Measuring affine quadric sectors at centre

Helmut Kahl
Munich University of Applied Sciences
80335 Munich
Germany

30 June 2010

Abstract

Sectors at centre of affine quadrics with point symmetry are investigated. As an application we demonstrate nice formulas for the area and the volume of such planar and spatial sectors, respectively.

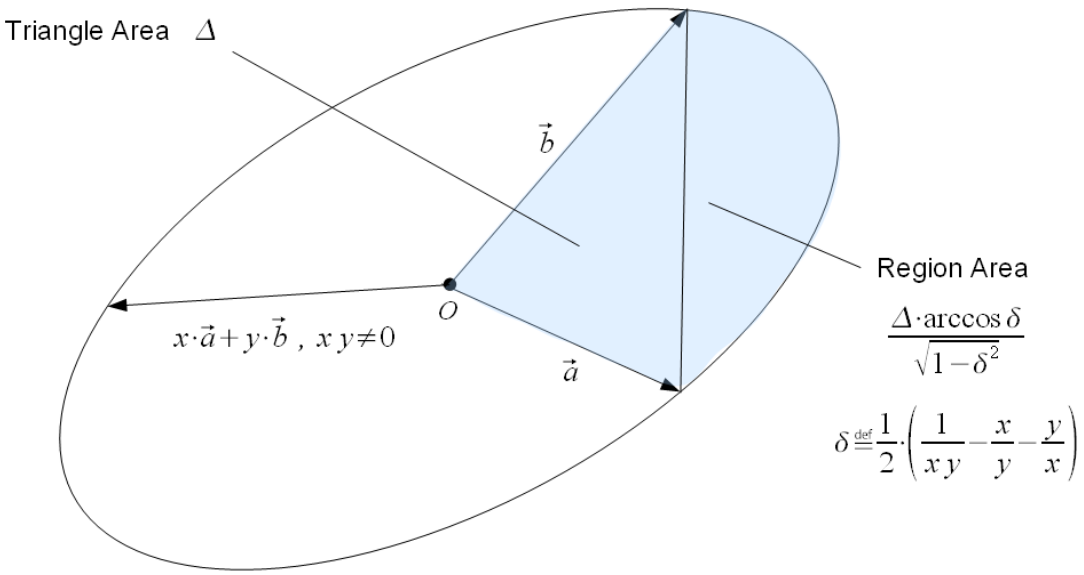
Mathematics Subject Classification (2010): 51N10, 28A75, 11E04

1. Introduction

Planar affine quadrics represent the most simple non-linear algebraic varieties and were investigated already in the Greek antiquity. Such quadrics are either two parallel lines or the conic sections but the parabolas when they are symmetric with respect to a point which does not belong to the quadric itself. Sectors of such quadrics centred at a point of symmetry[1] seem to have been of little interest in the history of geometry. Only the special case of circle sectors have been considered intensively, since it is essential for the notion of an angle. Nevertheless those sectors at centre deserve attention: Because of its simple geometric nature they allow a simple area formula (s. figure below and theorem 1 of section 3) and thus are useful for "second order approximations" of arbitrary areas along with Leibniz' sector formula (s. [MV, Satz 5.4, p.199]). An analogous assertion can be made for spatial affine quadric sectors in use of volume calculations (s. theorem 2 of section 3). In section 2 the theory of affine quadric sectors at centre of arbitrary finite dimension over arbitrary fields of characteristic different from two will be developed. In section 3 its measure theory over the field $\mathbb{R}$ of real numbers will be represented with general formulas in $\mathbb{R}^2$ and $\mathbb{R}^3$. Section 4 concludes with a summary and an open question concerning a formula in a special case of $\mathbb{R}^3$.

1 not to be confused with Kepler's sectors at a focus of an ellipse in his "Celestial Mechaniques"

Elliptic Sector Region (shaded) at symmetry centre O



2. Affine quadric sectors at centre

In this section $\boldsymbol{K}$ denotes a commutative field of characteristic different from two and A an affine space over $\boldsymbol{K}$. We consider an *affine quadric* p as a subset of A satisfying a polynomial equation of degree two over $\boldsymbol{K}$, i.e.

$$p: q\left(\overrightarrow{OP}\right)+l\left(\overrightarrow{OP}\right)=c \Leftrightarrow p=\left\{P\in A \mid q\left(\overrightarrow{OP}\right)+l\left(\overrightarrow{OP}\right)=c\right\}$$

for a point $O\in A$, a constant $c\in\boldsymbol{K}$, a non-zero quadratic form q and a linear form l of the linear space over $\boldsymbol{K}$ directing A (cf. [A, p.184]!). We concentrate on affine quadrics with *point symmetry*, i.e. there is a *centre* $O\in A$ with $O+\overrightarrow{PO}\in p$ for all $P\in p$. This property is an affine invariant, i.e. the point $\Psi(O)$ is a centre of the affine quadric $\Psi(p)$ for every affine mapping Ψ (s. def. 2.1. in [A, p.14]). An affine quadric p with point symmetry can be defined by the above equation with centre O and without linear form l, and the set of all centres of p conform an affine subspace of A (s. the argumentation for $\boldsymbol{K}=\mathbb{R}$ or $\boldsymbol{K}=\mathbb{C}$ in the proof of Theorem 1.11 and in Remarks 1.13 of [A, p.189], which works also over our more general $\boldsymbol{K}$!).

Remarks. a) For every $P\in p$ of an arbitrary set $p\subset A$ and every $O\in C$ of an affine space $C\subset A$ of centres of p the affine space $C+\overrightarrow{OP}\subset A$ is a subset of p. E.g.: Imagine p as an elliptic or hyperbolic cylinder in $\mathbb{R}^3$ with C as its symmetry axis.[2] Then for all $P\in p$ the line through P

2 Indeed every point of the symmetry axis is a centre. This is not true in general; s. e.g. a one-sheeted hyperboloid!

parallel to C belongs to p. For lack of references we proof the statement here: Since C is an affine space, for $O, O' \in C$ the point

$$O'' \overset{\text{def}}{=} O + \frac{1}{2} \cdot \overrightarrow{OO'}$$

is also a centre of p. Hence for $P \in p$ the "reflection point"

$$O'' + \overrightarrow{QO''} = O + \frac{1}{2} \cdot \overrightarrow{OO'} + \overrightarrow{QO} + \frac{1}{2} \cdot \overrightarrow{OO'} = O - \overrightarrow{PO} + \overrightarrow{OO'} = P + \overrightarrow{OO'}$$

of $Q \overset{\text{def}}{=} O + \overrightarrow{PO} \in p$ with respect to the symmetry centre O'' is an element of p. But due to the parallelogram rule (s. [A, p.9]!) this point equals $O' + \overrightarrow{OP}$, which proofs the statement.
b) If $C \cap p \neq \emptyset$ for p and C as in a), then $C \subset p$. By choosing $P = O \in p$ this follows from a).

The latter remark means $C \cap p = \emptyset$ (empty intersection) or $C \subset p$, when C denotes the affine space $C \subset A$ of centres[3] of an affine quadric $p \subset A$.

Lemma. For a centre $O \in p$ or $O \in A \setminus p$ of an affine quadric $p \subset A$ there exists a quadratic form q of the linear space directing A with

$$p : q\left(\overrightarrow{OP}\right) = 0 \quad \text{or} \quad p : q\left(\overrightarrow{OP}\right) = 1 \quad \text{, respectively.}$$

Proof: According to the assertion about vanishing of the linear form (in the beginning of this section) we have $p : q\left(\overrightarrow{OP}\right) = c$ for some quadratic form q and some constant c. Since $\overrightarrow{OO}$ equals the zero vector of the underlying linear space, we have $c = 0$ if and only if $O \in p$. Hence in case $O \notin p$ we can substitute q by the quadratic form q/c in order to obtain the demanded form.

The following proposition describes a geometric property of affine quadrics containing a centre.

Proposition. For an affine space A of finite dimension an affine quadric $p \subset A$ containing a centre is the union[4] of affine subspaces of A which intersect in the set of all centres of p.

Proof: For a centre $O \in p$, a point $P \in p$ and a field element $\lambda \in \boldsymbol{K}$ the point

$$Q \overset{\text{def}}{=} O + \lambda \cdot \overrightarrow{OP}$$

is also an element of $p : q\left(\overrightarrow{OP}\right) = 0$ (s. the lemma!), since $q\left(\overrightarrow{OQ}\right) = q\left(\lambda \cdot \overrightarrow{OP}\right) = \lambda^2 q\left(\overrightarrow{OP}\right) = 0$. In combination with remark a) this proofs $A(C, P) \subset p$ for the smallest affine space $A(C, P)$ containing the set C of all centres and the point P.[5] By running through all points P of p we get the

3 It may be empty as e.g. for a parabola.
4 not necessarily finite; s. e.g. a double cone in $\mathbb{R}^3$!
5 This can be seen easily by using an affine frame (s. def. below) of C and the polar form (s. [A, p.225] of q.

claimed affine subspaces $A(C,P)$. Their intersection is indeed *C*, since $A(C,O)=C$ for $O\in C\subset p$.

From a measure theoretic point of view the "sectors at centre" (s. def. below) are not very interesting in case $\emptyset\neq C\subset p$. So from now on we restrict ourselves to

$$C\neq\emptyset\ \wedge\ C\cap p=\emptyset\ .$$

The following notions and its properties are quite elementary. They will be used for foundation of the measure theory in the next section.

Definition. For an *affine frame* $(O,P_1,P_2,\ldots,P_n)\in A\times A\times\ldots\times A$ (*of dimension* $n\in\mathbb{N}$), i.e. $\overrightarrow{OP_1},\overrightarrow{OP_2},\ldots,\overrightarrow{OP_n}$ are linearly independent, we call $\Phi:\boldsymbol{K}^n\to A$ defined by

$$\Phi(x_1,x_2,\ldots,x_n)=O+x_1\cdot\overrightarrow{OP_1}+x_2\cdot\overrightarrow{OP_2}+\ldots+x_n\cdot\overrightarrow{OP_n}\ ,$$

its *frame affinity* (*with origin O*) and

$$\langle O,P_1,P_2,\ldots,P_n\rangle\stackrel{\text{def}}{=}\Phi(\boldsymbol{K}^n)$$

its *affine span*. For an affine quadric $p\subset A$ with centre $O\in A\setminus p$ and an affine frame

$$(O,P_1,P_2,\ldots,P_n)\ \text{ with }\ P_1,P_2,\ldots,P_n\in p$$

we call

$$(p,O,P_1,P_2,\ldots,P_n)$$

an (*n -dimensional affine quadric*) *sector at centre O of p with vertices* $P_1,P_2,\ldots,P_n$. For *planar sectors* (p,O,P,Q) at centre the planar affine quadric

$$s(p,O,P,Q)\stackrel{\text{def}}{=}p\cap\langle O,P,Q\rangle\subset A$$

is called *the* (*planar*) *section* and the field element $\chi(p,O,P,Q)\in\boldsymbol{K}$, defined by

$$\chi(p,O,P,Q)=\frac{1}{xy}-\frac{x}{y}-\frac{y}{x}$$

in case there is a point $R\in p$ such that $\overrightarrow{OR}=x\cdot\overrightarrow{OP}+y\cdot\overrightarrow{OQ}$ for some constants $x,y\in\boldsymbol{K}$ with $xy\neq0$ and otherwise $\chi(p,O,P,Q)=0$, *the sector coefficient*.[6]

The following theorem will show that all these definitions are very natural: The sector coefficients uniquely determine the corresponding sections (s. theorem f)!) and encode all the information about the affine quadric *p* at centre $O\notin p$ with respect to an affine frame $(O,P_1,P_2,\ldots,P_n)$ with $P_1,P_2,\ldots,P_n\in p$ (s. theorem d) and corollary c)!).

6 See example d) below for the non-existence of such a point *R*.

Theorem. a) The sector coefficient $\chi(p,O,P,Q)$ is well-defined.

b) It depends only on p and $\{\overrightarrow{OP},\overrightarrow{OQ}\}$, but not on the choice of centre $O\in A\setminus p$; i.e. for another centre O' the equation $\chi\left(p,O',P+\overrightarrow{OO'},Q+\overrightarrow{OO'}\right)=\chi(p,O,P,Q)$ holds.

c) It is an affine invariant; i.e. $\chi(\Psi p,\Psi O,\Psi P,\Psi Q)=\chi(p,O,P,Q)$ for an *affinity* Ψ, i.e a bijective affine map.

d) For an affine quadric sector $\left(p,O,P_1,P_2,\dots,P_n\right)$ (of dimension $n\in\mathbb{N}$) at centre $O\notin p$ there is only one symmetric $n\times n$ -matrix $S=\left(\delta_{ij}\right)$ over $\boldsymbol{K}$, called the *sector form*, with $\Phi(s)=p\cap\langle O,P_1,P_2,\dots,P_n\rangle$ for the frame affinity Φ and for the quadric

$$(1)\quad s:\vec{x}^T\cdot S\cdot\vec{x}=1 \quad \text{(column vectors } \vec{x}\in\boldsymbol{K}^n\text{)}.$$

Its entries are given by $s_{ii}=1$ for all $i\in\{1,\dots,n\}$ and

$$(2)\quad \delta_{ij}=\frac{1}{2}\cdot\chi(p,O,P_i,P_j) \quad \text{for } 1\leqslant i,j\leqslant n,\ i\neq j\ .$$

e) In case $n>1$ the affine quadric $p\cap\langle O,P_1,P_2,\dots,P_n\rangle$ is uniquely determined by the

$$\binom{n}{2} \text{ sections } s(p,O,P_i,P_j) \text{ with } 1\leqslant i<j\leqslant n\ .$$

f) Such a section is uniquely determined by (O,P_i,P_j) and $\chi(p,O,P_i,P_j)$; i.e. for two affine quadrics p,p' at centre O with points $P,Q\in p,p'$ such that $\overrightarrow{OP},\overrightarrow{OQ}$ are linearly independent and $\chi(p,O,P,Q)=\chi(p',O,P,Q)$ it follows

$$s(p,O,P,Q)=s(p',O,P,Q)\ .$$

Proof: a) The planar quadric $c\overset{\text{def}}{=}\Phi^{-1}(s(p,O,P,Q))\subset\boldsymbol{K}^2$ at centre $(0,0)=\Phi^{-1}(O)\notin c$ contains the points $(1,0)$ and $(0,1)$, hence it has the form $c:x^2+\beta xy+y^2=1$ for some $\beta\in\boldsymbol{K}$ due to the lemma. If there is a point

$$R\in s(p,O,P,Q) \text{ with } xy\neq 0$$

as in the definition of χ, then $(x,y)\in c$, and for

$$\delta\overset{\text{def}}{=}\frac{1}{2}\cdot\left(\frac{1}{xy}-\frac{x}{y}-\frac{y}{x}\right)$$

we have $\beta=2\delta$. This shows that χ is well defined.

b) The points $P+\overrightarrow{OO'},Q+\overrightarrow{OO'}$ are also elements of p according to the parallelogram rule and remark a). By definition of χ it suffices to show that in case there is no point R as described above the same holds for $s\left(p,O,P+\overrightarrow{OO'},Q+\overrightarrow{OO'}\right)$. But this is clear by the same argumentation as for P,Q with interchanged roles of O,O'.

c) Since $\overrightarrow{\Psi O\Psi P},\overrightarrow{\Psi O\Psi Q}$ are still linearly independent, the equality follows from the definition of χ.

d) The middle coefficient β of $\Phi^{-1}(s(p,O,P,Q))$ must equal $\chi(p,O,P,Q)$. This was shown already in a) in case of existence of *R* as above. In the other case β must vanish, since $(x,y)=(-\beta,1)$ is a point with $\Phi(x,y)\in s(p,O,P,Q)$. For arbitrary $n\in\mathbb{N}$ the quadric $s\overset{\text{def}}{=}\Phi^{-1}\left(p\cap\langle O,P_1,P_2,\ldots,P_n\rangle\right)$ contains the canonical unit vectors

$$\vec{e}_i\overset{\text{def}}{=}(0,\ldots,0,1,0,\ldots,0)\in\boldsymbol{K}^n \quad (\ i\in\{1,\ldots,n\}\ ),$$

hence has the form

$$s:\sum_{i=1}^{n}x_i^2+\sum_{1\leqslant i<j\leqslant n}\beta_{ij}x_ix_j=\vec{x}^T\cdot\begin{pmatrix}1&\delta_{12}&\ldots&\delta_{1n}\\ \delta_{12}&1&\ddots&\delta_{2n}\\ \vdots&\ddots&\ddots&\vdots\\ \delta_{1n}&\ldots&\delta_{n-1n}&1\end{pmatrix}\cdot\vec{x}=1\quad,\quad\vec{x}=(x_1,\ldots,x_n)\in\boldsymbol{K}^n$$

for some $\beta_{ij}\in\boldsymbol{K}$ and $\delta_{ij}\overset{\text{def}}{=}\beta_{ij}/2$ ($i<j$) due to the lemma. This shows the existence of the symmetric matrix *S*, even with diagonal elements equal to one. In case $n=1$ the uniqueness is already clear:

$$s:x^2=1\quad,\quad x\in\boldsymbol{K}\ .^{7}$$

In case $n>1$ we consider the planar quadrics $\Phi^{-1}\left(s(p,O,P_i,P_j)\right)\subset\boldsymbol{K}^n$ defined by

$$x_i^2+\beta_{ij}x_ix_j+x_j^2=1 \text{ and } x_k=0 \text{ for all } k\notin\{i,j\}\quad,\quad(x_1,x_2,\ldots,x_n)\in\boldsymbol{K}^n\quad(\ i<j\ ).$$

As above shown $\beta_{ij}=\chi(p,O,P_i,P_j)$ must hold. This shows the uniqueness of *S* together with the equations (1) and (2).

e) This assertion follows from the fact that the sector form *S* of $\left(p,O,P_1,P_2,\ldots,P_n\right)$ is already determined by the sector coefficients $\chi(p,O,P_i,P_j)$ (each for a section $s(p,O,P_i,P_j)$) and because of $\Phi(s)=p\cap\langle O,P_1,P_2,\ldots,P_n\rangle$ for the quadric *s* defined by *S*.

f) Because of (1) and (2) two planar sectors at centre (p,O,P,Q), (p',O,P,Q) with same sector coefficient β have the same sector form defining the planar quadric $s:x^2+\beta xy+y^2=1$. Since their sections are both images of *s* under the frame affinity of (O,P,Q), they must be equal.

The following corollary c) establishes a one-one-correspondence between the quadrics $\subset\boldsymbol{K}^n\setminus\{\vec{0}\}$ centred in the origin $\vec{0}$, which are not contained in a linear subspace of lower dimension than *n*, and its defining quadratic forms of *n* variables.

Corollary. a) For $\boldsymbol{K}=\mathbb{R}$ the section $s(p,O,P,Q)$ is

7 Then $p\cap\langle O,P_1\rangle$ consists of only two points.

- an ellipse in case $|\delta|<1$,[8]
- a pair of parallel lines in case $|\delta|=1$,
- a pair of hyperbola branches in case $|\delta|>1$, with

$$\delta \stackrel{\text{def}}{=} \frac{1}{2}\cdot\chi(p,O,P,Q) \quad .$$

b) For three linear dependent and pairwise linearly independent vectors $\vec{a},\vec{b},\vec{c}$ of a linear space L over $\boldsymbol{K}$ there is exactly one planar quadric $p\subset L\setminus\{\vec{0}\}$ with centre at the origin $\vec{0}$ and with $\vec{a},\vec{b},\vec{c}\in p$, namely $p=\Phi(s)$ for $\Phi(x,y)\stackrel{\text{def}}{=}x\cdot\vec{a}+y\cdot\vec{b}$ and the planar quadric s defined by

$$s: x^2+\beta x y+y^2=1 \quad (\ (x,y)\in\boldsymbol{K}^2\ )$$

with

$$\beta \stackrel{\text{def}}{=} \frac{1}{x\,y}-\frac{x}{y}-\frac{y}{x}$$

for the coordinates x,y of $\vec{c}=x\cdot\vec{a}+y\cdot\vec{b}$ with respect to $\vec{a},\vec{b}$.

c) For an affine quadric $q\subset\boldsymbol{K}^n\setminus\{\vec{0}\}$ with centre $\vec{0}$ which is not contained in a linear subspace of lower dimension[9] (than n) there is only one symmetric $n\times n$ -matrix Q defining q, i.e. with $q:\vec{x}^T\cdot Q\cdot\vec{x}=1$; namely

$$Q\stackrel{\text{def}}{=}J^{-T}\cdot S\cdot J^{-1} \quad ,$$

where S denotes the (symmetric) sector form of a sector $(q,\vec{0},\vec{p}_1,\ldots,\vec{p}_n)$ and where the i -th column vector of J is defined as $\vec{p}_i$ ($i=1,\ldots,n$).

d) The sector form of a sector at centre is an affine invariant, i.e. for every affinity Ψ the sector

$$(p,O,P_1,P_2,\ldots,P_n) \text{ at centre } O$$

has the same sector form as the sector

$$(\Psi p,\Psi O,\Psi P_1,\Psi P_2,\ldots,\Psi P_n) \text{ at centre } \Psi O \quad .$$

Proof: a) Since the type (ellipse, hyperbola, …) of a planar affine quadric is an affine invariant (s. e.g. [K, §30]), the type of $s(p,O,P,Q)$ is determined by the type of the corresponding sector form (s. theorem d)!) $c: x^2+2\delta x y+y^2=1$. The latter is well known to be characterized by its "discriminant" $4(\delta^2-1)$ according to the three cases of the statement.

b) Because of the pairwise linear independence we have $x\,y\neq 0$ for the coordinates of $\vec{c}$ with respect to $\vec{a},\vec{b}$, hence s and p are well defined. From $(1,0),(0,1),(x,y)\in s$ it follows $\vec{a},\vec{b},\vec{c}\in p$. This proofs the existence. By theorem d) we have $\beta=\chi(p,\vec{0},\vec{a},\vec{b})$. Every planar affine quadric $p'\subset L$ with centre $\vec{0}$ and $\vec{a},\vec{b},\vec{c}\in p'$ defines a section $s(p',\vec{0},\vec{a},\vec{b})=p'$

8 s. fig. of sect. 1 !

9 See (counter-) example c) below for the same assertion without this requirement.

with $\vec{c}\in p'$. It yields the same sector coefficient $\chi(p',\vec{0},\vec{a},\vec{b})=\beta$ by definition of χ . But due theorem f) this implies $p'=s(p',\vec{0},\vec{a},\vec{b})=s(p,\vec{0},\vec{a},\vec{b})=p$, hence the claimed uniqueness.

c) By the requirement there exist *n* linear independent vectors $\vec{p}_1,\ldots,\vec{p}_n\in q$. Let $\Phi:\boldsymbol{K}^n\to\boldsymbol{K}^n$ denote the frame affinity (here even an automorphism) of the affine frame $(\vec{0},\vec{p}_1,\ldots,\vec{p}_n)$, i.e. $\Phi(\vec{x})=J\cdot\vec{x}$ for all $\vec{x}\in\boldsymbol{K}^n$. It maps the quadric $s\subset\boldsymbol{K}^n$ defined by *S* onto

$$q\cap\langle\vec{p}_1,\ldots,\vec{p}_n\rangle=q\subset\boldsymbol{K}^n \quad .$$

Hence *q* is indeed defined by the symmetric matrix *Q*, which shows the existence. Assume, *q* is also defined by the symmetric matrix *R*. Then $T\overset{\text{def}}{=}J^T\cdot R\cdot J$ defines *s* by definition of *J*. Due to theorem d) *T* must equal *S*, and therefore $Q=J^{-T}\cdot(J^T\cdot R\cdot J)\cdot J^{-1}=R$.

d) This assertion follows from theorem c).

Examples. a) In every affine space $A_n(\boldsymbol{K})$ of dimension $n\in\mathbb{N}$ over $\boldsymbol{K}$ there exists an *n* -dimensional sector at centre, namely $(\Phi u,\Phi\vec{0},\Phi\vec{e}_1,\ldots,\Phi\vec{e}_n)$ for an arbitrary affinity $\Phi:\boldsymbol{K}^n\to A_n(\boldsymbol{K})$ and the quadric

$$u:\vec{x}^T\cdot\vec{x}=1 \quad \text{(column vectors } \vec{x}\in\boldsymbol{K}^n\text{)},$$

not containing $\vec{0}$.

b) The centres of the quadric $q:x^2=1$, $(x,y)\in\boldsymbol{K}^2$ are all the points $(0,y)\notin q$ with $y\in\boldsymbol{K}$, because for points

$$P=(\pm1,y)\in q \quad , \quad O=(0,z)\in\boldsymbol{K}^2\setminus q \quad \text{and} \quad Q=(x,z)\in\boldsymbol{K}^2 \quad \text{with} \quad x\neq0$$

we have

$$O+\overrightarrow{PO}=(0,z)+(0,z)-(\pm1,y)=(\mp1,z+z-y)\in q$$

and

$$Q+\overrightarrow{PQ}=(x,z)+(x,z)-(\pm1,y)=(x+x\mp1,z+z-y) \quad ,$$

whereby the the first coordinate can not equal ±1 , because $x+x-1=1$ contradicts $x+x+1=-1$. A two-dimensional sector is $(q,(0,0),(1,0),(1,1))$ with sector coefficient

$$-\frac{1}{2}+\frac{1}{2}+2=2$$

because of $(1,2)=2\cdot(1,1)-1\cdot(1,0)\in q$. Hence the corresponding sector form and the matrix *J* of corollary c) are

$$S=\begin{pmatrix}1&1\\1&1\end{pmatrix} \quad , \quad J=\begin{pmatrix}1&1\\0&1\end{pmatrix} \quad ,$$

yielding indeed (as claimed there)

$$J^{-T}\cdot S\cdot J^{-1}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad .$$

c) For the galois field $\boldsymbol{K}=GF(3)$ of three elements the quadric $q: x^2-y^2=1$ with centre $(0,0)\notin q$ consists only of the two linearly dependent vectors $(\pm 1,0)$. It is contained in a linear subspace of dimension one and can be represented also as $q: x^2+x\,y=1$.

d) For the quadric $q: x^2+y^2=1$ with centre $O\overset{\text{def}}{=}(0,0)\notin q$ and points $P\overset{\text{def}}{=}(1,0), Q\overset{\text{def}}{=}(0,1)\in q$ there is the planar sector (q, O, P, Q). Since the frame affinity of (O, P, Q) is the identity on $\boldsymbol{K}^2$, q is already defined by the sector form of (q, O, P, Q), i.e. $q=s$. Over $\boldsymbol{K}=GF(5)$ we have $q=\{\pm P, \pm Q\}$. So there is no point $R\in q$ linearly independent from *P* and from *Q*. Thus $\chi(q, O, P, Q)=0$, in accordance with the fact $q=s$ over arbitrary fields $\boldsymbol{K}$.

3. Measure theory of sectors at centre in euclidean space

In this section $\mu_N:\mathbb{R}^N\to\mathbb{R}_0\overset{\text{def}}{=}\{r\in\mathbb{R}\mid r\geqslant 0\}$ denotes the euclidean volume measure function. For an affine quadric $q\subset\mathbb{R}^N$ and a sector $\sigma\overset{\text{def}}{=}(q, O, P_1, \ldots, P_n)$ at centre $O\in\mathbb{R}^N\setminus q$ we define the *sector region* (to be measured) as

$$\sigma^+=\Phi\left(s^+\right)$$

with Φ being the frame affinity of $(O, P_1, \ldots, P_n)$ and $s^+\overset{\text{def}}{=}\left\{\vec{x}\in\mathbb{R}_0^n\mid \vec{x}^T\cdot S\cdot\vec{x}\leqslant 1\right\}$ with sector form $S=(\delta_{ij})$ (s. theorem d) of section 2). Now σ^+ possesses the vertices $O, P_1, \ldots, P_n$. It is bounded by the

$$\binom{n}{2}\ \text{affine planes}\ O+\left\langle\overrightarrow{OP_i}, \overrightarrow{OP_j}\right\rangle\subset\mathbb{R}^N\quad (\ 1\leqslant i, j\leqslant n\ ,\ i\neq j\ )$$

and by *q*, but not necessarily *finitely bounded*, i.e. bounded in the measure theoretical sense.

Examples. a) The elliptic sector region sketched in the figure of section 1 is finitely bounded.

b) When the three vectors of corollary b) lie on two different branches of a hyperbola (or on two different parallel lines), the corresponding planar sector region is not finitely bounded.

c) A spherical triangle with vertices P_1, P_2, P_3 together with the sphere's centre *O* determines a finitely bounded sector region in $\mathbb{R}^3$.

d) For a circle cylinder $q\subset\mathbb{R}^3$, vertices $P_1, P_2, P_3\in q$ of an equilateral triangle in a plane *E* orthogonal to the symmetry axis *Z* of *q* and a point $O\notin E\cap Z=\{B\}$ (different from the intersection point *B* of *E* and *Z*) the sector region of $\left(q, O, P_1, P_2, P_3\right)$ is not finitely bounded, since *B* is the barycentre of the triangle and therefore lying in its inner region. Nevertheless the

regions of the three planar sectors (q, O, P_i, P_j) for $1 \leqslant i < j \leqslant 3$ are elliptic and therefore finitely bounded.

Let J denote the coefficient matrix of the linear isomorphism

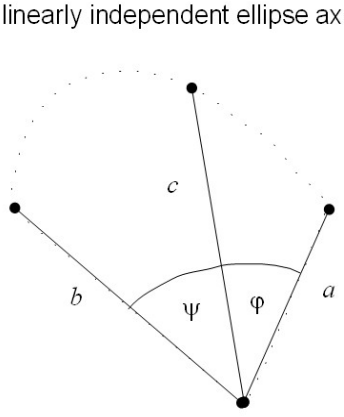
$$\vec{\Phi}: \mathbb{R}^n \to V \stackrel{\text{def}}{=} \langle \overrightarrow{OP_1}, \overrightarrow{OP_2}, \ldots, \overrightarrow{OP_n} \rangle \subset \mathbb{R}^N$$

belonging to Φ with respect to the canonical unit basis of $\mathbb{R}^n$ and an arbitrary orthonormal basis of the linear space V. Then for a bounded σ^+ we have $\mu_N(\sigma^+) = \|J\| \cdot \mu_n(s^+)$, due to the well known "change of variables formula" (s. e.g. [ABM, 7.1.2, p. 465]). If we denote by $\Sigma_n \subset \mathbb{R}^N$ the volume of the n-simplex with vertices $O, P_1, \ldots, P_n$, we will get

$$(3) \quad \mu_N(\sigma^+) = \Sigma_n \cdot n! \cdot \mu_n(s^+) \quad .$$

The first factor is easy to determine. We describe it in the following remarks b) and c) for $n = 2$ and $n = 3$, respectively, in dependence of the "geodesic data" length and angle. Remark a) shows how to compute a sector coefficient in dependence of these "geodesic data".

Remarks. a) In order to determine the sector coefficient of a planar sector determined by three linearly dependent and pairwise linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$ (s. corollary b) of section 2) one would measure (e.g. in geodesy) the lengths a, b, c of these vectors and the two angles φ, ψ between the vectors, whereby we take φ as the measure of the oriented angle (for definition s. [A, chap. III.1, pp. 68/69]) from $\vec{a}$ to $\vec{c}$ and ψ as the measure of the oriented angle from $\vec{c}$ to $\vec{b}$. In case of $\vec{c}$ lying between $\vec{a}$ and $\vec{b}$ on an elliptic arc the "geodesic data" are depicted in the following figure:



By help of some trigonometry we get

$$x=\frac{c\sin\psi}{a\sin(\varphi+\psi)}\quad,\quad y=\frac{c\sin\varphi}{b\sin(\varphi+\psi)}$$

for $\vec{c}=x\cdot\vec{a}+y\cdot\vec{b}$. Hence the sector coefficient reads

$$\frac{a\,b\sin^2(\varphi+\psi)}{c^2\sin\varphi\cdot\sin\psi}-\frac{b\sin\psi}{a\sin\varphi}-\frac{a\sin\varphi}{b\sin\psi}\quad.$$

b) The triangle area between two edges $a\,,b$ and their (non-oriented) angle $\omega\in(0\,,\pi)$ is

$$\Delta=\frac{1}{2}a\,b\sin(\omega)\quad.$$

c) The volume of a tetrahedron with edges $a\,,b\,,c$ at a common vertex and (non-oriented) angles $\varphi\,,\psi\,,\omega\in(0\,,\pi)$ between these edges is

$$\Theta=a\,b\,c\cdot\begin{vmatrix}1&\cos\varphi&\cos\psi\\\cos\varphi&1&\cos\omega\\\cos\psi&\cos\omega&1\end{vmatrix}\quad.$$

The third factor of formula (3) is determined by the sector coefficients $2\,\delta_{ij}$ ($1\leqslant i<j\leqslant n$) of the given sector. The main task is to express it as an analytic function of the δ_{ij} . In the following two theorems we investigate it again for the most practical cases $n=2$ and $n=3$.

Theorem 1. For vectors $\vec{a}\,,\vec{b}\,,\vec{c}$ of an euclidean space the sector between $\vec{a}\,,\vec{b}$ and the planar quadric *q* with centre at $\vec{0}$ defined by $\vec{a}\,,\vec{b}\,,\vec{c}$ (s. corollary b) of section 2) is (finitely)[10] bounded if and only if

$$\delta\overset{\text{def}}{=}\frac{1}{2}\cdot\chi\left(q\,,\vec{0}\,,\vec{a}\,,\vec{b}\right)>-1\quad.$$

In this case its area amounts to

- $\Delta\left(\vec{a}\,,\vec{b}\right)\cdot\dfrac{\arccos\delta}{\sqrt{1-\delta^2}}$ in the elliptic case $|\delta|<1$,[11]
- $\Delta\left(\vec{a}\,,\vec{b}\right)$ in the straight line case $\delta=1$,
- $\Delta\left(\vec{a}\,,\vec{b}\right)\cdot\dfrac{\operatorname{arcosh}\delta}{\sqrt{\delta^2-1}}$ in the hyperbolic case $\delta>1$,[12]

where $\Delta\left(\vec{a}\,,\vec{b}\right)\overset{\text{def}}{=}\Sigma_2$ is the area of the triangle between $\vec{a}$ and $\vec{b}$ (s. the figure in section1).

Proof: The region in question is the affine image $\sigma^+=\Phi\left(s^+\right)$ of

$$s^+:x^2+2\,\delta\,x\,y+y^2\leqslant 1\,;x\,,y\geqslant 0$$

10 From now on we skip this adverb.
11 The ratio can be interpreted as a “bump” factor with respect to the triangle area Δ .
12 The ratio can be interpreted as a “dent” factor with respect to the triangle area Δ .

under $\Phi(x,y)\overset{\text{def}}{=}x\cdot\vec{a}+y\cdot\vec{b}$ (s. corollary b) of section 2). Hence it is bounded if and only if s^+ is bounded. Since s^+ is symmetric with respect to the axis $y=x$, this is the case if and only if s^+ contains a point on this axis, i.e. $\delta>-1$. The area $\mu_2(s^+)$ turns out to be

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}}\sqrt{\frac{1+(\delta-1)x^2}{1+\delta}}\,dx-\frac{1}{2}$$

after rotation of s^+ by a positive angle of $\pi/4$. Now the claimed formula follows from equation (3) with help of elementary integration techniques and the formulas

$$2\arcsin\sqrt{\frac{1-\delta}{2}}=\arccos\delta \quad\text{for}\quad |\delta|\leqslant 1 \quad\text{and}\quad 2\,\mathrm{arsinh}\sqrt{\frac{\delta-1}{2}}=\mathrm{arcosh}\,\delta \quad\text{for}\quad \delta>1\ .$$

Numerical Example. For $a\overset{\text{def}}{=}30\,m$, $b\overset{\text{def}}{=}42\,m$, $c\overset{\text{def}}{=}45\,m$, $\varphi\overset{\text{def}}{=}35°$, $\psi\overset{\text{def}}{=}39°$ with any length m and $°\overset{\text{def}}{=}\pi/180$ we get $\delta\approx-0.29713$ according to the formula in remark a). Because of $|\delta|<1$ the corresponding points lie on an elliptic arc. For the triangle area we have

$$\frac{a\,b\sin(\varphi+\psi)}{2}=630\,m^2\cdot\sin 74°\approx 605.6\,m^2$$

and for the "bump" factor

$$\frac{\arccos\delta}{\sqrt{1-\delta^2}}\approx 1.961\ .$$

Hence the given elliptic sector at centre has an area of about $1190\ m^2$.

Theorem 2. For a *spatial* sector $\sigma\overset{\text{def}}{=}(q,O,P,Q,R)$ of an affine quadric $q\subset\mathbb{R}^3$ at centre O let $\Theta(O,P,Q,R)\overset{\text{def}}{=}\Sigma_3$ be the volume of its tetrahedron with vertices O,P,Q,R and

$$\eta\overset{\text{def}}{=}\delta_{11}\ ,\ \theta\overset{\text{def}}{=}\delta_{23}\ ,\ \kappa\overset{\text{def}}{=}\delta_{13}$$

for the 3×3 -sector form $S=(\delta_{ij})$ of σ. Then the volume of the sector region σ^+ of σ is

$$\Theta(O,P,Q,R)\cdot f(\eta,\theta,\kappa)$$

for the following analytic function f. Hereby we assume[13] without loss of generality, by suitable permutation of the vertices P,Q,R, the ordering $\eta\leqslant\theta\leqslant\kappa$:

$$f(\eta,\theta,\kappa)\overset{\text{def}}{=}\frac{2}{\sqrt{|S|}}\cdot\left(\arccos\frac{\eta-\theta\kappa}{\sqrt{(1-\theta^2)(1-\kappa^2)}}+\arccos\frac{\theta-\eta\kappa}{\sqrt{(1-\eta^2)(1-\kappa^2)}}+\arccos\frac{\kappa-\eta\theta}{\sqrt{(1-\eta^2)(1-\theta^2)}}-\pi\right)$$

in case $|S|=1+2\,\eta\theta\kappa-\eta^2-\theta^2-\kappa^2>0\ \wedge\ |\eta|,|\theta|,|\kappa|<1$ of an ellipsoid;

$$f(\eta,\theta,\kappa)\overset{\text{def}}{=}\frac{2}{1+\eta}\quad\text{or}\quad f(\eta,\theta,\kappa)\overset{\text{def}}{=}2\cdot(1-\eta)\cdot\left(\frac{\kappa+\theta-\eta-1}{(\theta-\eta\kappa)(\kappa-\eta\theta)}\right)$$

in case $|S|=0\ \wedge\ -1<\eta$ and $\kappa=1>\theta$ or $\theta>\eta\,\kappa$, respectively, of an elliptic cylinder;

13 Naturally this assumption is only necessary in those cases where f is not symmetric in its arguments.

$$f(1,1,1) \stackrel{\text{def}}{=} 1$$

in case of a plane;

$$f(\eta,\theta,\kappa) \stackrel{\text{def}}{=} \frac{2}{1+\kappa} \quad \text{or} \quad f(\eta,\theta,\kappa) \stackrel{\text{def}}{=} 2\cdot(\kappa-1)\cdot\left(\frac{1+\kappa-\theta-\eta}{(\eta\kappa-\theta)(\theta\kappa-\eta)}\right)$$

in case $|S|=0$ and $\eta=1<\theta$ or $\eta>1$, respectively, of a hyperbolic cylinder;

$$f(\eta,\theta,\kappa) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\kappa^2-1}}\cdot\int_0^1 \operatorname{arsinh}\frac{(\kappa^2-1)\cdot\left(\sqrt{1+\eta^2y^2-y^2}-\eta y\right)+(\kappa\theta-\eta)y}{\sqrt{\kappa^2-1+|S|y^2}} - \operatorname{arsinh}\frac{(\kappa\theta-\eta)y}{\sqrt{\kappa^2-1+|S|y^2}}\,dy$$

in case $|S|>0 \wedge -1<\eta$ of a hyperboloid of two sheets and in case $|S|<0 \wedge 1\leqslant\eta$ of a hyperboloid of one sheet;

$$f(\eta,\theta,\kappa) \stackrel{\text{def}}{=} \frac{2}{\sqrt{1-\eta^2}}\cdot\int_0^1 \arcsin\frac{(1-\eta^2)\cdot\left(\sqrt{1+\kappa^2y^2-y^2}-\kappa y\right)-(\eta\theta-\kappa)y}{\sqrt{1-\eta^2-|S|y^2}} + \arcsin\frac{(\eta\theta-\kappa)y}{\sqrt{1-\eta^2-|S|y^2}}\,dy$$

in case $|S|<0 \wedge 0\leqslant\eta<1$ of a hyperboloid of one sheet.

The sector region σ^+ is also bounded for

$$|S|<0 \wedge -1<\eta<0 \wedge (\theta>\eta\kappa \vee \eta+\kappa>0)$$, e.g. for $\eta=-\frac{3}{4}, \theta=0, \kappa=\frac{3}{4}$.

For this case[14] (of a hyperboloid of one sheet) a formula is outstanding. All other cases of η,θ,κ, which are not described for the above six types of quadrics, yield an unbounded sector region.

Proof: The sector form *S* defines the quadric

$$s: x^2+2\eta xy+y^2+2\theta yz+z^2+2\kappa zx=1 \quad , \quad x,y,z\in\mathbb{R} \quad .$$

Due to theorem 1 the inequalities $|\eta|,|\theta|,|\kappa|<1$ must hold in case of an ellipsoid. We transform *s* by the isomorphism $J(x,y,z) \stackrel{\text{def}}{=} T\cdot(x,y,z)^T$ with

$$T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \eta & \kappa \\ 0 & \sqrt{1-\eta^2} & \dfrac{\theta-\eta\kappa}{\sqrt{1-\eta^2}} \\ 0 & 0 & \sqrt{\dfrac{|S|}{1-\eta^2}} \end{pmatrix} ,$$

thus getting the quadric $n \stackrel{\text{def}}{=} J(s): x^2+y^2+z^2=1$, i.e. the unit sphere in $\mathbb{R}^3$. Under such transformations the sign of the determinant of the (ternary) quadratic form does not change. Hence we must have indeed $|S|>0$ in the ellipsoid case. Because of $|T|=\sqrt{|S|}$ the volume of the spherical triangle $J(s^+)$ is $\sqrt{|S|}$ times the volume of s^+ due to the "change of variables formula" (s. the beginning of this section). Its edge vectors are the columns $\vec{t}_1,\vec{t}_2,\vec{t}_3$ of *T*. The angles between these vectors are $\arccos\eta$, $\arccos\theta$ and $\arccos\kappa$. Now the formula follows

14 From a practical point of view this is no problem, since the sector coefficients can be made non-negative by choosing sectors of small enough angles; s. the formula in remark a)!

from the cosine theorem of spherical trigonometry (s. [MV, p.43]), Girard's formula for the area of a spherical triangle (s. [A, p.121]), the fact that the volume of the corresponding spherical sector is one third of that area and formula (3).

In case of an elliptic cylinder we transform *s* with

$$T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \eta & \kappa \\ 0 & \sqrt{1-\eta^2} & \frac{\theta-\eta\kappa}{\sqrt{1-\eta^2}} \\ 0 & 0 & \frac{1}{\sqrt{1-\eta^2}} \end{pmatrix}$$

in order to get $n: x^2+y^2=1$ because of $|S|=0$. The sector region between the columns $\vec{t}_1, \vec{t}_2, \vec{t}_3$ of *T* is determined by two linear functions of x, y each over a certain circle sector. Hence its volume can be computed by double integration of these functions. The normal vectors of the corresponding planes are given by

$$\vec{t}_1 \times \vec{t}_3 = \left(0, -\frac{1}{\sqrt{1-\eta^2}}, \frac{\theta-\eta\kappa}{\sqrt{1-\eta^2}}\right) \text{ and } \vec{t}_2 \times \vec{t}_3 = \left(1, -\frac{\eta}{\sqrt{1-\eta^2}}, \frac{\eta\theta-\kappa}{\sqrt{1-\eta^2}}\right) .$$

Hence the demanded functions are

$$z = \frac{y}{\theta-\eta\kappa} \text{ and } z = \frac{\sqrt{1-\eta^2}\cdot x - \eta y}{\kappa-\eta\theta}$$

over the unit circle sector from (angle) 0 to $\arccos\kappa$ and from $\arccos\kappa$ to $\arccos\eta$, respectively. Hereby the first function must be omitted in case of $\kappa=1$, since then the first plane is orthogonal to the *x y* -plane. Otherwise both denominators of these functions are different from zero in case of $-1<\eta\leqslant\theta\leqslant\kappa\leqslant 1 \wedge |S|=0$, which can be seen by the identity

$$(1-\eta^2)(1-\kappa^2)-(\theta-\eta\kappa)^2 = 1+2\eta\theta\kappa-\eta^2-\theta^2-\kappa^2 = |S| = 0$$

and the the same identity but with interchanged roles of θ and κ . Using polar coordinates we obtain

$$\mu_3(s^+) = \frac{1}{3}\left(\frac{1-\kappa}{\theta-\eta\kappa}+\frac{1-\theta}{\kappa-\eta\theta}\right) = \frac{1-\eta}{3}\left(\frac{\kappa+\theta-\eta-1}{(\theta-\eta\kappa)(\kappa-\eta\theta)}\right) ,$$

since a more detailed analysis shows that the denominators are even positive in case of a bounded sector region s^+ . This formula reduces to

$$\mu_3(s^+) = \frac{1}{3}\left(\frac{1-\theta}{1-\eta\theta}\right) = \frac{1}{3(1+\eta)}$$

in case of $\kappa=1$, since then $\eta=\theta\in(-1, 1)$ must hold because of $|S|=0$. Now the claimed formula follows from formula (3).

The case of a plane is trivial.

In case of a hyperbolic cylinder we transform with

$$T \stackrel{\text{def}}{=} \begin{pmatrix} 1 & \kappa & \eta \\ 0 & \sqrt{\kappa^2-1} & \frac{\eta\kappa-\theta}{\sqrt{\kappa^2-1}} \\ 0 & 0 & \frac{1}{\sqrt{\kappa^2-1}} \end{pmatrix}$$

in order to get $n: x^2-y^2=1$ because of $|S|=0$. In analogy with the case of an elliptic cylinder we double integrate the functions

$$z=\frac{y}{\eta\kappa-\theta} \quad \text{and} \quad z=\frac{\sqrt{\kappa^2-1}\cdot x-\kappa y}{\kappa\theta-\eta} \quad ,$$

using the coordinates $(x,y)=(r\cosh a, r\sinh a)$ with $0\leqslant r\leqslant 1$, $0\leqslant a\leqslant \operatorname{arcosh}\eta$ and $\operatorname{arcosh}\eta\leqslant a\leqslant \operatorname{arcosh}\kappa$, respectively. Hereby the first function must be omitted in case $\eta=1$. Then we get the desired result in analogy with the case of an elliptic cylinder.

In case $\theta,\kappa\geqslant 0$ the function

$$z(x,y)\stackrel{\text{def}}{=}\sqrt{l^2(x,y)+1-q(x,y)}-l(x,y)$$

with linear form $l(x,y)\stackrel{\text{def}}{=}\kappa x+\theta y$ and quadratic form $q(x,y)\stackrel{\text{def}}{=}x^2+2\eta x y+y^2$ is well defined over the planar sector region $q^+: q(x,y)\leqslant 1; x,y\geqslant 0$. It has the properties $z(x,y)\geqslant 0$ and $(x,y,z(x,y))\in s$ for all $(x,y)\in q^+$. Hence it defines the bounding quadric surface of s^+ . Under the additional condition $\eta\geqslant 0$ it follows

$$\mu_3(s^+)=\int_0^1 I(y)dy \quad \text{for} \quad I(y)\stackrel{\text{def}}{=}\int_0^{t(y)} z(x,y)dx \quad \text{with} \quad t(y)\stackrel{\text{def}}{=}\sqrt{1+\eta^2y^2-y^2}-\eta y \quad .$$

With help of the pendant in $\mathbb{R}^3$ (s. [AMR, Ex. 7.3E, p.511]) of Leibniz' sector rule it turns out

$$I(y)=\frac{1}{3}\cdot\int_0^{t(y)}\frac{dx}{\sqrt{r(x,y)}} \quad \text{with} \quad r(x,y)\stackrel{\text{def}}{=}l^2(x,y)+1-q(x,y) \quad .$$

Regarding *r* as a function of $x\in[0,t(y)]$ for fixed $y\in[0,1]$ it is quadratic of discriminant

$$D_\kappa(y)\stackrel{\text{def}}{=}4\left(\kappa^2-1+|S|y\right) \quad .$$

In case $1\leqslant\eta<\kappa$ it is always positive thus having the same positive sign as κ^2-1 . By looking up a standard integral table (s. e.g. [B, Nr. 241]) we get the first formula for the hyperboloid. In case $|S|<0\wedge 0\leqslant\eta<1$ we interchange the roles of η and κ in the integral, such that η^2-1 and $D_\eta(y)$ both have negative sign. By looking up the integral table (s. e.g. [B, Nr. 241] again) we find the second formula for the hyperboloid (of one sheet). The analysis of boundedness in case of a hyperboloid of one or two sheets is done via transformation to normal form in analogy with the elliptic or hyperbolic cylinder, respectively.

4. Conclusion

We presented an elementary theory of affine quadric sectors at centre and derived some fundamental formulas for their area and volume in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. No modern techniques are necessary for the proofs. The ingredients are some linear algebra, and some integration of real valued functions. All these means are well known. The key to the formulas is the well-definition of the sector coefficient, a certain affine invariant. It is desirable to find a formula for the outstanding case in theorem 2 and, furthermore, an integral free formula in case of a hyperboloid of one and of two sheets. Perhaps analytic volume formulas for higher dimensions are also a future task. It seems to the author that up to now there has not been any research in the field of affine quadric sectors, although their quadrics are represented elegantly with sector coefficients in their affine frames (s. theorem d) and corollary c) of section 2). This article might be a modest beginning of it.

5. References

[A] M.Audin, Geometry (2003) Springer-Verlag

[ABM] R.Abraham/J.E.Marsden/T.Ratiu, Manifolds, Tensor Analysis, and Applications, 2nd edition (1988) Springer-Verlag

[B] I.N.Bronstein, Taschenbuch der Mathematik, 5. Auflage (2000) Verlag Harri Deutsch AG

[K] H.-J.Kowalsky, Lineare Algebra, 9. Auflage (1979) Walter de Gruyter Verlag

[MV] K.Meyberg/P.Vachenauer, Höhere Mathematik 1, 6. Auflage (2001) Springer-Verlag