)H\big(\zeta_0(\xi)\big) \tag{2.13}$$

We will refer to G_m as the main term and to G_d as the diffractive term.

Define an operator $\tilde G_m$ with kernel

$$\int e^{i\theta(x,\xi)+it\xi_1-iy\cdot\xi}\Big((\rho A_+)\big(\zeta(x,\xi)\big)a(x,\xi) + (\rho A_+)'\big(\zeta(x,\xi)\big)b(x,\xi)\Big)\,d\xi$$

Then to control $I_\lambda(G_m)$, it suffices to show that $I_\lambda(\tilde{G}_m)$ satisfies (2.11) and (2.12), because

$$|H(s)| \le 2 \quad \text{for } s \in \mathbb{R}$$

By stationary phase,

$$\widehat{(\rho A_+)}(s) = 2\pi\, e^{i\frac{1}{3}s^3}\Psi_+(s)$$

where Ψ_+ is smooth and satisfies

$$\Big|\frac{d^k}{ds^k}\Psi_+(s)\Big| \le C_k$$

Applying the Fourier inversion formula and changing variables gives

$$(\rho A_+)(\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta + \frac{1}{3}s^3\xi_1^{-2})}\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)ds$$

Similarly,

$$(\rho A_+)'(\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta + \frac{1}{3}s^3\xi_1^{-2})}s\xi_1^{-4/3}\Psi_+(\xi_1^{-2/3}s)ds$$

So the kernel of $\tilde{G}_m$ is

$$\iint e^{i[\theta(x,\xi)+t\xi_1+s\xi_1^{-2/3}\zeta(x,\xi)+\frac{1}{3}s^3\xi_1^{-2}-y\cdot\xi]} \times \xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)\Big(a(x,\xi)+s\xi_1^{-2/3}b(x,\xi)\Big)\,dsd\xi$$

Here the symbol

$$\xi_1^{-2/3}\Psi_+(\xi_1^{-2/3}s)\Big(a(x,\xi)+s\xi_1^{-2/3}b(x,\xi)\Big)$$

is of type $(2/3,1/3)$ and order $-1/2$ on $\mathbb{R}^2_x \times \mathbb{R}^3_{s,\xi}$. Let ψ_0 be the function

$$\psi_0(x,t,\xi,s) = \theta(x,\xi) + t\xi_1 + s\xi_1^{-2/3}\zeta(x,\xi) + \frac{1}{3}s^3\xi_1^{-2}$$

Then $\tilde{G}_m$ is a Fourier integral operator of type $(2/3,1/3)$ and order zero associated to the canonical relation $\mathcal{C}$ given by

$$\mathcal{C} = \Big\{\Big(\big(x,t,\nabla_x\psi_0(x,t,\xi,s),\xi_1\big),\big(\nabla_\xi\psi_0(x,t,\xi,s),\xi\big)\Big) : \zeta(x,\xi) = -s^2\xi_1^{-4/3}\}$$

So we need to prove the following.

Lemma 2.3. *Let $\mathcal{G}$ be a Fourier integral operator of type $(2/3,1/3)$ and order zero associated to the canonical relation $\mathcal{C}$. Then for any broken geodesic γ in S_r of length $L \le 1$, the operators $I_\lambda(\mathcal{G})$ satisfies*

$$\|I_\lambda(\mathcal{G})f\|_{L^2(\gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)} \tag{2.14}$$

Also for any $\varepsilon > 0$, there is a constant C_ε such that the operators $I_\lambda(G)$ satisfies

$$\int_{S_r} |I_\lambda(\mathcal{G})f(x)|^2|g(x)|^2\,dx \le C_\varepsilon\lambda^{1/2}\|f\|^2_{L^2(\mathbb{R}^2)}\sup_{\gamma\in\Pi_0}\|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)} \tag{2.15}$$

The estimates for the main term will follow from Lemma 2.3. Before proving Lemma 2.3, we will show that is also implies the estimates for the diffractive term. First, we will show that for x in S_r and for ξ in a small conic neighborhood of the ξ_1-axis, we can write

$$H\big(\zeta_0(\xi)\big) = h\big(x, \xi_1, \zeta(x,\xi)\big)$$

where

$$\Big|\partial_{\xi_1}^m \partial_\zeta^j \partial_{x_1}^k \partial_{x_2}^\ell h(x,\xi_1,\zeta)\Big| \le C_{\alpha,j,k,\ell}\, \xi_1^{-m+2\ell/3} e^{-cx_2^{3/2}\xi_1 - |\zeta|^{3/2}} \tag{2.16}$$

By (2.9), there is a $c > 0$ such that

$$\zeta_0(\xi) \ge \zeta(x,\xi) + cx_2\xi_1^{2/3}$$

In the region $\zeta(x,\xi) \ge -2$, the asymptotics of the Airy functions now yield

$$\Big|H^{(m)}\big(\zeta_0(\xi)\big)\Big| \le C_m e^{-cx_2^{3/2}\xi_1 - |\zeta(x,\xi)|^{3/2}} \tag{2.17}$$

Define a new variable

$$\tau(x,\xi) = \xi_1^{1/3}\zeta(x,\xi)$$

When $x_2 = 0$, we have $\tau = -\xi_2$. It follows that we can write $\xi_2 = \sigma(x,\xi_1,\tau)$, where σ is homogeneous of degree 1 in (ξ_1,τ). Now we define

$$h(x,\xi_1,\zeta) = H\big(-\xi_1^{-1/3}\sigma(x,\xi_1,\xi_1^{1/3}\zeta)\big)$$

To prove (2.16) it suffices to show that

$$\begin{aligned}\Big|\partial_{\xi_1}^m \partial_\tau^j \partial_{x_1}^k \partial_{x_2}^\ell H\big(-\xi_1^{-1/3}\sigma(x,\xi_1,\tau)\big)\Big| \\ \le C_{m,j,k,\ell}\xi_1^{-m-j+2\ell/3} e^{-cx_2^{3/2}\xi_1 - |\tau|^{3/2}\xi_1^{-1/2}}\end{aligned} \tag{2.18}$$

If $x_2 = \tau = 0$, then $\sigma(x,\xi_1,\tau) = 0$. So the homogeneity of σ implies that

$$\Big|\partial_{\xi_1}^m \partial_\tau^j \partial_{x_1}^k \big(-\xi_1^{-1/3}\sigma(x,\xi_1,\tau)\big)\Big| \le C_{m,j,k}(x_2\xi_1^{2/3} + \xi_1^{-1/3}|\tau|)\xi_1^{-m-j}$$

Together with (2.17), this implies (2.18) when $\ell = 0$. It also follows for other values of ℓ because differentiating with respect to x_2 in (2.18) is similar to multiplying by a symbol of type $(1,0)$ and order $2/3$. Then (2.16) follows.

Now, for x in S_r and ξ in a small conic neighborhood of the ξ_1-axis, the symbol q from (2.13) can be written as $q(x,\xi) = q_0\big(x,\xi,\zeta(x,\xi)\big)$ where

$$q_0(x,\xi,\zeta) = \Big(\big((1-\rho)A_+\big)(\zeta)a(x,\xi) + \big((1-\rho)A_+\big)'(\zeta)b(x,\xi)\Big)h(x,\xi_1,\zeta)$$

By stationary phase,

$$\int e^{-is\zeta} q_0(x,\xi,\zeta)\, d\zeta = 2\pi\, e^{i\frac{1}{3}s^3} w(x,\xi,s)$$

where, for any $N > 0$,

$$\Big|\partial_\xi^m \partial_s^j \partial_{x_1}^k \partial_{x_2}^\ell w(x,\xi,s)\Big| \le C_{m,j,k,\ell,N}\xi_1^{-1/2-m+2\ell/3} e^{-cx_2^{3/2}\xi_1}(1+s)^{-N}$$

Applying the Fourier inversion formula and changing variables gives

$$q_0(x,\xi,\zeta) = \int e^{i(s\xi_1^{-2/3}\zeta + \frac{1}{3}s^3\xi_1^{-2})} w(x,\xi,\xi_1^{-2/3}s)\, ds$$

Now we can write the kernel of G_d as

$$\iint e^{i\mu(x,y,\xi,t,s)}c(x,\xi,s)\,dsd\xi$$

where

$$\mu(x,y,\xi,t,s)=\theta(x,\xi)+t\xi_1+s\xi_1^{-2/3}\zeta(x,\xi)+\frac{1}{3}s^3\xi_1^{-2}-y\cdot\xi$$

and

$$c(x,\xi,s)=w(x,\xi,\xi_1^{-2/3}s)$$

Here c satisfies

$$x_2^j\partial_{x_2}^k c(x,\xi,s)\in S^{1/2+2(k-j)/3}_{2/3,1/3}(\mathbb{R}_{x_1}\times\mathbb{R}^3_{\xi,s})$$

uniformly over x_2. In proving (2.11) and (2.12) for $I_\lambda(G_d)$, we may assume that c is supported by x in a small ball.

We have

$$c(x,\xi,s)=c(x_1,0,\xi,s)+\int_0^{x_2}\partial_{x_2}c(x_1,\sigma,\xi,s)\,d\sigma$$

So we can write $G_d=A_d+B_d$ where the kernel of A_d is

$$\iint e^{i\mu(x,y,\xi,t,s)}c(x_1,0,\xi,s)\,dsd\xi$$

The symbol $c(x_1,0,\xi,s)$ is of type $(2/3,1/3)$ and order $1/2$. So A_d is a Fourier integral operator of type $(2/3,1/3)$ and order zero associated to the canonical relation $\mathcal{C}$. Now $I_\lambda(A_d)$ satisfies (2.11) and (2.12) by Lemma 2.3.

The kernel of $I_\lambda(B_d)$ is

$$\int_0^{x_2}\iiint\hat\chi(t)e^{-it\lambda+i\mu(x,y,\xi,t,s)}\partial_{x_2}c(x_1,\sigma,\xi,s)\,dsd\xi dtd\sigma$$

Let β be a smooth function supported in $[1/3,3]$ with $\beta=1$ on $[1/2,2]$. Define operators B_λ with kernels

$$\int_0^{x_2}\iint e^{i\mu(x,y,\xi,t,s)}\beta\Big(\frac{\xi_1}{\lambda}\Big)\partial_{x_2}c(x_1,\sigma,\xi,s)\,dsd\xi d\sigma$$

The kernel of $I_\lambda(B_\lambda)$ is

$$\int_0^{x_2}\iiint\hat\chi(t)e^{-it\lambda+i\mu(x,y,\xi,t,s)}\beta\Big(\frac{\xi_1}{\lambda}\Big)\partial_{x_2}c(x_1,\sigma,\xi,s)\,dsd\xi dtd\sigma$$

Since $\partial_t\mu=\xi_1$, an integration by parts argument shows that $I_\lambda(B_d)$ differs from $I_\lambda(B_\lambda)$ by an operator whose kernel is uniformly bounded, independent of λ. Let

$$P_{\sigma,\lambda}(x,\xi,s)=\beta\Big(\frac{\xi_1}{\lambda}\Big)\partial_{x_2}c(x_1,\sigma,\xi,s)$$

Then

$$|I_\lambda(B_\lambda)f|\le\int\Big|\iiiint\hat\chi(t)e^{-it\lambda+i\mu(x,y,\xi,t,s)}P_{\sigma,\lambda}(x,\xi,s)f(y)\,dydsd\xi dt\Big|d\sigma$$

By Hölder's inequality, this is bounded by

$$\sup_\sigma\Big|\iiint\hat\chi(t)e^{-it\lambda+i\mu(x,y,\xi,t,s)}\lambda^{-2/3}(1+\lambda^{4/3}\sigma^2)P_{\sigma,\lambda}(x,\xi,s)f(y)\,dydsd\xi dt\Big|$$

That is,

$$|I_\lambda(B_\lambda)f|\le\sup_\sigma\Big|I_\lambda(B_{\sigma,\lambda})f\Big| \tag{2.19}$$

where $B_{\sigma,\lambda}$ is the operator with kernel

$$\iint e^{i\mu(x,y,\xi,t,s)}\lambda^{-2/3}(1+\lambda^{4/3}\sigma^2)P_{\sigma,\lambda}(x,\xi,s)\,dsd\xi$$

The amplitudes

$$\lambda^{-2/3}(1+\lambda^{4/3}\sigma^2)P_{\sigma,\lambda}(x,\xi,s)$$

are symbols of type $(2/3,1/3)$ and order $1/2$ over $\mathbb{R}^2_x\times\mathbb{R}^3_{\xi,s}$, uniformly in σ and λ. So the operators $B_{\sigma,\lambda}$ are Fourier integral operators of type $(2/3,1/3)$ and order zero associated to the canonical relation $\mathcal{C}$. By Lemma 2.3, the operators $I_\lambda(B_{\sigma,\lambda})$ satisfy (2.11) and (2.12), uniformly in σ and λ. Then $I_\lambda(B_\lambda)$ satisfies (2.11) and (2.12) because of (2.19). So Lemma 2.3 will imply the estimates for the diffractive term.

To proof Lemma 2.3, let $\mathcal{C}_0$ be the restriction of $\mathcal{C}$ to $t=0$. That is

$$\mathcal{C}_0=\Big\{\Big(\big(x,\nabla_x\psi_0(x,0,\xi,s)\big),\big(\nabla_\xi\psi_0(x,0,\xi,s),\xi\big)\Big):\zeta(x,\xi)=-s^2\xi_1^{-4/3}\}$$

It was shown in the proof of Lemma A.2 of Smith-Sogge [8] that $\mathcal{C}_0$ is the graph of a canonical transformation.

The projection of $\mathcal{C}$ onto $T^*(\mathbb{R}^3_{x,t})$ is contained in the characteristic variety of $\Box_0$, because of (2.10). So the canonical relation $\mathcal{C}\circ\mathcal{C}_0^{-1}$ is the flowout, under the bicharacteristic flow of $\Box_0$, of a conical subset of the diagonal at $t=0$. By the Lax construction, $\mathcal{C}\circ\mathcal{C}_0^{-1}$ can be parametrized by a phase function

$$\varphi(t,x,\xi)-y\cdot\xi$$

where φ satisfies

$$\varphi(0,x,\xi)=x\cdot\xi\quad\text{and}\quad\frac{\partial\varphi}{\partial t}=p_0\Big(x,\frac{\partial\varphi}{\partial x}\Big)\tag{2.20}$$

Here p_0 is the principal symbol of $\sqrt{-\Delta_0}$, that is

$$p_0(x,\xi)=\sqrt{\sum g^{jk}(x)\xi_j\xi_k}$$

Since $\varphi(t,x,\xi)-y\cdot\xi$ parametrizes $\mathcal{C}\circ\mathcal{C}_0^{-1}$, it follows that for small t,

$$y=\varphi'_\xi(t,x,\xi)\quad\text{implies}\quad t=d_0(x,y)\tag{2.21}$$

Now let J_0 and K_0 be classical Fourier integral operators of order zero, associated to the canonical relations $\mathcal{C}_0^{-1}$ and $\mathcal{C}_0$, respectively, such that $\mathcal{G}\circ J_0\circ K_0$ differs from $\mathcal{G}$ by a smoothing operator. To prove Lemma 2.3, we need to show that $I_\lambda(\mathcal{G}\circ J_0\circ K_0)$ satisfies (2.11) and (2.12). By the L^2 continuity of K_0, it suffices to show instead that $I_\lambda(\mathcal{G}\circ J_0)$ satisfies (2.11) and (2.12). Here $\mathcal{G}\circ J_0$ is a Fourier integral operator of type $(2/3,1/3)$ and order zero, associated to the canonical relation $\mathcal{C}\circ\mathcal{C}_0^{-1}$. So its kernel, modulo smoothing operators, is of the form

$$\int e^{i[\varphi(t,x,\xi)-y\cdot\xi]}a(t,x,\xi)\,d\xi$$

where a is a symbol of type $(2/3,1/3)$ and order zero on $\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi$. To show $I_\lambda(\mathcal{G}\circ J_0)$ satisfies (2.11), it now suffices to prove the following two lemmas.

Lemma 2.4. *Fix $a\in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$. Define an operator U_a by*

$$U_af=\iint e^{i\varphi(t,x,\xi)-iy\cdot\xi}a(t,x,\xi)f(y)\,d\xi dy$$

For any $\varepsilon > 0$ there is a constant C_ε such that for f with fixed compact support,

$$\int_{S_r} |I_\lambda(U_a)f(x)|^2 |g(x)|^2\, dx \le C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + \ C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)}$$

We will prove Lemma 2.4 in the next section. This will complete the proof of Theorem 1.3.

The next lemma will show that $I_\lambda(\mathcal{G}\circ J_0)$ satisfies (2.12).

Lemma 2.5. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$. Define an operator U_a by*

$$U_a f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi} a(t,x,\xi) f(y)\, d\xi dy$$

For any broken geodesic γ in S_r of length $L \le 1$, and for f with fixed compact support,

$$\|I_\lambda(U_a)f\|_{L^2(\gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)}$$

We will prove Lemma 2.5 in the fourth section. This will complete the proof of Theorem 1.1.

3. End of Proof of Theorem 1.3

To prove Theorem 1.3, it remains to prove Lemma 2.4. This will be a consequence of the following variant. To state it, let $\eta(x,y)$ be a smooth function supported by x and y with $\frac{1}{2}\delta \le d_0(x,y) \le \delta$. Also assume $\eta(x,y) = 1$ when $d_0(x,y)$ is in an open neighborhood of the support of $\hat\chi$.

Lemma 3.1. *Fix $b \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y}\times\mathbb{R}^2_\xi)$ supported by t in $[\frac{1}{2}\delta,\delta]$. Define an operator T_b by*

$$T_b f = \iint e^{i\varphi(t,x,\xi)-iy\cdot\xi}\eta(x,y) b(t,y,\xi) f(y)\, d\xi dy$$

For any $\varepsilon > 0$ there is a constant C_ε such that for f with fixed compact support,

$$\int_{S_r} |I_\lambda(T_b)f(x)|^2 |g(x)|^2\, dx \le C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} + \ \varepsilon\lambda^{1/4}\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^4(\mathbb{R}^2)} + \ C\|f\|^2_{L^2(\mathbb{R}^2)}\|g\|^2_{L^2(\mathbb{R}^2)}$$

Using Lemma 3.1, we can prove Lemma 2.4.

Proof of Lemma 2.4. Fix a symbol $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$. We can assume that a is supported by x in a small neighborhood of S_r and by t in $[\frac{1}{2}\delta,\delta]$. Moreover, we can assume that $\big(1-\eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of the set

$$\Sigma_0 = \Big\{(t,x,y,\xi) : t = d_0(x,y)\Big\}$$

We can make these assumptions because $I_\lambda(U_a)$ only depends on t in the support of $\hat\chi$. The kernel of U_a is

$$\int e^{i\varphi(t,x,\xi)-iy\cdot\xi} a(t,x,\xi)\, d\xi$$

Define an operator D_a with kernel

$$\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}\eta(x,y)a(t,x,\xi)\,d\xi$$

Define a set

$$\Sigma = \Big\{(t,x,y,\xi) : \varphi'_\xi(t,x,\xi) - y = 0\Big\}$$

By (2.21), the set Σ is contained in Σ_0. So the symbol $\big(1-\eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of Σ. By Proposition 1.2.4 of Hörmander [5], the difference between U_a and D_a is smoothing.

At $t=0$, the determinant of the matrix $[\varphi''_{\xi_i x_j}]$ is 1. We can assume a vanishes unless $t \in [\frac{1}{2}\delta, \delta]$. So if δ is small, we can apply the implicit function theorem to the equation

$$\varphi'_\xi(t,x,\xi) - y = 0$$

We can use a partition of unity to break up a into a finite sum $a = \sum a_j$, so that there are functions $\psi_j(t,y,\xi)$ that are homogeneous in ξ of degree zero. We can assume that, on the support of a_j, the set Σ is given by

$$x = \psi_j(t,y,\xi)$$

Define $b_0 \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y}\times\mathbb{R}^2_\xi)$ by

$$b_0(t,y,\xi) = \sum a_j\big(t,\psi_j(t,y,\xi),\xi\big)$$

Define an operator T_0 with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}b_0(t,y,\xi)\,d\xi$$

The difference between U_a and T_0 is an operator with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}\big(a(t,x,\xi)-b_0(t,y,\xi)\big)\,d\xi$$

The symbol $a(t,x,\xi)-b_0(t,y,\xi)$ vanishes on Σ, and the phase $\varphi(t,x,\xi)-y\cdot\xi$ is non-degenerate. It follows from Proposition 1.2.5 of Hörmander [5] that we can write this kernel in the form

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}a_0(t,x,y,\xi)\,d\xi$$

where a_0 is a symbol of order $-1/3$ and type $(2/3,1/3)$.

Iterating this argument yields symbols $b_k(t,y,\xi)$ of order $-k/3$ and type $(2/3,1/3)$. These symbols are such that if T_m is the operator with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}\sum_{k=0}^{m} b_k(t,y,\xi)\,d\xi$$

then the difference between U_a and T_m has a kernel of the form

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}a_m(t,x,y,\xi)\,d\xi$$

where a_m is a symbol of order $-(m+1)/3$ and type $(2/3,1/3)$. Let b be a symbol in $S^0_{2/3,1/3}(\mathbb{R}^3_{t,y}\times\mathbb{R}^2_\xi)$ with $b\sim\sum_{k=0}^\infty b_k$. Let T_b be the operator with kernel

$$\eta(x,y)\int e^{i\varphi(t,x,\xi)-iy\cdot\xi}b(t,y,\xi)\,d\xi$$

Then the difference between U_a and T_b is smoothing. We can assume b is supported by t in $[\frac{1}{2}\delta, \delta]$, so Lemma 2.4 will follow from Lemma 3.1. □

The next lemma will give a suitable description of the kernel of $I_\lambda(T_b)$. This description is sufficiently similar to the one used in Sogge [10], so that the same argument will yield Lemma 3.1.

Lemma 3.2. *Fix $b \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,y} \times \mathbb{R}^2_\xi)$ supported by t in $[\frac{1}{2}\delta, \delta]$. The kernel of $I_\lambda(T_b)$ is of the form*

$$\lambda^{1/2} e^{-i\lambda d_0(x,y)} A_\lambda(x,y) + R_\lambda(x,y) \tag{3.1}$$

Here the functions R_λ are uniformly bounded, independent of λ, and the functions A_λ are in $C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ satifying

$$|\partial_x^\alpha \partial_y^\beta A_\lambda| \le C_{\alpha,\beta} \lambda^{|\beta|/3}$$

Also the functions A_λ are supported by x and y satisfying $\frac{1}{2}\delta \le d_0(x,y) \le \delta$.

Proof. The kernel of $I_\lambda(T_b)$ is

$$\iint e^{i\varphi(t,x,\xi) - iy\cdot\xi - it\lambda} \hat\chi(t) \eta(x,y) b(t,y,\xi)\, d\xi dt$$

By (2.20),

$$\varphi(t,x,\xi) = x\cdot\xi + t|\xi|_x + Q(t,x,\xi)$$

where $|\cdot|_x$ is the norm from the Riemannian metric at x, and Q is homogeneous of degree 1 in the ξ-variable with

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta Q| \le C t^{2-k} |\xi|^{1-|\beta|} \tag{3.2}$$

Let β be a smooth function with $\beta(\xi) = 1$ when $|\xi| \in [C_0^{-1}, C_0]$ and $\beta(\xi) = 0$ when $|\xi| \notin [(2C_0)^{-1}, 2C_0]$, for some constant C_0. If C_0 is large and δ is small, then on the support of

$$\Big(1 - \beta\Big(\frac{\xi}{\lambda}\Big)\Big) \hat\chi(t) \eta(x,y) b(t,y,\xi)$$

we have

$$\Big|\frac{\partial}{\partial t}\Big(\varphi(t,x,\xi) - y\cdot\xi - t\lambda\Big)\Big| \;\gtrsim\; |\xi|_x + \lambda \;\gtrsim\; 1 + |\xi|$$

since $\lambda \ge 1$. So for any positive integer N,

$$\int e^{i\varphi(t,x,\xi) - iy\cdot\xi - it\lambda} \Big(1 - \beta\Big(\frac{\xi}{\lambda}\Big)\Big) \hat\chi(t) \eta(x,y) b(t,y,\xi)\, dt \lesssim (1+|\xi|)^{-N}$$

This implies that the difference between the kernel of $I_\lambda(T_b)$ and

$$\iint e^{i\varphi(t,x,\xi) - iy\cdot\xi - it\lambda} \beta\Big(\frac{\xi}{\lambda}\Big) \hat\chi(t) \eta(x,y) b(t,y,\xi)\, d\xi dt \tag{3.3}$$

is bounded uniformly in λ.

Now it suffices to show that (3.3) can be written as in (3.1). After changing variables (3.3) becomes

$$\lambda^2 \iint e^{i\lambda\Phi(t,x,y,\xi)} p_\lambda(t,x,y,\xi)\, d\xi dt$$

where the phase is

$$\Phi(t,x,y,\xi) = \varphi(t,x,\xi) - y\cdot\xi - t$$

and the amplitude is

$$p_\lambda(t,x,y,\xi) = \beta(\xi)\hat{\chi}(t)\eta(x,y)b(t,y,\lambda\xi)$$

Here p_λ is smooth and compactly supported with

$$|\partial_t^k \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma p_\lambda| \lesssim \lambda^{(k+|\beta|+|\gamma|)/3}$$

To apply stationary phase, the Hessian of Φ, with respect to the (t,ξ)-variables, must be non-degenerate on the support of p_λ. First note that its determinant is homogeneous of degree -1 in the ξ-variable. We have

$$\Phi(t,x,y,\xi) = (x-y)\cdot\xi + t|\xi|_x - t + Q(t,x,y,\xi)$$

where Q satisfies (3.2). We can compute explicitly the Hessian of

$$(x-y)\cdot\xi + t|\xi|_x - t$$

with respect to the (t,ξ)-variables. Its determinant is

$$-\frac{t}{|\xi|_x}\det g$$

Now it follows from (3.2) that the determinant of the Hessian of Φ, with respect to the (t,ξ)-variables, is

$$-\frac{t}{|\xi|_x}\det g + t^2 q(t,x,y,\xi)$$

where q is a smooth function, homogeneous of degree -1 in the ξ-variable. So if δ is small, then the Hessian of Φ, with respect to the (t,ξ)-variables, is non-degenerate on the support of p_λ.

The critical points of Φ, with respect to the (t,ξ)-variables, are the solutions of

$$\varphi'_\xi(t,x,\xi) = y \quad \text{and} \quad \varphi'_t(t,x,\xi) = 1$$

We can use the implicit function theorem at any critical point. By using a partition of unity and abusing notation, we can assume that there are smooth functions $t(x,y)$ and $\xi(x,y)$, such that if δ is small, then on the support of p_λ, the critical points are given by

$$\big(t(x,y),x,y,\xi(x,y)\big)$$

Because of (2.21), we have $t(x,y) = d_0(x,y)$. Applying Euler's homogeneity relation $\varphi = \varphi'_\xi\cdot\xi$ yields

$$\Phi\big(t(x,y),x,y,\xi(x,y)\big) = -t(x,y) = -d_0(x,y)$$

So Lemma 3.2 follows from the following stationary phase lemma. □

Lemma 3.3. *Consider the oscillatory integrals*

$$J_\lambda(x,y) = \int_{\mathbb{R}^3} e^{i\lambda\Psi(x,y,z)} q_\lambda(x,y,z)\,dz$$

where Ψ is a smooth function and the amplitudes q_λ are smooth with fixed compact support and satisfy

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma q_\lambda| \lesssim \lambda^{(|\beta|+|\gamma|)/3}$$

Assume that on the support of the symbols q_λ, the Hessian of Ψ with respect to the z-variable is non-degenerate and the solutions of $\Psi'_z(x,y,z) = 0$ are given by $(x,y,z(x,y))$ where $z(x,y)$ is a smooth function. Then

$$\Big|\partial_x^\alpha\partial_y^\beta\Big(e^{-i\lambda\Psi(x,y,z(x,y))}J_\lambda(x,y)\Big)\Big| \lesssim \lambda^{-3/2+|\beta|/3}$$

This lemma is similar to Corollary 1.1.8 in Sogge [11], which dealt with symbols q_λ with derivatives bounded independent of λ. Essentially the same proof as in Sogge [11] yields Lemma 3.3, and then Lemma 3.2 follows. We can now obtain Lemma 3.1 by using the argument in Sogge [10].

Argument from Sogge [10]**.** To finish the proof of Lemma 3.1 it suffices to show that for any $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned}(3.4)\quad &\int_{S_r} \Big|\lambda^{1/2} \int e^{-i\lambda d_0(x,y)} A_\lambda(x,y) f(y)\, dy\Big|^2 |g(x)|^2\, dx \\ &\qquad \le \varepsilon \lambda^{1/4} \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

By using a partition of unity and abusing notation, we can assume there are points x_0 and y_0 with x_0 in S_r and $\delta/2 \le d_0(x_0, y_0) \le \delta$ such that A_λ is supported by x in a small neighborhood $\mathcal{N}_x$ of x_0 and y in a small neighborhood $\mathcal{N}_y$ of y_0. In particular, we assume that $\mathcal{N}_x$ and $\mathcal{N}_y$ are, respectively, contained in $B(x_0, \delta/5)$ and $B(y_0, \delta/5)$, the geodesic balls of radius $\delta/5$ around x_0 and y_0, respectively.

We will work in Fermi normal coordinates $(\sigma, \tau)_F$ about γ_0, the geodesic going through x_0 which is orthogonal to the geodesic connecting x_0 and y_0. These coordinates are well defined on $B(x_0, 2\delta)$ if δ is small enough. These coordinates are such that γ_0 is given by a vertical line parallel to the τ-axis, and the geodesics which intersect γ_0 orthogonally are given by horizontal lines parallel to the σ-axis. Also x_0 lies on the negative σ-axis and y_0 on the positive σ-axis. Now by the Cauchy-Schwartz inequality, it suffices to prove

$$\begin{aligned}&\int \Big(\int_{S_r} \Big|\lambda^{1/2} \int e^{-i\lambda d_0(x,(\sigma,\tau)_F)} A_\lambda\big(x,(\sigma,\tau)_F\big) f(\sigma,\tau)\, d\tau\Big|^2 |g(x)|^2\, dx\Big)\, d\sigma \\ &\qquad \le \varepsilon \lambda^{1/4} \|f\|^2_{L^2(\mathbb{R}^2)} \|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(\mathbb{R}^2)} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

This will follow if we show

$$\begin{aligned}(3.5)\quad &\int_{S_r} \Big|\lambda^{1/2} \int e^{-i\lambda d_0(x,(\sigma,\tau)_F)} A_\lambda\big(x,(\sigma,\tau)_F\big) h(\tau)\, d\tau\Big|^2 |g(x)|^2\, dx \\ &\qquad \le \varepsilon \lambda^{1/4} \|h\|^2_{L^2(\mathbb{R})} \|g\|^2_{L^4(\mathbb{R}^2)} + C_\varepsilon \lambda^{1/2} \|h\|^2_{L^2(\mathbb{R})} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

where C_ε is now independent of σ as well as λ. To simplify the notation, we will only prove this for a fixed value of σ, which we may take to be zero by relabeling the coordinates. It will be clear how to adapt the argument to show uniformity in σ. Note that after relabeling, we can assume that the point $(0,0)_F$ is in $\mathcal{N}_y$. Then $x_0 = (-\sigma_0, 0)_F$ where $\sigma_0 > \delta/4$.

We take a smooth bump function $\eta \in C_0^\infty(\mathbb{R})$ supported in $[-1,1]$ and satisfying $\sum_{j\in\mathbb{Z}} \eta(\tau - j) = 1$. Define

$$\eta_{\lambda,j}(\tau) = \eta(\lambda^{1/2}\tau - j)$$

Let

$$z_j = z_j(\lambda, x, h) = \lambda^{1/2} \int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) h(\tau)\, d\tau$$

Then for $N = 1, 2, 3, \dots$,

$$\begin{aligned}\Big|\sum_{j,k\in\mathbb{Z}} z_j z_k\Big| &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + \Big|\sum_{|j-k|\le N} z_j z_k\Big| \\ &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + \sum_{|j-k|\le N} \frac{1}{2}\Big(|z_j|^2 + |z_k|^2\Big) \\ &\le \Big|\sum_{|j-k|>N} z_j z_k\Big| + (2N+1)\sum_{j\in\mathbb{Z}} |z_j|^2\end{aligned}$$

This means that

$$\begin{aligned}(3.6)\quad &\Big|\lambda^{1/2}\int e^{-i\lambda d_0(x,(0,\tau)_F)} A_\lambda\big(x,(0,\tau)_F\big) h(\tau)\, d\tau\Big|^2 \\ &\le \Big|\lambda \iint e^{-i\lambda[d_0(x,(0,\tau)_F)+d_0(x,(0,\tau')_F)]} B_{N,\lambda}(x,\tau,\tau') h(\tau) h(\tau')\, d\tau d\tau'\Big| \\ &\qquad + (2N+1)\sum_{j\in\mathbb{Z}} \lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) h(\tau)\, d\tau\Big|^2\end{aligned}$$

where

$$B_{N,\lambda}(x,\tau,\tau') = \sum_{|j-k|>N} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) \eta_{\lambda,k}(\tau') A_\lambda\big(x,(0,\tau')_F\big)$$

We will prove

$$\begin{aligned}(3.7)\quad &\Big\|\lambda \iint e^{-i\lambda[d_0(x,(0,\tau)_F)+d_0(x,(0,\tau')_F)]} B_{N,\lambda}(x,\tau,\tau') h(\tau) h(\tau')\, d\tau d\tau'\Big\|_{L^2_x(S_r)} \\ &\qquad\qquad \lesssim \lambda^{1/4} N^{-1/2} \|h\|^2_{L^2(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}(3.8)\quad &\int_{S_r} \lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) H(\tau)\, d\tau\Big|^2 |g(x)|^2\, dx \\ &\qquad\qquad \lesssim \lambda^{1/2} \|H\|^2_{L^2(\mathbb{R})} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

Let $\chi_{\lambda,j}$ be the characteristic function of supp $\eta_{\lambda,j}$. Then (3.8) will yield

$$\begin{aligned}(3.9)\quad &\sum_{j\in\mathbb{Z}} \int_{S_r} \lambda\Big|\int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,j}(\tau) A_\lambda\big(x,(0,\tau)_F\big) h(\tau)\, d\tau\Big|^2 dx \\ &\qquad \lesssim \sum_{j\in\mathbb{Z}} \lambda^{1/2} \|h\chi_{\lambda,j}\|^2_{L^2(\mathbb{R})} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))} \\ &\qquad \lesssim \lambda^{1/2} \|h\|^2_{L^2(\mathbb{R})} \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}\end{aligned}$$

Then (3.6), (3.7), and (3.9) will yield (3.5). So it remains to prove (3.7) and (3.8).

The inequality (3.7) will be a consequence of the following lemma.

Lemma 3.4. *Let $B_\lambda(x,\tau,\tau')$ be a smooth function over $\mathbb{R}^4$ with $|\partial_x^\alpha B_\lambda| \le C_\alpha$ and assume B_λ vanishes unless $|x| \le \delta_0$ and $|\tau - \tau'| \le \delta_0$. Assume that $\varphi(x,t)$ is a real*

smooth function over $\mathbb{R}^3$ satisfying the Carleson-Sjölin condition on the support of the amplitudes B_λ, that is

$$\det \begin{pmatrix} \varphi''_{x_1\tau} & \varphi''_{x_2\tau} \\ \varphi'''_{x_1\tau\tau} & \varphi'''_{x_2\tau\tau} \end{pmatrix} \neq 0$$

If $\delta_0 > 0$ is sufficiently small, then

$$\Big\| \iint_{|\tau-\tau'|\geq N\lambda^{-1/2}} e^{i\lambda[\varphi(x,\tau)+\varphi(x,\tau')]} B_\lambda(x,\tau,\tau') F(\tau,\tau')\, d\tau d\tau' \Big\|^2_{L^2_x(S_r)} \lesssim \lambda^{-3/2} N^{-1} \|F\|^2_{L^2(\mathbb{R}^2)} \tag{3.10}$$

Moreover, if the C_α are fixed and δ_0 is sufficiently small, this estimate is uniform over all functions B_λ which satisfy the hypotheses.

The functions $B_{N,\lambda}$ satisfy the hypotheses of Lemma 3.4 with C_α and δ_0 fixed, and it is well known that the function $\varphi(x,\tau) = -d_0(x,(0,\tau)_F)$ satisfies the Carleson-Sjölin condition. So Lemma 3.4 will imply (3.7).

Proof of Lemma 3.4. Let $\Upsilon(x,\tau,\tau') = \varphi(x,\tau) + \varphi(x,\tau')$. Then the determinant of the mixed Hessian of Υ satisfies

$$\Big| \det \Big(\frac{\partial^2 \Upsilon}{\partial x \partial(\tau,\tau')} \Big)(x,\tau,\tau') \Big| = \varphi''_{x_1\tau}(x,\tau)\varphi''_{x_2\tau'}(x,\tau') - \varphi''_{x_1\tau'}(x,\tau')\varphi''_{x_2\tau}(x,\tau)$$

By the Carleson-Sjölin condition, the τ' derivative of this function is nonzero on the diagonal $\tau = \tau'$. This implies that

$$\Big| \det \Big(\frac{\partial^2 \Upsilon}{\partial x \partial(\tau,\tau')} \Big) \Big| \geq c|\tau - \tau'|$$

for some $c > 0$ on the support of the amplitudes B_λ, if δ_0 is small. We use the change of variables

$$u = (\tau - \tau', \tau + \tau)$$

Since $|du/d(\tau,\tau')| = 2$, we obtain

$$\Big| \det \Big(\frac{\partial^2 \Upsilon}{\partial x \partial u} \Big) \Big| \geq c|u_1|$$

Now Υ is an even function in the u_1-variable, so it is a smooth function of u_1^2. We can make another change of variables

$$v = (\frac{1}{2}u_1^2, u_2).$$

Then $|dv/du| = |u_1|$, so

$$\Big| \det \Big(\frac{\partial^2 \Upsilon}{\partial x \partial v} \Big) \Big| \geq c$$

This implies that if v and $\tilde{v}$ are close then

$$\Big| \nabla_x [\Upsilon(x,v) - \Upsilon(x,\tilde{v})] \Big| \geq c'|v - \tilde{v}|$$

for some $c' > 0$. Since Υ is smooth as a function of x and v,

$$\Big| \partial_x^\alpha [\Upsilon(x,v) - \Upsilon(x,\tilde{v})] \Big| \lesssim C'_\alpha |v - \tilde{v}|$$

Now if we define

$$K_\lambda(v,\tilde{v}) = \int_{S_r} B_\lambda(x,\tau,\tau') \overline{B_\lambda(x,\tilde{\tau},\tilde{\tau}')} e^{i\lambda[\Upsilon(x,v)-\Upsilon(x,\tilde{v})]}\, dx$$

then for $j = 1, 2, 3, \ldots$, integrating by parts yields

$$|K_\lambda(v, \tilde{v})| \le C_j (1 + \lambda |v - \tilde{v}|)^{-2j} \tag{3.11}$$

For $a, b \ge 0$,

$$(1 + 2a)(1 + b) \le 2\Big(1 + (a^2 + b^2)^{1/2}\Big)^2$$

If we set $a = \lambda |v_1 - \tilde{v}_1|$ and $b = \lambda |v_2 - \tilde{v}_2|$, then (3.11) becomes

$$|K_\lambda(v, \tilde{v})| \le C_j' (1 + \lambda |(u_1^2 - \tilde{u}_1^2|)^{-j} (1 + \lambda |u_2 - \tilde{u}_2|)^{-j} \tag{3.12}$$

Let $E_{N,\lambda}$ be the characteristic function of the set

$$\{(u, \tilde{u}) \in \mathbb{R}^4 : |u_1|, |\tilde{u}_1| \ge N\lambda^{-1/2}\}$$

Then the left side of (3.10) equals

$$\iint E_{N,\lambda}(u, \tilde{u}) K_\lambda(u, \tilde{u}) F(u) \overline{F(\tilde{u})}\, du d\tilde{u}$$

By Hölder's inequality, it remains to prove that

$$\Big\| \int E_{N,\lambda}(u, \tilde{u}) K_\lambda(u, \tilde{u}) F(u)\, du \Big\|_{L^2_{\tilde{u}}(\mathbb{R}^2)} \lesssim \lambda^{-3/2} N^{-1} \|F\|_{L^2(\mathbb{R}^2)}$$

This will follow from Young's inequality, if we show that

$$\sup_{\tilde{u}} \int_{|u_1| \ge N\lambda^{-1/2}} |K_\lambda(u, \tilde{u})|\, du \lesssim \lambda^{-3/2} N^{-1}$$

and

$$\sup_{u} \int_{|\tilde{u}_1| \ge N\lambda^{-1/2}} |K_\lambda(u, \tilde{u})|\, d\tilde{u} \lesssim \lambda^{-3/2} N^{-1}$$

Because of (3.12), both of these inequalities will follow if we check that, for $c_0 \in \mathbb{R}$,

$$\sup_{c_1, c_2 \in \mathbb{R}} \int_{w_1 \ge N\lambda^{-1/2}} (1 + \lambda |w_1^2 - c_1|)^{-2} (1 + \lambda |w_2 - c_2|)^{-2}\, dw \lesssim \lambda^{-3/2} N^{-1} \tag{3.13}$$

By changing variables,

$$\sup_{c_2 \in \mathbb{R}} \int (1 + \lambda |w_2 - c_2|)^{-2}\, dw_2 = \lambda^{-1} \int (1 + |\tilde{w}_2|)^{-2}\, d\tilde{w}_2 \lesssim \lambda^{-1} \tag{3.14}$$

If we set $z = w_1^2$, then $dw_1 = \frac{1}{2} z^{-1/2} dz$, so we also have

$$\begin{aligned} \sup_{c_1 \in \mathbb{R}} \int_{w_1 \ge N\lambda^{-1/2}} & (1 + \lambda |w_1^2 - c_1|)^{-2}\, dw_1 \\ &= \frac{1}{2} \sup_{c_1 \in \mathbb{R}} \int_{z \ge N^2 \lambda^{-1}} (1 + \lambda |z - c_1|)^{-2} z^{-1/2}\, dz \\ &\le \lambda^{1/2} N^{-1} \sup_{c_1 \in \mathbb{R}} \int_{\sqrt{z} \ge N\lambda^{-1/2}} (1 + \lambda |z - c_1|)^{-2}\, dz \\ &\le \lambda^{-1/2} N^{-1} \int (1 + |\tilde{z}|)^{-2}\, d\tilde{z} \lesssim \lambda^{-1/2} N^{-1} \end{aligned} \tag{3.15}$$

Now (3.14) and (3.15) yield (3.13), completing the proof of Lemma 3.4. □

So we have proven (3.7), and it remains to show (3.8). To simplify the notation, we will only prove this for $j = 0$. It will be clear how to adapt the argument to show (3.8) holds uniformly over j in $\mathbb{R}$.

Let $p = (0,0)_F$. Let T be the tangent plane at p. The exponential map is a diffeomorphism from a ball of radius 2δ in T to $B\big(p, 2\delta\big)$ if δ is small. Let κ be the inverse function. We will identify T with $\mathbb{R}^2$ in such a way that the Riemannian metric on T agrees with the Euclidean metric on $\mathbb{R}^2$. We can make this identification in such a way that $\exp_p(\sigma, 0) = (\sigma, 0)_F$ for all σ. Let κ_1 and κ_2 denote the component functions of κ, so that $\kappa = (\kappa_1, \kappa_2)$. The inequality (3.8) will be a consequence of the following lemma.

Lemma 3.5. *Let $\psi(x,\tau) = -d_0\big(x, (0,\tau)_F\big)$ and let ρ_λ be a family of functions in $C^\infty(\mathbb{R}^3)$ satisfying*

$$|\partial_\tau^m \rho_\lambda(x,\tau)| \le C_m \lambda^{m/2} \tag{3.16}$$

and

$$\text{supp } \rho_\lambda \subset \Big\{(x,\tau) : |\tau| \le \lambda^{-1/2}, x \in \mathcal{N}_x, (0,\tau)_F \in \mathcal{N}_y\Big\} \tag{3.17}$$

Assume q_k are points in $\mathcal{N}_x$ satisfying

$$\Big|\frac{\kappa_2(q_k)}{|\kappa(q_k)|} - \frac{\kappa_2(q_\ell)}{|\kappa(q_\ell)|}\Big| \ge c\lambda^{-1/2}|k-\ell| \tag{3.18}$$

with $c > 0$, when $|k-\ell| \ge 2$. If $\mathcal{N}_x$ is sufficiently small, then

$$\lambda^{1/2} \int \Big|\sum_k e^{i\lambda\psi(q_j,\tau)} \rho_\lambda(q_k,\tau) p_k\Big|^2 d\tau \lesssim \sum |p_k|^2 \tag{3.19}$$

This estimate is uniform over different choices of the points q_k.

To see that Lemma 3.5 implies (3.8), let $\kappa_r(x)$ and $\kappa_\theta(x)$ be the polar coordinates of $\kappa(x)$. These functions are well defined and smooth on $\mathcal{N}_x$. Define

$$\rho_\lambda(x,\tau) = \eta_{\lambda,0}(\tau) A_\lambda\big(x, (0,\tau)_F\big)$$

Then (3.16) and (3.17) hold. Define the sets

$$V_k = \Big\{x \in \mathcal{N}_x : \lambda^{-1/2}k \le \kappa_\theta(x) < \lambda^{-1/2}(k+1)\Big\}$$

We have

$$\begin{aligned}
&\int_{S_r} \lambda \Big| \int e^{-i\lambda d_0(x,(0,\tau)_F)} \eta_{\lambda,0}(\tau) A_\lambda\big(x,(0,\tau)_F\big) H(\tau)\, d\tau\Big|^2 |g(x)|^2\, dx \\
&\qquad \le \sum_k \lambda \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau\Big\|^2_{L^\infty_x(V_k)} \|g\|^2_{L^2(V_k)} \\
&\qquad\qquad \le \sup_\ell \|g\|^2_{L^2(V_\ell)} \sum_k \lambda \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau\Big\|^2_{L^\infty_x(V_k)}
\end{aligned}$$

If $\mathcal{N}_x$ is small, then each V_ℓ is contained in $\mathcal{T}_\lambda(\tilde\gamma_k)$ for some $\tilde\gamma_k \in \Pi_0$. In fact, each $\tilde\gamma_k$ can be chosen to go through p. This yields

$$\sup_\ell \|g\|^2_{L^2(V_\ell)} \le \sup_{\gamma\in\Pi_0} \|g\|^2_{L^2(\mathcal{T}_\lambda(\gamma))}$$

Now to prove (3.8), it remains to show that

$$\sum_k \lambda^{1/2} \Big\| \int e^{i\lambda\psi(x,\tau)} \rho_\lambda(x,\tau) H(\tau)\, d\tau \Big\|^2_{L^\infty_x(V_k)} \lesssim \|H\|^2_{L^2(\mathbb{R})}$$

It suffices to check that for any choice of points q_k in V_k,

$$\sum_k \lambda^{1/2} \Big| \int e^{i\lambda\psi(q_k,\tau)} \rho_\lambda(q_k,\tau) H(\tau)\, d\tau \Big|^2 \lesssim \|H\|^2_{L^2(\mathbb{R})}$$

and that this holds uniformly over different choices of q_k. By duality, this inequality is equivalent to (3.19). To apply Lemma 3.5, we still need to check that any choice of points q_k in S_k satisfies (3.18). If $\mathcal{N}_x$ and $\mathcal{N}_y$ are sufficiently small, then $\kappa_\theta(\mathcal{N}_x)$ is contained in $[2\pi/3, 4\pi/3]$. When $|j-k| \geq 2$, we then have

$$\begin{aligned} \Big| \frac{\kappa_2(q_j)}{|\kappa(q_j)|} - \frac{\kappa_2(q_k)}{|\kappa(q_k)|} \Big| &= \Big| \sin\big(\kappa_\theta(q_j)\big) - \sin\big(\kappa_\theta(q_k)\big) \Big| \\ &\geq \frac{1}{2} \Big| \kappa_\theta(q_j) - \kappa_\theta(q_k) \Big| \geq \frac{1}{4} \lambda^{-1/2} |j-k| \end{aligned}$$

This is (3.18), so Lemma 3.5 will imply (3.8).

Proof of Lemma 3.5. We can write

$$\psi(x,\tau) = \psi(x,0) + \tau \partial_\tau \psi(x,0) + r(x,\tau)$$

where

$$|r(\tau,x)| \leq C_0 |\tau|^2 \quad |\partial_\tau r(\tau,x)| \leq C_1 |\tau|$$

and for $m = 2, 3, \ldots$

$$|\partial_\tau^m r(\tau,x)| \leq C_m$$

Fix x in $\mathcal{N}_x$ and let Θ be the geodesic sphere of radius $|\kappa(x)|$ around x. By Gauss' lemma, $\kappa(x)$ is normal to $\kappa(\Theta)$. Define a function G from $\mathbb{R}^2$ to $\mathbb{R}$ by

$$G(u) = -d_0(x, \exp_p(u))$$

Then $\kappa(\Theta)$ is a level set of G, so $\nabla G(0)$ is normal to $\kappa(\Theta)$. That is, $\nabla G(0)$ is a multiple of $\kappa(x)$. Define a curve c in T by $c(t) = t\kappa(x)$. Then $G(c(t)) = (t-1)|\kappa(x)|$ for t near 0, so the directional derivative of G at 0 in the direction $\kappa(x)$ is equals to $|\kappa(x)|$. We now have $\nabla G(0) \cdot \kappa(x) = |\kappa(x)|$. Since $\nabla G(0)$ is a multiple of $\kappa(x)$, this implies that

$$\nabla G(0) = \frac{\kappa(x)}{|\kappa(x)|}$$

This yields

$$\partial_\tau \psi(x,0) = \nu \cdot \frac{\kappa(x)}{|\kappa(x)|}$$

where

$$\nu = \partial_\tau \kappa\big((0,\tau)_F\big)\Big|_{\tau=0}$$

That is, ν is the pushforward under κ of $\partial/\partial\tau$ at p. It must be transverse to the pushforward under κ of $\partial/\partial\sigma$ at p, whose second component is zero. So the second component of ν is nonzero. By (3.18),

$$\Big| \partial_\tau \psi(q_k,0) - \partial_\tau \psi(q_\ell,0) \Big| \geq c' \lambda^{-1/2} |j-k|$$

for some $c' > 0$ when $|k-\ell| \geq 2$.

Now define

$$P_\lambda(q_k, q_\ell, \tau) = \rho_\lambda(q_k, \tau)\overline{\rho_\lambda(q_\ell, \tau)} e^{i\lambda[\psi(q_k,0)+r(q_k,\tau)]} e^{-i\lambda[\psi(q_\ell,0)+r(\ell,\tau)]}$$

Then $P_\lambda(q_k, q_\ell, \tau)$ vanishes when $|\tau| \geq \lambda^{-1/2}$ and satisfies

$$\Big|\partial_\tau^m P_\lambda(q_k, q_\ell, \tau)\Big| \leq C_m \lambda^{m/2}$$

The left side of (3.19) is equal to

$$\lambda^{1/2} \sum_{k,\ell} p_k \overline{p_\ell} \Big(\int e^{it\lambda[\partial_\tau \psi(q_k,0) - \partial_\tau \psi(q_\ell,0)]} P_\lambda(q_k, q_\ell, \tau)\, d\tau \Big)$$

We integrate by parts twice to control this by

$$\sum_{k,\ell} |p_k p_\ell| (1 + |k-\ell|)^{-2} \lesssim \sum_{k,\ell} (|p_k|^2 + |p_\ell|^2)(1+|k-\ell|)^{-2} \lesssim \sum_k |p_k|^2$$

This completes the proof of Lemma 3.5, and now Theorem 1.3 follows. □

4. Proof of Theorem 1.1

To complete the proof of Theorem 1.1, it remains prove Lemma 2.5. Since any unit length broken geodesic can be broken up into a fixed finite number of segments which are smooth, it suffices to prove the following.

Lemma 4.1. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$. Define an operator U_a by*

$$U_a f = \iint e^{i\varphi(t,x,\xi) - iy\cdot\xi} a(t,x,\xi) f(y)\, d\xi dy$$

For any smooth curve Γ in S_r of length $L \leq 1$, and for f with fixed compact support,

$$\|I_\lambda(U_a) f\|_{L^2(\Gamma)} \lesssim L^{1/4} \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)}$$

This will be a consequence of the following variant. To state it, recall $\eta(x,y)$ is a smooth function supported by x and y with $\frac{1}{2}\delta \leq d_0(x,y) \leq \delta$. Also $\eta(x,y) = 1$ when $d_0(x,y)$ is in an open neighborhood of the support of $\hat{\chi}$.

Lemma 4.2. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$ supported by t in $[\frac{1}{2}\delta, \delta]$. Define an operator D_a by*

$$D_a f = \iint e^{i\varphi(t,x,\xi) - iy\cdot\xi} \eta(x,y) a(t,x,\xi) f(y)\, d\xi dy$$

For any smooth curve Γ in S_r of length $L \leq 1$, and for f with fixed compact support,

$$\|I_\lambda(D_a) f\|_{L^2(\Gamma)} \lesssim L^{1/4} \lambda^{1/4} \|f\|_{L^2(\mathbb{R}^2)}$$

Using Lemma 4.2, we can now prove Lemma 4.1.

Proof of Lemma 4.1. Fix a symbol $a \in S^0_{\frac{2}{3},\frac{1}{3}}(\mathbb{R}^3_{t,x} \times \mathbb{R}^2_\xi)$. We can assume a is supported by x in a small neighborhood of S_r and by t in $[\frac{1}{2}\delta, \delta]$. Moreover, we can assume that $\big(1 - \eta(x,y)\big) a(t,x,\xi)$ vanishes on a neighborhood of the set

$$\Sigma_0 = \{(t,x,y,\xi) : t = d_0(x,y)\}$$

We can make these assumptions because $I_\lambda(U_a)$ only depends on t in the support of $\hat{\chi}$. The kernel of U_a is

$$\int e^{i\varphi(t,x,\xi) - iy\cdot\xi} a(t,x,\xi)\, d\xi$$

Define a set

$$\Sigma = \Big\{(t,x,y,\xi) : \varphi'_\xi(t,x,\xi) - y = 0\Big\}$$

Define an operator D_a with kernel

$$\int e^{i\varphi(t,x,\xi) - iy\cdot\xi}\eta(x,y)a(t,x,\xi)\,d\xi$$

By (2.21), the set Σ is contained in Σ_0. So the symbol $\big(1-\eta(x,y)\big)a(t,x,\xi)$ vanishes on a neighborhood of Σ. By Proposition 1.2.4 of Hörmander [5], the difference between U_a and D_a is smoothing. So Lemma 4.1 will follow from Lemma 4.2. □

The next lemma will give a suitable description of the kernel of $I_\lambda(D_a)$. This description is sufficiently similar to the one used in Burq-Gérard-Tzvetkov [3], so that the same argument will yield Lemma 4.2.

Lemma 4.3. *Fix $a \in S^0_{2/3,1/3}(\mathbb{R}^3_{t,x}\times\mathbb{R}^2_\xi)$ supported by t in $[\frac12\delta,\delta]$. The kernel of $I_\lambda(D_a)$ is of the form*

$$\lambda^{1/2}e^{-i\lambda d_0(x,y)}A_\lambda(x,y) + R_\lambda(x,y) \tag{4.1}$$

where R_λ is uniformly bounded in λ and A_λ is in $C^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ and satisfies

$$|\partial_x^\alpha\partial_y^\beta A_\lambda| \le C_{\alpha,\beta}\lambda^{|\alpha|/3}$$

Also A_λ is supported by x and y satisfying $\delta/2 \le d_0(x,y) \le \delta$.

Lemma 4.3 follows from the same proof as Lemma 3.2. Now we can follow the argument in Burq-Gérard-Tzvetkov [3] to finish the proof of Lemma 4.2.

Argument from Burq-Gérard-Tzvetkov [3]**.** Let T_λ be the operator with kernel

$$\lambda^{1/2}e^{-i\lambda d_0(x,y)}A_\lambda(x,y)$$

We will complete the proof of Lemma 4.2 by showing that for any curve Γ in S_r of length $L \le 1$,

$$\|T_\lambda f\|_{L^2(\Gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\mathbb{R}^2)} \tag{4.2}$$

By using a partition of unity and abusing notation, we can assume there is a point x_0 in S_r such that A_λ is supported by x in the geodesic ball $B(x_0, c_0\delta)$ of radius $c_0\delta$ around x_0, where $c_0 > 0$ is small. Then there are small constants $c_2 > c_1 > 0$ such that A_λ is supported by y in the geodesic annulus $B(x_0,c_2\delta)\smallsetminus B(x_0,c_1\delta)$.

We will use geodesic polar coordinates (ρ,ω) for the y-variable, with ω a unit vector in $T_{x_0}M_0$ and $\rho > 0$, so that $y = \exp_{x_0}(\rho\omega)$. Then we can write

$$(T_\lambda f)(x) = \int_{c_1\delta}^{c_2\delta}(T^\rho_\lambda f_\rho)(x)\,d\rho$$

with

$$(T^\rho_\lambda f)(x) = \int_{S^1} e^{-i\lambda d_{0,\rho}(x,\omega)}A_{\lambda,\rho}(x,\omega)f(\omega)\,d\omega$$

Here

$$d_{0,\rho}(x,\omega) = d_0(x,y), \quad f_\rho(\omega) = f(y), \quad \text{and} \quad A_{\lambda,\rho}(x,\omega) = J(\rho,\omega)A_\lambda(x,y)$$

where J is a smooth function satisfying $J(\rho,\omega) = \rho$ when $c_1\delta \le \rho \le c_2\delta$.

If we can prove the uniform estimates

$$\|T^\rho_\lambda f\|_{L^2(\Gamma)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(S^1)} \tag{4.3}$$

then (4.2) will follow, because we will have

$$\|T^\rho_\lambda f\|_{L^2(\Gamma)} \le \int_{c_1\delta}^{c_2\delta} \|T^\rho_\lambda f_\rho\|_{L^2(\Gamma)}\, d\rho \lesssim L^{1/4}\lambda^{1/4} \int_{c_1\delta}^{c_2\delta} \|f_\rho\|_{L^2(S^1)}\, d\rho \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\mathbb{R})}$$

So it suffices to prove (4.3). By duality, (4.3) is equivalent to

$$\|(T^\rho_\lambda)^* f\|_{L^2(S^1)} \lesssim L^{1/4}\lambda^{1/4}\|f\|_{L^2(\Gamma)} \tag{4.4}$$

We will prove

$$\|T^\rho_\lambda (T^\rho_\lambda)^* f\|_{L^2(\Gamma)} \lesssim L^{1/2}\lambda^{1/2}\|f\|_{L^2(\Gamma)} \tag{4.5}$$

This will imply (4.4), because if $\langle\cdot,\cdot\rangle$ is the inner product on $L^2(\Gamma)$ then

$$\|(T^\rho_\lambda)^* f\|^2_{L^2(S^1)} = \langle T^\rho_\lambda (T^\rho_\lambda)^* f, f\rangle \le \|T^\rho_\lambda (T^\rho_\lambda)^* f\|_{L^2(\Gamma)}\|f\|_{L^2(\Gamma)} \lesssim L^{1/2}\lambda^{1/2}\|f\|^2_{L^2(\Gamma)}$$

So it suffices to prove (4.5). Assume $x(t)$ parametrizes Γ by arc length with domain $0 \le t \le L$. The kernel of $T^\rho_\lambda (T^\rho_\lambda)^*$ is

$$K^\rho_\lambda(t,\tau) = \lambda \int_{S^1} e^{-i\lambda[d_{0,\rho}(x(t),\omega) - d_{0,\rho}(x(\tau),\omega)]} A_{\lambda,\rho}(x(t),\omega)\overline{A_{\lambda,\rho}(x(\tau),\omega)}\, d\omega$$

We will work in coordinates chosen so that $g_{ij}(x_0) = \delta^{ij}$. Then we have the following lemma, which we will use to control K^ρ_λ.

Lemma 4.4. *If $\rho > 0$ is small, then*

$$-\nabla_x d_{0,\rho}(x_0,\omega) = \omega \tag{4.6}$$

Proof. Let Θ be the geodesic sphere of radius ρ around $y = \exp_{x_0}(\rho\omega)$. By Gauss' lemma, the vector ω is normal to Θ at x_0. Define a function G by

$$G(x) = d_{0,\rho}(x,\omega)$$

Then Θ is a level set of G, so $\nabla G(x_0)$ is normal to Θ at x_0. That is, $\nabla G(x_0)$ is a multiple of ω. Let c be the geodesic satisfying $c(0) = x_0$ and $c'(0) = \omega$. Then for small s,

$$G\big(c(s)\big) = \rho - s$$

So the directional derivative of G at x_0 in the direction ω equals -1. That is,

$$\nabla G(x_0)\cdot\omega = -1$$

Since $\nabla G(x_0)$ is a multiple of ω, this implies that $\nabla G(x_0) = -\omega$, which is (4.6). □

Using Lemma 4.4, we can prove the following lemma.

Lemma 4.5. *There is a $\delta_0 > 0$ such that if $|t-\tau| < \delta_0$, then*

$$|K^\rho_\lambda(t,\tau)| \lesssim \lambda(1+\lambda|t-\tau|)^{-1/2}$$

Proof. Define

$$K^\rho_\lambda(x,x') = \lambda \int_{S^1} e^{-i\lambda[d_{0,\rho}(x,\omega) - d_{0,\rho}(x',\omega)]} A_{\lambda,\rho}(x,\omega)\overline{A_{\lambda,\rho}(x',\omega)}\, d\omega$$

Since Γ is smooth and parametrized by arc length, it suffices to show that

$$|K^\rho_\lambda(x,x')| \lesssim \lambda(1+\lambda|x-x'|)^{-1/2} \tag{4.7}$$

We can write

$$d_{0,\rho}(x,\omega) - d_{0,\rho}(x',\omega) = (x-x')\cdot \Psi_{0,\rho}(x,x',\omega)$$

where

$$\Psi_{0,\rho}(x,x',\omega) = \int_0^1 \nabla_x d_{0,\rho}\big(x' + s(x-x'),\omega\big)\, ds$$

For σ in S^1, define

$$\Phi_{0,\rho}(x,x',\sigma,\omega) = \sigma \cdot \Psi_{0,\rho}(x,x',\omega)$$

Now when $x \neq x'$,

$$d_{0,\rho}(x,\omega) - d_{0,\rho}(x',\omega) = |x-x'|\Phi_{0,\rho}(x,x',\sigma_{x,x'},\omega)$$

where

$$\sigma_{x,x'} = \frac{x-x'}{|x-x'|}$$

If we define

$$J^\rho_\mu(x,x',\sigma) = \int_{S^1} e^{-i\mu\Phi_{0,\rho}(x,x',\sigma,\omega)} A_{\lambda,\rho}(x,\omega)\overline{A_{\lambda,\rho}(x',\omega)}\, d\omega \tag{4.8}$$

then it suffices to show that

$$|J^\rho_\mu(x,x',\sigma)| \lesssim (1+\mu)^{-1/2} \tag{4.9}$$

Parametrize S^1 by

$$\omega(\theta) = (\cos\theta, \sin\theta)$$

for θ in $[0, 2\pi)$. Write

$$\sigma = (\cos\alpha, \sin\alpha)$$

where α is in $[0,2\pi)$. Then by Lemma 4.4,

$$\Phi_{0,\rho}(x_0,x_0,\sigma,\omega(\theta)) = -\sigma\cdot\omega(\theta) = -\cos(\theta-\alpha)$$

So we have

$$\partial_\theta \Phi_{0,\rho}(x_0,x_0,\sigma,\omega(\theta)) = \sin(\theta-\alpha)$$

and

$$\partial^2_\theta \Phi_{0,\rho}(x_0,x_0,\sigma,\omega(\theta)) = \cos(\theta-\alpha)$$

There are relatively open sets A and B, with $A\cup B = [0,2\pi)$, such that for θ in A,

$$|\partial_\theta \Phi_{0,\rho}(x_0,x_0,\sigma,\omega(\theta))| \geq c_A$$

and for θ in B,

$$|\partial^2_\theta \Phi_{0,\rho}(x_0,x_0,\sigma,\omega(\theta))| \geq c_B$$

Here c_A and c_B are positive constants. By continuity, if δ is sufficiently small and x, x' are in $B(x_0, c_0\delta)$, then for θ in A,

$$|\partial_\theta \Phi_{0,\rho}(x,x',\sigma,\omega(\theta))| \geq c_A/2 \tag{4.10}$$

and for θ in B

$$|\partial^2_\theta \Phi_{0,\rho}(x,x',\sigma,\omega(\theta))| \geq c_B/2 \tag{4.11}$$

By using a partition of unity on S^1 and abusing notation, it suffices to prove (4.9) in two cases. In the first case, we assume that (4.10) holds on the support of the amplitude in (4.8). This case can be handled by integrating by parts, which yields much stronger bounds than in (4.9). In the second case, we assume that (4.11) holds on the support of the amplitude in (4.8). This case can be handled by using stationary phase, which yields (4.9). □

We can use Lemma 4.5 and then Young's inequality to obtain

$$\|T^\rho_\lambda (T^\rho_\lambda)^* f\|_{L^2(\gamma)} \lesssim \Big\| \int_0^L \lambda(1+\lambda|t-\tau|)^{-1/2} f\big(x(\tau)\big)\, d\tau \Big\|_{L^2(0,L)}$$
$$\lesssim \Big(\int_0^L \lambda(1+\lambda t)^{-1/2}\, dt \Big) \|f\|_{L^2(\gamma)} \lesssim L^{1/2}\lambda^{1/2}\|f\|_{L^2(\gamma)}$$

This is (4.5), so we have proven Lemma 4.2. Now Theorem 1.1 follows.

5. Proof of Proposition 1.6

In proving Proposition 1.6, we may assume the length L of γ is small. For sufficiently small $\delta > 0$, we can break up γ into $\gamma \cap E_\delta$ and $\gamma \cap H_\delta$, where $\gamma \cap H_\delta$ is a broken geodesic with length at most $c_0\delta^{1/2}$ for some fixed constant $c_0 > 0$. This is because the boundary is strictly geodesically concave. By Theorem 1.1,

$$\limsup_{j\to\infty} \lambda_j^{-1/4} \|e_j\|_{L^2(\gamma\cap H_\delta)} \lesssim \delta^{1/8}$$

Now it suffices to prove

$$\limsup_{j\to\infty} \lambda_j^{-1/4} \|e_j\|_{L^2(\gamma\cap E_\delta)} = 0$$

That is, we may assume γ is a geodesic in M with $d_g(\gamma, \partial M) \geq \delta$. With this assumption, we can follow the proof by Sogge [10] for the boundaryless version of this problem, making only very minor modifications.

The proof will make use of Fermi normal coordinates about γ. These coordinates are well-defined on some neighborhood W of γ. In this coordinate system, γ becomes $\{(s,0) : s \in [0,L]\}$ and the metric satisfies

$$g_{ij}(s,0) = \delta^{ij}$$

In the Fermi coordinates, the principal symbol p of $\sqrt{-\Delta_g}$ satisfies

$$p\big((s,0),\xi\big) = |\xi|$$

Fix a real-valued $\chi \in \mathcal{S}(\mathbb{R})$ with $\chi(0) = 1$ and $\hat\chi$ supported on $[-1/2, 1/2]$. Then

$$\chi(N(\sqrt{-\Delta_g} - \lambda_j))e_j = e_j$$

So it suffices to prove

$$\|\chi(N(\sqrt{-\Delta_g} - \lambda))f\|_{L^2(\gamma)} \leq CN^{-1/2}\lambda^{1/4}\|f\|_{L^2(M)} + C_N\|f\|_{L^2(M)}$$

for all $N > 0$. Fix N. Then

$$\chi(N(\sqrt{-\Delta_g} - \lambda))f = N^{-1} \int \hat\chi(t/N) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f\, dt$$

Note the integrand is supported on $[-N/2, N/2]$.

The operator $Uf(t,x) = e^{it\sqrt{-\Delta_0}} f(x)$ is a Fourier integral operator from M to $M \times \mathbb{R}$. Its canonical relation is

$$\Big\{ (x,t,\xi,\tau; y,\eta) : (x,\xi) = \Phi_t(y,\eta), \pm\tau = p(x,\xi) \Big\}$$

where $\Phi_t : T^*M_0 \to T^*M_0$ is the geodesic flow on the cotangent bundle of M_0. The operator $Vf(t,x) = \left(e^{it\sqrt{-\Delta_0}}f\right)\big|_\gamma(x)$ is a Fourier integral operator from M to $\gamma \times \mathbb{R}$. Using the Fermi normal coordinates, we can write its canonical relation as

$$\mathcal{C} = \left\{ \big((s,0),t,\xi_1,\tau;y,\eta\big) : \big((s,0),(\xi_1,\xi_2)\big) = \Phi_t(y,\eta), \pm\tau = |\xi| \right\}$$

We can parametrize $\mathcal{C}$ with coordinates (s,t,ξ_1,ξ_2). Then the projection from $\mathcal{C}$ to $T^*(\gamma \times \mathbb{R})$ is given by the map

$$(s,t,\xi_1,\xi_2) \to (s,t,\xi_1,|\xi|)$$

This has surjective differential away from $\xi_2 = 0$.

Let $\psi \in C_0^\infty(M)$ be supported strictly inside W. Let A, B_1, and B_2 be pseudodifferential operators of order zero with symbols satisfying

$$\psi(x) = A(x,\xi) + B_1(x,\xi) + B_2(x,\xi)$$

In the Fermi coordinates, assume that A is essentially supported outside a conic neighborhood of the ξ_1-axis, B_1 is essentially supported in a conic neighborhood of the positive ξ_1-axis, and B_2 is essentially supported in a conic neighborhood of the negative ξ_1-axis.

If $|t| < \delta$, then

$$\left(A \circ e^{it\sqrt{-\Delta_g}}f\right)\big|_\gamma = \left(A \circ e^{it\sqrt{-\Delta_0}}f\right)\big|_\gamma$$

Define an operator J_A by

$$(J_A)f(t,x) = \left((A \circ e^{it\sqrt{-\Delta_0}})f\right)\big|_\gamma(x)$$

Then J_A is a non-degenerate Fourier integral operator of order zero, because A is essentially supported away from the ξ_1-axis. This implies that

$$\int_{-\frac{1}{2}\delta}^{\frac{1}{2}\delta} \|A \circ e^{it\sqrt{-\Delta_g}}f\|_{L^2(\gamma)}dt \le C_A\|f\|_{L^2(M)}$$

It follows that

$$\int_{-N}^{N} \|A \circ e^{it\sqrt{-\Delta_g}}f\|_{L^2(\gamma)}dt \le C_{N,A}\|f\|_{L^2(M)}$$

So if we define an operator $\chi_\lambda^{N,A}$ by

$$\chi_\lambda^{N,A}f = A \circ \chi(N(\sqrt{-\Delta_g} - \lambda))f = N^{-1}\int \hat{\chi}(t/N)e^{-it\lambda}(A \circ e^{it\sqrt{-\Delta_g}})f dt$$

then

$$\|\chi_\lambda^{N,A}f\|_{L^2(\gamma)} \le C'_{N,A}\|f\|_{L^2(M)}$$

It remains to control the operators χ_λ^{N,B_j} defined by

$$\chi_\lambda^{N,B_j}f = B_j \circ \chi(N(\sqrt{-\Delta_g} - \lambda))f = N^{-1}\int \hat{\chi}(t/N)e^{-it\lambda}\left(B_j \circ e^{it\sqrt{-\Delta_g}}\right)f dt$$

Define an operator V_j by

$$V_j f(t,x) = \left((B_j \circ e^{it\sqrt{-\Delta_g}} \circ B_j^*)f\right)(x)$$

Fix a distribution u supported in the interior of M. Assume that (t,x,τ,ξ) is in the wave front set of $V_j u$. Then (x,ξ) is in the essential support of B_j, and for some (y,η) in the essential support of B_j, there is a broken geodesic Γ satisfying $\Gamma(0) = y$, $\Gamma'(0) = \eta$, $\Gamma(t) = x$ and $\Gamma'(t) = \xi$. Since γ is not contained in a periodic

broken geodesic, the cutoffs ψ and B_j can be chosen with sufficiently small supports so that $V_j u$ is a smooth function over $2L \leq |t| \leq N+1$. That is, the operator V_j is smoothing over the region $2L \leq |t| \leq N+1$.

Define an operator U_j by

$$U_j f(t,x) = \Big((B_j \circ e^{it\sqrt{-\Delta_0}} \circ B_j^*) f \Big)(x)$$

Then the operator $V_j - U_j$ is smoothing over the region $|t| \leq 10L$, if L is small.

Let T be the operator $f \to (\chi_\lambda^{N,B_j} f)\big|_\gamma$. We want to show that

$$\|Tf\|_{L^2(M)} \leq (N^{-1/2}\lambda^{1/4} + C_{N,B_j})\|f\|_{L^2(\gamma)}$$

We will use the TT^* method. We have

$$\|T^*g\|^2_{L^2(M)} = \int_M T^*g\overline{T^*g}\, dx = \int_\gamma \big(TT^*g\big)\overline{g}\, ds \leq \|TT^*g\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)}$$

So by duality, it suffices to prove that

$$\|TT^*g\|_{L^2(\gamma)} \leq (N^{-1}\lambda^{1/2} + C_{N,B_j})\|g\|_{L^2(\gamma)} \tag{5.1}$$

Let $\rho(\tau) = (\chi(\tau))^2$. Then the kernel of TT^* is $K(\gamma(s),\gamma(s'))$ where $K(x,y)$ is the kernel of $B_j \circ \rho(N(\sqrt{-\Delta_g} - \lambda)) \circ B_j^*$. Also $\hat{\rho}$ is supported in $[-1,1]$, since $\hat{\rho} = \hat{\chi} * \hat{\chi}$. Now

$$B_j \circ \rho(N(\sqrt{-\Delta_g} - \lambda)) \circ B_j^* = N^{-1} \int \hat{\rho}(t/N) e^{-it\lambda} \Big(B_j \circ e^{it\sqrt{-\Delta_g}} \circ B_j^* \Big)\, dt$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ be supported on $[-1,1]$ with $\varphi = 1$ on $[-1/2, 1/2]$. Now, by the smoothing properties of the operators V_j and $V_j - U_j$, the difference between $B_j \circ \rho(N(\sqrt{-\Delta_g} - \lambda)) \circ B_j^*$ and

$$N^{-1} \int \varphi(t/5L)\hat{\rho}(t/N) e^{-it\lambda} \Big(B_j \circ e^{it\sqrt{-\Delta_0}} \circ B_j^* \Big)\, dt \tag{5.2}$$

has a kernel which is $\mathcal{O}(\lambda^{-m})$ for all m, so it remains to control the kernel of the operator (5.2). If $5L$ is less than the injectivity radius of M_0, then the Hadamard parametrix can be used here. Then by stationary phase arguments, it follows that the kernel of the operator (5.2) satisfies

$$|K(x,y)| \leq CN^{-1}\lambda^{1/2}(d_g(x,y))^{-1/2} + C_{B_j}$$

This yields (5.1), completing the proof of Proposition 1.6.

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