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DETERMINENTAL EQUATIONS FOR SECANT VARIETIES AND THE EISENBUD-KOH-STILLMAN CONJECTURE

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Abstract. We prove special cases of a conjecture of Eisenbud on the ideals of secant varieties of Veronese re-embeddings of arbitrary varieties. Eisenbud's conjecture generalizes a conjecture of Eisenbud, Koh and Stillman (EKS) for curves. We give explicit counter-examples to the EKS conjecture for singular curves. The techniques we use also allow us to prove a gap and uniqueness theorem for symmetric tensor rank. We put Eisenbud's conjecture in a more general context which we call *brpp*, and discuss conjectures coming from signal processing and complexity theory in this context.

1. Introduction

1.1. **Background and history.** D. Mumford [29] observed that if $X \subset \mathbb{P}V$ is a projective variety, and one takes a sufficiently large Veronese re-embedding of X, $v_d(X) \subset \mathbb{P}S^dV$, then $v_d(X)$ will be cut out set-theoretically by quadrics (in fact quadrics of rank at most four), and moreover, that if X is smooth, the ideal of $v_d(X)$ will be generated in degree two. P. Griffiths [17, Thm. p 271] remarked further that with d as above, and X smooth, if one considers $v_{2d}(X)$, it will be set-theoretically cut out by the two by two minors of a matrix of linear forms. These results were generalized to ideal-theoretic equations of minors for arbitrary varieties by J. Sidman and G. Smith [33].

More generally, let L_1, L_2 be ample line bundles on an abstract variety X. The map

$$\phi_{L_1^d \otimes L_2^e} : X \to \mathbb{P}(H^0(X, L_1^d \otimes L_2^e))^*$$

will be an embedding for d, e sufficiently large. Write $V_1 = H^0(X, L_1^d)^*$, $V_2 = H^0(X, L_2^e)^*$, so there is a map $V_1^* \otimes V_2^* \to H^0(X, L_1^d \otimes L_2^e)$ given on decomposable elements by multiplication of sections. Let W^* denote the image of the map, so there is an inclusion $W \subset V_1 \otimes V_2$, and $\phi_{L_1^d \otimes L_2^e}(X) \subset \mathbb{P}W$. Under this inclusion, the image of X lies in the Segre variety $Seg(\mathbb{P}V_1 \times \mathbb{P}V_2)$ of rank one elements intersected with $\mathbb{P}W$ (see, e.g. [10, 26]). The ideal of the Segre is generated in degree two by the two by two minors, i.e., $\Lambda^2 V_1^* \otimes \Lambda^2 V_2^*$, so these minors provide equations for $X \subset \mathbb{P}W$.

For an algebraic variety $X \subset \mathbb{P}V$, the r-th secant variety $\sigma_r(X)$ is defined by

$$\sigma_r(X) = \overline{\bigcup_{x_1,\dots,x_r \in X} \langle x_1, \dots, x_r \rangle} \subset \mathbb{P}V \tag{1.1}$$

where $\langle x_1, \dots, x_r \rangle \subset \mathbb{P}V$ denotes the linear span of the points $x_1, \dots, x_r$ and the overline denotes Zariski closure.

In the above setting, $\sigma_r(\phi_{L_1^d \otimes L_2^e}(X)) \subset \sigma_r(Seg(\mathbb{P}V_1 \times \mathbb{P}V_2))$, and thus equations for the latter give equations for the former. With this in mind, define

$$Rank_r(L_1^d, L_2^e) \subset Sym(W^*)$$

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the ideal generated by the image of the $r+1$ by $r+1$ minors. Note that in general $Rank_r(L_1^d, L_2^e)$ need not be radical, or even saturated.

The following conjecture is due to D. Eisenbud, J. Koh, and M. Stillman.

Conjecture 1.1.1 (EKS conjecture (1988) [10])**.** *Let C be a reduced, irreducible curve, let L_1, L_2 be line bundles on C. Then there exists a "good constant" r_0 depending only on the genus of C and the L_i such that there is an equality of ideals*

$$I(\sigma_r(\phi_{L_1^d\otimes L_2^e}(C))) = Rank_r(L_1^d, L_2^e)$$

for all $r \leq r_0(d,e)$. Moreover r_0 tends to infinity as $d, e \to \infty$.

Conjecture 1.1.1 was proved set-theoretically in the case C is a smooth curve in [31], and scheme-theoretically for smooth curves in [14]. In fact, sharp bounds on d, e were given in terms of genus of the curve.

To relate Conjecture 1.1.1 to the begining paragraph, take $C \subset \mathbb{P}V$, $L_1 = L_2 = \mathcal{O}_C(1)$ and $r = 1$. More generally, if $r \geq 1$, then $W \subset S^{d+e}V \subset S^dV \otimes S^eV$ and the corresponding equations are the so called, *symmetric flattenings* or *catalecticant minors* studied first by Sylvester, see [20] for a history.

Conjecture 1.1.1 was generalized to higher dimensions by D. Eisenbud in the form:

Conjecture 1.1.2 (Eisenbud, unpublished, also see [33])**.** *Let X be a reduced, irreducible variety, let L_1, L_2 be ample line bundles on X. Fix r, then there exist infinitely many sufficiently large d, e such that $I(\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))) = I(Rank_r(L_1^d, L_2^e))$.*

In particular, the ideal of $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))$ is generated in degree $r+1$.

In [33], a slightly different form of the conjecture is stated that involves only one line bundle which is required to be sufficiently ample.

Eisenbud's conjecture was stated in the ideal-theoretic setting, i.e., the two varieties in question had the same ideals. One could attempt to prove the weaker scheme, or set-theoretic conjectures. Another weaker form of the conjecture would be simply that the ideal of $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))$ is generated in degree $r+1$, or even weaker, that $\sigma_r(\phi_{L_1^d\otimes L_2^e}(X))$ is cut out set-theoretically by equations of degree $r+1$. It is this last statement, in the special case where one begins with $X \subset \mathbb{P}V$ and only considers Veronese re-embeddings, i.e., $L_1 = L_2 = \mathcal{O}_{\mathbb{P}V}(1)|_X$, that came to our attention because of its connections with conjectures originating in signal processing and theoretical computer science that we explain in §4.

For any nondegenerate variety, the ideal of its r-th secant variety is empty in degree r [24, Lemma 2.2] and for certain special examples, e.g. *sub-cominuscule varieties* (see [23]) which include quadratic Veronese varieties and two-factor Segre varieties, the ideal is known to be generated in degree $r+1$ for all r. M. Green has defined a sequence of properties N_p for algebraic varieties related to this property [16].

Remark 1.1.3. Already for $X = \mathbb{P}^n$, $n > 1$, and $r = 2$ Conjecture 1.1.2 is not known. Consider $L_1 = L_2 = \mathcal{O}(1)$ and $r = 2$. Then if d, e are large, the equations of $Rank_2(L_1^d, L_2^e)$ cut out $\sigma_2(v_{d+e}(\mathbb{P}^n))$ scheme-theoretically, but it is only known that if one adds the equations given by the 3×3 minors obtained from $L_1^1 \otimes L_2^{d+e-1}$ that one has generators of the ideal of $\sigma_2(v_{d+e}(\mathbb{P}^n))$, see [21]. A. Geramita [13, p. 155] conjectured that these additional equations are superflous.

Our first result gives counter-examples to Conjectures 1.1.1 and 1.1.2:

Theorem 1.1.4. *For any p and $r \geq 2$, there exist irreducible, singular varieties $X \subset \mathbb{P}V$ such that $\sigma_r(v_d(X))$ is not cut out set-theoretically by equations of degree at most p for all $d \geq 2r-1$. Explicit examples of curves with this property are given in §3.1 and §3.3.*

A more precise result is stated in Theorem 1.3.1 below. The following question remains:

Question 1.1.5. Let $X \subset \mathbb{P}V$ be a smooth variety. Given r, is $\sigma_r(v_d(X))$ set-theoretically defined by equations of degree $r+1$ for infinitely many sufficiently large d?

We split this question into two parts. The first part is just the question for projective space:

Question 1.1.6. Let $V = \mathbb{C}^{n+1}$ and fix a natural number r. Does there exist an integer $d_0 = d_0(n,r)$, such that for all $d \geq d_0$, there exists an ideal $I \subset \mathrm{Sym}(S^dV^*)$ generated in degrees at most $r+1$, such that the (reduced) subvariety in $\mathbb{P}(S^dV)$ consisting of the zero locus of I is $\sigma_r(v_d(\mathbb{P}V))$?

The equations of secants Veronese embeddings of $\mathbb{P}V$ are studied intensively, see [25] for the state of the art. Despite this, the only cases Question 1.1.6 is known to have an affirmative answer are (n,r): $(1,r)$, all r, $(2,r)$, $r \leq 6$, and all n when $r \leq 3$. Already the answer is not known for $(n,r) = (3,4)$. In general, $\sigma_r(v_d(\mathbb{P}V))$ is a component of a Rank locus for $r \leq \binom{\lfloor \frac{d}{2} \rfloor + n}{n}$, see [25, 20]

The second part requires the following definition:

Definition 1.1.7. For a scheme $Y \subset \mathbb{P}V$, $\langle Y \rangle \subset \mathbb{P}V$ denotes the scheme-theoretic span of Y. Given a variety $X \subset \mathbb{P}V$, let

$$\Sigma_r^d(X) := \big(\sigma_r(v_d(\mathbb{P}V)) \cap \langle v_d(X) \rangle\big)_{\mathrm{red}}$$

where $(\cdot)_{\mathrm{red}}$ denotes the reduced subscheme.

Note that $\sigma_r(v_d(X)) \subset \Sigma_r^d(X)$. If equality holds and $\sigma_r(v_d(\mathbb{P}V))$ is cut out set-theoretically by equations of degree $r+1$, then the answer to Question 1.1.5 is affirmative for $\sigma_r(X)$ as well. (Similarly for scheme, and ideal-theoretic versions.)

Question 1.1.8. Let $X \subset \mathbb{P}V$ be a smooth subvariety and fix $r \in \mathbb{N}$. Does there exist an integer $d_0 = d_0(r,X)$ such that for all $d \geq d_0$, the variety $\Sigma_r^d(X)$ coincides with $\sigma_r(v_d(X))$ as sets?

To our knowledge, Question 1.1.8 has not been studied before.

1.2. **Positive results.** In what follows $Got(h_X)$ denotes the Gotzmann number of the Hilbert polynomial of X, see Section 2.2. Recall that a variety $X \subset \mathbb{P}V$ is said to have *at most hypersurface singularities*, if the dimension of its Zariski tangent space at any point is at most one greater than the dimension of X. We prove the following cases of Question 1.1.8 have affirmative answers.

Theorem 1.2.1. *Let $X \subset \mathbb{P}V$ be a subvariety and let $r \in \mathbb{N}$. For all $d \geq r - 1 + Got(h_X)$ the following holds:*

(i) *If X is smooth and $r \leq 9$, then $\sigma_r(v_d(X)) = \Sigma_r^d(X)$.*
(ii) *If X is smooth and $\dim(X) \leq 2$, then $\sigma_r(v_d(X)) = \Sigma_r^d(X)$.*
(iii) *If X has only hypersurface singularities, and $r \leq 2$, then $\sigma_r(v_d(X)) = \Sigma_r^d(X)$,*
(iv) *If $d \geq \max\{2r-2, r-1+Got(h_X)\}$, then $\sigma_r(v_d(X))$ is an irreducible component of $\Sigma_r^d(X)$.*

See Proposition 2.3.2 for further, technical conditions that imply $\sigma_r(v_d(X)) = \Sigma_r^d(X)$.

Remark 1.2.2. Theorem 1.2.1 generalizes essentially verbatim to reducible X.

We summarize the theorem and the cases when the answer to Question 1.1.6 is known to be positive:

Corollary 1.2.3. *Suppose $X \subset \mathbb{P}V$ is smooth and $r \leq 3$, or X has at most hypersurface singularities and $r \leq 2$. Then there exists d_0 such that for all $d \geq d_0$ the secant variety $\sigma_r(v_d(X))$ is cut out (set-theoretically) by degree $r+1$ equations. In other words, Question 1.1.5 has affirmative answer for those (X,r).*

For arbitrary irreducible $X \subset \mathbb{P}V$, and arbitrary r, there exists d_0 such that $\sigma_r(v_d(X))$ is an irreducible component of a subvariety defined by rank conditions for all $d \geq d_0$.

We prove Theorem 1.2.1 in Section 2.2. The proof is based on the Gotzmann regularity property — see Proposition 2.1.2. The main new ingredient in the proof is the following lemma.

Lemma 1.2.4. *Let $X \subset \mathbb{P}V$ be a subscheme. Suppose $d \geq r - 1 + Got(h_X)$ and $p \in \langle v_d(X)\rangle$. If $R \subset \mathbb{P}V$ is a 0-dimensional scheme of degree at most r such that $p \in \langle v_d(R)\rangle$, then $p \in \langle v_d(R \cap X)\rangle$.*

The failure of the conjectures in the case of singular X, and the difficulty in extending the proof to all smooth X is the following: If we suppose that R in the Lemma is smoothable (i.e., a flat limit of r distinct points), then one needs to show that $R \cap X$ is also smoothable. Note however that in our case both R and $R \cap X$ are very special schemes so there is hope that this might hold in general for smooth X. See also Question 2.4.2.

Definition 1.2.5. Let $X \subset \mathbb{P}V$ be a subvariety and let $p \in \langle X\rangle$.

- Define $R_X(p)$ (the *X-rank of p*) to be the minimal number r, such that $p \in \langle p_1, \ldots, p_r\rangle$ for some points $p_i \in X$.
- Note that $\sigma_r(X) \subset \mathbb{P}V$ is the closure of the set of points in $\langle X\rangle$ of X-rank at most r.
- Define $\underline{\mathbf{R}}_X(p)$ (the *X-border rank of p*) to be the minimal number r, such that $p \in \sigma_r(X)$.

The following theorem is a consequence of Lemma 1.2.4. On one hand, it may be viewed as a generalization of the theorem of Comas and Seguir [8] that states if the rank of a point in $\mathbb{P}(S^d\mathbb{C}^2)$ is larger than its border rank, and the border rank is small, the rank must be at least $\lfloor \frac{d}{2} \rfloor + 2$. (Their theorem gives more precise information about ranks.) On the other hand, it also gives a criterion for uniqueness of an expression of a point as a sum of d-th powers that does not rely on a general point assumption (e.g. [6, 7]) or a Kruskal-type test [22].

Theorem 1.2.6. *Let $p \in \mathbb{P}S^dV$. If $R_{v_d(\mathbb{P}V)}(p) \leq \frac{d+1}{2}$, i.e., the rank of p is at most $\frac{d+1}{2}$, then $R_{v_d(\mathbb{P}V)}(p) = \underline{\mathbf{R}}_{v_d(\mathbb{P}V)}(p)$ and the expression of p as a sum of $R_{v_d(\mathbb{P}V)}(p)$ d-th powers is unique (up to trivialities).*

Remark 1.2.7. In contrast to the Veronese case, there is no such result for Segre varieties $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k) \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_k)$. When $k = 2$ rank equals border rank and there is no uniqueness, and when $k > 2$ elements of border rank two can have rank $2, 3, \ldots, k$.

1.3. **Negative results.** For a variety $X \subset \mathbb{P}V$ and a point $x \in X$, the *tangent star* of X at x, $T^\star_x X \subset \mathbb{P}V$ is defined to be the union of the points on the $\mathbb{P}^1$'s obtained as limits in the Grassmannian $\mathbb{G}(\mathbb{P}^1, \mathbb{P}V)$ of $\mathbb{P}^1_{x(t),y(t)}$'s spanned by points $x(t), y(t)$, with $x(t), y(t) \in X$ and $x(0) = y(0) = x$. Alternatively, consider the incidence correspondence

$$S_X := \overline{\{(x, y, z) \in X \times X \times \mathbb{P}V \mid z \in \langle x, y\rangle\}},$$

let $\psi : S_X \to X \times X$, $\mu : S_X \to \mathbb{P}V$ denote the projections, then $T^\star_x X = \mu(\psi^{-1}(x, x))$. Let $\tilde{T}_x X = \langle T^\star_x X\rangle \subset \mathbb{P}V$ denote its linear span. Note that if $x \in X$ is a smooth point, then $\tilde{T}_x X$ is the embedded tangent projective space and $T^\star_x X = \tilde{T}_x X$ (but the converse does not hold).

Here is a more precise version of Theorem 1.1.4:

Theorem 1.3.1. *Let $X \subset \mathbb{P}V$ be a subvariety and let $x \in X$ be a singular point. Suppose $r \geq 2$ and let $d \geq 2r - 1$.*

(i) *If $T^\star_x X \neq \tilde{T}_x X$, then $\sigma_r(v_d(X)) \neq \Sigma^d_r(X)$.*

(ii) *Suppose $I(T^\star_x X)$ is the defining ideal of the tangent star in $\tilde{T}_x X$. Suppose for some q, the homogeneous parts $\bigoplus_{i=0}^q I_i(T^\star_x X)$ define a subscheme Z of $\tilde{T}_x X$, such that no component of Z_{red} is equal to $T^\star_x X$ (this happens for instance if the ideal $I(T^\star_x X)$ is*

trivial in degrees $\leq q$ and $T^\star_x X \neq \tilde{T}_x X$). Then $\sigma_r(v_d(X))$ is not defined set-theoretically by equations of degree $\leq q$.

We prove Theorem 1.3.1 in §3.2.

Overview. In §2 we first review facts about Hilbert schemes, including Gotzmann's regularity result, and explain the relationship between the study of the smoothable component of the Hilbert scheme with the study of secant varieties. We then give proofs of the positive results, and discuss what would be needed to resolve the questions in general. In §3 we prove our negative results and construct explicit counter examples to the EKS conjecture. We conclude by discussing questions in signal processing and computer science that led us to study Eisenbud's conjecture in §4. The purpose of this section is to introduce these beautiful problems to the community of algebraic geometers.

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2. Proofs of positive results

2.1. **Hilbert schemes and regularity.** Throughout this paper we work over the field of complex numbers $\mathbb{C}$. Let $X \subset \mathbb{P}V$ be a subscheme.

Standard Notations.

- $\langle X \rangle \subset \mathbb{P}V$ denotes the scheme-theoretic linear span of X.
- X_{red} denotes the reduced subscheme of X.
- $I(X) \subset \mathrm{Sym}(V^*)$ denotes the homogeneous, saturated ideal defining of X. The d-th homogeneous piece of $I(X)$ is denoted $I_d(X) \subset S^dV^*$. The ideal sheaf of X is denoted by $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}V}$, so that $H^0(\mathcal{I}_X(d)) = I_d(X)$.
- For a positive integer d the *d-th Veronese reembedding* of X, denoted $v_d(X) \subset \mathbb{P}S^dV$, is the subscheme defined by the ideal in $\mathrm{Sym}(S^dV^*)$ which is the kernel of the following composition:

$$Sym(S^dV^*) = \bigoplus_{k=0}^{\infty} S^k(S^dV^*) \twoheadrightarrow \bigoplus_{k=0}^{\infty} S^{kd}V^* \hookrightarrow \bigoplus_{k=0}^{\infty} S^dV^* = Sym(V^*) \twoheadrightarrow Sym(V^*)/I(Y).$$

Note that for a scheme $X \subset \mathbb{P}V$ the linear span $\langle X \rangle$ is equal to $\mathbb{P}(I_1(X)^\perp)$, i.e., the projective zero locus of the linear part of $I(X)$. In particular, $\langle v_d(X) \rangle = \mathbb{P}(I_d(X)^\perp) \subset \mathbb{P}S^dV$. It is a standard fact that $v_d(X)$ is isomorphic to X as an abstract scheme, see, e.g., [18, Chapter 2, Theorem 2.4.7] or [19, Ex. II 5.13].

For a review on Hilbert schemes, Hilbert polynomials, Hilbert functions, and regularity see, e.g., [30] and references therein.

Given a subscheme $X \subset \mathbb{P}V$ with ideal sheaf $\mathcal{I}_X$, we say $\mathcal{I}_X$ is *δ-regular*, if $H^i(\mathcal{I}_X(\delta - i)) = 0$ for all $i > 0$. Serre's vanishing theorem implies that every $X \subset \mathbb{P}V$ has δ-regular ideal sheaf for sufficiently large δ.

Proposition 2.1.1. *Suppose $\mathcal{I}_X$ is δ-regular.*

(i) *X is also d-regular for all $d \geq \delta$.*
(ii) *$H^i(\mathcal{O}_X(d)) = 0$ for $d \geq \delta - i$ and $i > 0$.*
(iii) *If h_X is the Hilbert polynomial of X, then $h^0(\mathcal{O}_X(d)) = h_X(d)$ for all $d \geq \delta - 1$.*

Proof. Part (i) is explained in [30, Lem. 2.1(b)]. Part (ii) follows from the long exact cohomology sequence of

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}V}(d) \to \mathcal{O}_X(d) \to 0. \tag{2.1}$$

Part (iii) follows from (ii), keeping in mind that the Hilbert polynomial is also an Euler characteristic. □

Gotzmann's regularity theorem gives a bound on how large δ must be for $\mathcal{I}_X$ to be δ-regular. This bound depends only on the Hilbert polynomial of X, which is essential for our purposes.

Proposition 2.1.2 (Gotzmann's regularity, [15])**.** *Suppose P is the Hilbert polynomial of a subscheme $X \subset \mathbb{P}V$ such that*

$$P(d) = \sum_{i=1}^{Got(P)} \binom{d + a_i - i + 1}{a_i}$$

where $a_1 \geq a_2 \geq .. \geq a_{Got(P)} \geq 0$. Then $\mathcal{I}_X$ is $Got(P)$-regular. In particular:

- *if $X' \subset \mathbb{P}V$ is another scheme with the same Hilbert polynomial P, then $\mathcal{I}_{X'}$ is also $Got(P)$-regular.*
- *if $R \subset \mathbb{P}V$ is a zero dimensional scheme of degree r, then $\mathcal{I}_R$ is r-regular.*

For an exposition of the proof, see [4, Thm 4.3.2].

Lemma 2.1.3. *Suppose $X \subset \mathbb{P}V$ is a δ-regular subscheme with Hilbert polynomial h_X. Then for $d \geq \delta - 1$:*

$$\dim \langle v_d(X) \rangle + 1 = h^0(\mathcal{O}_X(d)) = h_X(d).$$

In particular, if R is a zero dimensional scheme of degree r, and $d \geq r-1$ then $\dim \langle v_d(R) \rangle = r-1$.

Proof. Since all the higher cohomologies vanish, the short exact sequence (2.1) gives rise to a short exact sequence of sections. Thus $\langle v_d(X) \rangle$ is non-degenerate and the claim follows. The $+1$ is just the difference between projective and vector space dimensions. □

Lemma 2.1.4 (Additivity of Hilbert polynomials)**.** *Suppose $X, R \subset \mathbb{P}V$ are two subschemes. Suppose h_X, h_R, $h_{X\cap R}$ and $h_{X\cup R}$ are respectively the Hilbert polynomials of X, R, $X \cap R$ and $X \cup R$. Then:*

$$h_{X\cap R} = h_X + h_R - h_{X\cup R}.$$

If in addition R is zero dimensional of degree r, and h_X is the Hilbert polynomial of X, then $X \cup R$ is $(Got(h_X) + t)$-regular where $t = \deg R - \deg(X \cap R)$.

Proof. For d sufficiently large, all the higher cohomologies vanish in the following short exact sequence:

$$0 \to \mathcal{O}_{X\cup R}(d) \to \mathcal{O}_X(d) \oplus \mathcal{O}_R(d) \to \mathcal{O}_{X\cap R}(d) \to 0.$$

The additivity claim follows. The second claim follows by Gotzmann regularity. □

For a projective variety X, let $\mathcal{H}_r(X)$ denote the union of all the irreducible components of the Hilbert scheme $Hilb_r(X)$ of degree r dimension 0 subschemes of X, which contain r distinct points. In case X is irreducible, $\mathcal{H}_r(X)$ is irreducible too. Also if $Y \subset X$, then $\mathcal{H}_r(Y) \subset \mathcal{H}_r(X)$. Schemes that are in $\mathcal{H}_r(X)$ are called *smoothable* (because there exists a flat irreducible deformation to a smooth scheme). It is an interesting and non-trivial problem to determine when $Hilb_r(X) = \mathcal{H}_r(X)$, and to identify the schemes that are in $\mathcal{H}_r(X)$ if the equality does not hold— see, e.g., [5], [11] and references therein.

The smoothable component $\mathcal{H}_r(X)$ is relevant to our study because of its relation to secant varieties:

Lemma 2.1.5. *Let $X \subset \mathbb{P}V$ be a subvariety that is not a set of less than r points, and let $d \geq r-1$. For $p \in \mathbb{P}S^dV$, $p \in \sigma_r(v_d(X))$ if and only if there exists a scheme $R \in \mathcal{H}_r(X)$ such that $p \in \langle v_d(R)\rangle$.*

Proof. By Lemma 2.1.3 any scheme R of degree at most r, is linearly independent after a d-th Veronese re-embedding. With this in mind, the claim becomes [2, Prop. 2.8]. □

2.2. **Proofs of Lemma 1.2.4 and Theorem 1.2.6.** Fix an integer r and a subscheme $X \subset \mathbb{P}V$. Let $d_0 = Got(h_X) + r - 1$. Then, if $R \subset \mathbb{P}V$ is a zero dimensional scheme of degree at most r, X, R, $X \cap R$ and $X \cup R$ are (d_0+1)-regular by Propositions 2.1.1(i), 2.1.2 and Lemma 2.1.4.

Proof of Lemma 1.2.4. Let $d \geq d_0$, let $R \subset \mathbb{P}V$ be a subscheme of degree at most r. To prove the lemma it is enough to prove

$$\langle v_d(X \cap R)\rangle = \langle v_d(X)\rangle \cap \langle v_d(R)\rangle.$$

Since $\subset$ trivially holds, it is enough to prove that the dimension of the left hand side equals the dimension of the right hand side.

Since X, $X \cup R$, R and $X \cap R$ are d-regular, it follows from Lemmas 2.1.3 and 2.1.4 that

$$\begin{aligned}\dim \langle v_d(X \cap R)\rangle &= h_{X\cap R}(d) - 1 = h_X(d) + h_R(d) - h_{X\cup R}(d) - 1\\ &= (\dim \langle v_d(X)\rangle + 1) + (\dim \langle v_d(R)\rangle + 1) - (\dim \langle v_d(X \cup R)\rangle + 1) - 1\\ &= \dim \langle v_d(X)\rangle + \dim \langle v_d(R)\rangle - \dim \langle v_dX \cup v_dR\rangle\\ &= \dim (\langle v_d(X)\rangle \cap \langle v_d(R)\rangle).\end{aligned}$$

□

The following is a consequence of Lemma 1.2.4 (with $X = Q$):

Corollary 2.2.1. *Suppose $p \in \mathbb{P}S^dV$ and that $R, Q \subset \mathbb{P}V$ are two zero dimensional schemes of degree at most r, such that $p \in \langle v_d(R)\rangle$ and $p \in \langle v_d(Q)\rangle$ for some $d \geq \deg(R \cup Q) - 1$. Suppose also, that R is minimal in the following sense: for any $R' \subsetneq R$ we have $p \notin \langle v_d(R')\rangle$. Then $R \subset Q$.*

Proof. Apply Lemma 1.2.4 with $X = Q$, and $d_0 = \deg(R \cup Q) - 1$. Thus $p \in \langle v_d(R \cap Q)\rangle$, and by the assumption that R is minimal, $R \cap Q = R$. □

Theorem 1.2.6 follows, because given two different expressions via schemes R, Q, $\deg(R \cup Q) \leq \deg(R) + \deg(Q) \leq 2r \leq d+1$, thus Corollary 2.2.1 applies.

2.3. **Situations when $X \cap R$ is smoothable.** Suppose $Q \subset \mathbb{P}V$ is a zero dimensional scheme of degree q. We remark that $\dim (Hilb_{q-1}(Q)) = 0$ says that there are only a finite number of subschemes $Q' \subset Q$ with $\deg Q' = \deg Q - 1$.

For a variety $X \subset \mathbb{P}V$ and $x \in X$, the *embedded affine Zariski tangent* space $\hat{T}_xX \subset V$ may be defined by recalling that the (abstract) Zariski tangent space T_xX is a linear subspace of $T_x\mathbb{P}V$, and $T_x\mathbb{P}V = \hat{x}^* \otimes V/\hat{x}$. The affine Zariski tangent space is the inverse image of T_xX in V. The *projective Zariski tangent space* $\mathbb{P}\hat{T}_xX \subset \mathbb{P}V$ is its associated projective space. Note that $\tilde{T}_xX \subseteq \mathbb{P}\hat{T}_xX$.

Proposition 2.3.1. *Suppose X is an algebraic variety and $Q \subset X$ is a zero dimensional subscheme of degree r. Then:*

(i) *If X is smooth and either*
- *$r \leq 7$,*
- *$r \leq 8$ and $\dim X \leq 3$,*
- *or r is arbitrary and $\dim X \leq 2$,*

then Q is smoothable.

(ii) *If X is smooth, $r = 8$, $\dim X \geq 4$, and $\dim Hilb_7(Q) = 0$, then Q is smoothable.*
(iii) *If $T^\star_x X = \mathbb{P}\hat{T}_x X$ for all $x \in X$ and $r = 2$, then Q is smoothable.*

Proof. Case (i) is covered in [28, 5, 12], respectively.

By [5, Thm 1.1], if X is smooth, $r = 8$, and $\dim X \geq 4$, then Q is either smoothable or Q is supported at one point and the local algebra of Q is isomorphic to a homogeneous algebra with Hilbert function $(1, 4, 3)$. However any subscheme $Q' \subset Q$ with Hilbert function $(1, 4, 2)$ is a point of $Hilb_7(Q)$, which are parametrized by $Gr(2, 3) \simeq \mathbb{P}^2$, which has dimension two. Since we are assuming $\dim Hilb_7(Q) = 0$, Hilbert function $(1, 4, 3)$ is not possible and thus Q is smoothable, proving (ii).

If $r = 2$, then either Q is a disjoint union of 2 points (which is trivially smoothable) or Q is a double point supported at $x \in X$. In the second case, x together with any line in $\ell \subset \mathbb{P}\hat{T}_x Q$ such that $T_x Q = \ell$ uniquely determines Q. Q is smoothable if and only if it is a limit of 2 points on X with both points converging to x, which is (by definition of the tangent star) equivalent to $\ell \subset T^\star_x X$. Thus, if $T^\star_x X = \mathbb{P}\hat{T}_x X$, then Q is smoothable, proving (iii). □

Note that if X has a hypersurface singularity at x, then $T^\star_x X = \mathbb{P}\hat{T}_x X$.

The following proposition provides technical conditions under which the answer to Question 1.1.8 is positive.

Proposition 2.3.2. *Suppose $X \subset \mathbb{P}V$ is a (not necessarily irreducible) subvariety and r is an integer such that:*

A. *All zero dimensional subschemes $R \subset X$ of degree r which are smoothable in $\mathbb{P}V$, are also smoothable in X.*
B. *$\dim\left(Hilb_{q-1}(Q)\right) > 0$ for all non-smoothable zero dimensional subschemes $Q \subset X$ of degree q with $q < r$.*

Then $\sigma_r(v_d(X)) = \Sigma^d_r(X)$ for all $d \geq d_0 = Got(h_x) + r - 1$.

Proof. Suppose $p \in \Sigma^d_r(X) := \left(\langle v_d(X)\rangle \cap \sigma_r(v_d(\mathbb{P}V))\right)_{\text{red}}$, so that by Lemma 2.1.5 there exists a zero dimensional smoothable subscheme $R \subset \mathbb{P}V$ of degree at most r, such that $p \in \langle v_d(X)\rangle \cap \langle v_d(R)\rangle$. By Lemma 1.2.4, also $p \in \langle v_d(X \cap R)\rangle$. If $X \cap R$ is smoothable in X (i.e., if $X \cap R \in \mathcal{H}_t(X)$ for $t = \deg(X \cap R)$), then $p \in \sigma_r(X)$ by Lemma 2.1.5. So suppose $Q := X \cap R$ is not smoothable in X and set $q := \deg Q$.

By condition A, $q < r$. By condition B, the Hilbert scheme $Hilb_{q-1}Q$ is positive dimensional. Since $d \geq q$, and the Hilbert scheme is projective

$$\bigcup_{Q' \subsetneq Q} \langle v_d(Q')\rangle = \langle v_d(Q)\rangle.$$

Thus there exists a subscheme $Q' \subsetneq Q$, such that $p \in \langle v_d(Q')\rangle$. Without loss of generality, suppose Q' is a minimal such scheme and suppose $q' := \deg Q'$. Then $\dim Hilb_{q'-1}Q' = 0$. By condition B, this means Q' is smoothable in X and by Lemma 2.1.5 $p \in \sigma_{q'}(v_d(X)) \subset \sigma_r(v_d(X))$. □

We list situations when conditions A and B are satisfied.

Lemma 2.3.3. *Suppose either $X \subset \mathbb{P}V$ is a smooth subvariety and r is arbitrary, or X has at worst hypersurface singularities and $r \leq 2$. Then condition A is satisfied.*

Proof. For X smooth, the embedded deformation theory of a zero dimensional scheme $R \subset X$ is smooth over the abstract deformation theory of R, see [1, p.4] and also [11, p.2]. Thus if R is smoothable in $\mathbb{P}V$, then obviously R is abstractly smoothable, and thus it is smoothable in X as well.

For X with at most hypersurface singularities and $r \leq 2$ see Proposition 2.3.1(iii). □

Lemma 2.3.4. *Suppose either*

- *X is a smooth variety of any dimension and $r \leq 9$,*
- *X is a smooth surface and r is arbitrary,*
- *or X is arbitrary and $r \leq 3$.*

Then condition B is satisfied.

Proof. When X is smooth the statement follows from Proposition 2.3.1(i) and (ii). When $r \leq 3$, the condition is always satisfied, as there is only a finite number of subschemes of a scheme of degree at most 2. □

2.4. **Proof of Theorem 1.2.1, consequences and a question.**

Proof of Theorem 1.2.1. Cases (i)–(iii) follow from Proposition 2.3.2 as conditions A and B hold by Lemmas 2.3.3 and 2.3.4.

To prove (iv), $\sigma_r(v_d(X))$ is irreducible because X is. Let Σ be an irreducible component of $\Sigma_r^d(X)$ containing $\sigma_r(v_d(X))$. For a general point $p \in \sigma_r(v_d(X))$, let $p \in \langle v_d(R)\rangle$, where $R \subset X$ consists of r distinct points, and p is not in the span of any of $r-1$ of those points. We claim $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$. Suppose to the contrary that $Q \subset \mathbb{P}V$ is a zero dimensional scheme of degree $\leq r-1$, such that $p \subset \langle v_d(Q)\rangle$. Then Corollary 2.2.1 implies $R \subset Q$, a contradiction, since $\deg R > \deg Q$.

The set of points with $v_d(\mathbb{P}V)$-rank r is open in $\sigma_r(v_d(\mathbb{P}V)) \setminus \sigma_{r-1}(v_d(\mathbb{P}V))$. Thus since $\Sigma \subset \sigma_r(v_d(\mathbb{P}V))$, $p \in \Sigma$, $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$, and p has $v_d(\mathbb{P}V)$-rank r, a general point p' in Σ also has $v_d(\mathbb{P}V)$-rank r. Let R' be the union of r distinct points of $\mathbb{P}V$ such that $p' \in \langle v_d(R')\rangle$. By Lemma 1.2.4, $p' \in \langle v_d(X \cap R')\rangle$, and $X \cap R'$ is smooth (hence trivially smoothable). Thus $p' \in \sigma_r(v_d(X))$ and $\sigma_r(v_d(X)) = \Sigma$ as claimed in (iv). □

Here is another consequence of Corollary 2.2.1.

Corollary 2.4.1. *Suppose Condition B of Proposition 2.3.2 holds for some r and $X = \mathbb{P}V$. Then for all $p \in \sigma_r(v_d(\mathbb{P}V))$, $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$ and $d \geq 2r-1$, the scheme R of degree r such that $p \in \langle v_d(R)\rangle$ is unique.*

The hypotheses of Corollary 2.4.1 hold in all dimensions when $r \leq 9$ by Lemma 2.3.4.

Proof. Let $R \subset \mathbb{P}V$ be a smoothable zero-dimensional scheme of degree r such that $p \in \langle v_d(R)\rangle$. Suppose $R' \subset R$ is a subscheme such that $p \in \langle v_d(R)\rangle$ and suppose R' is minimal with this property. Condition B implies that R' is smoothable. Since $p \notin \sigma_{r-1}(v_d(\mathbb{P}V))$, we have $R' = R$ and R is minimal. Thus by Corollary 2.2.1, the scheme R is unique as claimed. □

Question 2.4.2. Suppose $Q \subset \mathbb{P}V$ is a zero dimensional scheme of degree q, with $\dim Hilb_{q-1}Q = 0$ (i.e., there is only a finite number of $Q' \subset Q$ with $\deg Q' = \deg Q - 1$). Is Q necessarily smoothable?

If the answer is positive, then for any smooth $X \subset \mathbb{P}V$ and any r, the answer to Question 1.1.8 is positive too. This follows from Proposition 2.3.2, as this guarantees condition B, and condition A is provided by Lemma 2.3.3. We expect that if the answer is negative, then it should be possible to construct a smooth example, for which Question 1.1.8 fails.

3. Singular counter-examples to the conjectures

3.1. **Explicit examples of curves.** Let $X \subset \mathbb{P}V$ be a (possibly reducible) variety. Recall the incidence correspondence S_X from §1.3 and note that $\sigma_2(X) = \mu(S_X)$, and $\dim T_x^\star X \leq 2 \dim X$.

Consider the case X is the union of two lines that intersect in a point y. Then $T_y^\star X = \mathbb{P}\hat{T}_y X$ is a $\mathbb{P}^2$.

Now let X be the union of three lines in $\mathbb{P}^3 = \mathbb{P}V$ that intersect in a point y and are otherwise in general linear position, e.g., the lines corresponding to coordinate axes in affine space. (That is, give $\mathbb{P}^3$ coordinates $[x_0, x_1, x_2, x_3]$ and take the union of lines given by $x_i x_j = 0$, $1 \leq i < j \leq 3$.) Then $T^\star_y X$ is the union of three $\mathbb{P}^2$'s, but $\tilde{T}_y X = \mathbb{P}^3$. Consider $v_d(X)$, for $d \geq 3$. If we label the coordinates in $S^d V$ in order $x_0^d, x_0^{d-1}x_1, x_0^{d-1}x_2, x_0^{d-1}x_3, x_0^{d-2}x_1^2, \ldots$, then $\tilde{T}_{v_d(y)}v_d(X) = \tilde{T}_{v_d(x)}\mathbb{P}V$ is the span of the first four coordinate points and $T^\star_{v_d(y)}v_d(X)$ is the union of the $\mathbb{P}^2$'s spanned by the duals of $x_0^d, x_0^{d-1}x_i, x_0^{d-1}x_j$, $1 \leq i < j \leq 3$. Consider the point $z = [1, 1, 1, 1, 0, \ldots, 0] \in \tilde{T}_{v_d(y)}v_d(X) = \langle T^\star_{v_d(y)}v_d(X)\rangle$. It lies in $\sigma_2(v_d(\mathbb{P}V))$, but it is not in $\sigma_2(v_d(X))$. To prove this, note that the scheme R of degree two defining z as an element of $\sigma_2(v_d(\mathbb{P}V))$ is unique by Corollary 2.4.1, but R is obtained by x_0^d and the tangent vector in the direction of $x_0^{d-1}(x_1 + x_2 + x_3)$ and the latter is not in $T^\star_{v_d(y)}v_d(X)$.

Thus $\sigma_2(v_d(X))$ is not defined by the equations inherited from $\sigma_2(v_d(\mathbb{P}^3))$. However, it is defined by cubics, namely the cubics inherited from $\sigma_2(v_d(\mathbb{P}^3))$ and those defining the union of the three $\mathbb{P}^2$'s in $\mathbb{P}S^d V$.

Now let X_k be the union of $k \geq 4$ lines in $\mathbb{P}^3 = \mathbb{P}V$ that intersect at a point y but are otherwise in general linear position. Then $T^\star_y X_k$ is a union of $\binom{k}{2}$ $\mathbb{P}^2$'s, and thus is a hypersurface of degree $\binom{k}{2}$ in $\tilde{T}_y X = \mathbb{P}V$. As before $T^\star_{v_d(y)}v_d(X_k)$ will also be a union of $\binom{k}{2}$ $\mathbb{P}^2$'s, namely the linear spaces whose tangent spaces are the images of the tangent spaces to the $\mathbb{P}^2$'s in $T^\star_y X_k$ under the differential of the Veronese. And as above, $\tilde{T}_{v_d(y)}v_d(X) = \tilde{T}_{v_d(y)}v_d(\mathbb{P}^3)$ will be the $\mathbb{P}^3 \subset \mathbb{P}S^d V$ that they span, and a general point of $\tilde{T}_{v_d(y)}v_d(X)$ will not be in $\sigma_2(v_d(X))$.

This provides an explicit construction of a sequence of reducible varieties such that the ideal of $\sigma_2(v_d(X_k))$ is generated in degrees at least $\binom{k}{2}$ for all $d \geq 3$.

To obtain irreducible varieties, it is sufficient that they locally look like the above example near a point y. To be explicit, take for example, a rational normal curve $C \subset \mathbb{P}^n$ (with $n \geq k+2$) and a linear subspace $W \simeq \mathbb{P}^{k-1} \subset \mathbb{P}^n$, spanned by k general points on C. Then choose a general hyperplane $H \simeq \mathbb{P}^{k-2} \subset W$ and let $\pi\colon \mathbb{P}^n \setminus H \to \mathbb{P}^{n-k+1}$ be the projection away from H. Then W is mapped to a single point and $X := \pi(C)$ is a degree n irreducible curve with singularity isomorphic to k transversal lines and for any $d \geq 3$, one needs equations of degree at least $\binom{k}{2}$ to define $\sigma_2(v_d(X))$, even set-theoretically.

In the next section we show that for d sufficiently large, the same examples work for all r.

In §3.3 we present further counter-examples, which are complete intersections.

3.2. **Proof of Theorem 1.3.1.**

Lemma 3.2.1. *Let $X \subset \mathbb{P}V$ be a variety, let $r \geq 2$, let $d \geq 1$, and let $x, y_1, \ldots, y_{r-2} \in v_d(X)$ be $r-1$ disjoint points. Then*

$$W := \langle \mathbb{P}\hat{T}_x v_d(X) \cup \{y_1, \ldots, y_{r-2}\}\rangle \subset \Sigma^d_r(X).$$

Proof. By definition $\Sigma^d_r(X) = (\sigma_r(v_d\mathbb{P}V) \cap \langle v_d(X)\rangle)_{\text{red}}$. Note that

$$\mathbb{P}\hat{T}_x v_d(X) \subset \mathbb{P}\hat{T}_x(v_d(\mathbb{P}V)) \subset \sigma_2(v_d\mathbb{P}V)$$

and thus $W \subset \sigma_r(v_d\mathbb{P}V)$. Also $W \subset \langle v_d(X)\rangle$. Since W is reduced, the claim follows. □

Lemma 3.2.2. *In the setup of Lemma 3.2.1, suppose $d \geq 2r-1$. Let $p \in W$ be a point which is not contained in $\langle \mathbb{P}\hat{T}_x v_d(X) \cup (\{y_1, \ldots, y_{r-2}\} \setminus \{y_i\})\rangle$ for any i. Then $p \in \sigma_r(v_d(X))$ if and only if $p \in \langle x, z, y_1, \ldots, y_{r-2})\rangle$ for some $z \in T^\star_x v_d(X)$.*

In other words $(\sigma_r(v_d(X)) \cap W)_{\text{red}}$ consists of the cone over $T^\star_{v_d(x)}v_d(X)$ with vertex $\langle y_1, \ldots, y_{r-2}\rangle$ and possibly other components contained in $\langle \mathbb{P}\hat{T}_x v_d(X) \cup (\{y_1, \ldots, y_{r-2}\} \setminus \{y_i\})\rangle$ for some i.

Proof. The tangent star is always contained in $\sigma_2(v_d(X))$, thus one implication is easy as $\sigma_r(v_d(X))$ is the join of $\sigma_2(v_d(X))$ and $\sigma_{r-2}(v_d(X))$.

To prove the other implication, suppose $p \in \sigma_r(v_d(X))$. If $p \in \langle x, y_1, \dots, y_{r-2}\rangle$, then z can be taken to be x. Otherwise, let $R \subset \mathbb{P}V$ be a smoothable scheme of degree at most r, such that $p \in \langle v_d(R)\rangle$ and R is minimal with this property. By the uniqueness provided by Corollary 2.2.1, $R = R_z \cup \{y_1, \dots, y_{r-2}\}$, where R_z is the degree 2 scheme supported at x, and contained in the line $\langle x, z\rangle$ for some $z \in \mathbb{P}\hat{T}_x(v_d(X))$. Since $p \in \sigma_r(v_d(X))$, R is smoothable in X, and also R_z is smoothable in X. So z is in the tangent star of $v_d(X)$ at x. □

Proof of Theorem 1.3.1. If $T^\star_x X \neq \mathbb{P}\hat{T}_x X$, then Lemmas 3.2.1 and 3.2.2 imply (i).

Suppose equations of degree at most q are not enough to define $T^\star_x X$ plus some other components. Thus the same holds for $T^\star_{v_d(x)} v_d(X) \subset \mathbb{P}\hat{T}_{v_d(x)} v_d(X)$, because the derivative of v_d at x maps isomorphically the embedded pair: $T^\star_x X \subset \mathbb{P}\hat{T}_x X$ onto $T^\star_{v_d(x)} v_d(X) \subset \mathbb{P}\hat{T}_{v_d(x)} v_d(X)$. And equations of degree at most q are not enough to define any linear cone over $T^\star_{v_d(x)} v_d(X)$ (plus other components).

Fix points $r-2$ distinct points $y_1, \dots, y_{r-2} \in X \setminus \{x\}$. Let W be as in Lemma 3.2.1

An ideal I defining $\sigma_r(v_d(X))$ must contain an ideal I' defining $\Sigma^d_r(X)$. Let J be the ideal generated by linear equations of W. By Lemmas 3.2.1 and 3.2.2, $I' + J = J$, but $I + J$ defines a cone over $T^\star_{v_d(x)} v_d(X)$ in W with vertex $\langle y_1, \dots, y_{r-2}\rangle$ and possibly some other components as in the Lemma above. Thus I needs more equations than there are in I', and our assumptions imply that equations of degrees at most q are not enough. □

We conclude that the counter-examples illustrated in §3.1 also work for $r \geq 3$.

3.3. **Singular complete intersection counter-examples.** The examples in §3.1 are not local complete intersections, and one could try to restrict the EKS conjecture only to Cohen-Macauley curves. Yet, even singular complete intersections fail to satisfy the conjecture.

Suppose h and h' are two general cubics in three variables x_1, x_2, x_3 and let

$$f := x_0(x_1x_2 - x_2x_3) + h \text{ and } f' := x_0(x_1x_3 - x_2x_3) + h'.$$

In $\mathbb{P}^3$ consider the scheme X defined by $f = f' = 0$. It is a reduced complete intersection of two cubics, as can be easily verified, for instance, by intersecting with the hyperplane $x_0 = 0$. The curve X is singular at $x := [1, 0, 0, 0]$. The tangent cone at x is given by

$$x_1x_2 - x_2x_3 = x_1x_3 - x_2x_3 = 0,$$

so it is the union of four lines through $[1, 0, 0, 0]$ and one of the four points $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$, $[0, 1, 1, 1]$. The tangent star in this case is the secant variety of the tangent cone, and thus it is a union of 6 planes. Hence it is defined by a single equation of degree 6 and thus by Theorem 1.3.1 the secant varieties $\sigma_r(v_d(X))$ cannot be defined by equations of degree ≤ 5 when d is sufficiently large.

Similarly, consider $X \subset \mathbb{P}^3$ to be a complete intersection of:

$$f := {x_0}^s g + h \text{ and } f' := {x_0}^{s'} g' + h',$$

where g, g', h, h' are general homogeneous polynomials in x_1, x_2, x_3 of degrees, respectively, $t, t', (s + t), (s' + t')$, with $t, t' \geq 2$. If t, t' grow, then the degree of the defining equation of the tangent star will grow too, thus one has complete intersection counter-examples to the EKS conjecture for arbitrary $r \geq 2$.

4. BRPP and connections with signal processing and computer science

The purpose of this section is to present several conjectures in algebraic geometry that arose in applications. It is primarily expository in nature.

4.1. **Conjectures of Strassen and Comon.** As mentioned in §1.1, Conjecture 1.1.2 came to our attention because of its relation with conjectures arising from computer science and signal processing. The conjectures are as follows:

Strassen's conjecture. In complexity theory one is interested in finding upper and lower bounds for the number of operations required to execute a bilinear map. One is especially interested the particular bilinear map matrix multiplication. V. Strassen [34] asked if there exists an algorithm that simultaneously computes two different matrix multiplications, that costs less than the sum of the best algorithms for the individual matrix multiplications. If not, one says that *additivity* holds for matrix multiplication. Similarly, define additivity for arbitrary bilinear maps.

Conjecture 4.1.1 (Strassen)**.** [34] *Additivity holds for bilinear maps.*

Comon's conjecture. In signal processing one is interested expressing a given tensor as sum of a minimal number of decomposable tensors. Often the tensors that arise have symmetry or at least partial symmetry. Much more is known about symmetric tensors than general tensors so it would be convenient to be able to reduce questions about tensors to questions about symmetric tensors. In particular, if one is handed a symmetric tensor, which has rank r as a symmetric tensor, can it have lower rank as a tensor?

Conjecture 4.1.2 (P. Comon)**.** [9] *The tensor rank of a symmetric tensor equals its symmetric tensor rank. That is, for $p \in \mathbb{P}S^dV$, considering $S^dV \subset V^{\otimes d}$, $R_{v_d(\mathbb{P}V)}(p) = R_{Seg(\mathbb{P}V\times\cdots\times\mathbb{P}V)}(p)$.*

We consider R_X and $\underline{\mathbf{R}}_X$ as functions $\langle X\rangle \to \mathbb{N}$ and if $L \subset \langle X\rangle$, then $R_X|_L$ and $\underline{\mathbf{R}}_X|_L$ denote the restricted functions.

4.2. **Uniform formulation of conjectures.** Using the following definitions, one obtains a uniform formulation of the conjectures of Eisenbud, Strassen and Comon:

Definition 4.2.1. Let $X \subset \mathbb{P}V$ be a variety and $L \subset \mathbb{P}V$ be a linear subspace. Let $Y := (X \cap L)_{\text{red}}$.

- We say (X, L) is a *rank preserving pair* or *rpp* for short, if $\langle Y\rangle = L$ and $R_X|_L = R_Y$ as functions.
- We say (X, L) is a *border rank preserving pair* or *brpp* for short, if $\langle Y\rangle = L$ and $\underline{\mathbf{R}}_X|_L = \underline{\mathbf{R}}_Y$ as functions, i.e., $\sigma_r(X) \cap L = \sigma_r(Y)$ for all r.
- Similarly we say (X, L) is rpp_r (respectively, $brpp_r$) if $R_X(p) = R_{X\cap L}(p)$ for all $p \in L$ with $R_X(p) \le r$ (respectively, $\sigma_s(X) \cap L = \sigma_s(Y)$ for all $s \le r$).

Note that one always has $R_X(p) \le R_{X\cap L}(p)$ and $\underline{\mathbf{R}}_X(p) \le \underline{\mathbf{R}}_{X\cap L}(p)$.

Question 4.2.2 (rephrasing of Question 1.1.8)**.** For all smooth subvarieties $X \subset \mathbb{P}V$ and all $r \in \mathbb{N}$, does there exist an integer d_0 such that for all $d \ge d_0$, the pair $(v_d(\mathbb{P}V), \langle v_d(X)\rangle)$ is $brpp_r$?

Conjecture 4.2.3 (rephrasing of, and extending to multi-linear maps, Conjecture 4.1.1)**.** *Let A_j be vector spaces Write $A_j = A'_j \oplus A''_j$ and let $L = \mathbb{P}((A'_1\otimes\cdots\otimes A'_k) \oplus (A''_1\otimes\cdots\otimes A''_k))$ Then*

$$(X, L) = (Seg(\mathbb{P}A_1 \times \cdots \times \mathbb{P}A_k), \mathbb{P}((A'_1\otimes\cdots\otimes A'_k) \oplus (A''_1\otimes\cdots\otimes A''_k)))$$

is rpp.

Conjecture 4.2.4 (rephrasing of Conjecture 4.1.2)**.** *Let* $\dim A_j = \mathbf{a}$ *for each* j *and identify each* A_j *with a vector space* A*. Consider* $L = \mathbb{P}(S^k A) \subset A_1 \otimes \cdots \otimes A_k$*. Then*

$$(X, L) = (Seg(\mathbb{P}A \times \cdots \times \mathbb{P}A), \mathbb{P}S^k A)$$

is rpp.

Border rank versions. In the cases of the the conjectures of Comon and Strassen, it is natural to ask the corresponding questions for border rank. For Strassen's conjecture, this has already been answered negatively:

Theorem 4.2.5 (Schönhage, [32])**.** *brpp fails for*

$$(X, L) = (Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C), (A' \otimes B' \otimes C') \oplus (A'' \otimes B'' \otimes C''))$$

starting with the case $\mathbf{a} \geq 5 = 2 + 3$*,* $\mathbf{b} \geq 6 = 3 + 3$*,* $\mathbf{c} \geq 7 = 6 + 1$ *, where the splittings into sums give the dimensions of the subspaces.*

We discuss Schönhage's theorem below.

4.3. **General facts about rpp and brpp.** Let $\mathbb{G}(k, \mathbb{P}V)$ denote the Grassmannian of $\mathbb{P}^k$'s in $\mathbb{P}V$.

Proposition 4.3.1. *Suppose* $L \in \mathbb{G}(k, \mathbb{P}V)$ *is in general position with respect to an irreducible* $X \subset \mathbb{P}V$*, and* $\dim(L) \geq codim(X)$*. If* (X, L) *is rpp then it is brpp.*

Proof. The set of points of X-rank at most r, contains an open subset U_r of $\sigma_r(X)$. Thus by our assumptions $U_r \cap L$ is not empty and consists of points of $(X \cap L)$-rank at most r. Moreover, $\sigma_r(X) \cap L$ is the closure of $U_r \cap L$, because $\sigma_r(X) \cap L$ is irreducible. Therefore $\sigma_r(X) \cap L \subset \sigma_r(X \cap L)$ and since the other inclusion always holds, we have *brpp.* □

Recall that for a variety $X \subset \mathbb{P}V$, $\dim \sigma_r(X) \leq r(\dim X + 1) - 1$ and equality generally holds. Write $\delta_r(X) = r(\dim X + 1) - 1 - \dim \sigma_r(X)$, for the r-th secant defect of X.

Proposition 4.3.2. *Let* $X \subset \mathbb{P}V$ *with* $\dim X \geq k$ *and assume* $\delta_r(X) \leq k(r-1) - 1$*. Let* $L \in G(\dim V - k, V)$ *be general. Then* (X, L) *is neither rpp nor brpp.*

Proof. The dimensions have been arranged such that $\dim \sigma_r(X \cap L) < \dim[\sigma_r(X) \cap L]$, so there is $p \in \sigma_r(X) \cap L$ such that $p \notin \sigma_r(X \cap L)$, showing the failure of *brpp.* Moreover since L is general, it will have a non-empty intersection with $\sigma_r(X)_{general}$ showing the failure of *rpp* as well. □

4.4. **Examples.** The reader can easily verify the following:

Example 4.4.1. $(X, L) = (v_3(\mathbb{P}^1), \mathbb{P}^2)$ where $\mathbb{P}^2$ is general is neither *rpp* nor *brpp.*

Example 4.4.2. Let $X \subset \mathbb{P}V$ be a hypersurface, and let L be such that $\langle (X \cap L)_{\text{red}} \rangle = L$. Then (X, L) is both *rpp* and *brpp.*

Example 4.4.3. Let $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ and let $L \subset \mathbb{P}^5$ be a general hyperplane. Then (X, L) is *brpp* but not *rpp.*

To see this, note that a general hyperplane section of $v_2(\mathbb{P}^2)$ is a $v_2(v_2(\mathbb{P}^1)) = v_4(\mathbb{P}^1)$. In coordinates, $v_4(\mathbb{P}^1)$ may be described as set of symmetric 3×3 matrices (x^i_j) of rank 1 with $x^1_3 = x^2_2$. The hypersurfaces $\sigma_2(X) \subset \mathbb{P}^5$ and $\sigma_2(X \cap L) \subset L$ are both given by the vanishing of the determinant, and the 3rd secant variety is the ambient space, hence (X, L) is *brpp.* On the other hand $x^3 y \in S^4\mathbb{C}^2$ has rank 4, but the maximal rank of any point in $S^2\mathbb{C}^3$ is three.

Proposition 4.4.4. *Strassen's conjecture and its border rank version hold for* $Seg(\mathbb{P}^1 \times \mathbb{P}B \times \mathbb{P}C)$*.*

Proof. In this case $X \cap L = \mathbb{P}^0 \times \mathbb{P}B' \times \mathbb{P}C' \sqcup \mathbb{P}^0 \times \mathbb{P}B'' \times \mathbb{P}C''$. So any element in the span of $X \cap L$ is of the form:

$$p := e_1 \otimes (f_1 \otimes g_1 + \cdots + f_k \otimes g_k) + e_2 \otimes (f_{k+1} \otimes g_{k+1} + \cdots + f_{k+l} \otimes g_{k+l}).$$

We can assume that the f_i's are linearly independent and the g_i's as well so that $\underline{\mathbf{R}}_{X\cap L}(p) = R_{X\cap L}(p) = k + l$. After projection $\mathbb{P}^1 \to \mathbb{P}^0$ which maps both e_1 and e_2 to a single generator of $\mathbb{C}^1$, this element therefore becomes clearly of rank $k + l$. Hence both rank and border rank of p are at least $k + l$. □

Example 4.4.5 (Cases where brpp version of Comon's conjecture holds)**.** If $\sigma_r(v_d(\mathbb{P}^n))$ is defined by flattenings, or more generally by equations inherited from the tensor product space, such as the Aronhold invariant (which is a symmetrized version of Strassen's equations) then $brpp_r$ will hold. However defining equations for $\sigma_r(v_d(\mathbb{P}^n))$ are only known in very few cases. In the known cases, including $\sigma_r(v_d(\mathbb{P}^1)$ for all r, d, the equations are indeed inherited. Fixing r and allowing d to grow, the asymptotic version of the conjecture would follow from a version of Conjecture 1.1.2.

Regarding the rank version, it holds trivially for general points (as the brpp version holds) and for points in $\sigma_2(v_d(\mathbb{P}^n))$, as point not of honest rank two is if the form $x^{d-1}y$, which gives rise to $x\otimes\cdots\otimes x\otimes y + x\otimes\cdots\otimes x\otimes y\otimes x + \cdots + y\otimes x\otimes\cdots\otimes x$. By examining the flattenings of this point and using induction one concludes.

If one would like to look for counter-examples, it might be useful to look for linear spaces M such that $M\cap Seg(\mathbb{P}^n\times\cdots\times\mathbb{P}^n)$ contains more than $\dim M+1$ points but $L\cap M\cap Seg(\mathbb{P}^n\times\cdots\times\mathbb{P}^n)$ contains the expected number of points as these give rise to counter-examples to the brpp version of Strassen's conjecture.

4.5. **Failure of the border rank version of Strassen's conjecture.** Before giving an actual example, we present a near counter-example to the border rank version of Strassen's conjecture that captures the essential idea, which is that one chooses $M \subset A\otimes B\otimes C$ such that $M\cap Seg(\mathbb{P}A\times \mathbb{P}B\times\mathbb{P}C)$ contains more than $\dim M+1$ points. Then taking $\dim M+2$ points in the intersection, one can take any point in the sum of the $M + 2$ tangent spaces (see [27, §10.1]).

Here is an example of this phenomenon:

Example 4.5.1 (Bini et. al's example)**.** An "approximate algorithm" for multiplying 2×2 matrices where the first matrix has a zero in the $(2, 2)$ slot is presented in [3]. We show how the algorithm corresponds to a point of $\sigma_5(Seg(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3))$ of the nature above. Namely consider $Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1) \subset \mathbb{P}^7$. Any $\mathbb{P}^4$ will intersect $Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ in at least $\deg(Seg(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)) = 6$ points. Even better, the following five points on the Segre span a $\mathbb{P}^3$. Let the three $\mathbb{C}^2$'s respectively have bases a_1, a_2, b_1, b_2, c_1, c_2. Write

$$x_1 = a_1\otimes b_1\otimes c_1,\ x_2 = a_2\otimes b_2\otimes c_2,\ x_3 = a_1\otimes b_2\otimes(c_1 + c_2),\ x_4 = a_2\otimes(b_1 + b_2)\otimes c_2.$$

The lines $\langle x_1, x_3\rangle$ and $\langle x_2, x_4\rangle$ are contained in the Segre, so there are two lines worth of points of intersection of the Segre with the $\mathbb{P}^3$ spanned by these four points, but to use [27, §10.1] to be able to get any point in the span of the tangent spaces to these four points, we need a fifth point that is not in the span of any three points. Consider

$$x_5 = -x_1 - x_2 + x_3 + x_4 = (a_1 + a_2)\otimes b_1\otimes c_2$$

which is not in the span of any three of the points.

Regarding the matrix multiplication operator

$$\sum a^i_s\otimes b^s_u\otimes c^u_i \in (\mathbb{C}^2\otimes\mathbb{C}^{2*})\otimes(\mathbb{C}^2\otimes\mathbb{C}^{2*})\otimes(\mathbb{C}^2\otimes\mathbb{C}^{2*}),$$

then the relevant points are

$$x_1 = a_2^1\otimes b_2^1\otimes c_2^1,\ x_2 = a_1^2\otimes b_1^1\otimes c_1^1,\ x_3 = a_2^1\otimes b_2^1\otimes(c_1^1+c_2^1),\ x_4 = a_1^2\otimes(b_1^1+b_2^1)\otimes c_1^1.$$

Then taking

$$\begin{aligned} x_1' &= a_1^1\otimes b_2^1\otimes c_2^1 + a_2^1\otimes b_2^2\otimes c_2^1 - a_2^1\otimes b_1^2\otimes c_2^1, & x_2' &= a_1^1\otimes b_1^1\otimes c_1^1 + a_1^2\otimes b_1^1\otimes c_1^2 - a_1^2\otimes b_1^1\otimes c_2^2,\\ x_3' &= a_2^1\otimes b_1^2\otimes(c_1^1+c_2^1), & x_4' &= a_1^2\otimes(b_1^1+b_2^1)\otimes c_2^2,\\ x_5' &= 0 \end{aligned}$$

the matrix multiplication operator for the partially filled matrices is $x_1'+x_2'+x_3'+x_4'$. The fact that we didn't use any of the initial points is not surprising as the derivatives can always be altered to incorporate the initial points.

Note that if one chooses four points not in special position, a general point in the sum of their tangent spaces will be in the 8-th secant variety.

While this is not an example of the failure of brpp, it illustrates the method that is used in Schönhage's example, which is more complicated, because it arranges that all first order terms cancel so one can use second order data.

Example 4.5.2 (Schönhage's example)**.** Let

$$M_{s,t,u} : (\mathbb{C}^s\otimes\mathbb{C}^{t*})\times(\mathbb{C}^t\otimes\mathbb{C}^{u^*})\to(\mathbb{C}^s\otimes\mathbb{C}^{u^*})$$

denote the matrix multiplication operator. Schönhage [32] proves that, while $\underline{\mathbf{R}}(M_{e,1,\ell}) = e\ell$ and $\underline{\mathbf{R}}(M_{1,h,1}) = h$, nevertheless:

Proposition 4.5.3. *Let $h=(e-1)(\ell-1)$, then $\underline{\mathbf{R}}(M_{e,1,\ell}\oplus M_{1,h,1})\leq e\ell+1$.*

Proof. Write $A = A'\oplus A''$, $B = B'\oplus B''$, $C = C'\oplus C''$. Let $\dim A' = e$, $\dim B' = \ell$, $\dim C' = e\ell$, $\dim A'' = h$, $\dim B'' = h$, and $\dim C'' = 1$. Fix index ranges $1\leq i\leq e$, $1\leq s\leq\ell$, $1\leq u\leq e-1$, $1\leq v\leq\ell-1$. Give the vector spaces bases $\{a_i\}$, $\{b_s\}$, $\{c_{is}\}$, $\{\alpha_{uv}\}$, $\{\beta_{uv}\}$, and γ.

Take $e\ell+1$ points to all lie on $Seg(\mathbb{P}A'\times\mathbb{P}B'\times\mathbb{P}C'') = Seg(\mathbb{P}^{e-1}\times\mathbb{P}^{\ell-1}\times\mathbb{P}^0)\subset\mathbb{P}(A'\otimes B'\otimes C') = \mathbb{P}^{e\ell-1}$ so they must be linearly dependent.

Moreover, there is a clever choice of first order terms so that the t coefficient cancels, and the t^2 coefficient gives the desired tensor.

In the above bases,

$$M_{e,1,\ell} = \sum_{1\leq i\leq e,1\leq s\leq\ell} a_i\otimes b_s\otimes c_{si} \text{ and } M_{1,h,1} = \sum_{1\leq u\leq e-1,1\leq v\leq\ell-1}\alpha_{u,v}\otimes\beta_{u,v}\otimes\gamma.$$

Set $\alpha_{i\ell} = -\sum_v\alpha_{v,u}$, $\beta_{e,v} = -\sum_u\beta_{u,v}$, $\alpha_{e,i}=0$, $\beta_{j,\ell}=0$, which can be illustrated with the following picture:

$$\begin{pmatrix} & & 0\\ & (\alpha_{u,v}) & \vdots\\ & & 0\\ -\sum_v\alpha_{v,1},\ldots,-\sum_v\alpha_{v,e-1} & & 0 \end{pmatrix}\quad\begin{pmatrix} & & -\sum_u\beta_{1,u}\\ & (\beta_{u,v}) & \vdots\\ & & -\sum_u\beta_{\ell-1,u}\\ 0,\ldots,0 & & 0 \end{pmatrix}$$

Now let

$$T(t) = \sum_{1\leq i\leq e,1\leq s\leq\ell}(a_i+t\alpha_{i,s})\otimes(b_s+t\beta_{i,s})\otimes(\gamma+t^2c_{si}) - (\sum_i a_i)\otimes(\sum_s b_s)\otimes\gamma$$

Note that for $t\neq 0$, $R(T(t))\leq e\ell+1$, and that

$$T(t) = t^2T + O(t^4)$$

where $T = M_{e,1,\ell}+M_{1,h,1}$. □

The first non-trivial case is when $e = 3, \ell = 2$, where the rank is 10 and the border rank is at most seven. Using this case three times, Schönhage showed the border rank of the multiplication of 3×3 matrices is at most 21. Note that setting $e = \ell$, the gap between the rank and border rank grows quadratically in e.

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