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DETERMINANTAL EQUATIONS FOR SECANT VARIETIES AND THE EISENBUD-KOH-STILLMAN CONJECTURE

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Abstract. We address special cases of a question of Eisenbud on the ideals of secant varieties of Veronese re-embeddings of arbitrary varieties. Eisenbud's question generalizes a conjecture of Eisenbud, Koh and Stillman (EKS) for curves. We prove that set-theoretic equations of secant variety to high degree Veronese re-embedding of smooth variety are determined by equations of the ambient Veronese variety and some linear equations. However this is false for singular varieties, and we give explicit counter-examples to the EKS conjecture for singular curves. The techniques we use also allow us to prove a gap and uniqueness theorem for symmetric tensor rank. We put Eisenbud's question in a more general context which we call *brpp*, and discuss conjectures coming from signal processing and complexity theory in this context.

1. Introduction

Throughout the paper we work over the base field of complex numbers $\mathbb{C}$. The starting point of this paper was the observation that aspects of conjectures and questions originating in signal processing, computer science, and algebraic geometry all amounted to assertions regarding linear sections of secant varieties of Segre and Veronese varieties. In this paper we focus on linear sections of Veronese varieties to (i) reduce a question of Eisenbud (Question 1.2.3) regarding arbitrary varieties to the case of projective space, (ii) give explicit counter-examples to a 20 year old conjecture of Eisenbud, Koh and Stillmann (Conjecture 1.2.1), and (iii) prove a uniqueness theorem for tensor decomposition (Theorem 1.5.1) that should be useful for applications to signal processing (more precisely, blind source separation, see, e.g. [13]).

1.1. **Secant varieties of Veronese re-embeddings.** Fix a projective variety $X \subset \mathbb{P}V$, an integer $r \geq 1$ and choose a sufficiently large $d \in \mathbb{N}$. The main objective of this paper is to compare the r-th secant variety of d-th Veronese embeddings of X and $\mathbb{P}V$, denoted, respectively, $\sigma_r(v_d(X))$ and $\sigma_r(v_d(\mathbb{P}V))$. Here and throughout the article, for $Y \subset \mathbb{P}^N$, the *r-th secant variety* $\sigma_r(Y)$ is defined as

$$\sigma_r(Y) = \overline{\bigcup_{y_1,\dots,y_r \in Y} \langle y_1, \dots, y_r \rangle} \subset \mathbb{P}^N \tag{1.1}$$

where $\langle y_1, \dots, y_r \rangle \subset \mathbb{P}^N$ denotes the linear span of the points $y_1, \dots, y_r$ and the overline denotes Zariski closure. The *d-th Veronese embedding* is denoted $v_d \colon \mathbb{P}V \to \mathbb{P}(S^dV)$.

Since $\sigma_r(v_d(X))$ is contained in $\sigma_r(v_d(\mathbb{P}V))$ and also in $\langle v_d(X) \rangle$, the linear span of $v_d(X)$, it is contained in the intersection:

$$\sigma_r(v_d(X)) \subset \sigma_r(v_d(\mathbb{P}V)) \cap \langle v_d(X) \rangle \tag{1.2}$$

Our first result is that for smooth X and d sufficiently large, the above inclusion is (set-theoretically) an equality. Let $Got(h_X)$ denote the Gotzmann number of the Hilbert polynomial of X, see Section 2.2.

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Theorem 1.1.1. *Let $X \subset \mathbb{P}^n$ be a smooth subvariety and let $r \in \mathbb{N}$. For all $d \geq r-1+Got(h_X)$, one has the equality of sets*

$$\sigma_r(v_d(X)) = \big(\sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X)\rangle\big)_{\rm red},$$

where $(\cdot)_{\rm red}$ denotes the reduced subscheme.

The theorem is motivated by a question of Eisenbud, which we review in Section 1.2, and questions arising in applications comparing, for a projective variety Z, $\sigma_r(Z)\cap L$ with $\sigma_r(Z\cap L)$ where L is a linear space, see §1.3 below. We prove Theorem 1.1.1 in Section 2.2. The proof is based on the Gotzmann regularity property — see Proposition 2.1.2. The main new ingredient in the proof is the following lemma.

Lemma 1.1.2 (Main Lemma)**.** *Let $X \subset \mathbb{P}^n$ be a subscheme. Suppose $d \geq r-1+Got(h_X)$ and $R \subset \mathbb{P}^n$ is a 0-dimensional scheme of degree at most r. Then $\langle v_d(R)\rangle \cap \langle v_d(X)\rangle = \langle v_d(R\cap X)\rangle$.*

More generally, one can ask:

Question 1.1.3. Does Theorem 1.1.1 remain true for singular varieties?

We first show that Question 1.1.3 fails in general:

Theorem 1.1.4. *For any q and $r \geq 2$, there exist irreducible, singular varieties $X \subset \mathbb{P}V$ such that $\sigma_r(v_d(X)) \neq \big(\sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X)\rangle\big)_{\rm red}$ as sets, and moreover $\sigma_r(v_d(X))$ is not cut out set-theoretically by equations of degree at most q for all $d \geq 2r-1$. Explicit examples of curves with this property are given in §3.1 and §3.3.*

A more precise result is stated in Theorem 1.4.1 below. There we explain what type of singularities are needed to obtain the inequality $\sigma_r(v_d(X)) \neq \sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X)\rangle$, and what type of singularities are needed to have $\sigma_r(v_d(X))$ defined by equations of high degrees.

However we show Question 1.1.3 has a positive answer when X has at worst hypersurface singularities and $r \leq 2$ in Theorem 2.4.2. We also show that the question has a positive answer "locally", in the sense that $\sigma_r(v_d(X))$ is an irreducible component of $\sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X)\rangle$ in Theorem 2.4.1. These results generalize essentially verbatim to reducible X.

1.2. **Background and history.** D. Mumford [32] observed that if $X \subset \mathbb{P}V$ is a projective variety, and one takes a sufficiently large Veronese re-embedding of X, $v_d(X) \subset \mathbb{P}S^dV$, then $v_d(X)$ will be cut out set-theoretically by quadrics (in fact quadrics of rank at most four), and moreover, that if X is smooth, the ideal of $v_d(X)$ will be generated in degree two. P. Griffiths [21, Thm. p 271] remarked further that with d as above, and X smooth, the embedding $v_{2d}(X)$ will be cut out set-theoretically by the two by two minors of a matrix of linear forms. These results were generalized to ideal-theoretic equations of minors for arbitrary varieties by J. Sidman and G. Smith [37].

More generally, let L_1, L_2 be ample line bundles on an abstract variety X. The map

$$\phi_{L_1^d\otimes L_2^e} : X \to \mathbb{P}\big(H^0(X, L_1^d\otimes L_2^e)^*\big)$$

will be an embedding for d, e sufficiently large. Write $V_1 = H^0(X, L_1^d)^*$, $V_2 = H^0(X, L_2^e)^*$, so there is a map $V_1^*\otimes V_2^* \to H^0(X, L_1^d\otimes L_2^e)$ given on decomposable elements by multiplication of sections. Let W^* denote the image of the map, so there is an inclusion $W \subset V_1\otimes V_2$, and $\phi_{L_1^d\otimes L_2^e}(X) \subset \mathbb{P}W$. Under this inclusion, the image of X lies in the Segre variety $Seg(\mathbb{P}V_1 \times \mathbb{P}V_2)$ of rank one elements intersected with $\mathbb{P}W$ (see, e.g. [15, 30]). The ideal of the Segre is generated in degree two by the two by two minors, i.e., $\Lambda^2V_1^*\otimes\Lambda^2V_2^*$, so these minors provide equations for $X \subset \mathbb{P}W$.

In the above setting, $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X)) \subset \sigma_r(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2))$, and thus equations for the latter give equations for the former. With this in mind, define

$$I(Rank_r(L_1^d, L_2^e)) \subset \mathrm{Sym}(W^*)$$

the ideal generated by the image of the $r+1$ by $r+1$ minors. Note that in general $I(Rank_r(L_1^d, L_2^e))$ need not be radical, or even saturated.

The following conjecture is due to D. Eisenbud, J. Koh, and M. Stillman.

Conjecture 1.2.1 (EKS conjecture (1988) [15])**.** *Let C be a reduced, irreducible curve, let L_1, L_2 be line bundles on C. Then there exists a "good constant" r_0 depending only on the genus of C and the L_i such that there is an equality of ideals*

$$I(\sigma_r(\phi_{L_1^d\otimes L_2^e}(C))) = I(Rank_r(L_1^d, L_2^e))$$

for all $r \leq r_0(d, e)$. Moreover r_0 tends to infinity as $d, e \to \infty$.

Conjecture 1.2.1 was proved set-theoretically in the case C is a smooth curve in [35], and scheme-theoretically for smooth curves in [18]. In fact, sharp bounds on d, e were given in terms of genus of the curve.

To relate Conjecture 1.2.1 to the begining paragraph of this subsection, take $C \subset \mathbb{P}V$, $L_1 = L_2 = \mathcal{O}_C(1)$ and $r = 1$. More generally, if $r \geq 1$, then $W \subset S^{d+e}V \subset S^dV \otimes S^eV$ and the corresponding equations are the so called, *symmetric flattenings* or *catalecticant minors* studied first by Sylvester, see [24] for a history.

Conjecture 1.2.1 was generalized to higher dimensions by D. Eisenbud in the form of a question:

Question 1.2.2 (Eisenbud, unpublished, also see [37])**.** Let X be a reduced, irreducible variety, let L_1, L_2 be ample line bundles on X. Fix r. Does there exist infinitely many sufficiently large d, e such that $I(\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))) = Rank_r(L_1^d, L_2^e)$?

Already for $X = \mathbb{P}^n$, $n > 1$, and $r = 2$, the answer to Question 1.2.2 was not known when we this paper was first posted on arXiv. Consider $L_1 = L_2 = \mathcal{O}(1)$ and $r = 2$. Then if d, e are large, the equations of $Rank_2(L_1^d, L_2^e)$ cut out $\sigma_2(v_{d+e}(\mathbb{P}^n))$ scheme-theoretically, but it was only known that if one adds the equations given by the 3×3 minors obtained from $L_1^1 \otimes L_2^{d+e-1}$, then one has generators of the ideal of $\sigma_2(v_{d+e}(\mathbb{P}^n))$, see [25]. A. Geramita [17, p. 155] conjectured that these additional equations are superfluous, which was recently proven by C. Raicu [34]. For $X = \mathbb{P}^n$, $n > 1$, and $r \geq 3$, the answer to Question 1.2.2 is still not known.

In [37], a slightly different form of the question is stated as a conjecture that involves only one line bundle which is required to be sufficiently ample.

The Eisenbud-Koh-Stillman conjecture and the question of Eisenbud were stated in the ideal-theoretic setting, i.e., the two varieties in question had the same ideals. One could attempt to address the weaker scheme, or set-theoretic problems. Another weaker form of the problem would be simply to determine, if the ideal of $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))$ is generated in degree $r + 1$, or even weaker, that $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))$ is cut out set-theoretically by equations of degree $r + 1$. It is this last statement, in the special case where one begins with $X \subset \mathbb{P}V$ and only considers Veronese re-embeddings, i.e., $L_1 = L_2 = \mathcal{O}_{\mathbb{P}V}(1)|_X$, that came to our attention because of its connections with conjectures originating in signal processing and theoretical computer science that we explain in §4. By Theorem 1.1.4, we should restrict attention to smooth varieties. Thus we focus on the following special case:

Question 1.2.3 (Restricted Eisenbud Question)**.** Let $X \subset \mathbb{P}V$ be an irreducible variety, and fix $r \in \mathbb{N}$. Does there exists infinitely many d such that $\sigma_r(v_d(X))$ is cut out set-theoretically by equations of degree $r + 1$?

Theorem 1.1.1 implies:

Corollary 1.2.4. *The answer to the restricted Eisenbud Question is affirmative for smooth $X \subset \mathbb{P}V$ if it is affirmative for $\mathbb{P}V$.*

Thus it remains to resolve the following question:

Question 1.2.5. Let $V = \mathbb{C}^{n+1}$ and fix a natural number r. Does there exist an integer $d_0 = d_0(n, r)$, such that for infinitely many (or even all) $d \geq d_0$, there exists an ideal $I \subset \mathrm{Sym}(S^dV^*)$ generated in degrees at most $r+1$, such that the (reduced) subvariety in $\mathbb{P}(S^dV)$ consisting of the zero locus of I is $\sigma_r(v_d(\mathbb{P}V))$?

There is little information known about ideals of secant varieties: for any nondegenerate variety, the ideal of its r-th secant variety is empty in degree r [28, Lemma 2.2] and for certain special examples, e.g. *sub-cominuscule varieties* (see [27]) which include quadratic Veronese varieties and two-factor Segre varieties, the ideal is known to be generated in degree $r+1$ for all r. M. Green has defined a sequence of properties N_p for algebraic varieties related to this property [20].

Theorem 1.1.4 provides examples for which the degrees of equations necessary to define $\sigma_r(v_d(X))$ are very high. In particular, these examples are counter-examples to Conjecture 1.2.1 and examples where the answer to Question 1.2.2 is negative, but both only for singular varieties.

The equations of secant varieties of Veronese embeddings of $\mathbb{P}V$ are studied intensively, see [29] for the state of the art. Despite this, the only cases Question 1.2.5 is known to have an affirmative answer are (n, r): $(1, r)$, all r, $(2, r)$, $r \leq 6$, and all n when $r \leq 3$. Already the answer is not known for $(n, r) = (3, 4)$. In general, $\sigma_r(v_d(\mathbb{P}V))$ is a component of a Rank locus for $r \leq \binom{\lfloor \frac{d}{2} \rfloor + n}{n}$, see [29], [24].

We summarize by stating the cases where the answer to Question 1.2.3 is positive:

Corollary 1.2.6. *Suppose $X \subset \mathbb{P}V$ is smooth and $r \leq 3$, or X has at most hypersurface singularities and $r \leq 2$. Then there exists d_0 such that for all $d \geq d_0$ the secant variety $\sigma_r(v_d(X))$ is cut out (set-theoretically) by degree $r+1$ equations.*

For arbitrary irreducible $X \subset \mathbb{P}V$, and arbitrary r, there exists d_0 such that $\sigma_r(v_d(X))$ is an irreducible component of a subvariety defined by rank conditions for all $d \geq d_0$.

Remark 1.2.7. After this article was posted on arXiv, in [5], W. Buczyńska and the first named author proved that the catalecticant minors are not enough in general enough to generate the ideal I as in Question 1.2.5. In particular, it provides a negative answer to Question 1.2.2 for smooth varieties X of sufficiently large dimension and sufficiently large r. However, they also prove that for small values of r or n, the catalecticant minors are enough to generate I, and thus in these cases Questions 1.2.2 and 1.2.5 have affirmative answer. See [5] for details.

1.3. **How we were led to these questions.** For many applications, one needs defining equations for secant varieties to Segre and Veronese varieties. A typical problem that arises in applications is one is handed a tensor and would like to decompose it to a minimal sum of rank one tensors. It is natural to generalize this decomposition to arbitrary varieties as follows:

Definition 1.3.1. Let $X \subset \mathbb{P}V$ be a subvariety and let $p \in \langle X \rangle$.

- Define $R_X(p)$ (the *X-rank of p*) to be the minimal number r, such that $p \in \langle p_1, \ldots, p_r \rangle$ for some points $p_i \in X$. (Note that $\sigma_r(X) \subset \mathbb{P}V$ is the closure of the set of points in $\langle X \rangle$ of X-rank at most r.)
- Define $\underline{\mathbf{R}}_X(p)$ (the *X-border rank of p*) to be the minimal number r, such that $p \in \sigma_r(X)$.

When X is a Segre or Veronese variety, the X-rank is the smallest number r of rank one terms needed for a decomposition, and the X-border rank is the smallest r needed if one is statisfied with a decomposition accurate within an ϵ of one's choosing.

Definition 1.3.2. Let $X \subset \mathbb{P}V$ be a variety and $L \subset \mathbb{P}V$ be a linear subspace. Let $Y := (X \cap L)_{\text{red}}$.

- We say (X, L) is a *rank preserving pair* or *rpp* for short, if $\langle Y \rangle = L$ and $R_X|_L = R_Y$ as functions.
- We say (X, L) is a *border rank preserving pair* or *brpp* for short, if $\langle Y \rangle = L$ and $\underline{\mathbf{R}}_X|_L = \underline{\mathbf{R}}_Y$ as functions, i.e., $\sigma_r(X) \cap L = \sigma_r(Y)$ for all r.
- Similarly we say (X, L) is rpp_r (respectively, $brpp_r$) if $R_X(p) = R_{X\cap L}(p)$ for all $p \in L$ with $R_X(p) \leq r$ (respectively, $\sigma_s(X) \cap L = \sigma_s(Y)$ for all $s \leq r$).

Note that one always has $R_X(p) \leq R_{X\cap L}(p)$ and $\underline{\mathbf{R}}_X(p) \leq \underline{\mathbf{R}}_{X\cap L}(p)$.

Theorem 1.1.1 may be rephrased in this language:

Theorem 1.3.3 (rephrasing of Theorem 1.1.1)**.** *For all smooth subvarieties $X \subset \mathbb{P}V$ and all $r \in \mathbb{N}$, there exist an integer d_0 such that for all $d \geq d_0$, the pair $(v_d(\mathbb{P}V), \langle v_d(X)\rangle)$ is $brpp_r$.*

As explained in §4, V. Strassen and P. Comon asked the following similar questions:

Conjecture 1.3.4. *Let A_j be vector spaces Write $A_j = A'_j \oplus A''_j$ and let $L = \mathbb{P}((A'_1 \otimes \cdots \otimes A'_k) \oplus (A''_1 \otimes \cdots \otimes A''_k))$ Then*

$$(X, L) = (Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k), \mathbb{P}((A'_1 \otimes \cdots \otimes A'_k) \oplus (A''_1 \otimes \cdots \otimes A''_k)))$$

is rpp.

Conjecture 1.3.5. *Let $\dim A_j = \mathbf{a}$ for each j and identify each A_j with a vector space A. Consider $L = \mathbb{P}(S^k A) \subset A_1 \otimes \cdots \otimes A_k$. Then*

$$(X, L) = (Seg(\mathbb{P}A \times \cdots \times \mathbb{P}A), \mathbb{P}S^k A)$$

is rpp.

Our project began with the idea to study these problems simultaneously. We have included a discussion of these related conjectures in the hope of bringing them to the attention of the community of algebraic geometers. In particular, we expect our geometric interpretation of Schönhage's result mentioned below will be useful in the study of these questions. A few general results on rank and border rank preserving pairs are given in §4.

Border rank versions. In the cases of the the conjectures of Comon and Strassen, it is natural to ask the corresponding questions for border rank. For Strassen's conjecture, this has already been answered negatively:

Theorem 1.3.6 (Schönhage, [36])**.** *brpp fails for*

$$(X, L) = (Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), (A' \otimes B' \otimes C') \oplus (A'' \otimes B'' \otimes C''))$$

starting with the case $\mathbf{a} \geq 5 = 2 + 3$, $\mathbf{b} \geq 6 = 3 + 3$, $\mathbf{c} \geq 7 = 6 + 1$, where the splittings into sums give the dimensions of the subspaces.

We discuss Schönhage's theorem in §4.4.

1.4. **A more precise version of Theorem 1.1.4.** For a variety $X \subset \mathbb{P}V$ and a point $x \in X$, the *tangent star* of X at x, $T^\star_x X \subset \mathbb{P}V$ is defined to be the union of the points on the $\mathbb{P}^1$'s obtained as limits in the Grassmannian $\mathbb{G}(1, \mathbb{P}V)$ of $\mathbb{P}^1_{x(t),y(t)}$'s spanned by points $x(t), y(t)$, with $x(t), y(t) \in X$ and $x(0) = y(0) = x$. Alternatively, consider the incidence correspondence

$$S_X := \overline{\{(x, y, z) \in X \times X \times \mathbb{P}V \mid z \in \langle x, y\rangle\}},$$

let $\psi : S_X \to X \times X$, $\mu : S_X \to \mathbb{P}V$ denote the projections, then $T^\star_x X = \mu(\psi^{-1}(x, x))$. Note that if $x \in X$ is a smooth point, then $\langle T^\star_x X\rangle$ is the embedded tangent projective space and $T^\star_x X = \langle T^\star_x X\rangle$ (but the converse does not hold).

Theorem 1.4.1. *Let $X \subset \mathbb{P}V$ be a subvariety and let $x \in X$ be a singular point. Suppose $r \geq 2$ and let $d \geq 2r-1$.*

(i) *If $T^\star_x X \neq \langle T^\star_x X\rangle$, then $\sigma_r(v_d(X)) \neq \big(\sigma_r(v_d(\mathbb{P}^n)) \cap \langle v_d(X)\rangle\big)_{\rm red}$.*
(ii) *Suppose $I(T^\star_x X)$ is the defining ideal of the tangent star in $\langle T^\star_x X\rangle$. Suppose for some q, the homogeneous parts $\bigoplus_{i=0}^q I_i(T^\star_x X)$ define a subscheme Z of $\langle T^\star_x X\rangle$, such that no component of $Z_{\rm red}$ is equal to $T^\star_x X$ (this happens for instance if the ideal $I(T^\star_x X)$ is trivial in degrees $\leq q$ and $T^\star_x X \neq \langle T^\star_x X\rangle$). Then $\sigma_r(v_d(X))$ is not defined set-theoretically by equations of degree $\leq q$.*

We prove Theorem 1.4.1 in §3.2.

1.5. **A uniqueness theorem for symmetric tensor rank.** The following theorem is a consequence of Lemma 1.1.2. On one hand, it may be viewed as a generalization of the theorem of Comas and Seguir [11] that states if the rank of a point in $\mathbb{P}(S^d\mathbb{C}^2)$ is larger than its border rank, and the border rank is small, the rank must be at least $\lfloor \frac{d}{2}\rfloor + 2$. (Their theorem gives more precise information about ranks.) On the other hand, it also gives a criterion for uniqueness of an expression of a point as a sum of d-th powers that does not rely on a general point assumption (e.g. [9, 10]) or a Kruskal-type test [26].

Theorem 1.5.1. *Let $p \in \mathbb{P}S^dV$. If $R_{v_d(\mathbb{P}V)}(p) \leq \frac{d+1}{2}$, i.e., the symmetric tensor rank of p is at most $\frac{d+1}{2}$, then $R_{v_d(\mathbb{P}V)}(p) = \underline{\mathbf{R}}_{v_d(\mathbb{P}V)}(p)$ and the expression of p as a sum of $R_{v_d(\mathbb{P}V)}(p)$ d-th powers is unique (up to trivialities).*

Remark 1.5.2. In contrast to the Veronese case, such a result does not hold for Segre varieties $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k) \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$. When $k = 2$ rank equals border rank and there is no uniqueness, and when $k > 2$ elements of border rank two can have rank $2, 3, \ldots, k$.

Overview. In §2 we first review facts about Hilbert schemes, including Gotzmann's regularity result, and explain the relationship between the study of the smoothable component of the Hilbert scheme with the study of secant varieties. We then give proofs of the positive results, and discuss what would be needed to resolve the questions in general. In §3 we construct explicit counter examples to the EKS conjecture. We conclude by discussing questions in signal processing and computer science that led us to study Eisenbud's conjecture in §4. The purpose of this section is to introduce these beautiful problems to the community of algebraic geometers.

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2. Proofs of positive results

2.1. **Hilbert schemes and regularity.** Let $X \subset \mathbb{P}V$ be a subscheme.
Standard Notations.

- $\langle X\rangle \subset \mathbb{P}V$ denotes the scheme-theoretic linear span of X.
- $X_{\rm red}$ denotes the reduced subscheme of X.
- $I(X) \subset {\rm Sym}(V^*)$ denotes the homogeneous, saturated ideal defining of X. The d-th homogeneous piece of $I(X)$ is denoted $I_d(X) \subset S^dV^*$. The ideal sheaf of X is denoted by $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}V}$, so that $H^0(\mathcal{I}_X(d)) = I_d(X)$.

- For a positive integer d the *d-th Veronese reembedding* of X, denoted $v_d(X) \subset \mathbb{P}S^dV$, is the subscheme defined by the ideal in $\mathrm{Sym}(S^dV^*)$ which is the kernel of the following composition:

$$\mathrm{Sym}(S^dV^*) = \bigoplus_{k=0}^{\infty} S^k(S^dV^*) \twoheadrightarrow \bigoplus_{k=0}^{\infty} S^{kd}V^* \hookrightarrow \bigoplus_{k=0}^{\infty} S^dV^* = \mathrm{Sym}(V^*) \twoheadrightarrow \mathrm{Sym}(V^*)/I(Y).$$

Note that for a scheme $X \subset \mathbb{P}V$ the linear span $\langle X\rangle$ is equal to $\mathbb{P}(I_1(X)^\perp)$, i.e., the projective zero locus of the linear part of $I(X)$. In particular, $\langle v_d(X)\rangle = \mathbb{P}(I_d(X)^\perp) \subset \mathbb{P}S^dV$. It is a standard fact that $v_d(X)$ is isomorphic to X as an abstract scheme, see, e.g., [22, Chapter 2, Theorem 2.4.7] or [23, Ex. II 5.13].

For a review on Hilbert schemes, Hilbert polynomials, Hilbert functions, and regularity see, e.g., [33] and references therein.

Given a subscheme $X \subset \mathbb{P}V$ with ideal sheaf $\mathcal{I}_X$, we say $\mathcal{I}_X$ is *δ-regular*, if $H^i(\mathcal{I}_X(\delta - i)) = 0$ for all $i > 0$. Serre's vanishing theorem implies that every $X \subset \mathbb{P}V$ has a δ-regular ideal sheaf for sufficiently large δ.

Proposition 2.1.1. *Suppose $\mathcal{I}_X$ is δ-regular.*

(i) *X is also d-regular for all $d \geq \delta$.*
(ii) *$H^i(\mathcal{O}_X(d)) = 0$ for $d \geq \delta - i$ and $i > 0$.*
(iii) *If h_X is the Hilbert polynomial of X, then $h^0(\mathcal{O}_X(d)) = h_X(d)$ for all $d \geq \delta - 1$.*

Proof. Part (i) is explained in [33, Lem. 2.1(b)]. Part (ii) follows from the long exact cohomology sequence of

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}V}(d) \to \mathcal{O}_X(d) \to 0. \tag{2.1}$$

Part (iii) follows from (ii), keeping in mind that the Hilbert polynomial is also an Euler characteristic. □

Gotzmann's regularity theorem gives a bound on how large δ must be for $\mathcal{I}_X$ to be δ-regular. This bound depends only on the Hilbert polynomial of X, which is essential for our purposes.

Proposition 2.1.2 (Gotzmann's regularity, [19])**.** *Suppose P is the Hilbert polynomial of a subscheme $X \subset \mathbb{P}V$ such that*

$$P(d) = \sum_{i=1}^{Got(P)} \binom{d + a_i - i + 1}{a_i}$$

where $a_1 \geq a_2 \geq .. \geq a_{Got(P)} \geq 0$. Then $\mathcal{I}_X$ is $Got(P)$-regular. In particular:

- *if $X' \subset \mathbb{P}V$ is another scheme with the same Hilbert polynomial P, then $\mathcal{I}_{X'}$ is also $Got(P)$-regular.*
- *if $R \subset \mathbb{P}V$ is a zero dimensional scheme of degree r, then $\mathcal{I}_R$ is r-regular.*

For an exposition of the proof, see [4, Thm 4.3.2].

Lemma 2.1.3. *Suppose $X \subset \mathbb{P}V$ is a δ-regular subscheme with Hilbert polynomial h_X. Then for $d \geq \delta - 1$:*

$$\dim \langle v_d(X)\rangle + 1 = h^0(\mathcal{O}_X(d)) = h_X(d).$$

In particular, if R is a zero dimensional scheme of degree r, and $d \geq r-1$ then $\dim \langle v_d(R)\rangle = r-1$.

Proof. Since all the higher cohomologies vanish, the short exact sequence (2.1) gives rise to a short exact sequence of sections. The codimension of $\langle v_d(X)\rangle$ in $\mathbb{P}(S^dV)$ is equal to $h^0(I_d(X))$. Thus $\dim \langle v_d(X)\rangle + 1 = h^0(\mathcal{O}_{\mathbb{P}V}(d)) - h^0(I_d(X)) = h^0(\mathcal{O}_X(d))$ and the claim follows. The $+1$ is just the difference between projective and vector space dimensions. □

Lemma 2.1.4 (Additivity of Hilbert polynomials)**.** *Suppose $X, R \subset \mathbb{P}V$ are two subschemes. Suppose h_X, h_R, $h_{X\cap R}$ and $h_{X\cup R}$ are respectively the Hilbert polynomials of X, R, $X \cap R$ and $X \cup R$. Then:*

$$h_{X\cap R} = h_X + h_R - h_{X\cup R}.$$

If in addition R is zero dimensional, then $X \cup R$ is $(Got(h_X) + t)$-regular where $t = \deg R - \deg(X \cap R)$.

Proof. For d sufficiently large, all the higher cohomologies vanish in the following short exact sequence:

$$0 \to \mathcal{O}_{X\cup R}(d) \to \mathcal{O}_X(d) \oplus \mathcal{O}_R(d) \to \mathcal{O}_{X\cap R}(d) \to 0.$$

The additivity claim follows. The second claim follows by Gotzmann regularity, see Proposition 2.1.2. □

For a projective variety X, let $\mathcal{H}_r(X)$ denote the union of all the irreducible components of the Hilbert scheme $Hilb_r(X)$ of degree r dimension 0 subschemes of X, which contain r distinct points. In case X is irreducible, $\mathcal{H}_r(X)$ is irreducible too. Also if $Y \subset X$, then $\mathcal{H}_r(Y) \subset \mathcal{H}_r(X)$. Schemes that are in $\mathcal{H}_r(X)$ are called *smoothable* It is an interesting and non-trivial problem to determine when $Hilb_r(X) = \mathcal{H}_r(X)$, and to identify the schemes that are in $\mathcal{H}_r(X)$ if the equality does not hold— see, e.g., [6], [16] and references therein.

The smoothable component $\mathcal{H}_r(X)$ is relevant to our study because of its relation to secant varieties:

Lemma 2.1.5. *Let $X \subset \mathbb{P}V$ be a subvariety that is not a set of less than r points, and let $d \geq r - 1$. Let $p \in \mathbb{P}S^dV$. Then $p \in \sigma_r(v_d(X))$ if and only if there exists a scheme $R \in \mathcal{H}_r(X)$ such that $p \in \langle v_d(R)\rangle$.*

Proof. By Lemma 2.1.3 any scheme R of degree at most r, is linearly independent after a d-th Veronese re-embedding. With this in mind, the claim becomes [2, Prop. 2.8]. □

We conclude with a simple criterion for flatness and smoothability of zero-dimensional schemes, that we are going to use in the proof of Theorem 1.1.1.

Lemma 2.1.6. *Let $D \subset \mathbb{C}$ be an analytic disk around 0, let $D^\bullet := D \setminus 0$ and $\mathcal{R} \subset \mathbb{P}V \times D$ be a closed subscheme. Suppose for $t \in D^\bullet$ the fibre $\mathcal{R}_t$ is reduced and consists of r distinct points. Let $R \subset \mathbb{P}V$ be the scheme such that the special fibre $\mathcal{R}_0 = R \times \{0\}$. In addition let $\tilde{\mathcal{R}} := \overline{\mathcal{R}|_{D^\bullet}} \subset \mathbb{P}V \times D$, that is $\tilde{\mathcal{R}}$ is the smallest reduced closed subscheme of $\mathbb{P}V \times D$ containing $\mathcal{R}|_{D^\bullet}$. Then:*

(i) *$\tilde{\mathcal{R}} \to D$ is flat;*
(ii) *$\mathcal{R} \to D$ is flat, if and only if $\dim R = 0$ and $\deg R = r$, if and only if $\mathcal{R} = \tilde{\mathcal{R}}$;*

Proof. By [23, Prop. III.9.8] and its proof the map $\tilde{\mathcal{R}} \to D$ is flat, thus (i) is proved. By the same proposition $\mathcal{R}$ is flat if and only if $\mathcal{R} = \tilde{\mathcal{R}}$. Thus (ii) follows from the observation that always $\tilde{\mathcal{R}} \subset \mathcal{R}$ and $\dim \tilde{\mathcal{R}}_0 = 0$ and $\deg \tilde{\mathcal{R}}_0 = r$. □

Remark 2.1.7. In the lemma above and its proof one can replace the analytic disk D with the formal scheme $\operatorname{Spec}\mathbb{C}[[t]]$, where $\mathbb{C}[[t]]$ is the power series ring and the punctured disk $D^\bullet$ with $\operatorname{Spec}\mathbb{C}[[t]][t^{-1}]$ (which is the spectrum of the field of fractions of $\mathbb{C}[[t]]$).

2.2. **Proofs of Main Lemma and the uniqueness Theorem.** Fix an integer r and a subscheme $X \subset \mathbb{P}V$. Let $d_0 = Got(h_X) + r - 1$. Then, if $R \subset \mathbb{P}V$ is a zero dimensional scheme of degree at most r, X, R, $X \cap R$ and $X \cup R$ are $(d_0 + 1)$-regular by Propositions 2.1.1(i), 2.1.2 and Lemma 2.1.4.

Recall that Lemma 1.1.2 states that for $d \geq d_0$ the following equality of linear spans holds:

$$\langle v_d(R)\rangle \cap \langle v_d(X)\rangle = \langle v_d(R \cap X)\rangle.$$

Proof of Lemma 1.1.2. Let $d \geq d_0$, let $R \subset \mathbb{P}V$ be a subscheme of degree at most r. Since $\langle v_d(X \cap R)\rangle \subseteq \langle v_d(X)\rangle \cap \langle v_d(R)\rangle$ trivially holds, to prove equality, it is enough to prove that the dimension of the left hand side equals the dimension of the right hand side.

Since X, $X \cup R$, R and $X \cap R$ are d-regular, it follows from Lemmas 2.1.3 and 2.1.4 that

$$\begin{aligned}\dim \langle v_d(X \cap R)\rangle &= h_{X\cap R}(d) - 1 = h_X(d) + h_R(d) - h_{X\cup R}(d) - 1\\ &= (\dim \langle v_d(X)\rangle + 1) + (\dim \langle v_d(R)\rangle + 1) - (\dim \langle v_d(X \cup R)\rangle + 1) - 1\\ &= \dim \langle v_d(X)\rangle + \dim \langle v_d(R)\rangle - \dim \langle v_d X \cup v_d R\rangle\\ &= \dim (\langle v_d(X)\rangle \cap \langle v_d(R)\rangle).\end{aligned}$$

□

The following is a consequence of Lemma 1.1.2 (with $X = Q$):

Corollary 2.2.1. *Suppose $p \in \mathbb{P}S^dV$ and that $R, Q \subsetneqq \subset \mathbb{P}V$ are two zero dimensional schemes of degree at most r, such that $p \in \langle v_d(R)\rangle$ and $p \in \langle v_d(Q)\rangle$ for some $d \geq \deg(R\cup Q)-1$. Suppose furthermore, that R is minimal in the following sense: for any $R' \subsetneqq R$ we have $p \notin \langle v_d(R')\rangle$. Then $R \subset Q$.*

Proof. Apply Lemma 1.1.2 with $X = Q$, and $d_0 = \deg(R \cup Q) - 1$. Thus $p \in \langle v_d(R \cap Q)\rangle$, and by the assumption that R is minimal, $R \cap Q = R$. □

Theorem 1.5.1 follows, because given two different expressions via schemes R, Q, $\deg(R\cup Q) \leq \deg(R) + \deg(Q) \leq 2r \leq d + 1$, so Corollary 2.2.1 applies.

2.3. **Proof of Theorem 1.1.1.** We start by introducing the following notation.

Notation 2.2. Given a variety $X \subset \mathbb{P}V$, let

$$\Sigma^d_r(X) := \big(\sigma_r(v_d(\mathbb{P}V)) \cap \langle v_d(X)\rangle\big)_{\mathrm{red}}$$

where $(\cdot)_{\mathrm{red}}$ denotes the reduced subscheme.

Theorem 1.1.1 states that $\Sigma^d_r(X) = \sigma_r(v_d(X))$ for $d \geq d_0 = Got(h_x) + r - 1$.

Proof of Theorem 1.1.1. Suppose X is smooth and $p \in \Sigma^d_r(X)$, so that by Lemma 2.1.5 there exists a zero dimensional smoothable subscheme $R \subset \mathbb{P}V$ of degree at most r, such that $p \in \langle v_d(X)\rangle \cap \langle v_d(R)\rangle$. Lemma 1.1.2 implies $p \in \langle v_d(X \cap R)\rangle$. Since smoothability of a zero dimensional scheme is a local property, the components of R which have support away from X are redundant, in the sense that we can replace R with the union of only those components of R that have support on X. Thus, without loss of generality, assume $R_{\mathrm{red}} \subset X$ and also $\deg R = r$ for simplicity of notation.

Since R is smoothable in $\mathbb{P}V$, consider a reducible curve $\mathcal{R} \subset \mathbb{P}V \times D$ approximating R. Here $D \subset \mathbb{C}$ is a sufficiently small analytic disk, the special fibre $\mathcal{R}_0 = R \times \{0\}$ and for $D^\bullet = D \setminus \{0\}$ and $t \in D^\bullet$, the fibre $\mathcal{R}_t$ consists of r distinct points. Moreover $\mathcal{R}$ is reduced by Lemma 2.1.6.

Let $U_1 \subset \mathbb{P}V$ and $U_2 \subset X$ be two sufficiently small open analytic neighborhoods of the support of R (which by our assumption is contained in X). Without loss of generality, assume D is small enough so that $\mathcal{R} \subset U_1$. Suppose also $\pi : U_1 \to U_2$ is a fibration, such that $\pi|_{U_2} = \mathrm{id}_{U_2}$ and such that the curves of $\mathcal{R} \setminus \mathcal{R}_0$ under $\pi \times \mathrm{id}_{D^\bullet}$ are mapped to disjoint curves over $D^\bullet$. (The existence of U_2 and π is implied by the smoothness of X.)

Let $\mathcal{Q} \subset X \times D$ be the image $(\pi \times \mathrm{id}_D)(\mathcal{R})$. Then:

- the general fiber $\mathcal{Q}_t$ is r distinct points on X,

- $\mathcal{Q}$ is reduced, because it is an image of reduced $\mathcal{R}$,
- if $\tilde{\mathcal{Q}} := \overline{\mathcal{Q}|_{D^\bullet}}$, then supports of $\tilde{\mathcal{Q}}$ and $\mathcal{Q}$ agree; thus by the previous item $\tilde{\mathcal{Q}} = \mathcal{Q}$ and $\mathcal{Q} \to D$ is flat by Lemma 2.1.6,
- $(R \cap X) \times \{0\} \subset \mathcal{Q}$, because:

$$(R \cap X) \times \{0\} = \pi(R \cap X) \times \{0\} \subset (\pi \times \mathrm{id}_D)(\mathcal{R}) = \mathcal{Q}.$$

Let $Q \subset X$ be the scheme such that $Q \times \{0\} := \mathcal{Q}_0$ is the special fibre. Since $\mathcal{Q}$ is flat, $\dim Q = 0$ and $\deg Q = r$. Therefore the scheme Q is a smoothable subscheme of X. Moreover $R \cap X \subset Q$. Thus $p \in \langle v_d(Q)\rangle$ and by Lemma 2.1.5, $p \in \sigma_r(v_d(X))$ as claimed. The other inclusion $\sigma_r(v_d(X)) \subset \Sigma^d_r(X)$ always holds. □

2.4. **Extensions to singular varieties.** Throughout this section we continue to use Notation 2.2.

For singular varieties the inclusion $\Sigma^d_r(X) \subset \sigma_r(v_d(X))$ may fail to be an equality (see Section 3). In this section we study the inclusion in detail. First we prove $\sigma_r(v_d(X))$ is an irreducible component of $\Sigma^d_r(X)$.

Theorem 2.4.1. *If $d \geq \max\{2r-2, r-1+Got(h_X)\}$, then $\sigma_r(v_d(X))$ is an irreducible component of $\Sigma^d_r(X)$.*

Proof. The variety $\sigma_r(v_d(X))$ is irreducible because X is. Let Σ be an irreducible component of $\Sigma^d_r(X)$ containing $\sigma_r(v_d(X))$. For a general point $p \in \sigma_r(v_d(X))$, let $p \in \langle v_d(R)\rangle$, where $R \subset X$ consists of r distinct points, and p is not in the span of any of $r-1$ of those points. We claim $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$. Suppose to the contrary that $Q \subset \mathbb{P}V$ is a zero dimensional scheme of degree $\leq r-1$, such that $p \in \langle v_d(Q)\rangle$. Then Corollary 2.2.1 implies $R \subset Q$, a contradiction, since $\deg R > \deg Q$.

The set of points with $v_d(\mathbb{P}V)$-rank r is open in $\sigma_r(v_d(\mathbb{P}V)) \setminus \sigma_{r-1}(v_d(\mathbb{P}V))$. Thus since $\Sigma \subset \sigma_r(v_d(\mathbb{P}V))$, $p \in \Sigma$, $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$, and p has $v_d(\mathbb{P}V)$-rank r, a general point p' in Σ also has $v_d(\mathbb{P}V)$-rank r. Let R' be the union of r distinct points of $\mathbb{P}V$ such that $p' \in \langle v_d(R')\rangle$. By Lemma 1.1.2, $p' \in \langle v_d(X \cap R')\rangle$, and $X \cap R'$ is smooth (hence trivially smoothable). Thus $p' \in \sigma_r(v_d(X))$ and $\sigma_r(v_d(X)) = \Sigma$ as claimed. □

Recall that a variety $X \subset \mathbb{P}V$ is said to have *at most hypersurface singularities*, if the dimension of its Zariski tangent space at any point is at most one greater than the dimension of X.

Theorem 2.4.2. *If X has only hypersurface singularities, and $r = 2$, then $\sigma_r(v_d(X)) = \Sigma^d_r(X)$ for all $d \geq d_0 = Got(h_x) + 1$.*

We postpone the proof of the theorem until later in this subsection. Proposition 2.4.3 below gives further, technical conditions that imply $\sigma_r(v_d(X)) = \Sigma^d_r(X)$, i.e., conditions under which the answer to Question 1.1.3 is positive.

Proposition 2.4.3. *Suppose $X \subset \mathbb{P}V$ is a (not necessarily irreducible) subvariety and r is an integer such that:*

A. *All zero dimensional subschemes $R \subset X$ of degree r which are smoothable in $\mathbb{P}V$ are also smoothable in X.*
B. *All Gorenstein zero dimensional subschemes $Q \subset X$ of degree q with $q < r$ are smoothable in $\mathbb{P}V$.*

Then $\sigma_r(v_d(X)) = \Sigma^d_r(X)$ for all $d \geq d_0 = Got(h_x) + r - 1$.

Condition A is quite strong. We observe in Lemma 2.4.6 that the condition is satisfied in the situation of Theorem 2.4.2. On contrary, in Section 3 we use failure of condition A to produce counter-examples to the EKS Conjecture.

On the other hand, condition B is much milder — we list several cases when it is known to hold in Lemma 2.4.7. In particular, in the situation of Theorem 2.4.2 it holds trivially, as here $q \leq 1$. We are also unaware of any situation when X is singular, condition A is satisfied, but condition B fails to be satisfied. In fact, condition B might not be needed. For example, it fails for $X = \mathbb{P}^N$, with $N \geq 6$ and $r \geq 15$ (i.e., there exist non-smoothable zero-dimensional Gorenstein schemes with embedding dimension 6 and of degree 14, see [24]) yet for smooth X the equality holds (see Theorem 1.1.1).

Before proving Proposition 2.4.3 we explain the relation of condition B with our problem in the following lemma.

Lemma 2.4.4. *Suppose $Q \subset \mathbb{P}V$ is a zero-dimensional subscheme of degree q. Let $d \geq q-1$ be an integer. Then the following conditions are equivalent:*

(i) *Q is Gorenstein;*
(ii) $\dim Hilb_{q-1}Q = 0$*;*
(iii) $\langle v_d(Q)\rangle \neq \bigcup_{Q' \subsetneqq Q} \langle v_d(Q')\rangle$.

Schemes satisfying (i) are studied intensively, see for instance [14, Chap. 21], [8], [7], [24]. Condition (iii) says that Q is minimal in the following sense: for a general $p \in \langle v_d(Q)\rangle$ there exists no smaller $Q' \subset Q$ such that $p \in \langle v_d(Q')\rangle$. We thank to Frank-Olaf Schreyer and Vivek Shende for (independently) pointing out to us the equivalence of (i) and (ii).

Proof. Conditions (i) and (ii) are local, i.e., they hold for Q if and only if they hold for all connected components of Q. Thus to prove the equivalence of (i) and (ii) we may assume Q is supported at one point and hence the structure ring $\mathcal{O}_Q$ is a local algebra of finite dimension over $\mathbb{C}$.

Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_Q$ and $\mathfrak{s}$ be the *socle* of $\mathcal{O}_Q$, that is the annihilator of $\mathfrak{m}$ in $\mathcal{O}_Q$ (see [14, p. 522]). Now a subscheme of length n is defined by an ideal of dimension $q-n$; consequently, $Hilb_{q-1}Q = \mathbb{P}\big(\mathrm{Hom}_{\mathcal{O}_Q}(\mathbb{C}, \mathcal{O}_Q)\big)$. Here $\mathbb{C} = \mathcal{O}_Q/\mathfrak{m}$, thus the image of any homomorphism $\mathbb{C} \to \mathcal{O}_Q$ is contained in the socle $\mathfrak{s}$. On the other hand, any $f \in \mathfrak{s}$ determines a homomorphism $\mathbb{C} \to \mathcal{O}_Q$, by sending $1 \mapsto f$. Thus $\dim Hilb_{q-1}Q = 0$, if and only if $\dim \mathrm{Hom}_{\mathcal{O}_Q}(\mathbb{C}, \mathcal{O}_Q) = 1$ if and only if the socle of $\mathcal{O}_Q$ is one-dimensional, if and only if Q is Gorenstein (see [14, Prop. 21.5a&c]).

To prove the equivalence of (ii) and (iii) note that:

(a) $Hilb_{q-1}Q$ is a projective scheme;
(b) if $Q'' \subsetneqq Q$ is a non-trivial subscheme, then there exists a subscheme Q' of degree $q-1$ such that $Q'' \subset Q' \subset Q$ and thus $\bigcup_{Q' \subsetneqq Q} \langle v_d(Q')\rangle$ is the same if we restrict the union to only Q' of degree $q-1$;
(c) if $Q', Q'' \subset Q$ are two subschemes and $Q' \neq Q''$, then $\langle v_d(Q')\rangle \neq \langle v_d(Q'')\rangle$;
(d) Since $d \geq q-1$, for all $Q' \subset Q$ (including $Q' = Q$), we have $\dim \langle v_d(Q')\rangle = \deg Q' - 1$.

Thus, if $\dim Hilb_{q-1}Q = 0$, then $\dim \bigcup_{Q' \subsetneqq Q} \langle v_d(Q')\rangle$ is $q-2$, so this union cannot be equal to $\langle v_d(Q')\rangle$.

On contrary, if $\dim Hilb_{q-1}Q > 0$, then $\bigcup_{Q' \subsetneqq Q} \langle v_d(Q')\rangle$ is swept by a projective, positive dimensional family of distinct linear subspaces of dimension $q-2$, thus it is closed and of dimension at least $q-1$. Since it is always contained in $\langle v_d(Q)\rangle$ (which is irreducible and of dimension $q-1$), it follows that

$$\bigcup_{Q' \subsetneqq Q} \langle v_d(Q')\rangle = \langle v_d(Q)\rangle.$$

□

Proof of Proposition 2.4.3. Suppose

$$p \in \Sigma_r^d(X) := \big(\langle v_d(X)\rangle \cap \sigma_r(v_d(\mathbb{P}V))\big)_{\rm red},$$

so that by Lemma 2.1.5 there exists a zero dimensional smoothable subscheme $R \subset \mathbb{P}V$ of degree at most r, such that $p \in \langle v_d(X)\rangle \cap \langle v_d(R)\rangle$. By Lemma 1.1.2, also $p \in \langle v_d(X \cap R)\rangle$. If $X \cap R$ is smoothable in X, then $p \in \sigma_r(X)$ by Lemma 2.1.5. So suppose Q is the minimal subscheme of $X \cap R$, such that $p \in \langle v_d(Q)\rangle$ and set $q := \deg Q$.

By the minimality of Q, the hypotheses of Lemma 2.4.4(iii) hold for Q, thus Q is Gorenstein by Lemma 2.4.4(i). Now either $Q = R$, and then it is smoothable in X by A, or $q < r$, and then Q is smoothable in $\mathbb{P}V$ by B. Note that condition A also holds for R replaced with $Q \cup \{x_1, \dots, x_{r-q}\}$, where the x_i are distinct points, disjoint from the support of Q. Thus Q is smoothable in X. □

The following Lemmas determine situations when conditions A and B are satisfied.

Lemma 2.4.5. *Suppose $X \subset \mathbb{P}V$ is a smooth subvariety and r is arbitrary. Then condition A is satisfied.*

Proof. The embedded deformation theory of a zero dimensional scheme $R \subset X$ is smooth over the abstract deformation theory of R, see [1, p.4] and also [16, p.2]. Thus if R is smoothable in $\mathbb{P}V$, then obviously R is abstractly smoothable, and thus it is smoothable in X as well. □

For a variety $X \subset \mathbb{P}V$ and $x \in X$, the *embedded affine Zariski tangent* space $\hat{T}_x X \subset V$ may be defined by recalling that the (abstract) Zariski tangent space $T_x X$ is a linear subspace of $T_x\mathbb{P}V$, and $T_x\mathbb{P}V = \hat{x}^* \otimes V/\hat{x}$. The affine Zariski tangent space is the inverse image of $T_x X$ in V. The *projective Zariski tangent space* $\mathbb{P}\hat{T}_x X \subset \mathbb{P}V$ is its associated projective space. Note that $\langle T_x^\star X\rangle \subseteq \mathbb{P}\hat{T}_x X$.

Lemma 2.4.6. *Suppose $X \subset \mathbb{P}V$ is an algebraic variety and $r = 2$. If $T_x^\star X = \mathbb{P}\hat{T}_x X$ for all $x \in X$ (for instance, X has at worst hypersurface singularities), then condition A is satisfied.*

Proof. A scheme R of degree 2 is either a disjoint union of 2 points (which is trivially smoothable) or R is a double point supported at $x \in X$. In the second case, x together with the line $\ell \subset \mathbb{P}\hat{T}_x X$ such that $T_x R = \ell$ uniquely determines R. The scheme R is smoothable if and only if it is a limit of 2 points on X with both points converging to x, which is (by definition of the tangent star) equivalent to $\ell \subset T_x^\star X$. Thus, if $T_x^\star X = \mathbb{P}\hat{T}_x X$, then R is smoothable. □

Lemma 2.4.7. *Suppose $X \subset \mathbb{P}V$ is a subscheme and $r \leq 11$ or X can be locally embedded into a smooth 3-fold. Then condition B is satisfied.*

Proof. If $r \leq 11$, then $q \leq 10$, and zero-dimensional Gorenstein schemes of degree at most 10 are smoothable, see [8]. If X can be locally embedded into a smooth 3-fold, then also the embedding dimension of any $Q \subset X$ is at most 3 at each point, and a zero-dimensional Gorenstein scheme that can be embedded in $\mathbb{P}^3$ is smoothable by [7, Cor. 2.4] and thus Q is also smoothable in $\mathbb{P}V$ by Lemma 2.4.5. □

Theorem 2.4.2 follows from Proposition 2.4.3 as conditions A and B hold by Lemmas 2.4.6 and 2.4.7.

Here is another consequence of Corollary 2.2.1.

Corollary 2.4.8. *Suppose Condition B of Proposition 2.4.3 holds for some r and $X = \mathbb{P}V$. Then for all $p \in \sigma_r(v_d(\mathbb{P}V))$, $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$ and $d \geq 2r-1$, the scheme R of degree r such that $p \in \langle v_d(R)\rangle$ is unique.*

The hypotheses of Corollary 2.4.8 hold in all dimensions when $r \leq 11$ and for all r when $\dim \mathbb{P}V \leq 3$ by Lemma 2.4.7.

Proof. Let $R \subset \mathbb{P}V$ be a smoothable zero-dimensional scheme of degree r such that $p \in \langle v_d(R)\rangle$. Suppose $R' \subset R$ is a subscheme such that $p \in \langle v_d(R)\rangle$ and suppose R' is minimal with this property. Condition B implies that R' is smoothable. Since $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$, we have $R' = R$ and R is minimal. Thus by Corollary 2.2.1, the scheme R is unique as claimed. □

3. Singular counter-examples

3.1. **Explicit examples of curves.** Let $X \subset \mathbb{P}V$ be a (possibly reducible) variety. Recall the incidence correspondence S_X from §1.4 and note that $\sigma_2(X) = \mu(S_X)$, and $\dim T^\star_x X \leq 2\dim X$.

Consider the case X is the union of two lines that intersect in a point y. Then $T^\star_y X = \mathbb{P}\hat{T}_y X$ is a $\mathbb{P}^2$.

Now let X be the union of three lines in $\mathbb{P}^3 = \mathbb{P}V$ that intersect in a point y and are otherwise in general linear position, e.g., the lines corresponding to coordinate axes in affine space. (That is, give $\mathbb{P}^3$ coordinates $[x_0, x_1, x_2, x_3]$ and take the union of lines given by $x_i x_j = 0$, $1 \leq i < j \leq 3$.) Then $T^\star_y X$ is the union of three $\mathbb{P}^2$'s, but $\langle T^\star_y X\rangle = \mathbb{P}^3$. Consider $v_d(X)$, for $d \geq 3$. If we label the coordinates in $S^d V$ in order $x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, x_0^{d-1}x_3, x_0^{d-2}x_1^2, \ldots$, then $\langle T^\star_{v_d(y)} v_d(X)\rangle = \langle T^\star_{v_d(x)}\mathbb{P}V\rangle$ is the span of the first four coordinate points and $T^\star_{v_d(y)}v_d(X)$ is the union of the $\mathbb{P}^2$'s spanned by the duals of $x_0^d, x_0^{d-1}x_i, x_0^{d-1}x_j$, $1 \leq i < j \leq 3$. Consider the point $z = [1, 1, 1, 1, 0, \ldots, 0] \in \langle T^\star_{v_d(y)} v_d(X)\rangle$. It lies in $\sigma_2(v_d(\mathbb{P}V))$, but it is not in $\sigma_2(v_d(X))$. To prove this, note that the scheme R of degree two defining z as an element of $\sigma_2(v_d(\mathbb{P}V))$ is unique by Corollary 2.4.8, but R is obtained by x_0^d and the tangent vector in the direction of $x_0^{d-1}(x_1 + x_2 + x_3)$ and the latter is not in $T^\star_{v_d(y)}v_d(X)$.

Thus $\sigma_2(v_d(X))$ is not defined by the equations inherited from $\sigma_2(v_d(\mathbb{P}^3))$. However, it is defined by cubics, namely the cubics inherited from $\sigma_2(v_d(\mathbb{P}^3))$ and those defining the union of the three $\mathbb{P}^2$'s in $\mathbb{P}S^d V$.

Finally let X_k be the union of $k \geq 4$ lines in $\mathbb{P}^3 = \mathbb{P}V$ that intersect at a point y but are otherwise in general linear position. Then $T^\star_y X_k$ is a union of $\binom{k}{2}$ $\mathbb{P}^2$'s, and thus is a hypersurface of degree $\binom{k}{2}$ in $\langle T^\star_y X_k\rangle = \mathbb{P}V$. As above, $T^\star_{v_d(y)}v_d(X_k)$ is also a union of $\binom{k}{2}$ $\mathbb{P}^2$'s, namely the linear spaces whose tangent spaces are the images of the tangent spaces to the $\mathbb{P}^2$'s in $T^\star_y X_k$ under the differential of the Veronese. And as above, $\langle T^\star_{v_d(y)} v_d(X_k)\rangle = \langle T^\star_{v_d(y)} v_d(\mathbb{P}^3)\rangle$ will be the $\mathbb{P}^3 \subset \mathbb{P}S^d V$ that they span, and a general point of $\langle T^\star_{v_d(y)} v_d(X_k)\rangle$ will not be in $\sigma_2(v_d(X_k))$.

This provides an explicit construction of a sequence of reducible varieties such that the ideal of $\sigma_2(v_d(X_k))$ has generators in degree $\binom{k}{2}$ for all $d \geq 3$.

To obtain irreducible varieties, it is sufficient that they locally look like the above example near a point y. To be explicit, take for example, a rational normal curve $C \subset \mathbb{P}^n$ (with $n = k+2$) and a linear subspace $W \simeq \mathbb{P}^{k-1} \subset \mathbb{P}^n$, spanned by k general points on C. Then choose a general hyperplane $H \simeq \mathbb{P}^{k-2} \subset W$ and let $\pi\colon \mathbb{P}^n \setminus H \to \mathbb{P}^3$ be the projection away from H. Then W is mapped to a single point and $X := \pi(C)$ is a degree n irreducible curve with singularity isomorphic to k general intersecting lines and for any $d \geq 3$, one needs equations of degree at least $\binom{k}{2}$ to define $\sigma_2(v_d(X))$, even set-theoretically.

In the next section we show that for d sufficiently large, the same examples work for all r.

In §3.3 we present further counter-examples, which are complete intersections.

3.2. **Proof of Theorem 1.4.1.** Recall that in Theorem 1.4.1 we give conditions on singularities of X that force $\sigma_r(v_d(X)) \neq \Sigma^d_r(X)$ and conditions that force some of the defining equations of $\sigma_r(v_d(X))$ to be of high degree. In the proof we intersect both $\sigma_r(v_d(X))$ and $\Sigma^d_r(X)$ with a linear space W, which for $r = 2$ is just the projective tangent space at a sufficiently singular

point $\mathbb{P}\hat{T}_x v_d(X)$. We show there is enough of difference between $\sigma_r(v_d(X)) \cap W$ and $\Sigma^d_r(X) \cap W$ to prove the theorem.

In the first lemma below we describe $\Sigma^d_r(X) \cap W$, while in the next lemma we describe $\sigma_r(v_d(X)) \cap W$.

Lemma 3.2.1. *Let $X \subset \mathbb{P}V$ be a variety, let $r \geq 2$, let $d \geq 1$, and let $x, y_1, \ldots, y_{r-2} \in v_d(X)$ be $r-1$ disjoint points. Then*

$$W := \langle \mathbb{P}\hat{T}_x v_d(X) \cup \{y_1, \ldots, y_{r-2}\} \rangle \subset \Sigma^d_r(X).$$

Proof. By definition $\Sigma^d_r(X) = (\sigma_r(v_d\mathbb{P}V) \cap \langle v_d(X)\rangle)_{\text{red}}$. Note that

$$\mathbb{P}\hat{T}_x v_d(X) \subset \mathbb{P}\hat{T}_x(v_d(\mathbb{P}V)) \subset \sigma_2(v_d\mathbb{P}V)$$

and thus $W \subset \sigma_r(v_d\mathbb{P}V)$. Also $W \subset \langle v_d(X)\rangle$. Since W is reduced, the claim follows. □

Lemma 3.2.2. *In the setup of Lemma 3.2.1, suppose $d \geq 2r-1$. Let $p \in W$ be a point which is not contained in $\langle \mathbb{P}\hat{T}_x v_d(X) \cup (\{y_1, \ldots, y_{r-2}\} \setminus \{y_i\})\rangle$ for any i. Then $p \in \sigma_r(v_d(X))$ if and only if $p \in \langle x, z, y_1, \ldots, y_{r-2})\rangle$ for some $z \in T^\star_x v_d(X)$.*

In other words $(\sigma_r(v_d(X)) \cap W)_{\text{red}}$ consists of the cone over $T^\star_{v_d(x)} v_d(X)$ with vertex $\langle y_1, \ldots, y_{r-2}\rangle$ and possibly other components contained in $\langle \mathbb{P}\hat{T}_x v_d(X) \cup (\{y_1, \ldots, y_{r-2}\} \setminus \{y_i\})\rangle$ for some i.

Proof. The tangent star is always contained in $\sigma_2(v_d(X))$, thus one implication is easy as $\sigma_r(v_d(X))$ is the join of $\sigma_2(v_d(X))$ and $\sigma_{r-2}(v_d(X))$.

To prove the other implication, suppose $p \in \sigma_r(v_d(X))$. If $p \in \langle x, y_1, \ldots, y_{r-2}\rangle$, then z can be taken to be x. Otherwise, let $R \subset \mathbb{P}V$ be a smoothable scheme of degree at most r, such that $p \in \langle v_d(R)\rangle$ and R is minimal with this property. By the uniqueness provided by Corollary 2.2.1, $R = R_z \cup \{y_1, \ldots, y_{r-2}\}$, where R_z is the degree 2 scheme supported at x, and contained in the line $\langle x, z\rangle$ for some $z \in \mathbb{P}\hat{T}_x(v_d(X))$. Since $p \in \sigma_r(v_d(X))$, R is smoothable in X, and also R_z is smoothable in X. So z is in the tangent star of $v_d(X)$ at x. □

Proof of Theorem 1.4.1. If $T^\star_x X \neq \mathbb{P}\hat{T}_x X$, then Lemmas 3.2.1 and 3.2.2 imply (i).

Suppose equations of degree at most q are not enough to define $T^\star_x X$ plus some other components. Thus the same holds for $T^\star_{v_d(x)} v_d(X) \subset \mathbb{P}\hat{T}_{v_d(x)} v_d(X)$, because the derivative of v_d at x maps isomorphically the embedded pair: $T^\star_x X \subset \mathbb{P}\hat{T}_x X$ onto $T^\star_{v_d(x)} v_d(X) \subset \mathbb{P}\hat{T}_{v_d(x)} v_d(X)$. And equations of degree at most q are not enough to define any linear cone over $T^\star_{v_d(x)} v_d(X)$ (plus other components).

Fix $r-2$ distinct points $y_1, \ldots, y_{r-2} \in X \setminus \{x\}$. Let W be as in Lemma 3.2.1

An ideal I defining $\sigma_r(v_d(X))$ must contain an ideal I' defining $\Sigma^d_r(X)$. Let J be the ideal generated by linear equations of W. By Lemmas 3.2.1 and 3.2.2, $I' + J = J$, but $I + J$ defines a cone over $T^\star_{v_d(x)} v_d(X)$ in W with vertex $\langle y_1, \ldots, y_{r-2}\rangle$ and possibly some other components as in the Lemma above. Thus I needs more equations than there are in I', and our assumptions imply that equations of degrees at most q are not enough. □

We conclude that the counter-examples illustrated in §3.1 also work for $r \geq 3$.

3.3. **Singular complete intersection counter-examples.** The examples in §3.1 are not local complete intersections, and one could try to restrict the EKS conjecture only to such curves. Yet, even singular complete intersections fail to satisfy the conjecture.

Suppose h and h' are two general cubics in three variables x_1, x_2, x_3 and let

$$f := x_0(x_1x_2 - x_2x_3) + h \text{ and } f' := x_0(x_1x_3 - x_2x_3) + h'.$$

In $\mathbb{P}^3$ consider the scheme X defined by $f = f' = 0$. It is a reduced complete intersection of two cubics, as can be easily verified, for instance, by intersecting with the hyperplane $x_0 = 0$. The curve X is singular at $x := [1, 0, 0, 0]$. The tangent cone at x is given by

$$x_1x_2 - x_2x_3 = x_1x_3 - x_2x_3 = 0,$$

so it is the union of four lines through $[1, 0, 0, 0]$ and one of the four points $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$, $[0, 1, 1, 1]$. The tangent star in this case is the secant variety of the tangent cone, and thus it is a union of 6 planes. Hence it is defined by a single equation of degree 6 and thus by Theorem 1.4.1 the secant varieties $\sigma_r(v_d(X))$ cannot be defined by equations of degree ≤ 5 when d is sufficiently large.

Similarly, consider $X \subset \mathbb{P}^3$ to be a complete intersection of:

$$f := {x_0}^s g + h \text{ and } f' := {x_0}^{s'} g' + h',$$

where g, g', h, h' are general homogeneous polynomials in x_1, x_2, x_3 of degrees, respectively, $t, t', (s + t), (s' + t')$, with $t, t' \geq 2$. If t, t' grow, then the degree of the defining equation of the tangent star will grow too, thus one has complete intersection counter-examples to the EKS conjecture for arbitrary $r \geq 2$.

4. Border Rank Preserving Pairs (BRPP) and connections with signal processing and computer science

The purpose of this section is to present several conjectures in algebraic geometry that arose in applications. It is primarily expository in nature.

4.1. **Conjectures of Strassen and Comon.** As mentioned in §1.2, Question 1.2.2 came to our attention because of its relation with conjectures arising from computer science and signal processing. The conjectures are as follows:

Strassen's conjecture. In complexity theory one is interested in finding upper and lower bounds for the number of operations required to execute a bilinear map. One is especially interested the particular bilinear map matrix multiplication. V. Strassen [38] asked if there exists an algorithm that simultaneously computes two different matrix multiplications, that costs less than the sum of the best algorithms for the individual matrix multiplications. If not, one says that *additivity* holds for matrix multiplication. Similarly, define additivity for arbitrary bilinear maps.

Conjecture 4.1.1 (Strassen)**.** [38] *Additivity holds for bilinear maps.*

Comon's conjecture. In signal processing one is interested expressing a given tensor as sum of a minimal number of decomposable tensors. Often the tensors that arise have symmetry or at least partial symmetry. Much more is known about symmetric tensors than general tensors so it would be convenient to be able to reduce questions about tensors to questions about symmetric tensors. In particular, if one is handed a symmetric tensor which has symmetric rank r, can it have lower rank as a tensor?

Conjecture 4.1.2 (P. Comon)**.** [12] *The tensor rank of a symmetric tensor equals its symmetric tensor rank. That is, for $p \in \mathbb{P}S^dV$, considering $S^dV \subset V^{\otimes d}$, $R_{v_d(\mathbb{P}V)}(p) = R_{Seg(\mathbb{P}V\times\cdots\times\mathbb{P}V)}(p)$.*

We consider R_X and $\underline{\mathbf{R}}_X$ as functions $\langle X\rangle \to \mathbb{N}$ and if $L \subset \langle X\rangle$, then $R_X|_L$ and $\underline{\mathbf{R}}_X|_L$ denote the restricted functions.

Uniform formulations of the conjectures of Eisenbud, Strassen and Comon were presented in §1.3 using the language of rpp and brpp.

4.2. **General facts about rpp and brpp.** Let $\mathbb{G}(k, \mathbb{P}V)$ denote the Grassmannian of $\mathbb{P}^k$'s in $\mathbb{P}V$.

Proposition 4.2.1. *Suppose $L \in \mathbb{G}(k, \mathbb{P}V)$ is in general position with respect to an irreducible $X \subset \mathbb{P}V$, and $\dim(L) \geq codim(X)$. If (X, L) is rpp then it is brpp.*

Proof. The set of points of X-rank at most r, contains an open subset U_r of $\sigma_r(X)$. By our assumptions $U_r \cap L$ is not empty and consists of points of $(X \cap L)$-rank at most r. Moreover, $\sigma_r(X) \cap L$ is the closure of $U_r \cap L$, because $\sigma_r(X) \cap L$ is irreducible. Therefore $\sigma_r(X) \cap L \subset \sigma_r(X \cap L)$ and since the other inclusion always holds, we have *brpp*. □

Recall that for a variety $X \subset \mathbb{P}V$, $\dim \sigma_r(X) \leq r(\dim X + 1) - 1$ and equality generally holds. Write $\delta_r(X) = r(\dim X + 1) - 1 - \dim \sigma_r(X)$, for the r-th secant defect of X.

Proposition 4.2.2. *Let $X \subset \mathbb{P}V$ with $\dim X \geq k$ and assume $\delta_r(X) \leq k(r-1) - 1$. Let $L \in \mathbb{G}(\dim V - k - 1, \mathbb{P}V)$ be general. Then (X, L) is neither rpp nor brpp.*

Proof. The dimensions have been arranged such that $\dim \sigma_r(X \cap L) < \dim[\sigma_r(X) \cap L]$, so there is $p \in \sigma_r(X) \cap L$ such that $p \notin \sigma_r(X \cap L)$, showing the failure of *brpp*. Moreover since L is general, it will have a non-empty intersection with $\sigma_r(X)_{general}$ showing the failure of *rpp* as well. □

4.3. **Examples.** The reader can easily verify the following:

Example 4.3.1. $(X, L) = (v_3(\mathbb{P}^1), \mathbb{P}^2)$ where $\mathbb{P}^2$ is general is neither *rpp* nor *brpp*.

Example 4.3.2. Let $X \subset \mathbb{P}V$ be a hypersurface, and let L be such that $\langle (X \cap L)_{\text{red}} \rangle = L$. Then (X, L) is both *rpp* and *brpp*.

Example 4.3.3. Let $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ and let $L \subset \mathbb{P}^5$ be a general hyperplane. Then (X, L) is *brpp* but not *rpp*.

To see this, note that a general hyperplane section of $v_2(\mathbb{P}^2)$ is a $v_2(v_2(\mathbb{P}^1)) = v_4(\mathbb{P}^1)$. In coordinates, $v_4(\mathbb{P}^1)$ may be described as set of symmetric 3×3 matrices (x^i_j) of rank 1 with $x^1_3 = x^2_2$. The hypersurfaces $\sigma_2(X) \subset \mathbb{P}^5$ and $\sigma_2(X \cap L) \subset L$ are both given by the vanishing of the determinant, and the 3rd secant variety is the ambient space, hence (X, L) is *brpp*. On the other hand $x^3y \in S^4\mathbb{C}^2$ has rank 4, but the maximal rank of any point in $S^2\mathbb{C}^3$ is three.

Proposition 4.3.4. *Strassen's conjecture and its border rank version hold for $Seg(\mathbb{P}^1 \times \mathbb{P}B \times \mathbb{P}C)$.*

Proof. In this case $X \cap L = \mathbb{P}^0 \times \mathbb{P}B' \times \mathbb{P}C' \sqcup \mathbb{P}^0 \times \mathbb{P}B'' \times \mathbb{P}C''$. So any element in the span of $X \cap L$ is of the form:

$$p := e_1 \otimes (f_1 \otimes g_1 + \cdots + f_k \otimes g_k) + e_2 \otimes (f_{k+1} \otimes g_{k+1} + \cdots + f_{k+l} \otimes g_{k+l}).$$

We can assume that the f_i's are linearly independent and the g_i's as well so that $\underline{\mathbf{R}}_{X \cap L}(p) = R_{X \cap L}(p) = k + l$. After projection $\mathbb{P}^1 \to \mathbb{P}^0$ which maps both e_1 and e_2 to a single generator of $\mathbb{C}^1$, this element therefore becomes clearly of rank $k + l$. Hence both rank and border rank of p are at least $k + l$. □

Example 4.3.5 (Cases where brpp version of Comon's conjecture holds)**.** If $\sigma_r(v_d(\mathbb{P}^n))$ is defined by flattenings, or more generally by equations inherited from the tensor product space, such as the Aronhold invariant (which is a symmetrized version of Strassen's equations) then $brpp_r$ will hold. However defining equations for $\sigma_r(v_d(\mathbb{P}^n))$ are only known in very few cases. In the known cases, including $\sigma_r(v_d(\mathbb{P}^1))$ for all r, d, the equations are indeed inherited.

Regarding the rank version, it holds trivially for general points (as the brpp version holds) and for points in $\sigma_2(v_d(\mathbb{P}^n))$, as a point not of honest rank two is of the form $x^{d-1}y$, which gives

rise to $x\otimes\cdots\otimes x\otimes y + x\otimes\cdots\otimes x\otimes y\otimes x + \cdots + y\otimes x\otimes\cdots\otimes x$. By examining the flattenings of this point and using induction one concludes.

If one would like to look for counter-examples, it might be useful to look for linear spaces M such that $M\cap Seg(\mathbb{P}^n\times\cdots\times\mathbb{P}^n)$ contains more than $\dim M+1$ points but $L\cap M\cap Seg(\mathbb{P}^n\times\cdots\times\mathbb{P}^n)$ contains the expected number of points as these give rise to counter-examples to the brpp version of Strassen's conjecture.

4.4. **Failure of the border rank version of Strassen's conjecture.** Before giving an actual example, we present a near counter-example to the border rank version of Strassen's conjecture that captures the essential idea, which is that one chooses $M\subset A\otimes B\otimes C$ such that $M\cap Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C)$ contains more than $\dim M+1$ points. Then taking $\dim M+2$ points in the intersection, one can take any point in the sum of the $M+2$ tangent spaces (see [31, §10.1]).

Here is an example of this phenomenon:

Example 4.4.1 (Bini et. al's example)**.** An "approximate algorithm" for multiplying 2×2 matrices where the first matrix has a zero in the $(2,2)$ slot is presented in [3]. We show how the algorithm corresponds to a point of $\sigma_5(Seg(\mathbb{P}^2\times\mathbb{P}^3\times\mathbb{P}^3))$ of the nature above. Namely consider $Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)\subset\mathbb{P}^7$. Any $\mathbb{P}^4$ will intersect $Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ in at least $\deg(Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)) = 6$ points. Even better, the following five points on the Segre span a $\mathbb{P}^3$. Let the three $\mathbb{C}^2$'s respectively have bases a_1,a_2, b_1,b_2, c_1,c_2. Write

$$x_1 = a_1\otimes b_1\otimes c_1,\ x_2 = a_2\otimes b_2\otimes c_2,\ x_3 = a_1\otimes b_1\otimes(c_1+c_2),\ x_4 = a_2\otimes(b_1+b_2)\otimes c_2.$$

The lines $\langle x_1,x_3\rangle$ and $\langle x_2,x_4\rangle$ are contained in the Segre, so there are two lines worth of points of intersection of the Segre with the $\mathbb{P}^3$ spanned by these four points, but to use [31, §10.1] to be able to get any point in the span of the tangent spaces to these four points, we need a fifth point that is not in the span of any three points. Consider

$$x_5 = -x_1 - x_2 + x_3 + x_4 = (a_1+a_2)\otimes b_1\otimes c_2$$

which is not in the span of any three of the points.

Regarding the matrix multiplication operator

$$\sum a^i_s\otimes b^s_u\otimes c^u_i \in (\mathbb{C}^2\otimes\mathbb{C}^{2*})\otimes(\mathbb{C}^2\otimes\mathbb{C}^{2*})\otimes(\mathbb{C}^2\otimes\mathbb{C}^{2*}),$$

then the relevant points are

$$x_1 = a^1_2\otimes b^1_2\otimes c^1_2,\ x_2 = a^2_1\otimes b^1_1\otimes c^1_1,\ x_3 = a^1_2\otimes b^1_2\otimes(c^1_1+c^1_2),\ x_4 = a^2_1\otimes(b^1_1+b^1_2)\otimes c^1_1.$$

Then taking

$$x'_1 = a^1_1\otimes b^1_2\otimes c^1_2 + a^1_2\otimes b^2_2\otimes c^1_2 - a^1_2\otimes b^2_1\otimes c^1_2, \qquad x'_2 = a^1_1\otimes b^1_1\otimes c^1_1 + a^2_1\otimes b^1_1\otimes c^2_1 - a^2_1\otimes b^1_1\otimes c^2_2,$$
$$x'_3 = a^1_2\otimes b^2_1\otimes(c^1_1+c^1_2), \qquad x'_4 = a^2_1\otimes(b^1_1+b^1_2)\otimes c^2_2,$$
$$x'_5 = 0$$

the matrix multiplication operator for the partially filled matrices is $x'_1+x'_2+x'_3+x'_4$. The fact that we didn't use any of the initial points is not surprising as the derivatives can always be altered to incorporate the initial points.

Note that if one chooses four points not in special position, a general point in the sum of their tangent spaces will be in the 8-th secant variety.

While this is not an example of the failure of brpp, it illustrates the method that is used in Schönhage's example, which is more complicated, because it arranges that all first order terms cancel so one can use second order data.

Example 4.4.2 (Schönhage's example)**.** Let

$$M_{s,t,u} : (\mathbb{C}^s \otimes \mathbb{C}^{t*}) \times (\mathbb{C}^t \otimes \mathbb{C}^{u^*}) \to (\mathbb{C}^s \otimes \mathbb{C}^{u^*})$$

denote the matrix multiplication operator. Schönhage [36] proves that, while $\underline{\mathbf{R}}(M_{e,1,\ell}) = e\ell$ and $\underline{\mathbf{R}}(M_{1,h,1}) = h$, nevertheless:

Proposition 4.4.3. *Let $h = (e-1)(\ell-1)$, then $\underline{\mathbf{R}}(M_{e,1,\ell} \oplus M_{1,h,1}) \leq e\ell + 1$.*

Proof. Write $A = A' \oplus A''$, $B = B' \oplus B''$, $C = C' \oplus C''$. Let $\dim A' = e$, $\dim B' = \ell$, $\dim C' = e\ell$, $\dim A'' = h$, $\dim B'' = h$, and $\dim C'' = 1$. Fix index ranges $1 \leq i \leq e$, $1 \leq s \leq \ell$, $1 \leq u \leq e-1$, $1 \leq v \leq \ell - 1$. Give the vector spaces bases $\{a_i\}$, $\{b_s\}$, $\{c_{is}\}$, $\{\alpha_{uv}\}$, $\{\beta_{uv}\}$, and γ.

Take $e\ell+1$ points to all lie on $Seg(\mathbb{P}A' \times \mathbb{P}B' \times \mathbb{P}C'') = Seg(\mathbb{P}^{e-1} \times \mathbb{P}^{\ell-1} \times \mathbb{P}^0) \subset \mathbb{P}(A' \otimes B' \otimes C') = \mathbb{P}^{e\ell-1}$ so they must be linearly dependent.

Moreover, there is a clever choice of first order terms so that the t coefficient cancels, and the t^2 coefficient gives the desired tensor.

In the above bases,

$$M_{e,1,\ell} = \sum_{1\leq i\leq e, 1\leq s\leq \ell} a_i \otimes b_s \otimes c_{si} \text{ and } M_{1,h,1} = \sum_{1\leq u\leq e-1, 1\leq v\leq \ell-1} \alpha_{u,v} \otimes \beta_{u,v} \otimes \gamma.$$

Set $\alpha_{i\ell} = -\sum_v \alpha_{v,u}$, $\beta_{e,v} = -\sum_u \beta_{u,v}$, $\alpha_{e,i} = 0$, $\beta_{j,\ell} = 0$, which can be illustrated with the following picture:

$$\begin{pmatrix} & & 0 \\ & (\alpha_{u,v}) & \vdots \\ & & 0 \\ -\sum_v \alpha_{v,1}, \ldots, -\sum_v \alpha_{v,e-1} & & 0 \end{pmatrix} \quad \begin{pmatrix} & & -\sum_u \beta_{1,u} \\ & (\beta_{u,v}) & \vdots \\ & & -\sum_u \beta_{\ell-1,u} \\ 0, \ldots, 0 & & 0 \end{pmatrix}$$

Now let

$$T(t) = \sum_{1\leq i\leq e, 1\leq s\leq \ell} (a_i + t\alpha_{i,s}) \otimes (b_s + t\beta_{i,s}) \otimes (\gamma + t^2 c_{si}) - (\sum_i a_i) \otimes (\sum_s b_s) \otimes \gamma$$

Note that for $t \neq 0$, $R(T(t)) \leq e\ell + 1$, and that

$$T(t) = t^2 T + O(t^4)$$

where $T = M_{e,1,\ell} + M_{1,h,1}$. □

The first non-trivial case is when $e = 3, \ell = 2$, where the rank is 10 and the border rank is at most seven. Using this case three times, Schönhage showed the border rank of the multiplication of 3×3 matrices is at most 21. Note that setting $e = \ell$, the gap between the rank and border rank grows quadratically in e.

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