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Disorder relevance for the random walk pinning model in $d = 3$

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Abstract

We study the continuous time version of the *random walk pinning model*, where conditioned on a continuous time random walk $(Y_s)_{s\geq 0}$ on $\mathbb{Z}^d$ with jump rate $\rho > 0$, which plays the role of disorder, the law up to time t of a second independent random walk $(X_s)_{0\leq s\leq t}$ with jump rate 1 is Gibbs transformed with weight $e^{\beta L_t(X,Y)}$, where $L_t(X,Y)$ is the collision local time between X and Y up to time t. As the inverse temperature β varies, the model undergoes a localization-delocalization transition at some critical $\beta_c \geq 0$. A natural question is whether or not there is disorder relevance, namely whether or not β_c differs from the critical point β_c^{ann} for the annealed model. In [BS09], it was shown that there is disorder irrelevance in dimensions $d = 1$ and 2, and disorder relevance in $d \geq 4$. For $d \geq 5$, disorder relevance was first proved in [BGdH08]. In this paper, we prove that if X and Y have the same jump probability kernel, which is irreducible and symmetric with finite second moments, then there is also disorder relevance in the critical dimension $d = 3$. Our proof employs coarse graining and fractional moment techniques, which have recently been successfully applied by Giacomin, Lacoin and Toninelli [GLT09] to establish disorder relevance for the random pinning model in the critical dimension, and by Lacoin [L09] to the directed polymer model in random environment. Along the way, we also prove a continuous time version of Doney's local limit theorem [D97] for renewal processes with infinite mean.

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1 Model and result

Let us recall the continuous time *random walk pinned to random walk model* studied in [BS09], which we will abbreviate from now on as the *random walk pinning model*. Let X and Y be two continuous time random walks on $\mathbb{Z}^3$ starting from the origin, such that X and Y have respectively jump rates 1 and $\rho \geq 0$, and identical irreducible symmetric jump probability kernels on $\mathbb{Z}^3$ with finite second moments. Let μ_t denote the law of $(X_s)_{0\leq s\leq t}$. Then given $\beta \in \mathbb{R}$ and conditioned on $(Y_s)_{s\geq 0}$, which we interpret as a random environment or disorder, we define a Gibbs transform $\mu_{t,Y}^{\beta}$ of the path measure μ_t via Radon-Nikodym derivative

$$\frac{\mathrm{d}\mu_{t,Y}^{\beta}}{\mathrm{d}\mu_{\mathrm{t}}}(X) = \frac{e^{\beta L_t(X,Y)}}{Z_{t,Y}^{\beta}}, \tag{1.1}$$

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where $L_t(X,Y) = \int_0^t 1_{\{X_s=Y_s\}} \mathrm{d}s$, and

$$Z_{t,Y}^{\beta} = \mathbb{E}_0^X \left[e^{\beta L_t(X,Y)} \right] \tag{1.2}$$

is the *quenched partition function* with $\mathbb{E}_x^X[\cdot]$ denoting expectation w.r.t. X starting from $x \in \mathbb{Z}^3$. We can interpret X as a polymer which is attracted to a random defect line Y. A more commonly studied model is to consider a constant defect line $Y \equiv 0$, but with random strength of interaction between X and Y at different time points. This is known as the *random pinning model*, the discrete time analogue of which was the subject of many recent papers (see e.g. [DGLT09, GLT08, GLT09]), as well as a book [G07].

A common variant of the Gibbs measure $\mu_{N,Y}^{\beta}$ is to introduce pinning of path at the end point t, i.e., we define the Gibbs measure $\mu_{t,Y}^{\beta,\mathrm{pin}}$ with

$$\frac{\mathrm{d}\mu_{t,Y}^{\beta,\mathrm{pin}}}{\mathrm{d}\mu_t}(X) = 1_{\{X_t=Y_t\}} \frac{e^{\beta L_t(X,Y)}}{Z_{t,Y}^{\beta,\mathrm{pin}}} \tag{1.3}$$

with $Z_{t,Y}^{\beta,\mathrm{pin}} = \mathbb{E}_0^X \left[e^{\beta L_t(X,Y)} 1_{\{X_t=Y_t\}} \right]$. It was shown in [BS09] that, almost surely w.r.t. Y, the limit

$$F(\beta,\rho) = \lim_{t\to\infty} \frac{1}{t} \log Z_{t,Y}^{\beta} = \lim_{t\to\infty} \frac{1}{t} \log Z_{t,Y}^{\beta,\mathrm{pin}} \tag{1.4}$$

exists and is independent of the disorder Y, which we call the *quenched free energy* of the model. There exists a critical inverse temperature $\beta_c = \beta_c(\rho)$, such that $F(\beta,\rho) > 0$ if $\beta > \beta_c$ and $F(\beta) = 0$ if $\beta < \beta_c$. The supercritical region $\beta \in (\beta_c, \infty)$ is the localized phase where given Y, and with respect to either $\mu_{t,Y}^{\beta}$ or $\mu_{t,Y}^{\beta,\mathrm{pin}}$, the contact fraction $L_t(X,Y)/t$ between X and Y up to time t typically remains positive as $t \to \infty$, so that the walk X is pinned to Y. The subcritical region $\beta \in (-\infty, \beta_c)$ is the de-localized phase, where the contact fraction $L_t(X,Y)/t$ is typically of order $o(1)$ as $t \to \infty$, so that X becomes delocalized from Y.

An important tool in the study of models with disorder is to compare the quenched free energy with the annealed free energy, which is defined by

$$F_{\mathrm{ann}}(\beta,\rho) := \lim_{t\to\infty} \frac{1}{t} \log Z_{t,\mathrm{ann}}^{\beta} = \lim_{t\to\infty} \frac{1}{t} \log Z_{t,\mathrm{ann}}^{\beta,\mathrm{pin}}, \tag{1.5}$$

where

$$Z_{t,\mathrm{ann}}^{\beta} = \mathbb{E}_{0,0}^{X,Y}[e^{\beta L_t(X,Y)}] \qquad \text{and} \qquad Z_{t,\mathrm{ann}}^{\beta,\mathrm{pin}} = \mathbb{E}_{0,0}^{X,Y}[e^{\beta L_t(X,Y)} 1_{\{X_t=Y_t\}}]$$

are the free (resp. constrained) versions of the annealed partition function for the random walk pinning model. Since $X - Y$ is also a random walk, we see that $Z_{t,\mathrm{ann}}^{\beta}$ and $Z_{t,\mathrm{ann}}^{\beta,\mathrm{pin}}$ are the partition functions of a random walk pinning model, where the random walk $X - Y$ is attracted to the constant defect line 0. This defines the annealed model. In particular, there also exists a critical point $\beta_c^{\mathrm{ann}} = \beta_c^{\mathrm{ann}}(\rho)$ such that $F_{\mathrm{ann}}(\beta,\rho) > 0$ when $\beta > \beta_c^{\mathrm{ann}}$ and $F_{\mathrm{ann}}(\beta,\rho) = 0$ when $\beta < \beta_c^{\mathrm{ann}}$. By Jensen's inequality, it is easily seen that $F(\beta,\rho) \le F_{\mathrm{ann}}(\beta,\rho)$, and hence $\beta_c \ge \beta_c^{\mathrm{ann}}$. A fundamental question is then to determine whether the disorder is sufficient to shift the critical point of the model so that $\beta_c < \beta_c^{\mathrm{ann}}$, which is called *disorder relevance*. If $\beta_c = \beta_c^{\mathrm{ann}}$, then we say there is *disorder irrelevance*, and it is generally believed that the quenched model's behavior in this case is similar to that of the annealed model. It turns out that disorder relevance/irrelevance has an interesting dependence on the spatial dimension d.

In [BS09], it was shown that if X and Y are continuous time simple random walks, then the random walk pinning model is disorder irrelevant in $d = 1$ and 2, and disorder relevant in $d \ge 4$. Furthermore, it was shown that in $d \ge 5$, there exists $a > 0$ such that $\beta_c - \beta_c^{\mathrm{ann}} > a\rho$ for all $\rho \in [0,1]$; while in $d = 4$, for any $\delta > 0$, there exists $a_\delta > 0$ such that $\beta_c - \beta_c^{\mathrm{ann}} \ge a_\delta \rho^{1+\delta}$ for

all $\rho \in [0,1]$. It is easy to check that the proof of these results in [BS09] apply equally well to continuous time random walks X and Y with the same irreducible symmetric jump probability kernel with finite second moments. In this paper, we resolve the marginal case $d = 3$ and show that there is disorder relevance.

Theorem 1.1 [Annealed vs quenched critical points]
Let X and Y be two continuous time random walks with respective jump rates 1 and $\rho > 0$ and identical irreducible symmetric jump probability kernel $q(\cdot)$ on $\mathbb{Z}^3$ with finite second moments. Assume $X_0 = Y_0 = 0$. Then the associated random walk pinning model has $\beta_c < \beta_c^{\mathrm{ann}}$.

We will in fact prove a stronger version of Theorem 1.1.

Theorem 1.2 [Non-coincidence of critical points strengthened]
Assume the same conditions as in Theorem 1.1. Then we have $\beta_c^{\mathrm{ann}} < \beta_c^*$, where

$$\beta_c^* = \sup\Big\{\beta \in \mathbb{R} : \sup_{t>0} Z_{t,Y}^{\beta} < \infty \ \textit{a.s. w.r.t. } Y\Big\}. \tag{1.6}$$

Note that $\beta_c^* \le \beta_c$.

Theorem 1.2 confirms a conjecture of Greven and den Hollander [GdH07, Conj. 1.8] that in $d = 3$, the parabolic Anderson model with Brownian noise could admit an equilibrium measure with an infinite second moment. We refer to [BS09, Sec. 1.4] for a more detailed discussion on the connection between the random walk pinning model and the parabolic Anderson model, as well as the connection of the discrete time random pinning and random walk pinning models with the directed polymer model in random environment.

We will follow the general approach developed by Giacomin, Lacoin, and Toninelli in [GLT08, GLT09] for proving the marginal relevance of disorder for the random pinning model, as well as by Lacoin in [L09] for the study of the directed polymer model in random environment. The basic ingredients are change of measure arguments for bounding fractional moments of the partition function $Z_{t,Y}^{\beta}$, coupled with a coarse grain splitting of $Z_{t,Y}^{\beta}$. These techniques have proven to be remarkably powerful, and they apply to a wide range of models: in particular, to weighted renewal processes in random environments, including the random pinning, the random walk pinning, and the copolymer models (see [BS09, Sec. 1.4] for a more detailed discussion), as well as to weighted random walks in random environments, including the directed polymer model [L09] and random walk in random environments [YZ09]. We will recall in detail the fractional moment techniques and the coarse graining procedure and formulate them for the random walk pinning model, which will constitute the model independent part of our analysis. A key element of the fractional moment technique involves a change of measure, and more generally, the choice of a suitable test function. This is the model dependent part of the analysis, which in general is far from trivial, since disorder relevance in the critical dimension is a rather subtle effect. The bulk of this paper is thus dedicated to the choice of a suitable test function for the random walk pinning model and its analysis. Compared to the random pinning model and the directed polymer model, new complications arise due to the particular nature of the disorder for the random walk pinning model.

The rest of the paper is organized as follows. In Section 2, we recall from [BGdH08] and [BS09] a representation of $Z_{t,Y}^{\beta}$ as the partition function of a weighted renewal process in random environment. In Section 3, we recall the coarse graining procedure and fractional moment techniques developed in [GLT08, GLT09] and [L09]. To prove the stronger result, Theorem 1.2, we apply the coarse graining procedure to $Z_{t,Y}^{\beta}$ instead of the constrained partition function $Z_{t,Y}^{\beta,\mathrm{pin}}$ as done in [GLT08, GLT09]. The proof of disorder relevance is reduced in Section 3 to two key lemmas: Lemma 3.1, which is model dependent and needs to be proved for any new weighted renewal process in random environment one is interested in, and Lemma 3.2, which is model independent.

Compared to analogues of Lemma 3.1 for the random pinning model, see [GLT09, Lemma 3.1], our weaker formulation allows a more direct comparison with renewal processes without boundary constraints, for which the law of large numbers and the ergodic theorem can be applied. In Section 4, we identify a suitable test function H_L for the disorder Y and state some essential properties. Assuming these properties, we then prove in Section 5 the key Lemma 3.1, which is further reduced to a model dependent Lemma 5.1 by extracting some model independent renewal calculations. We then deduce Lemma 3.2 from Lemma 3.1 in Section 6, which is again model independent. The properties of the test function H_L are established in Sections 7–8. Lastly, in Appendices A and B, we prove some renewal and random walk estimates which we need for our proof. In particular, we prove in Lemma A.1 a continuous time version of Doney's local limit theorem [D97, Thm. 3] for renewal processes with infinite mean.

Note. During the preparation of this manuscript, we became aware of a preprint by Q. Berger and F.L. Toninelli [BT09], in which they proved the analogue of Theorem 1.1 for the discrete time random walk pinning model in dimension 3 under the assumption that the random walk increment is symmetric with sub-Gaussian tails. An inspection shows that the main difference between our two approaches lies in the choice of the test function $H_L(\cdot)$ in (4.1), which results in completely different model dependent analysis as well as different assumptions on the model. In principle, both approaches should be applicable to both discrete and continuous time models. Most results in this paper carry over directly to the discrete time case. The only exception is Lemma B.5, for which we do not have a proof for its discrete time analogue. Lemma B.5 is used to prove Lemma 4.2 (4.7). In light of [BT09], we will not pursue this further in this paper.

Notation: Throughout the rest of this paper, unless stated otherwise, we will use C, C_1 and C_2 to denote generic constants whose precise values may change from line to line. However, their values all depend only on the jump probability kernel $q(\cdot)$ and are uniform in $\rho \in (0, 1]$.

2 Representation as a weighted renewal process in random environment

First we recall from [BGdH08] and [BS09] a representation of $Z^{\beta}_{t,Y}$ as the partition function of a weighted renewal process in random environment. Let $p_s(\cdot) = p^X_s(\cdot)$ denote the transition probability kernel of X at time s. Then Y and $X - Y$ have respective transition kernels $p^Y_s(\cdot) := p_{\rho s}(\cdot)$ and $p^{X-Y}_s(\cdot) := p_{(1+\rho)s}(\cdot)$. Let

$$G = \int_0^\infty p_s(0)\mathrm{d}s, \quad G^{X-Y} = \int_0^\infty p_{(1+\rho)s}(0)\mathrm{d}s = \frac{G}{1+\rho}, \quad K(t) = \frac{p^{X-Y}_t(0)}{G^{X-Y}} = \frac{(1+\rho)p_{(1+\rho)t}(0)}{G},$$

where $K(t)dt$ is to be interpreted as the renewal time distribution of a recurrent renewal process $\sigma = \{\sigma_0 = 0 < \sigma_1 < \cdots\} \subset [0,\infty)$. Let $z = \beta G^{X-Y} = \beta G/(1+\rho)$ and $\mathcal{Z}^z_{t,Y} := Z^\beta_{t,Y}$. Then

$$\begin{aligned}
\mathcal{Z}^z_{t,Y} &= \mathbb{E}^X_0\big[e^{\beta L_t(X,Y)}\big] = \mathbb{E}^X_0\left[1+\sum_{m=1}^\infty \frac{\beta^m}{m!}\Big(\int_0^t 1_{\{X_s=Y_s\}}ds\Big)^m\right] \\
&= \mathbb{E}^X_0\Big[1+\sum_{m=1}^\infty \beta^m \int\cdots\int_{0<\sigma_1\cdots<\sigma_m<t} 1_{\{X_{\sigma_1}=Y_{\sigma_1},\cdots,X_{\sigma_m}=Y_{\sigma_m}\}}d\sigma_1\cdots d\sigma_m\Big] \\
&= 1+\sum_{m=1}^\infty \beta^m \int\cdots\int_{0<\sigma_1\cdots<\sigma_m<t} p_{\sigma_1}(Y_{\sigma_1})p_{\sigma_2-\sigma_1}(Y_{\sigma_2}-Y_{\sigma_1})\cdots p_{\sigma_m-\sigma_{m-1}}(Y_{\sigma_m}-Y_{\sigma_{m-1}})d\sigma_1\cdots d\sigma_m \\
&= 1+\sum_{m=1}^\infty z^m \int\cdots\int_{\sigma_0=0<\sigma_1\cdots<\sigma_m<t} \prod_{i=1}^m K(\sigma_i-\sigma_{i-1})W(\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}})\,d\sigma_1\cdots d\sigma_m, \qquad (2.1)
\end{aligned}$$

where

$$W(\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}}) = \frac{p^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p^{X-Y}_{\sigma_i-\sigma_{i-1}}(0)}. \qquad (2.2)$$

We can thus interpret $\mathcal{Z}^z_{t,Y}$ as the partition function of a weighted renewal process σ in the random environment Y, where the renewal time distribution is given by $K(\cdot)$, and each renewal return incurs a weight factor of $zW(\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}})$.

Similarly, for any $0 \le U \le V$, we can define $\mathcal{Z}^{z,\mathrm{pin}}_{[U,V],Y} := 1$ when $U = V$, and otherwise

$$\mathcal{Z}^{z,\mathrm{pin}}_{[U,V],Y} := \sum_{m=1}^\infty \int\cdots\int_{\sigma_0=U<\sigma_1\cdots<\sigma_m=V} z^m \prod_{i=1}^m K(\sigma_i-\sigma_{i-1})W(\sigma_i-\sigma_{i-1},Y_{\sigma_i}-Y_{\sigma_{i-1}})\,d\sigma_1\cdots d\sigma_{m-1}, \qquad (2.3)$$

where the term for $m=1$ is defined to be $zK(V-U)W(V-U,Y_V-Y_U)$. Note that $\mathcal{Z}^{z,\mathrm{pin}}_{[0,t],Y} = \beta Z^{\beta,\mathrm{pin}}_{t,Y}$, which we will simply denote by $\mathcal{Z}^{z,\mathrm{pin}}_{t,Y}$.

Since K is the renewal time distribution of a recurrent renewal process σ on $[0,\infty)$, and note that $\mathbb{E}^Y_0[W(v-u,Y_v-Y_u)] = 1$ for any $u < v$, the critical point z^{ann}_c of the annealed model with partition function $\mathbb{E}^Y_0[\mathcal{Z}^z_{t,Y}]$ is exactly 1. By the mapping $z = \beta G^{X-Y}$, we deduce that $\beta^{\mathrm{ann}}_c = 1/G^{X-Y} = (1+\rho)/G$. The mapping to a weighted renewal process in random environment casts the random walk pinning model in the same framework as the random pinning model, which paves the way for the application of general approaches developed in [GLT08, GLT09].

3 Fractional moment techniques and coarse graining

We now recall the fractional moment techniques and the coarse graining procedure, which were developed in a series of papers for the random pinning model that culminated in [GLT08, GLT09], where marginal relevance of disorder was established, as well as in [L09] where the same techniques were applied to the directed polymer model in random environment.

By (1.2), $Z^\beta_{t,Y} = \mathcal{Z}^z_{t,Y}$ is monotonically increasing in t for every realization of Y. Therefore, to prove Theorem 1.2, it suffices to show that for some $z > 1$ and $\gamma \in (0,1)$, we have

$$\sup_{t>0} \mathbb{E}^Y_0[(\mathcal{Z}^z_{t,Y})^\gamma] < \infty. \qquad (3.1)$$

To bound the fractional moment $\mathbb{E}^Y_0[(\mathcal{Z}^z_{t,Y})^\gamma]$, we apply the inequality

$$(\sum_{i=1}^n a_i)^\gamma \le \sum_{i=1}^n a_i^\gamma \qquad \text{for} \quad a_i \ge 0,\ 1 \le i \le n, \quad \text{and} \quad \gamma \in (0,1). \qquad (3.2)$$

This seemingly trivial inequality turns out to be exceptionally powerful in bounding fractional moments. However, the success of such a bound depends crucially on how $\mathcal{Z}_{t,Y}^z$ is split into a sum of terms. This is where coarse graining comes in, which was used in [GLT08, L09, GLT09]. We remark that in the earlier paper [DGLT09] on the random pinning model, and later in the analysis [BS09] of the random walk pinning model in $d \geq 4$, $\mathcal{Z}_{t,Y}^z$ is partitioned according to the values of the consecutive renewal times $\sigma_i < \sigma_{i+1}$ which straddle a fixed time $L > 0$. The coarse graining procedure we recall below uses a more refined partition of $\mathcal{Z}_{t,Y}^z$.

Fix a large constant $L > 0$, which will be the coarse graining scale, and assume that $t = mL$ for some $m \in \mathbb{N}$. Then we partition $(0, t]$ into m blocks $B_1, \cdots, B_m$ with $B_i := ((i-1)L, iL]$. The coarse graining procedure simply groups terms in (2.1) according to which blocks B_i does the renewal configuration $\sigma := \{\sigma_0 = 0 < \sigma_1 < \cdots\}$ intersect. More precisely, the set of blocks in $\{B_i\}_{1 \leq i \leq m}$ which σ intersects can be represented by a set $I \subset \{1, \cdots, m\}$. Then we can decompose $\mathcal{Z}_{t,Y}^z$ in (2.1) as

$$\mathcal{Z}_{t,Y}^z = \sum_{I \subset \{1,\cdots,m\}} \mathcal{Z}_{t,Y}^{z,I},$$

where $\mathcal{Z}_{t,Y}^{z,\emptyset} := 1$, and for each $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq m\} \neq \emptyset$,

$$\mathcal{Z}_{t,Y}^{z,I} = \int_{\substack{a_1 < b_1 \\ a_1, b_1 \in B_{i_1}}} \cdots \int_{\substack{a_k < b_k \\ a_k, b_k \in B_{i_k}}} \prod_{j=1}^{k} K(a_j - b_{j-1}) z W(a_j - b_{j-1}, Y_{a_j} - Y_{b_{j-1}}) \mathcal{Z}_{[a_j, b_j], Y}^{z,\mathrm{pin}} \prod_{j=1}^{k} \mathrm{d}a_j \, \mathrm{d}b_j, \tag{3.3}$$

where $b_0 := 0$. By (3.2), for any $\gamma \in (0, 1)$, we have

$$\mathbb{E}_0^Y \left[(\mathcal{Z}_{t,Y}^z)^\gamma \right] \leq \sum_{I \subset \{1,\cdots,m\}} \mathbb{E}_0^Y \left[(\mathcal{Z}_{t,Y}^{z,I})^\gamma \right]. \tag{3.4}$$

We will prove (3.1) by comparing $\mathbb{E}_0^Y \left[(\mathcal{Z}_{t,Y}^{z,I})^\gamma \right]$ with the probability that a subcritical renewal process on $\mathbb{N} \cup \{0\}$ intersects $\{1, \cdots, m\}$ exactly at I.

To bound $\mathbb{E}_0^Y \left[(\mathcal{Z}_{t,Y}^{z,I})^\gamma \right]$, one introduces a change of measure. Let $f_I(Y)$ be a non-negative function of the disorder Y. By Hölder's inequality,

$$\mathbb{E}_0^Y \left[\left(\mathcal{Z}_{t,Y}^{z,I} \right)^\gamma \right] = \mathbb{E}_0^Y \left[f_I(Y)^\gamma f_I(Y)^{-\gamma} \left(\mathcal{Z}_{t,Y}^{z,I} \right)^\gamma \right] \leq \mathbb{E}_0^Y \left[f_I(Y)^{-\frac{\gamma}{1-\gamma}} \right]^{1-\gamma} \mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right]^\gamma. \tag{3.5}$$

To decouple different blocks B_i, we will let $f_I(Y) = \prod_{i \in I} f((Y_s - Y_{(i-1)L})_{s \in B_i})$ with

$$\mathbb{E}_0^Y [f((Y_s)_{0 \leq s \leq L})^{-\frac{\gamma}{1-\gamma}}] \leq 2. \tag{3.6}$$

To make $\mathbb{E}_0^Y \left[\left(\mathcal{Z}_{t,Y}^{z,I} \right)^\gamma \right]$ small, f should be chosen to make $\mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right]$ small. There have been two approaches in bounding $\mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right]$ in the literature.

The first approach is to choose $f_I(Y)$ to be a probability density so that $\mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right]$ becomes the annealed partition function of a random walk pinning model with a new law for the disorder Y. This approach was used in [DGLT09] to prove disorder relevance for the random pinning model, where the laws of the disorder at different time points are independently tilted to favor delocalization. It was later adapted to the random walk pinning model in dimensions $d \geq 4$ in [BS09], where the change of measure for Y increases its jump rate, which turns out to favor delocalization. To prove disorder relevance for the random pinning model at the critical dimension, which borderlines the known disorder relevance/irrelevance regimes, a more sophisticated change of measure was introduced in [GLT08] for the random pinning model with Gaussian disorder, which induces negative correlation between the disorder at different time points, and Gaussian

calculations are used to estimate the annealed partition function under the new disorder. For the random walk pinning model in the critical dimension $d = 3$, the analogue would be to introduce correlation between the increments of Y at different time steps. However the presence of correlation makes it unfeasible to estimate the annealed partition function under the new disorder.

A variant approach to estimate $\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]$ was then introduced in [L09] for the directed polymer model, and in [GLT09] for the random pinning model at the critical dimension with general disorder. The function f_I will be taken to be a test function on the disorder Y instead of as a probability density that changes the law of Y. For its simplicity, f in (3.6) is taken to be of the form

$$f((Y_j)_{0\le s\le L}) = 1_{\{H_L(Y)\le M\}} + \epsilon_M\, 1_{\{H_L(Y)>M\}}, \tag{3.7}$$

where $H_L(Y)$ is a functional of the disorder Y, positively correlated with $\mathcal{Z}_{t,Y}^{z,I}$, and we choose

$$\epsilon_M = \mathbb{P}_0^Y\left(H_L(Y) > M\right)^{\frac{1-\gamma}{\gamma}} \tag{3.8}$$

to guarantee that (3.6) holds. We will make ϵ_M small by choosing M large. To bound $\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]$, we use the representation (3.3) to write

$$\begin{aligned}
&f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\\
&=\int_{\substack{a_1<b_1\\ a_1,b_1\in B_{i_1}}}\cdots\int_{\substack{a_k<b_k\\ a_k,b_k\in B_{i_k}}}\prod_{j=1}^{k}K(a_j-b_{j-1})zW(a_j-b_{j-1},Y_{a_j}-Y_{b_{j-1}})\mathcal{Z}^{z,\mathrm{pin}}_{[a_j,b_j],Y}f((Y_s-Y_{(i_j-1)L})_{s\in B_{i_j}})\prod_{j=1}^{k}\mathrm{d}a_j\,\mathrm{d}b_j,
\end{aligned}$$

where $b_0 := 0$. By Lemma B.1, the local central limit theorem for X and $X-Y$, there exists $C > 0$ such that uniformly in $t > 0$ and Y, we have

$$W(t, Y_t - Y_0) = \frac{p_t^X(Y_t - Y_0)}{p_t^{X-Y}(0)} = \frac{p_t(Y_t - Y_0)}{p_{(1+\rho)t}(0)} \le C. \tag{3.9}$$

Therefore

$$\begin{aligned}
&\mathbb{E}_0^Y\big[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\big]\\
&\qquad\le\int_{\substack{a_1<b_1\\ a_1,b_1\in B_{i_1}}}\cdots\int_{\substack{a_k<b_k\\ a_k,b_k\in B_{i_k}}}(Cz)^k\prod_{j=1}^{k}K(a_j-b_{j-1})\mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a_j,b_j],Y}f((Y_s-Y_{(i_j-1)L})_{s\in B_{i_j}})\big]\prod_{j=1}^{k}\mathrm{d}a_j\,\mathrm{d}b_j,
\end{aligned} \tag{3.10}$$

where we used the independence of $(Y_s - Y_{(i-1)L})_{s\in B_i}$, $i \in \mathbb{N}$. The goal is to show that

Lemma 3.1 *Assume $\rho > 0$. For every $\epsilon > 0$ and $\delta > 0$, we can find suitable choices of $H_L(\cdot)$ and M in (3.7), such that for all L sufficiently large, and for all $z \in (1, 1 + L^{-1}]$, $0 \le a < (1 - 3\epsilon)L$, and $c > L$, we have*

$$\begin{aligned}
\int_{a+\epsilon L}^{(1-\epsilon)L}\mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y}f((Y_s)_{s\in[0,L]})\big]db &\le \delta\int_a^L P(b-a)db, &(3.11)\\
\int_{a+\epsilon L}^{(1-\epsilon)L}\mathbb{E}_0^Y\big[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y}f((Y_s)_{s\in[0,L]})\big]K(c-b)db &\le \delta\int_a^L P(b-a)K(c-b)db, &(3.12)
\end{aligned}$$

where

$$P(t) = \sum_{m=1}^{\infty}\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t}\cdots\int\prod_{i=1}^{m}K(\sigma_i-\sigma_{i-1})\prod_{i=1}^{m-1}d\sigma_i, \tag{3.13}$$

where the term for $m = 1$ is defined to be $K(t)$.

Lemma 3.1 implies the following:

Lemma 3.2 *Assume $\rho > 0$. For every $\eta > 0$, we can find suitable choices of $H_L(\cdot)$ and M in (3.7), such that for all L sufficiently large, and for all $z \in (1, 1+L^{-1}]$, $m \in \mathbb{N}$, and $I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \cdots, m\}$, we have*

$$\mathbb{E}_0^Y \big[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \big] \le C_L \prod_{j=1}^{k} \frac{\eta}{(i_j - i_{j-1})^{\frac{3}{2}}} \tag{3.14}$$

for some $C_L > 1$ depending only on L.

By (3.4) and (3.5), Lemma 3.2 implies that uniformly in $t = mL$, $m \in \mathbb{N}$, we have

$$\mathbb{E}_0^Y \big[(\mathcal{Z}_{t,Y}^{z})^\gamma \big] \le \sum_{k=0}^{\infty} \sum_{\substack{I \subset \mathbb{N} \\ |I|=k}} \mathbb{E}_0^Y \big[(\mathcal{Z}_{t,Y}^{z,I})^\gamma \big] \le \sum_{k=0}^{\infty} \sum_{I=\{i_1<\cdots<i_k\}} C_L^\gamma \prod_{j=1}^{k} \frac{\eta^\gamma 2^{(1-\gamma)}}{(i_j - i_{j-1})^{\frac{3\gamma}{2}}} \le C_L^\gamma \sum_{k=0}^{\infty} \Big(\sum_{n=1}^{\infty} \frac{\eta^\gamma 2^{1-\gamma}}{n^{\frac{3\gamma}{2}}} \Big)^k,$$

which is finite if we choose $\gamma \in (2/3, 1)$, and $\eta > 0$ sufficiently small such that $\sum_{n=1}^{\infty} \frac{\eta^\gamma 2^{1-\gamma}}{n^{\frac{3\gamma}{2}}} < 1$. By the monotonicity of $\mathcal{Z}_{t,Y}^{z}$ in t, this implies (3.1) and hence Theorem 1.2.

The key is therefore Lemma 3.1, which is the model dependent part and whose proof will be the focus of the rest of this paper. The new idea developed in [L09] and [GLT09] to bound quantities like $\mathbb{E}_0^Y[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f((Y_s)_{s\in[0,L]})]$ is to use the renewal representation (2.3) to write

$$\begin{aligned}
&\mathbb{E}_0^Y \Big[\mathcal{Z}_{[a,b],Y}^{z,\mathrm{pin}} f((Y_s)_{s\in[0,L]}) \Big] \\
&= \sum_{k=1}^{\infty} \int\!\!\cdots\!\!\int_{\sigma_0=a<\cdots<\sigma_k=b} z^k \prod_{i=1}^{k} K(\sigma_i - \sigma_{i-1}) \mathbb{E}_0^Y \Big[f((Y_s)_{s\in[0,L]}) \prod_{i=1}^{k} W(\sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}}) \Big] \prod_{i=1}^{k-1} \mathrm{d}\sigma_i \\
&= \sum_{k=1}^{\infty} \int\!\!\cdots\!\!\int_{\sigma_0=a<\cdots<\sigma_k=b} z^k \prod_{i=1}^{k} K(\sigma_i - \sigma_{i-1}) \mathbb{E}_0^{Y^\sigma} [f((Y_s^\sigma)_{s\in[0,L]})] \prod_{i=1}^{k-1} \mathrm{d}\sigma_i,
\end{aligned} \tag{3.15}$$

where $\prod_{i=1}^{k} W(\sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}})$ has been interpreted as the density for a change of measure for Y, and $\mathbb{E}_0^{Y^\sigma}[\cdot]$ denotes expectation with respect to a random path $(Y_s^\sigma)_{0\le s\le t}$ whose law is absolutely continuous with respect to that of $(Y_s)_{0\le s\le t}$ with density $\prod_{i=1}^{k} W(\sigma_i - \sigma_{i-1}, Y_{\sigma_i - \sigma_{i-1}})$. Recall the form of f in (3.7): the key point is to choose the functional H_L such that for typical realizations of σ and Y^σ, $H_L((Y_s^\sigma)_{s\in[0,L]})$ is much larger than typical values of $H_L\big((Y_s)_{s\in[0,L]}\big)$. Then in (3.7), we can choose M large such that $\epsilon_M << 1$ and $\mathbb{E}_0^{Y^\sigma}[f((Y_s^\sigma)_{s\in[0,1]})] << 1$. The factor z^k in (3.15) can be bounded by a constant of order 1 if $z \in (1, 1+L^{-1}]$, since conditioned on the renewal process σ with $a < b \in \sigma$, the number of renewal returns in $[a, b]$ is typically of the order $\sqrt{b-a} \le \sqrt{L}$.

The above procedure applies to general weighted renewal processes in random environments, whose partition functions can be represented in the form of (2.3) and (2.1), where given a random environment $(\Omega_s)_{s\ge0}$ with stationary independent increments and a renewal configuration $\sigma := \{\sigma_0 = 0 < \sigma_1 < \cdots\}$, each two consecutive renewal times $\sigma_i < \sigma_{i+1}$ give rise to a weight factor $zW(\sigma_{i+1} - \sigma_i, (\Omega_s - \Omega_{\sigma_i})_{\sigma_i<s\le\sigma_{i+1}})$. See e.g. [BS09, Section 1.3] for a more detailed exposition on how random pinning, random walk pinning, and copolymer models can all be seen as renewal processes in random environments with different weight factors W. With proper normalization, $W(\sigma_{i+1} - \sigma_i, (\Omega_s - \Omega_{\sigma_i})_{\sigma_i<s\le\sigma_{i+1}})$ can always be interpreted as a change of measure for the disorder Ω.

4 Mean and variance of $H_L(Y)$ and $H_L(Y^\sigma)$

We will now choose the functional $H_L(\cdot)$ in (3.7), state its essential properties, and briefly outline how these properties may lead to Lemma 3.1. Given a renewal configuration $\sigma := \{\sigma_0 = a < \cdots < \sigma_k = b\}$, the new disorder random walk Y^σ introduced in (3.15) has heuristically smaller fluctuations than Y due to the density $\prod W(\sigma_i - \sigma_{i-1}, Y_{\sigma_i} - Y_{\sigma_{i-1}}) = \prod \frac{p^X(Y_{\sigma_i} - Y_{\sigma_{i-1}})}{p^{X-Y}(\sigma_i - \sigma_{i-1})}$ which favors values of Y_{σ_i} that are close to $Y_{\sigma_{i-1}}$. A natural choice for H_L in (3.7) is then the following. Fix $e << A_1 << A_2 < \infty$, and define

$$H_L(Y) = H_L((Y_s - Y_0)_{0\le s\le L}) := \iint\limits_{\substack{0<r<s<L\\ A_1<s-r<A_2}} \frac{1_{\{Y_r = Y_s\}}}{\log(s-r)} dr ds. \tag{4.1}$$

We will choose A_1 large in order to apply the local central limit theorem, and A_2 large but finite to ensure boundedness. We have the following bound on the mean and variance of $H_L(Y)$.

Lemma 4.1 *Let Y be as in Theorem 1.1, and let $H_L(Y)$ be defined as in (4.1). Then*

$$\mathbb{E}_0^Y[H_L(Y)] = \iint\limits_{\substack{0<r<s<L\\ A_1<s-r<A_2}} \frac{p_{\rho(s-r)}(0)}{\log(s-r)} dr ds \le L \int_{A_1}^{A_2} \frac{p_{\rho r}(0)}{\log r} dr, \tag{4.2}$$

and there exists some $0 < C < \infty$ such that uniformly for all $e < A_1 < A_2 < \infty$,

$$\mathrm{Var}(H_L(Y)) \le C\rho^{-3} L. \tag{4.3}$$

To show that in (3.15), $\mathbb{E}_0^{Y^\sigma}[f((Y_s^\sigma)_{s\in[0,L]})]$ is small for typical realizations of σ, it then suffices to show that for typical realizations of σ and $(Y_s^\sigma)_{s\in[0,L]}$, $H_L(Y^\sigma) > \mathbb{E}_0^Y[H_L(Y)] + C_L\sqrt{L}$, where C_L can be made arbitrarily large by choosing L large. Thus we need to bound the mean and variance of $H_L(Y^\sigma)$ conditioned on σ. Recall that given $\sigma = \{\sigma_0 = a < \sigma_1 < \cdots < \sigma_k = b\} \subset [0, L]$, the law of $(Y_s^\sigma)_{s\ge 0}$ is absolutely continuous with respect to the law of $(Y_s)_{s\ge 0}$ with density $\prod_{i=1}^k \frac{p^X_{\sigma_i - \sigma_{i-1}}(Y_{\sigma_i} - Y_{\sigma_{i-1}})}{p^{X-Y}_{\sigma_i - \sigma_{i-1}}(0)}$. It is thus problematic to get direct bounds on the mean and variance of $H_L(Y^\sigma)$ due to the dependency structure of Y^σ. Therefore, we decompose $H_L(Y^\sigma)$ in (4.1) according to whether or not the variables of integration $r < s$ satisfy $(r, s) \cap \{\sigma_0 = a < \cdots < \sigma_k = b\} = \emptyset$. Namely,

$$H_L(Y^\sigma) = H^{\mathrm{int}}_{[0,a]}(Y^\sigma) + H^{\mathrm{int}}_{[b,L]}(Y^\sigma) + \sum_{i=1}^k H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) + H^{\mathrm{ext}}_{[0,a]}(Y^\sigma) + \sum_{i=1}^k H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) - C_{\sigma,Y^\sigma}, \tag{4.4}$$

where for any $s < t$,

$$\begin{aligned} H^{\mathrm{int}}_{[s,t]}(Y^\sigma) &= \iint\limits_{\substack{s<s_1<s_2<t\\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1} = Y^\sigma_{s_2}\}}}{\log(s_2 - s_1)} ds_2 ds_1,\\ H^{\mathrm{ext}}_{[s,t]}(Y^\sigma) &= \iint\limits_{\substack{s<s_1<t<s_2\\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1} = Y^\sigma_{s_2}\}}}{\log(s_2 - s_1)} ds_2 ds_1, \end{aligned} \tag{4.5}$$

and

$$C_{\sigma,Y^\sigma} = \iint\limits_{\substack{0<s_1<\sigma_k, s_2>L\\ A_1<s_2-s_1<A_2}} \frac{1_{\{Y^\sigma_{s_1} = Y^\sigma_{s_2}\}}}{\log(s_2 - s_1)} ds_2 ds_1$$

arises because $H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}$ may include pair correlation terms $1_{\{Y^\sigma_{s_1}=Y^\sigma_{s_2}\}}$ with $s_1 < L < s_2$, which is excluded in the definition of H_L. Note that

$$C_{\sigma,Y^\sigma} \le A_2^2 \qquad \text{and} \qquad H^{\mathrm{ext}}_{[s,t]}(Y^\sigma) \le A_2^2 \qquad \text{for all } s<t,\ \sigma, \text{ and } Y^\sigma. \tag{4.6}$$

Conditioned on σ, for any two consecutive renewal times $\sigma_i < \sigma_{i+1}$, we then have the following bounds on the mean of $H^{\mathrm{ext}}_{[\sigma_i,\sigma_{i+1}]}(Y^\sigma)$ and $H^{\mathrm{int}}_{[\sigma_i,\sigma_{i+1}]}(Y^\sigma)$, and the variance of $H^{\mathrm{int}}_{[\sigma_i,\sigma_{i+1}]}(Y^\sigma)$.

Lemma 4.2 *For any $e < A_1 < A_2 < \infty$ and $\sigma := \{\sigma_0 = a < \sigma_1 < \cdots \sigma_k = b\} \subset (0, L]$, we have*

$$\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)] - \mathbb{E}_0^{Y}[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)] > 0, \qquad 1 \le i \le k, \tag{4.7}$$

$$\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)] - \mathbb{E}_0^{Y}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)] > 0, \qquad 1 \le i \le k. \tag{4.8}$$

If we now let $\mathbb{E}^\sigma_a[\cdot]$ denote expectation w.r.t. a renewal process σ conditioned on $\sigma_0 = a \in \sigma$, and let Y^σ have a law which is absolutely continuous w.r.t. the law of Y with density $\frac{p^X_{\sigma_1}(Y_{\sigma_1}-Y_{\sigma_0})}{p^{X-Y}_{\sigma_1-\sigma_0}(0)}$, then for any $A_1 > 0$ and $\lambda > 0$, we can choose A_2 sufficiently large such that

$$\mathbb{E}^\sigma_a\big[\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)] - \mathbb{E}_0^{Y}[H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y)]\big] > \lambda. \tag{4.9}$$

Furthermore,

$$\mathrm{Var}(H^{\mathrm{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)|\sigma) \le C\rho^{-3}(\sigma_1 - \sigma_0), \tag{4.10}$$

where $\mathrm{Var}(\cdot|\sigma)$ denotes variance w.r.t. the law of Y^σ conditioned on σ, and C is uniform in A_1 and A_2.

Our basic strategy is to apply Lemma 4.2 with the ergodic theorem. We sketch the key points here. If we only condition on $\sigma_0 = a \in \sigma$ and define Y^σ accordingly by changing the measure of Y independently on each renewal interval (σ_{i-1}, σ_i), then $\big(H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)\big)_{i\in\mathbb{N}}$ is an i.i.d. sequence while $\big(H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)\big)_{i\in\mathbb{N}}$ is an ergodic sequence with respect to the joint law of σ and Y^σ. We compare the decomposition (4.4) with the corresponding decomposition for $\mathbb{E}_0^Y[H_L(Y)]$. Applying the ergodic theorem to the sequence $\big(H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) - \mathbb{E}_0^Y[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)]\big)_{i\in\mathbb{N}}$ together with (4.7) shows that for typical realizations of σ and Y^σ and for large L, $\sum_{i=1}^{|\sigma\cap(a,L]|} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ dominates $\sum_{i=1}^{|\sigma\cap(a,L]|} \mathbb{E}_0^Y[H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)]$. The law of large numbers applied to the sequence $\big(\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)] - \mathbb{E}_0^Y[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)]\big)_{i\in\mathbb{N}}$ together with (4.9) shows that for typical realizations of σ,

$$\sum_{i=1}^{|\sigma\cap(a,L]|} \big(\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)] - \mathbb{E}_0^Y[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)]\big) \ge \lambda|\sigma\cap(a,L]|,$$

where $|\sigma\cap(a,L]|$ is typically of the order $\sqrt{L}$ if $L-a$ is of order L. This effectively shifts the mean of $H_L(Y^\sigma)$ upward from $\mathbb{E}_0^Y[H_L(Y)]$ by an order of $\lambda\sqrt{L}$. On the other hand, Lemma 4.2 (4.10) shows that conditional on σ, the fluctuation of $\sum_{i=1}^{|\sigma\cap(a,L]|} H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ is only of the order $\sqrt{L}$. Therefore by choosing $\lambda > 0$ sufficiently large and by choosing L large, we are guaranteed that typically $H_L(Y^\sigma)$ takes on much larger values than the typical values of $H_L(Y)$.

5 Proof of Lemma 3.1

We now prove Lemma 3.1 using the functional H_L defined in (4.1) and Lemmas 4.1 and 4.2. We remark that Lemma 3.1 is the analogue of [GLT09, Lemma 3.1] formulated for the discrete time random pinning model. The main difference is that [GLT09, Lemma 3.1] involves a comparison

of the integrands on both sides of (3.12) for each b with $b-a \geq \epsilon L$. Our integral formulation of (3.12) allows us to reduce more easily estimates involving renewal configurations pinned at two points $a < b$ to renewal configurations pinned only at a, for which we can apply the law of large numbers and the ergodic theorem. We need (3.11) because we are studying $\mathcal{Z}^z_{t,Y}$, instead of the constrained partition function $\mathcal{Z}^{z,\mathrm{pin}}_{t,Y}$ as in [GLT09]. First we establish a variant of Lemma 3.1 which corresponds to renewal configurations pinned only at one point a.

Lemma 5.1 *For every $\epsilon > 0$ and $\delta > 0$, we can find a suitable choice of $A_1 < A_2$ in the definition of H_L in (4.1) and a suitable choice of $M > 0$ in (3.7), such that for all L sufficiently large, and for all $z \in (1, 1+L^{-1}]$ and $0 \leq a < (1-3\epsilon)L$, we have*

$$\int_a^L \mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]}) \right] K([L-b,\infty))db \leq \delta, \tag{5.1}$$

where $K([x,\infty)) = \int_x^\infty K(t)dt$.

Proof. Let $\sigma = \{\sigma_0 = a < \sigma_1 < \cdots\}$ be a renewal process on $[a,\infty)$ with renewal time distribution $K(t)dt$, and denote $k = k(\sigma, L) = |\sigma \cap (a, L]|$ and expectation w.r.t. σ by $\mathbb{E}^\sigma_a[\cdot]$. Let $(\tilde{Y}^\sigma_s)_{s\geq 0}$ be defined such that $\tilde{Y}^\sigma_s = Y_s$ on $[0,a]$, and for each $i \in \mathbb{N}$, $(\tilde{Y}^\sigma_s - \tilde{Y}^\sigma_{\sigma_{i-1}})_{\sigma_{i-1}\leq s\leq \sigma_i}$ is independently distributed with a law that is absolutely continuous w.r.t. the law of $(Y_s - Y_{\sigma_{i-1}})_{\sigma_{i-1}\leq s\leq\sigma_i}$ with density $\frac{p^X_{\sigma_i-\sigma_{i-1}}(Y_{\sigma_i}-Y_{\sigma_{i-1}})}{p^{X-Y}_{\sigma_i-\sigma_{i-1}}(0)}$. Given L, we define $Y^\sigma_s = \tilde{Y}^\sigma_s$ for $s \in [0,\sigma_k]$, and $Y^\sigma_s - Y^\sigma_{\sigma_k} = Y_s - Y_{\sigma_k}$ for $s \geq \sigma_k$. By (3.15), we may rewrite the LHS of (5.1) as

$$\begin{aligned}
&\int_a^L \mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]}) \right] K([L-b,\infty))db = \mathbb{E}^\sigma_a \Big[1_{\{k\geq 1\}} z^k \mathbb{E}_0^{Y^\sigma} \left[f((Y^\sigma_s)_{s\in[0,L]}) \right] \Big] \\
\leq\ & \mathbb{E}^\sigma_a[(1+L^{-1})^k 1_{\{k>L\}}] + C\mathbb{E}^\sigma_a \left[\mathbb{E}_0^{Y^\sigma} \left[f((Y^\sigma_s)_{s\in[0,L]}) \right] \right] \\
\leq\ & C_1 e^{-C_2 L} + C\mathbb{P}^\sigma_a \mathbb{P}_0^{Y^\sigma}(H_L(Y^\sigma) \leq M) + C\epsilon_M \mathbb{P}^\sigma_a \mathbb{P}_0^{Y^\sigma}(H_L(Y^\sigma) > M),
\end{aligned} \tag{5.2}$$

where the bound for $\mathbb{E}^\sigma_a[(1+L^{-1})^k 1_{\{k>L\}}]$ follows from standard large deviation estimates for the i.i.d. random variables $(\sigma_i - \sigma_{i-1})_{i\in\mathbb{N}}$.

Recall the decomposition of $H_L(Y^\sigma)$ in (4.4). By construction, $\big(H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\big)_{i\in\mathbb{N}}$ is an i.i.d. sequence, while $\big(H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)\big)_{i\in\mathbb{N}}$ is an ergodic sequence with respect to the joint law of σ and $\tilde{Y}^\sigma$. Note that by the coupling between $\tilde{Y}^\sigma$ and Y^σ and by the same reasoning as in (4.6), $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)$ and $\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)$ differ by at most A_2^2. Therefore by the pointwise ergodic theorem and Lemma 4.2 (4.7), we have for fixed $A_1 < A_2$

$$\lim_{L\to\infty} \inf_{a\in[0,(1-2\epsilon)L]} \mathbb{P}^\sigma_a \mathbb{P}_0^{Y^\sigma} \Big(\sum_{i=1}^{k(\sigma,L)} H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y [H^{\mathrm{ext}}_{[\sigma_{i-1},\sigma_i]}(Y)] > 0 \Big) = 1. \tag{5.3}$$

Let us denote the event in $\mathbb{P}^\sigma_a \mathbb{P}_0^{Y^\sigma}(\cdot)$ above by $E_{(5.3)}$. Similarly, by Lemma 4.2 (4.8)–(4.9) and the law of large numbers, for any $\lambda > 0$, we can choose $A_1 < A_2$ such that

$$\lim_{L\to\infty} \inf_{a\in[0,(1-2\epsilon)L]} \mathbb{P}^\sigma_a \Big(\sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^{Y^\sigma} [H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)] - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y [H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)] > \lambda k(\sigma,L) \Big) = 1. \tag{5.4}$$

We will denote the event in $\mathbb{P}^\sigma_a(\cdot)$ above by $E_{(5.4)}$. We remark that we could not apply the law of large numbers directly to $(H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma) - \mathbb{E}_0^Y[H^{\mathrm{int}}_{[\sigma_{i-1},\sigma_i]}(Y)])_{i\in\mathbb{N}}$ due to the lack of control on its

L_1 norm with respect to the joint law of σ and $\tilde{Y}^\sigma$. However, (4.8) allows us to apply the law of large numbers to $(\mathbb{E}_0^{\tilde{Y}^\sigma}[H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(\tilde{Y}^\sigma)] - \mathbb{E}_0^Y[H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y)])_{i\in\mathbb{N}}$.

By Lemma 4.2 (4.10) and the independence of $\big(H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)\big)_{i\in\mathbb{N}}$, for any realization of σ, we have

$$\mathrm{Var}\Big(\sum_{i=1}^{k(\sigma,L)} H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma)\big|\sigma\Big) \le C\rho^{-3}L,$$

where C is independent of $A_1 < A_2$ in the definition of H_L. Therefore conditioned on σ for which the event $E_{(5.4)}$ holds, and conditioned on $k(\sigma,L) > h\sqrt{L}$ for some $h>0$ to be determined later, by Markov inequality, we have

$$\mathbb{P}_0^{Y^\sigma}\Big(\sum_{i=1}^{k(\sigma,L)} H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y[H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y)] < \frac{4\lambda h\sqrt{L}}{5}\Big) \le \frac{25C}{\rho^3\lambda^2h^2},$$

and hence

$$\begin{aligned}
&\mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}\Big(\sum_{i=1}^{k(\sigma,L)} H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y^\sigma) - \sum_{i=1}^{k(\sigma,L)} \mathbb{E}_0^Y[H^{\rm int}_{[\sigma_{i-1},\sigma_i]}(Y)] < \frac{4\lambda h\sqrt{L}}{5}\Big)\\
\le\ & \mathbb{P}_a^\sigma(k\le h\sqrt{L}) + \mathbb{P}_a^\sigma(E^c_{(5.4)}) + \frac{25C}{\rho^3\lambda^2h^2}.
\end{aligned} \tag{5.5}$$

In the decomposition of $H_L(Y^\sigma)$ in (4.4), by (4.6), we have

$$H^{\rm ext}_{[0,a]}(Y^\sigma) - \mathbb{E}_0^Y[H^{\rm ext}_{[0,a]}(Y)] - C_{\sigma,Y^\sigma} + \mathbb{E}_0^Y[C_{\sigma,Y}] \ge -2A_2^2, \tag{5.6}$$

where $C_{\sigma,Y}$ is defined exactly as C_{σ,Y^σ} with Y^σ replaced by Y. By the same calculation as in the proof of Lemma 4.1 (4.3), we have $\mathrm{Var}(H^{\rm int}_{[0,a]}(Y)) \le C\rho^{-3}L$ and $\mathrm{Var}(H^{\rm int}_{[b,L]}(Y)) \le C\rho^{-3}L$. Since by construction, Y^σ have the same increments as Y on $[0,a]$ and $[b,L]$, we have

$$\begin{aligned}
\mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}\Big(H^{\rm int}_{[0,a]}(Y^\sigma) - \mathbb{E}_0^Y[H^{\rm int}_{[0,a]}(Y)] \ &\le\ -\frac{\lambda h\sqrt{L}}{5}\Big) \le \frac{25C}{\rho^3\lambda^2h^2},\\
\mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}\Big(H^{\rm int}_{[b,L]}(Y^\sigma) - \mathbb{E}_0^Y[H^{\rm int}_{[b,L]}(Y)] \ &\le\ -\frac{\lambda h\sqrt{L}}{5}\Big) \le \frac{25C}{\rho^3\lambda^2h^2}.
\end{aligned} \tag{5.7}$$

Now in (3.7), we let $M = \frac{\lambda h\sqrt{L}}{5}$ with L large such that $M \ge 2A^2$. Then combining (5.2)–(5.7), we obtain

$$\mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}(H_L(Y^\sigma)\le M) \le \mathbb{P}_a^\sigma\mathbb{P}_0^{Y^\sigma}(E^c_{(5.3)}) + \mathbb{P}_a^\sigma(E^c_{(5.4)}) + \mathbb{P}_a^\sigma(k\le h\sqrt{L}) + \frac{75C}{\rho^3\lambda^2h^2},$$

while by (3.8) and Lemma 4.1 (4.3),

$$\epsilon_M = \mathbb{P}_0^Y(H_L(Y) > M)^{\frac{1-\gamma}{\gamma}} \le \big(\frac{25C}{\rho^3\lambda^2h^2}\big)^{\frac{1-\gamma}{\gamma}}.$$

For each $\delta>0$, because $L-a\ge 3\epsilon L$, by Lemma A.2, we can choose $h>0$ sufficiently small such that $\mathbb{P}_a^\sigma(k\le h\sqrt{L}) < \delta/2$ for all L large. Then by sequentially choosing λ large, A_2 large, and finally L sufficiently large, the sum of the terms in (5.2) can be bounded by δ, and we are done. ∎

Proof of Lemma 3.1. The deduction of Lemma 3.1 from Lemma 5.1 is model independent and depends only on $K(\cdot)$. Since $K(t)\sim\frac{C}{t^{\frac{3}{2}}}$, we have $K([t,\infty)) = \int_t^\infty K(s)ds \sim \frac{C}{\sqrt{t}}$. By Lemma A.1,

we also have $P(t) \sim \frac{C}{\sqrt{t}}$. Therefore, given $\epsilon > 0$ and L large, there exist C_1 and C_2 depending only on $\epsilon > 0$, such that uniformly for all $\epsilon L \le a + \epsilon L \le b_1, b_2 \le (1-\epsilon)L$ and $c > L$, we have

$$C_1 \le \frac{P(b_1 - a)}{P(b_2 - a)} \le C_2, \qquad C_1 \le \frac{K(c - b_1)}{K(c - b_2)} \le C_2. \tag{5.8}$$

Under the assumptions of Lemma 3.1, by Lemma 5.1, we have

$$\delta \ge \int_{a+\epsilon L}^{(1-\epsilon)L} \frac{\mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})\right]}{P(b-a)} P(b-a)K([L-b,\infty))db \ge \frac{C}{L}\int_{a+\epsilon L}^{(1-\epsilon)L} \frac{\mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})\right]}{P(b-a)} db,$$

where C depends only on ϵ and $K(\cdot)$. Together with (5.8), this implies that

$$\begin{aligned}
& \int_{a+\epsilon L}^{(1-\epsilon)L} \mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})\right] K(c-b)db \\
= & \int_{a+\epsilon L}^{(1-\epsilon)L} \frac{\mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})\right]}{P(b-a)} P(b-a)K(c-b)db \\
= & \int_{a+\epsilon L}^{(1-\epsilon)L} \frac{\mathbb{E}_0^Y \left[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y} f((Y_s)_{s\in[0,L]})\right]}{P(b-a)} \frac{P(b-a)K(c-b)\,db}{\int_{a+\epsilon L}^{(1-\epsilon)L} P(b_2-a)K(c-b_2)db_2} \int_{a+\epsilon L}^{(1-\epsilon)L} P(b-a)K(c-b)db \\
\le & \left(\frac{C_2}{C_1}\right)^2 \frac{1}{(1-\epsilon)L - (a+\epsilon L)} \frac{\delta L}{C} \int_a^L P(b-a)K(c-b)db \le \frac{\delta C_2^2}{\epsilon C C_1^2} \int_a^L P(b-a)K(c-b)db,
\end{aligned}$$

where we used the assumption that $a \le (1-3\epsilon)L$. Since given $\epsilon > 0$, we can choose $\delta > 0$ arbitrarily small by Lemma 5.1, (3.12) then follows. The proof of (3.11) is similar. ∎

6 Proof of Lemma 3.2

The deduction of Lemma 3.2 from Lemma 3.1 is model independent. Part of the proof is similar to its discrete time analogue (see e.g. the proofs of Proposition 2.3 and Lemma 2.4 in [GLT09]), with the main difference being that in the integrals in (3.11)–(3.12), we have excluded not only contributions from $b \in [a, a+\epsilon L]$, but also from $b \in [L-\epsilon L, L]$. The latter requires new bounds.

First note that by (2.3), for any $0 < a < b < L$ and $z \in [1, 1+L^{-1}]$, we have

$$\begin{aligned}
\mathbb{E}_0^Y[\mathcal{Z}^{z,\mathrm{pin}}_{[a,b],Y}] &= \sum_{m=1}^{\infty} \int\cdots\int_{\sigma_0=a<\sigma_1\cdots<\sigma_m=b} z^m \prod_{i=1}^m K(\sigma_i - \sigma_{i-1})d\sigma_1\cdots d\sigma_{m-1} \\
&= z(G^{X-Y})^{-1}\mathbb{E}_{0,0}^{X,Y}\left[e^{z(G^{X-Y})^{-1}\int_a^b 1_{\{X_s=Y_s\}}ds} 1_{\{X_b=Y_b\}} \big| X_a = Y_a\right] \\
&\le z(G^{X-Y})^{-1}\mathbb{E}_{0,0}^{X,Y}\left[e^{(G^{X-Y})^{-1}(1+\int_a^b 1_{\{X_s=Y_s\}}ds)} 1_{\{X_b=Y_b\}} \big| X_a = Y_a\right] \\
&= ze^{(G^{X-Y})^{-1}} P(b-a) \le CP(b-a),
\end{aligned} \tag{6.1}$$

where $P(t)$ is defined in (3.13), and $C > 0$ is uniform in $L \ge 1$ and Y's jump rate $\rho \in [0,1]$. For the equality in the last line set $z = 1$, i.e. $\beta = 1/G^{X-Y}$, in (2.3) and average over Y. Let

$I = \{i_1 < i_2 < \cdots < i_k\} \subset \{1, \cdots, m\}$ for some $m \in \mathbb{N}$. Then by (3.10), Lemma 3.1, and (6.1),

$$\begin{aligned} &\mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right] \qquad (6.2) \\ \leq &\int_{\substack{a_j < b_j \in B_{i_j} \\ 1 \leq j \leq k}} \cdots \int (Cz)^k \prod_{j=1}^k K(a_j - b_{j-1}) (C 1_{\{b_j - a_j < \epsilon L\}} + C 1_{\{b_j \geq (i_j - \epsilon)L\}} + \delta) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j \, \mathrm{d}b_j. \end{aligned}$$

We will show that for any $\eta > 0$, if $\epsilon > 0$ and $\delta > 0$ are chosen sufficiently small, then we have

$$\mathbb{E}_0^Y \left[f_I(Y) \mathcal{Z}_{t,Y}^{z,I} \right] \leq \int_{\substack{a_j < b_j \in B_{i_j} \\ 1 \leq j \leq k}} \cdots \int (Cz\eta)^k \prod_{j=1}^k K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j \, \mathrm{d}b_j. \qquad (6.3)$$

Lemma 3.2 then follows from the bound

$$P_L(I) := \int_{\substack{a_j < b_j \in B_{i_j} \\ 1 \leq j \leq k}} \cdots \int \prod_{j=1}^k K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^k \mathrm{d}a_j \, \mathrm{d}b_j \leq C_L \prod_{j=1}^{|I|} \frac{C}{(i_j - i_{j-1})^{\frac{3}{2}}}, \qquad (6.4)$$

where C_L depends only on L.

First we give a proof of (6.4), which is similar to its discrete time counterpart, [GLT09, Lemma 2.4]. We partition I into blocks of consecutive integers $\{u_1, u_1 + 1, \cdots, v_1\}$, $\{u_2, u_2 + 1, \cdots, v_2\}$, $\ldots$, $\{u_l, u_l + 1, \cdots, v_l\}$, where $u_j - v_{j-1} \geq 2$ for all $2 \leq j \leq l$. When substituting the definition of $P(b_j - a_j)$ in (3.13) into (6.4), the resulting multifold expansion is the probability of a set of renewal configurations, where the variables of integration constitute the renewal configuration σ. By only retaining the constraint that σ intersects B_{u_i} and B_{v_i} for $1 \leq i \leq l$, we obtain

$$P_L(I) \leq \int_{\substack{a_1 < b_1 \\ a_1 \in B_{u_1}, b_1 \in B_{v_1}}} \cdots \int_{\substack{a_l < b_l \\ a_l \in B_{u_l}, b_l \in B_{v_l}}} \prod_{j=1}^l K(a_j - b_{j-1}) P(b_j - a_j) \prod_{j=1}^l \mathrm{d}a_j \, \mathrm{d}b_j, \qquad (6.5)$$

where $b_0 := 0$. We integrate out one pair of variables a_j, b_j at a time. For $j = l$,

$$\iint_{\substack{a_l < b_l \\ a_l \in B_{u_l}, b_l \in B_{v_l}}} K(a_l - b_{l-1}) P(b_l - a_l) db_l da_l \leq \frac{C}{(u_l - v_{l-1})^{\frac{3}{2}} L^{\frac{3}{2}}} \iint_{\substack{a_l < b_l \\ a_l \in B_{u_l}, b_l \in B_{v_l}}} P(b_l - a_l) db_l da_l, \qquad (6.6)$$

since $a_l - b_{l-1} \geq (u_l - v_{l-1} - 1)L \geq L$, and hence $K(a_l - b_{l-1}) \leq \frac{C_1}{(u_l - v_{l-1} - 1)^{\frac{3}{2}} L^{\frac{3}{2}}} \leq \frac{C}{(u_l - v_{l-1})^{\frac{3}{2}} L^{\frac{3}{2}}}$. If $u_l = v_l$, then by Lemma A.1,

$$\iint_{\substack{a_l < b_l \\ a_l, b_l \in B_{u_l}}} P(b_l - a_l) db_l da_l \leq \iint_{\substack{a_l < b_l \\ a_l, b_l \in B_{u_l}}} \frac{C}{\sqrt{b_l - a_l}} db_l da_l \leq C L^{\frac{3}{2}}.$$

If $v_l - u_l = 1$, again by Lemma A.1,

$$\iint_{\substack{a_l < b_l \\ a_l \in B_{u_l}, b_l \in B_{u_l + 1}}} P(b_l - a_l) db_l da_l \leq \iint_{\substack{a_l < b_l \\ a_l \in B_{u_l}, b_l \in B_{u_l + 1}}} \frac{C}{\sqrt{b_l - a_l}} db_l da_l \leq C L^{\frac{3}{2}}.$$

If $v_l - u_l \geq 2$, then Lemma A.1 again leads to a bound of $CL^{\frac{3}{2}}$ for the integral on the RHS of (6.6). Integrating out a_l, b_l in (6.5) thus gives a factor $C(u_l - v_{l-1})^{-\frac{3}{2}}$. Iterating this procedure

then gives the bound in (6.4), where a prefactor $C_L = L^{\frac{3}{2}}$ arises when we integrate out a_1 and b_1 in the case $u_1 = 1$. This proves (6.4).

To deduce (6.3) from (6.2), we first bound the contributions from $C1_{\{b_j - a_j < \epsilon L\}}$. We claim that there exists some $C > 0$ depending only on $K(\cdot)$ and uniform in $\rho \in [0,1]$, such that for all L sufficiently large, $\epsilon \in (0, 1/4)$, and $a \le 0 < L \le b$, we have

$$\int\limits_{\substack{0\le s<t\le L\\ t-s<\epsilon L}} K(s-a)P(t-s)K(b-t)dtds \;\le\; C\sqrt{\epsilon}\int\limits_{0\le s<t\le L} K(s-a)P(t-s)K(b-t)dtds, \tag{6.7}$$

$$\int\limits_{\substack{0\le s<t\le L\\ t-s<\epsilon L}} K(s-a)P(t-s)dtds \;\le\; C\sqrt{\epsilon}\int\limits_{0\le s<t\le L} K(s-a)P(t-s)dtds. \tag{6.8}$$

To prove (6.7), note that either $s \le L/2$ or $s > L/2$ in the integral. Using the fact that $K(t) \sim \frac{C}{t^{\frac{3}{2}}}$ by the local central limit theorem and the fact that $\int_0^t P(s)ds \sim C\sqrt{t}$ by Lemma A.1, we have

$$\int\limits_{\substack{0\le s<t\le L\\ s\le \frac{L}{2}, t-s<\epsilon L}} K(s-a)P(t-s)K(b-t)dtds \le \frac{C\sqrt{\epsilon L}}{(b-\frac{3L}{4})^{\frac{3}{2}}}\int\limits_{0\le s\le L/2} K(s-a)ds \le \frac{C\sqrt{\epsilon L}}{b^{\frac{3}{2}}}\int\limits_{0\le s\le L/2} K(s-a)ds,$$

where we used $b - t > b - 3L/4$ and $b \ge L$. On the other hand,

$$\int\limits_{0\le s<t\le L} K(s-a)P(t-s)K(b-t)dtds \ge \int\limits_{\substack{0\le s\le \frac{L}{2}\\ 0<t-s\le \frac{L}{4}}} K(s-a)P(t-s)K(b-t)dtds \ge \frac{C\sqrt{L}}{b^{\frac{3}{2}}}\int\limits_{0\le s\le L/2} K(s-a)ds.$$

Together with a similar bound for the LHS of (6.7) integrated over $s > L/2$, this implies (6.7). The proof of (6.8) is similar and will be omitted. Substituting (6.7) and (6.8) into (6.2) then gives

$$\mathbb{E}_0^Y\left[f_I(Y)\mathcal{Z}_{t,Y}^{z,I}\right] \le (Cz)^k \int\limits_{\substack{a_j<b_j\in B_{i_j}\\ 1\le j\le k}}\cdots\int \prod_{j=1}^k K(a_j - b_{j-1})(C1_{\{b_j\ge(i_j-\epsilon)L\}} + \tilde{\eta})P(b_j - a_j)\prod_{j=1}^k \mathrm{d}a_j\,\mathrm{d}b_j, \tag{6.9}$$

where $\tilde{\eta} = C\sqrt{\epsilon} + \delta$, which can be made arbitrarily small by choosing ϵ and δ small. By expanding the product $\prod_{j=1}^k (C1_{\{b_j\ge(i_j-\epsilon)L\}} + \tilde{\eta})$, we note that (6.3) follows once we show that there exists some C such that for any $J \subset \{1,\dots,k\}$, we have

$$\int\limits_{\substack{1\le j\le k:\ a_j<b_j\in B_{i_j}\\ j\in J:\ b_j\ge(i_j-\epsilon)L}}\cdots\int \prod_{j=1}^k K(a_j - b_{j-1})P(b_j - a_j)\prod_{j=1}^k \mathrm{d}a_j\,\mathrm{d}b_j \le (C\sqrt{\epsilon})^{|J|}P_L(I), \tag{6.10}$$

where $P_L(I)$ was defined in (6.4).

If $J = \emptyset$, then (6.10) is trivial; otherwise, let l be the largest element in J. It suffices to show that we can replace the indicator $1_{\{b_l\ge(i_l-\epsilon)L\}}$ by the factor $C\sqrt{\epsilon}$. We can then apply the argument inductively to deduce (6.10). There are three cases: either (1) $l = k$; or (2) $i_{l+1} - i_l \ge 2$; or (3) $i_{l+1} - i_l = 1$. For the case $l = k$, it suffices to show that uniformly in $b_{k-1} \in B_{i_{k-1}}$, we have

$$\iint\limits_{\substack{a_k<b_k\in B_{i_k}\\ b_k\ge(i_k-\epsilon)L}} K(a_k - b_{k-1})P(b_k - a_k)da_k db_k \le C\sqrt{\epsilon}\iint\limits_{a_k<b_k\in B_{i_k}} K(a_k - b_{k-1})P(b_k - a_k)da_k db_k. \tag{6.11}$$

Note that by Lemma A.1, uniformly in $u > 0$, we have $\int_u^{u+\epsilon L} P(s)ds \le C\sqrt{\epsilon L}$ for L large. Uniformly in $u > 0$, we also have $\int_u^{u+L} K(s)ds \le 2\int_u^{u+\frac{L}{2}} K(s)ds$. It follows that

$$\iint\limits_{\substack{a_k<b_k\in B_{i_k}\\ b_k\ge(i_k-\epsilon)L}} K(a_k-b_{k-1})P(b_k-a_k)da_kdb_k \le 2C\sqrt{\epsilon L}\int\limits_{(i_k-1)L\le a_k\le(i_k-\frac{1}{2})L} K(a_k-b_{k-1})da_k.$$

On the other hand, by Lemma A.1, $\int_0^t P(s)ds \sim C\sqrt{t}$. Therefore for L sufficiently large,

$$\iint\limits_{a_k<b_k\in B_{i_k}} K(a_k-b_{k-1})P(b_k-a_k)da_kdb_k \ge \int\limits_{\substack{(i_k-1)L\le a_k\le(i_k-\frac{1}{2})L\\ 0\le b_k-a_k\le\frac{L}{2}}} K(a_k-b_{k-1})P(b_k-a_k)da_kdb_k \ge C\sqrt{L}\int\limits_{(i_k-1)L\le a_k\le(i_k-\frac{1}{2})L} K(a_k-b_{k-1})da_k.$$

The above two estimates together imply (6.11).

For case (2), $i_{l+1}-i_l \ge 2$, it suffices to show that uniformly in $b_{l-1}\in B_{i_{l-1}}$ and $a_{l+1}\in B_{i_{l+1}}$, we have

$$\iint\limits_{\substack{a_l<b_l\in B_{i_l}\\ b_l\ge(i_l-\epsilon)L}} K(a_l-b_{l-1})P(b_l-a_l)K(a_{l+1}-b_l)da_ldb_l \le C\sqrt{\epsilon}\iint\limits_{a_l<b_l\in B_{i_l}} K(a_l-b_{l-1})P(b_l-a_l)K(a_{l+1}-b_l)da_ldb_l.$$

This follows from the same proof as for (6.11) once we note that, because $a_{l+1}-b_l \ge L$, uniformly in $s_1,s_2\in B_{i_l}$ and $t_1,t_2\in B_{i_{l+1}}$, we have $C \le \frac{K(t_1-s_1)}{K(t_2-s_2)} \le C^{-1}$ for some $C\in(0,\infty)$ depending only on $K(\cdot)$.

For case (3), $i_{l+1}-i_l = 1$, there are two subcases: either $l+1=k$ or $l+1<k$. We only examine the case $l+1<k$, since the case $l+1=k$ is similar and simpler. To simplify notation, we will shift coordinates and assume $l=1$ and $i_l=1$. Since l is the largest element in J, it suffices to show that uniformly in $b_0\le 0$ and $a_3\ge 2L$, we have

$$\begin{aligned}&\int\limits_{\substack{b_1\ge(1-\epsilon)L\\ 0<a_1<b_1<L<a_2<b_2<2L}}\cdots\int K(a_1-b_0)P(b_1-a_1)K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)da_1db_1da_2db_2\\ \le\ & C\sqrt{\epsilon}\int\limits_{0<a_1<b_1<L<a_2<b_2<2L}\cdots\int K(a_1-b_0)P(b_1-a_1)K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)da_1db_1da_2db_2. \quad (6.12)\end{aligned}$$

By restricting the region of integration to $a_1\in[0,L/4]$, $b_1\in[3L/4,L]$, $a_2\in[L,5L/4]$ and $b_2\in[7L/4,2L]$, and using the fact that $P(t)\sim\frac{C}{\sqrt{t}}$, $K(t)\sim\frac{C}{t^{\frac{3}{2}}}$, $\int_t^\infty K(s)ds\sim\frac{C}{\sqrt{t}}$, we find

$$\begin{aligned}&\int\limits_{0<a_1<b_1<L<a_2<b_2<2L}\cdots\int K(a_1-b_0)P(b_1-a_1)K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)da_1db_1da_2db_2\\ \ge\ & \frac{C}{\sqrt{L}}\int_0^{\frac{L}{4}} K(a_1-b_0)da_1\int_{\frac{7L}{4}}^{2L} K(a_3-b_2)db_2. \quad (6.13)\end{aligned}$$

To upper bound the LHS of (6.12), we claim that uniformly in all $b_1\le L<2L\le a_3$, we have

$$\int_L^{2L}\int_{a_2}^{2L} K(a_2-b_1)P(b_2-a_2)K(a_3-b_2)db_2da_2 \le \frac{C}{\sqrt{L}}\int_L^{\frac{5L}{4}} K(a_2-b_1)da_2\int_{\frac{7L}{4}}^{2L} K(a_3-b_2)db_2, \quad (6.14)$$

and uniformly for all $b_0 \le 0$ and $(1-\epsilon)L < b_1 < L$, we have

$$\int_0^{b_1} K(a_1 - b_0)P(b_1 - a_1)da_1 \le \frac{C}{\sqrt{L}} \int_0^{\frac{L}{4}} K(a_1 - b_0)da_1, \tag{6.15}$$

which when substituted into the LHS of (6.12) imply that

$$\begin{aligned}
&\int_{\substack{b_1 \ge (1-\epsilon)L \\ 0<a_1<b_1<L<a_2<b_2<2L}} \cdots \int K(a_1 - b_0)P(b_1 - a_1)K(a_2 - b_1)P(b_2 - a_2)K(a_3 - b_2)da_1db_1da_2db_2 \\
\le\ & \frac{C}{L} \int_{(1-\epsilon)L}^{L} \int_{L}^{\frac{5L}{4}} K(a_2 - b_1)da_2db_1 \int_0^{\frac{L}{4}} K(a_1 - b_0)da_1 \int_{\frac{7L}{4}}^{2L} K(a_3 - b_2)db_2 \\
\le\ & \frac{C\sqrt{\epsilon}}{\sqrt{L}} \int_0^{\frac{L}{4}} K(a_1 - b_0)da_1 \int_{\frac{7L}{4}}^{2L} K(a_3 - b_2)db_2,
\end{aligned} \tag{6.16}$$

where we used the fact that $\int_L^{\frac{5L}{4}} K(a_2 - b_1)da_2 \le \frac{C}{\sqrt{L-b_1}}$. Together with (6.13), this implies (6.12).

To prove (6.14), note that the bound therein certainly holds if we restrict integration to $a_2 \in [L, \frac{5L}{4}]$ and $b_2 \in [\frac{7L}{4}, 2L]$. If either of the constraints on a_2 and b_2 fails, without loss of generality, say $a_2 \in [\frac{5L}{4}, 2L]$, then because $K(t) \le \frac{C}{t^{\frac{3}{2}}}$ and $\int_0^t P(s)ds \le C\sqrt{t}$, we have

$$\begin{aligned}
&\int_{\frac{5L}{4}}^{2L} \int_{a_2}^{2L} K(a_2 - b_1)P(b_2 - a_2)K(a_3 - b_2)db_2da_2 \le \frac{C\sqrt{L}}{(\frac{5L}{4} - b_1)^{\frac{3}{2}}} \int_L^{2L} K(a_3 - b_2)db_2 \\
\le\ & \frac{C\sqrt{L}}{(\frac{5L}{4} - b_1)^{\frac{3}{2}}} \int_{\frac{7L}{4}}^{2L} K(a_3 - b_2)db_2 \le \frac{C}{\sqrt{L}} \int_L^{\frac{5L}{4}} K(a_2 - b_1)da_2 \int_{\frac{7L}{4}}^{2L} K(a_3 - b_2)db_2,
\end{aligned}$$

since $\int_L^{\frac{5L}{4}} K(a_2 - b_1)da_2 \ge \frac{CL}{4}(\frac{5L}{4} - b_1)^{-\frac{3}{2}}$. This proves (6.14). The proof of (6.15) is similar and will be omitted. This completes the proof of (6.10) as well as of Lemma 3.2. ∎

7 Proof of Lemma 4.1

Note that (4.2) is obvious. For $s \in [0, L]$, let us denote

$$h_L(s, Y) := \int_{\substack{s<t<L \\ A_1<t-s<A_2}} \frac{1_{\{Y_s = Y_t\}}}{\log(t-s)}dt.$$

Then

$$\begin{aligned}\mathrm{Var}(H_L(Y)) &= 2\int_{0<s_1<s_2<L}\big(\mathbb{E}_0^Y[h_L(s_1,Y)h_L(s_2,Y)]-\mathbb{E}_0^Y[h_L(s_1,Y)]\mathbb{E}_0^Y[h_L(s_2,Y)]\big)ds_1ds_2\\ &\le 2\iiiint_{\substack{0<s_1<s_2<L,\,t_1,t_2<L\\ A_1<t_1-s_1,t_2-s_2<A_2}}\frac{\big|\mathbb{P}_0^Y(Y_{s_1}=Y_{t_1},Y_{s_2}=Y_{t_2})-\mathbb{P}_0^Y(Y_{s_1}=Y_{t_1})\mathbb{P}_0^Y(Y_{s_2}=Y_{t_2})\big|}{\log(t_1-s_1)\log(t_2-s_2)}dt_1dt_2ds_1ds_2\\ &\le 2\int_{0<s_1<s_2<L}\phi(s_2-s_1)ds_1ds_2\le 2L\int_0^\infty\phi(w)dw,\end{aligned}$$

where

$$\phi(w) = \int_e^\infty\int_{w+e}^\infty\frac{\big|\mathbb{P}_0^Y(Y_0=Y_{s_1},Y_w=Y_{s_2})-\mathbb{P}_0^Y(Y_0=Y_{s_1})\mathbb{P}_0^Y(Y_w=Y_{s_2})\big|}{\log s_1\log(s_2-w)}ds_2ds_1. \quad (7.1)$$

To prove (4.3), it suffices to show that $\int_0^\infty\phi(w)dw<\infty$.

Note that in (7.1), s_1,s_2 fall into three regions: (0) $0<s_1<w$; (1) $w<s_1<s_2$; (2) $w<s_2<s_1$. In case (0), the integrand in (7.1) is 0 by the independent increment properties of Y. In case (1), let $r_1=s_1-w$ and $r_2=s_2-s_1$, while in case (2) let $r_1=s_2-w$ and $r_2=s_1-s_2$, then

$$\phi(w)=\mathrm{I}(w)+\mathrm{II}(w) \quad (7.2)$$

with

$$\mathrm{I}(w)=\iint_{[0,\infty)^2}1_{\{w+r_1>e,r_1+r_2>e\}}\frac{\big|\mathbb{P}_0^Y(Y_0=Y_{w+r_1},Y_w=Y_{w+r_1+r_2})-p_{\rho(w+r_1)}(0)p_{\rho(r_1+r_2)}(0)\big|}{\log(w+r_1)\log(r_1+r_2)}\,dr_1dr_2,$$

$$\mathrm{II}(w)=\iint_{[0,\infty)^2}1_{\{r_1>e\}}\frac{\big|\mathbb{P}_0^Y(Y_0=Y_{w+r_1+r_2},Y_w=Y_{w+r_1})-p_{\rho(w+r_1+r_2)}(0)p_{\rho r_1}(0)\big|}{\log(w+r_1+r_2)\log r_1}\,dr_1dr_2.$$

We establish (4.3) once we show that there exists $C>0$ such that

$$\begin{aligned}\mathrm{I}(w),\mathrm{II}(w) &\le \frac{C}{\rho^2} && \text{for all } w>0,\\ \mathrm{I}(w),\mathrm{II}(w) &\le \frac{C}{\rho^3 w(\log w)^2} && \text{for all } w>e.\end{aligned} \quad (7.3)$$

In $\mathrm{I}(w)$, by Lemmas B.1 and B.2,

$$\begin{aligned}&\mathbb{P}_0^Y(Y_0=Y_{w+r_1},Y_w=Y_{w+r_1+r_2})\\ =\;&\sum_{x\in\mathbb{Z}^3}p_{\rho w}(x)p_{\rho r_1}(-x)p_{\rho r_2}(x)\le\min\Big\{p_{\rho r_1}(0)p_{\rho r_2}(0),\frac{C}{\rho^3(wr_1+wr_2+r_1r_2)^{\frac32}}\Big\},\end{aligned}$$

from which we easily deduce that $\mathrm{I}(w)\le 2(\int_0^\infty p_{\rho r}(0)dr)^2=2G^2\rho^{-2}$. Similarly, $\mathrm{II}(w)\le 2G^2\rho^{-2}$. On the other hand, by the local central limit theorem, Lemma B.1, we have

$$\mathrm{I}(w)\le\frac{C}{\rho^3}\iint_{[0,\infty)^2}1_{\{w+r_1>e,r_1+r_2>e\}}\frac{\frac{1}{(wr_1+wr_2+r_1r_2)^{\frac32}}+\frac{1}{(w+r_1)^{\frac32}(r_1+r_2)^{\frac32}}}{\log(w+r_1)\log(r_1+r_2)}\,dr_1dr_2. \quad (7.4)$$

Let $r_1 = wt_1$ and $r_2 = wt_2$, and assume $w > e$, then (7.4) becomes

$$\begin{aligned}
\mathrm{I}(w) &\le \frac{C}{\rho^3 w} \iint_{[0,\infty)^2} 1_{\{1+t_1>ew^{-1}, t_1+t_2>ew^{-1}\}} \frac{\frac{1}{(t_1+t_2+t_1t_2)^{\frac{3}{2}}} + \frac{1}{(1+t_1)^{\frac{3}{2}}(t_1+t_2)^{\frac{3}{2}}}}{\log(w(1+t_1))\log(w(t_1+t_2))}\, dt_1 dt_2 \\
&\le \frac{C}{\rho^3 w(\log w)^2} \iint\limits_{\substack{t_1,t_2\ge 0\\ t_1+t_2\ge 1}} \Big(\frac{1}{(t_1+t_2+t_1t_2)^{\frac{3}{2}}} + \frac{1}{(1+t_1)^{\frac{3}{2}}(t_1+t_2)^{\frac{3}{2}}} \Big) dt_1 dt_2 \qquad (7.5) \\
&\quad + \frac{C}{\rho^3 w \log w} \iint\limits_{\substack{t_1,t_2\ge 0\\ t_1+t_2\le 1}} 1_{\{t_1+t_2>ew^{-1}\}} \frac{2}{(t_1+t_2)^{\frac{3}{2}} \log(w(t_1+t_2))} dt_1 dt_2. \qquad (7.6)
\end{aligned}$$

The integral in (7.5) is finite. Letting $y = t_1 + t_2$, the integral in (7.6) equals

$$\begin{aligned}
\int_{\frac{e}{w}}^1 \frac{2}{\sqrt{y}\log(wy)} dy &= \frac{1}{\sqrt{w}} \int_e^w \frac{2}{\sqrt{x}\log x} dx \qquad (7.7) \\
&= \frac{4\sqrt{x}}{\sqrt{w}\log x}\Big|_e^w + \frac{4}{\sqrt{w}} \int_e^w \frac{1}{\sqrt{x}(\log x)^2} dx \le \frac{C}{\log w},
\end{aligned}$$

which proves that $\mathrm{I}(w) \le \frac{C}{\rho^3 w(\log w)^2}$.

In II(w),

$$\mathbb{P}_0^Y(Y_0 = Y_{w+r_1+r_2}, Y_w = Y_{w+r_1}) = \sum_{x\in\mathbb{Z}^3} p_{\rho w}(x) p_{\rho r_1}(0) p_{\rho r_2}(-x) = p_{\rho r_1}(0) p_{\rho(w+r_2)}(0).$$

Therefore

$$\mathrm{II}(w) = \iint\limits_{[0,\infty)^2} 1_{\{r_1>e\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{\log(w+r_1+r_2)\log r_1}\, dr_1 dr_2. \qquad (7.8)$$

We separate the integral in (7.8) according to whether $r_1 > w$ or $r_1 < w$. When $r_1 > w$, we have

$$\begin{aligned}
&\int_w^\infty \int_0^\infty 1_{\{r_1>e\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{\log(w+r_1+r_2)\log r_1} dr_2 dr_1 \\
\le\ & \frac{C\sqrt{w}}{\rho^{\frac{3}{2}}\log w} \int_1^\infty \int_0^\infty 1_{\{t_1>ew^{-1}\}} \frac{p_{\rho w(1+t_2)}(0) + p_{\rho w(1+t_1+t_2)}(0)}{t_1^{\frac{3}{2}}\log(wt_1)}\, dt_2 dt_1 \\
\le\ & \frac{C\sqrt{w}}{\rho^3 \log w} \int_1^\infty \int_0^\infty \frac{\frac{1}{(w(1+t_2))^{\frac{3}{2}}}}{t_1^{\frac{3}{2}}\log w} dt_2 dt_1 \\
\le\ & \frac{C}{\rho^3 w(\log w)^2},
\end{aligned}$$

where we used the local central limit theorem to bound $p_s(0) \le Cs^{-\frac{3}{2}}$ and made the change of

variables $r_1 = wt_1$ and $r_2 = wt_2$. When $0 < r_1 < w$ in (7.8), by Lemma B.3, we have

$$\begin{aligned}
&\int_0^w \int_0^\infty 1_{\{r_1>e\}} \frac{p_{\rho r_1}(0)|p_{\rho(w+r_2)}(0) - p_{\rho(w+r_1+r_2)}(0)|}{\log(w+r_1+r_2)\log r_1} dr_2 dr_1 \\
\leq\ & \frac{C}{\log w} \int_0^w \int_0^\infty 1_{\{r_1>e\}} \frac{p_{\rho r_1}(0) \frac{r_1}{\rho^{\frac{3}{2}}(w+r_2)^{\frac{5}{2}}}}{\log r_1} dr_2 dr_1 \\
\leq\ & \frac{C}{\rho^3 \log w} \int_0^w \int_0^\infty 1_{\{r_1>e\}} \frac{1}{\sqrt{r_1}(w+r_2)^{\frac{5}{2}} \log r_1} dr_2 dr_1 \\
=\ & \frac{C}{\rho^3 w \log w} \int_0^1 \int_0^\infty 1_{\{t_1>ew^{-1}\}} \frac{1}{\sqrt{t_1}(1+t_2)^{\frac{5}{2}} \log(wt_1)} dt_2 dt_1 \\
\leq\ & \frac{C}{\rho^3 w \log w} \int_{\frac{e}{w}}^1 \frac{1}{\sqrt{t_1}\log(wt_1)} dt_1 \leq \frac{C}{\rho^3 w (\log w)^2},
\end{aligned}$$

where the last inequality follows from the same calculation as in (7.7). This proves that $\mathrm{II}(w) \leq \frac{C}{\rho^3 w(\log w)^2}$ and concludes the proof of Lemma 4.1. ∎

8 Proof of Lemma 4.2

Proof of Lemma 4.2 (4.7)–(4.8). By definition, conditioned on σ,

$$\mathbb{E}_0^{Y^\sigma}[H^{\mathrm{ext}}_{[\sigma_i,\sigma_{i+1}]}(Y^\sigma)] - \mathbb{E}_0^{Y}[H^{\mathrm{ext}}_{[\sigma_i,\sigma_{i+1}]}(Y)] = \iint\limits_{\substack{\sigma_i<s_1<\sigma_{i+1}<s_2 \\ A_1<s_2-s_1<A_2}} \frac{\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) - \mathbb{P}(Y_{s_1} = Y_{s_2})}{\log(s_2-s_1)} ds_2 ds_1.$$

To prove (4.7), it suffices to show that

$$\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) > \mathbb{P}(Y_{s_1} = Y_{s_2}).$$

This follows from Lemma B.5. Indeed, we can decompose $Y^\sigma_{s_2} - Y^\sigma_{s_1}$ as the sum of independent random variables $Z_1, Z_2, \cdots, Z_{n+1}$, where n is such that $\sigma_{i+n} \leq s_2 < \sigma_{i+n+1}$, $Z_1 = Y^\sigma_{\sigma_{i+1}} - Y^\sigma_{s_1}$, $Z_j = Y^\sigma_{\sigma_{i+j}} - Y^\sigma_{\sigma_{i+j-1}}$ for $2 \leq i \leq n$, and $Z_{n+1} = Y^\sigma_{s_2} - Y^\sigma_{\sigma_{i+n}}$. From the definition of Y^σ,

$$\mathbb{P}(Z_1 = y) = \frac{\sum_x p^Y_{s_1-\sigma_i}(x) p^Y_{\sigma_{i+1}-s_1}(y) p^X_{\sigma_{i+1}-\sigma_i}(x+y)}{p^{X-Y}_{\sigma_{i+1}-\sigma_i}(0)} = \frac{p_{\rho(\sigma_{i+1}-s_1)}(y) p_{\sigma_{i+1}-\sigma_i+\rho(s_1-\sigma_i)}(y)}{p_{(1+\rho)(\sigma_{i+1}-\sigma_i)}(0)},$$

where we used the fact that X and Y have the same symmetric jump probability kernel with respective rates 1 and ρ. Therefore Z_1 is distributed as $X_{\rho(\sigma_{i+1}-s_1)}$ conditioned on $X_{(1+\rho)(\sigma_{i+1}-\sigma_i)} = 0$. Similarly, Z_j for $2 \leq j \leq n$ is distributed as $X_{\rho(\sigma_{i+j}-\sigma_{i+j-1})}$ conditioned on $X_{(1+\rho)(\sigma_{i+j}-\sigma_{i+j-1})} = 0$, and Z_{n+1} is distributed as $X_{\rho(s_2-\sigma_{i+n})}$ conditioned on $X_{(1+\rho)(\sigma_{i+n+1}-\sigma_{i+n})} = 0$. Therefore Lemma B.5 applies. The proof of (4.8) is analogous and simpler. ∎

Proof of Lemma 4.2 (4.9). Without loss of generality, assume that $\sigma_0 = a = 0$, and let $\sigma_1 = \Delta$. For $0 \leq s_1 \leq s_2 \leq \Delta$, we have

$$\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) = \frac{\sum_{x,y\in\mathbb{Z}^3} p_{\rho s_1}(x) p_{\rho(s_2-s_1)}(0) p_{\rho(\Delta-s_2)}(y) p_\Delta(x+y)}{p_{(1+\rho)\Delta}(0)} = \frac{p_{(1+\rho)\Delta-\rho(s_2-s_1)}(0) p_{\rho(s_2-s_1)}(0)}{p_{(1+\rho)\Delta}(0)}.$$

Therefore, conditioned on $\sigma_0 = 0$ and $\sigma_1 = \Delta$,

$$\mathbb{E}_0^{Y^\sigma}[H^{\text{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)] - \mathbb{E}_0^{Y}[H^{\text{int}}_{[\sigma_0,\sigma_1]}(Y)] = \iint\limits_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}) - \mathbb{P}(Y_{s_1} = Y_{s_2})}{\log(s_2 - s_1)} ds_2 ds_1$$

$$= \iint\limits_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{p_{\rho(s_2-s_1)}(0)(p_{(1+\rho)\Delta-\rho(s_2-s_1)}(0) - p_{(1+\rho)\Delta}(0))}{p_{(1+\rho)\Delta}(0)\log(s_2-s_1)} ds_2 ds_1$$

$$\geq C \iint\limits_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{((1+\rho)\Delta)^{\frac{3}{2}} \frac{\rho(s_2-s_1)}{((1+\rho)\Delta)^{\frac{5}{2}}}}{\rho^{\frac{3}{2}}(s_2-s_1)^{\frac{3}{2}}\log(s_2-s_1)} ds_2 ds_1$$

$$\geq \frac{C}{\Delta\sqrt{\rho}} \iint\limits_{\substack{0<s_1<s_2<\Delta \\ A_1<s_2-s_1<A_2}} \frac{ds_2 ds_1}{\sqrt{s_2-s_1}\log(s_2-s_1)} \geq \frac{C\sqrt{\Delta}}{\sqrt{\rho}\log\Delta} \iint\limits_{\substack{0<t_1<t_2<1 \\ A_1\Delta^{-1}<t_2-t_1<A_2\Delta^{-1}}} \frac{dt_2 dt_1}{\sqrt{t_2-t_1}}, \tag{8.1}$$

where we have applied Lemma B.3, used the local central limit theorem to bound $p_t(0) \leq C_1 t^{-\frac{3}{2}}$ for all $t \geq 0$ and $p_t \geq C_2 t^{-\frac{3}{2}}$ for all $t \geq 1$, and lastly made the change of variable $s_i = \Delta t_i$. We can now take expectation with respect to $\Delta = \sigma_1 - \sigma_0$, which has density $K(t)dt$ with $K(t) \sim \frac{C}{t^{\frac{3}{2}}}$ as $t \to \infty$. Note that for any $a > 0$,

$$\mathbb{E}^\Delta\Big[\frac{\sqrt{\Delta}}{\log\Delta} 1_{\{\Delta > a\}}\Big] = \infty.$$

For all $\Delta > 2A_1$, we have

$$\lim_{A_2\to\infty} \iint\limits_{\substack{0<t_1<t_2<1 \\ A_1\Delta^{-1}<t_2-t_1<A_2\Delta^{-1}}} \frac{dt_2 dt_1}{\sqrt{t_2-t_1}} > \iint\limits_{\substack{0<t_1<t_2<1 \\ t_2-t_1>\frac{1}{2}}} \frac{dt_2 dt_1}{\sqrt{t_2-t_1}} > \frac{1}{8}.$$

Therefore by the monotone convergence theorem, we have

$$\lim_{A_2\to\infty} \mathbb{E}_0^\sigma \mathbb{E}_0^{Y^\sigma}[H^{\text{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)] - \mathbb{E}_0^\sigma \mathbb{E}_0^{Y}[H^{\text{int}}_{[\sigma_0,\sigma_1]}(Y)] = \infty,$$

which proves (4.9). ∎

Proof of Lemma 4.2 (4.10). Without loss of generality, assume that $\sigma_0 = a = 0$, and let $\sigma_1 = \Delta$. We have

$$\text{Var}(H^{\text{int}}_{[\sigma_0,\sigma_1]}(Y^\sigma)|\sigma)$$

$$= \iint\limits_{\substack{0<s_1<s_2<\Delta, A_1<s_2-s_1<A_2, \\ 0<s_1'<s_2'<\Delta, A_1<s_2'-s_1'<A_2}} \frac{\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}, Y^\sigma_{s_1'} = Y^\sigma_{s_2'}) - \mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2})\mathbb{P}(Y^\sigma_{s_1'} = Y^\sigma_{s_2'})}{\log(s_2-s_1)\log(s_2'-s_1')} ds_1 ds_2 ds_1' ds_2'$$

$$\leq 2 \iint\limits_{\substack{0<s_1<s_2<\Delta, A_1<s_2-s_1, \\ s_1<s_1'<s_2'<\Delta, A_1<s_2'-s_1'}} \frac{\big|\mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2}, Y^\sigma_{s_1'} = Y^\sigma_{s_2'}) - \mathbb{P}(Y^\sigma_{s_1} = Y^\sigma_{s_2})\mathbb{P}(Y^\sigma_{s_1'} = Y^\sigma_{s_2'})\big|}{\log(s_2-s_1)\log(s_2'-s_1')} ds_1 ds_2 ds_1' ds_2'. \tag{8.2}$$

In the integral above, s_1, s_2, s_1' and s_2' fall into three regions: (1) $s_1 < s_2 < s_1' < s_2'$; (2) $s_1 < s_1' < s_2 < s_2'$; (3) $s_1 < s_1' < s_2' < s_2$. In region (1), let $r_1 = s_2 - s_1$, $r_2 = s_1' - s_2$, $r_3 = s_2' - s_1'$, and similarly in regions (2) and (3), let r_1, r_2 and r_3 be the successive increments of the ordered

variables. Let (1), (2) and (3) also denote the respective contributions to the integral in (8.2) from the three regions. Then for (1), we have

$$\mathbb{P}(Y^\sigma_{s_1}=Y^\sigma_{s_2}, Y^\sigma_{s_1'}=Y^\sigma_{s_2'}) = \frac{\sum_{x,y,z\in\mathbb{Z}^d} p_{\rho s_1}(x)p_{\rho r_1}(0)p_{\rho r_2}(y)p_{\rho r_3}(0)p_{\rho(\Delta-s_1-r_1-r_2-r_3)}(z)p_\Delta(x+y+z)}{p_{(1+\rho)\Delta}(0)}$$
$$= \frac{p_{\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0)}{p_{(1+\rho)\Delta}(0)},$$

and

$$\mathbb{P}(Y^\sigma_{s_1}=Y^\sigma_{s_2})\mathbb{P}(Y^\sigma_{s_1'}=Y^\sigma_{s_2'}) = \frac{p_{\rho r_1}(0)p_{(1+\rho)\Delta-\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho r_3}(0)}{p_{(1+\rho)\Delta}(0)^2}.$$

Therefore

$$(1) = \tag{8.3}$$
$$\iint_{\substack{0<s_1,r_1,r_2,r_3<\Delta,\ A_1<r_1,r_3 \\ s_1+r_1+r_2+r_3<\Delta}} \frac{\Big|\frac{p_{\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0)}{p_{(1+\rho)\Delta}(0)} - \frac{p_{\rho r_1}(0)p_{(1+\rho)\Delta-\rho r_1}(0)p_{\rho r_3}(0)p_{(1+\rho)\Delta-\rho r_3}(0)}{p_{(1+\rho)\Delta}(0)^2}\Big|}{\log r_1 \log r_3} ds_1 dr_1 dr_2 dr_3.$$

By similar considerations, we have

$$(2) = \tag{8.4}$$
$$\iint_{\substack{0<s_1,r_1,r_2,r_3<\Delta, \\ A_1<r_1+r_2, r_2+r_3, s_1+r_1+r_2+r_3<\Delta}} \frac{ds_1dr_1dr_2dr_3}{\log(r_1+r_2)\log(r_2+r_3)}\Big|\frac{\sum_{x\in\mathbb{Z}^3} p_{\rho r_1}(x)p_{\rho r_2}(x)p_{\rho r_3}(x)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(x)}{p_{(1+\rho)\Delta}(0)}$$
$$- \frac{p_{\rho(r_1+r_2)}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2)}(0)p_{\rho(r_2+r_3)}(0)p_{(1+\rho)\Delta-\rho(r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)^2}\Big|,$$

$$(3) = \tag{8.5}$$
$$\iint_{\substack{0<s_1,r_1,r_2,r_3<\Delta, \\ A_1<r_2,\ s_1+r_1+r_2+r_3<\Delta}} \frac{ds_1dr_1dr_2dr_3}{\log(r_1+r_2+r_3)\log r_2}\Big|\frac{p_{\rho(r_1+r_3)}(0)p_{\rho r_2}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)}$$
$$- \frac{p_{\rho(r_1+r_2+r_3)}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)p_{\rho r_2}(0)p_{(1+\rho)\Delta-\rho r_2}(0)}{p_{(1+\rho)\Delta}(0)^2}\Big|.$$

We will show that (1), (2), (3) are all bounded by $C_\rho\Delta$ for some C_ρ depending only on ρ.

For (1), we have

$$(1) \le \Delta^2 \iint_{\substack{A_1<r_1,r_3<\Delta \\ r_1+r_3<\Delta}} \frac{p_{\rho r_1}(0)p_{\rho r_3}(0)\big|p_{(1+\rho)\Delta}(0)p_{(1+\rho)\Delta-\rho(r_1+r_3)}(0) - p_{(1+\rho)\Delta-\rho r_1}(0)p_{(1+\rho)\Delta-\rho r_3}(0)\big|}{p_{(1+\rho)\Delta}(0)^2\log r_1\log r_3} dr_1dr_3$$
$$\le \frac{C\Delta^5}{\rho^3} \iint_{A_1<r_1,r_3<\Delta} \frac{\frac{\rho^2 r_1 r_3}{\Delta^5}}{r_1^{\frac{3}{2}} r_3^{\frac{3}{2}}\log r_1\log r_3} dr_1dr_3$$
$$\le \frac{C\Delta}{\rho},$$

where we used the local central limit theorem to bound $p_s(0)$ and we applied Lemma B.4 with $t=(1+\rho)\Delta-\rho(r_1+r_3)$.

We can bound (2) by passing the absolute value in (8.4) inside. By Lemma B.1, $p_s(0)\le\frac{C}{(1+s)^{\frac{3}{2}}}$ for some $C>0$ for all $s\ge 0$, and for Δ large, we have

$$\frac{p_{\Delta+s}(x)}{p_{(1+\rho)\Delta}(0)} < (1+\rho)^2 \qquad \text{for all } \ 0\le s\le\rho\Delta \ \text{ and } \ x\in\mathbb{Z}^3. \tag{8.6}$$

Therefore

$$\begin{aligned}
&\iint\limits_{\substack{0<s_1,r_1,r_2,r_3<\Delta,\\ A_1<r_1+r_2,r_2+r_3,\, s_1+r_1+r_2+r_3<\Delta}} \frac{\frac{p_{\rho(r_1+r_2)}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2)}(0)p_{\rho(r_2+r_3)}(0)p_{(1+\rho)\Delta-\rho(r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)^2}}{\log(r_1+r_2)\log(r_2+r_3)} ds_1 dr_1 dr_2 dr_3 \\
\le\ & \frac{C\Delta}{\rho^3} \iint\limits_{0\le r_1,r_2,r_3<\infty} \frac{1}{(1+r_1+r_2)^{\frac{3}{2}}(1+r_2+r_3)^{\frac{3}{2}}\log(e+r_1+r_2)\log(e+r_2+r_3)} dr_1 dr_2 dr_3 \\
\le\ & \frac{C\Delta}{\rho^3}, \qquad (8.7)
\end{aligned}$$

where the last integral is finite since integrating out r_1 and r_3 leads to a bound of the form $\int_0^\infty \frac{C}{(1+r_2)(\log(e+r_2))^2} dr_2 < \infty$. For the remaining term in (8.4), we have

$$\begin{aligned}
&\iint\limits_{\substack{0<s_1,r_1,r_2,r_3<\Delta,\\ A_1<r_1+r_2,r_2+r_3,\, s_1+r_1+r_2+r_3<\Delta}} \frac{\sum_{x\in\mathbb{Z}^3} p_{\rho r_1}(x)p_{\rho r_2}(x)p_{\rho r_3}(x)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(x)}{p_{(1+\rho)\Delta}(0)\log(r_1+r_2)\log(r_2+r_3)} ds_1 dr_1 dr_2 dr_3 \\
\le\ & \frac{C\Delta}{\rho^3} \iint\limits_{0\le r_1,r_2,r_3<\infty} \frac{dr_1 dr_2 dr_3}{(1+r_1r_2+r_1r_3+r_2r_3)^{\frac{3}{2}}\log(e+r_1+r_2)\log(e+r_2+r_3)}, \\
\le\ & \frac{C\Delta}{\rho^3} \iint\limits_{0\le r_1,r_2<\infty} \frac{dr_1 dr_2}{(\log(e+r_2))^2\,(r_1+r_2)\sqrt{1+r_1r_2}} \\
=\ & \frac{C\Delta}{\rho^3} \iint\limits_{0\le t,r_2<\infty} \frac{dt dr_2}{(\log(e+r_2))^2\,(1+t)\sqrt{1+tr_2^2}}, \qquad (8.8)
\end{aligned}$$

where we used (8.6), applied Lemma B.2, and made a change of variable $r_1 = tr_2$. The integral in (8.8) is clearly finite when integrated over $r_2 > 1$, since we can bound $\frac{1}{\sqrt{1+tr_2^2}}$ by $\frac{1}{r_2\sqrt{t}}$. For $0 < r_2 < 1$, note that

$$\int_0^\infty \frac{dt}{(1+t)\sqrt{1+tr_2^2}} = \int_0^\infty \frac{dw}{(w+r_2^2)\sqrt{1+w}} \le C - 2\ln r_2,$$

which is integrable over $r_2 \in [0,1]$. Therefore the integral in (8.8) is finite, and together with (8.7), this shows that $(2) \le C\rho^{-3}\Delta$.

For (3), we have

$$\begin{aligned}
(3)\ \le\ & \Delta \iint\limits_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2,r_1+r_2+r_3<\Delta}} dr_1 dr_2 dr_3 \frac{p_{\rho r_2}(0)p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)^2(\log(e+r_2))^2} \\
&\qquad \times \ \big|p_{\rho(r_1+r_3)}(0)p_{(1+\rho)\Delta}(0) - p_{\rho(r_1+r_2+r_3)}(0)p_{(1+\rho)\Delta-\rho r_2}(0)\big| \\
\le\ & \frac{C\Delta^{\frac{5}{2}}}{\rho^{\frac{3}{2}}} \iint\limits_{\substack{0<r_1,r_3<\Delta\\ A_1<r_2<\Delta}} dr_1 dr_2 dr_3 \Big(\frac{p_{\rho(r_1+r_3)}(0)|p_{(1+\rho)\Delta}(0) - p_{(1+\rho)\Delta-\rho r_2}(0)|}{(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} \\
&\qquad + \ \frac{p_{(1+\rho)\Delta-\rho r_2}(0)|p_{\rho(r_1+r_3)}(0) - p_{\rho(r_1+r_2+r_3)}(0)|}{(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} \Big),
\end{aligned}$$

where we applied (8.6) to $\frac{p_{(1+\rho)\Delta-\rho(r_1+r_2+r_3)}(0)}{p_{(1+\rho)\Delta}(0)}$. Using Lemma B.3, we have

$$\begin{aligned}
&\frac{C\Delta^{\frac{5}{2}}}{\rho^{\frac{3}{2}}} \iint\limits_{\substack{0<r_1,r_3<\Delta \\ A_1<r_2<\Delta}} \frac{p_{\rho(r_1+r_3)}(0)|p_{(1+\rho)\Delta}(0)-p_{(1+\rho)\Delta-\rho r_2}(0)|}{(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} dr_1dr_2dr_3 \\
\leq\ & \frac{C\Delta^{\frac{5}{2}}}{\rho^3} \iint\limits_{0<r_1,r_2,r_3<\Delta} \frac{\frac{\rho r_2}{\Delta^{\frac{5}{2}}}}{(1+r_1+r_3)^{\frac{3}{2}}(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} dr_1dr_2dr_3 \\
\leq\ & \frac{C}{\rho^2}\int_0^\Delta \frac{1}{\sqrt{r_2}}dr_2 \iint_{0\leq r_1+r_3\leq 2\Delta} \frac{1}{(r_1+r_3)^{\frac{3}{2}}} dr_1dr_3 \\
\leq\ & \frac{C\Delta}{\rho^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{C\Delta^{\frac{5}{2}}}{\rho^{\frac{3}{2}}} \iint\limits_{\substack{0<r_1,r_3<\Delta \\ A_1<r_2<\Delta}} \frac{p_{(1+\rho)\Delta-\rho r_2}(0)|p_{\rho(r_1+r_3)}(0)-p_{\rho(r_1+r_2+r_3)}(0)|}{(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} dr_1dr_2dr_3 \\
\leq\ & \frac{C\Delta}{\rho^{\frac{3}{2}}} \iint\limits_{\substack{0<r_1,r_3<\infty \\ A_1<r_2<\infty}} \frac{\frac{r_2}{\rho^{\frac{3}{2}}(r_1+r_3)^{\frac{3}{2}}(r_1+r_2+r_3)}}{(1+r_2)^{\frac{3}{2}}(\log(e+r_2))^2} dr_1dr_2dr_3 \\
\leq\ & \frac{C\Delta}{\rho^3}\int_{A_1}^\infty\int_0^\infty \frac{1}{\sqrt{r_2}(\log(e+r_2))^2}\frac{1}{\sqrt{w}(r_2+w)} dwdr_2 \\
=\ & \frac{C\Delta}{\rho^3}\int_{A_1}^\infty\int_0^\infty \frac{1}{r_2(\log(e+r_2))^2}\frac{1}{\sqrt{t}(1+t)} dtdr_2 \\
\leq\ & \frac{C\Delta}{\rho^3},
\end{aligned}$$

where we used the fact that $\iint_{[0,\infty)^2} f(r_1+r_3)dr_1dr_3 = \int_0^\infty wf(w)dw$, and made a change of variable $w = r_2t$. Thus we have proved (3) $\leq C\rho^{-3}\Delta$, which concludes the proof of (4.10). ∎

A Renewal estimates

Consider a renewal process $\sigma := \{\sigma_0 = 0 < \sigma_1 < \cdots\}$ on $[0,\infty)$, where $(\sigma_i - \sigma_{i-1})_{i\in\mathbb{N}}$ are i.i.d. with distribution $K(t)dt$ for a bounded density K on $\mathbb{R}_+$ satisfying

$$K(t) \sim c_K t^{-1-\alpha}, \quad t\to\infty \tag{A.1}$$

for some $\alpha\in(0,1)$ and $c_K\in(0,\infty)$. Let K^{*n} denote the n-fold convolution of K with itself, and let $P(t) = \sum_{n=1}^\infty K^{*n}(t)$, as defined in (3.13), be the corresponding renewal density.

We prove in Lemma A.1 a special case of the continuous time version of Doney's local limit theorem for renewal processes with infinite mean [D97, Thm. 3]. Note that [D97] allows a general regularly varying function in the right-hand side of (A.1). We stick to the narrower class, which suffices for our purposes, for the sake of a less cumbersome proof.

Lemma A.1 *We have*

$$\lim_{t\to\infty} c_K t^{1-\alpha}P(t) = \frac{\alpha\sin(\alpha\pi)}{\pi}. \tag{A.2}$$

Lemma A.2 *There exists a positive stable random variable G with exponent α, such that*

$$\lim_{t\to\infty} \mathbb{P}\big(|\sigma \cap [0,t]| \geq at^\alpha\big) = \mathbb{P}\Big(G \leq \frac{1}{a^{1/\alpha}}\Big) \qquad \text{for all } a > 0. \tag{A.3}$$

It is well known that

$$\frac{\sigma_n}{n^{\frac{1}{\alpha}}} = \frac{(\sigma_1 - \sigma_0) + \cdots + (\sigma_n - \sigma_{n-1})}{n^{\frac{1}{\alpha}}} \xrightarrow{d} G \quad \text{as } n \to \infty, \tag{A.4}$$

where G is a one-sided stable random variable of index α. Note that $\int_t^\infty K(s)\,ds \sim (c_K/\alpha)t^{-\alpha}$, thus the normalisation is chosen here in such a way that $\mathbb{E}[e^{-\lambda G}] = \exp\big(-\frac{c_K\Gamma(1-\alpha)}{\alpha}\lambda^\alpha\big)$, $\lambda \geq 0$, i.e., G is $(c_K\Gamma(1-\alpha)/\alpha)^{1/\alpha}$ times a "standard" one-sided stable random variable of index α (see, e.g., [F66, Thm. XIII.6.2]). Since the characteristic function of G decays faster than any polynomial at infinity, G has a C^∞ density g, see, e.g., [IL71, p. 48]. As G is a limit of non-negative random variables, we must have $g(x) = 0$ for $x < 0$, implying $g(0) = 0$ by continuity. Furthermore, $g(x) \sim c_G x^{-1-\alpha}$ for $x \to \infty$ with some $c_G \in (0,\infty)$, see, e.g., [IL71, Thm. 2.4.1]. In particular, $x \mapsto x^{-\alpha}g(x)$ is bounded and uniformly continuous with $x^{-\alpha}g(x) \leq c/(1+x^{1+2\alpha})$ for some $c < \infty$. We have

$$\int_0^\infty x^{-\alpha} g(x)\,dx = \frac{\alpha}{c_K\Gamma(1-\alpha)} \cdot \frac{1}{\Gamma(1+\alpha)} = \frac{1}{c_K\Gamma(1-\alpha)\Gamma(\alpha)} = \frac{\sin(\alpha\pi)}{c_K\pi}. \tag{A.5}$$

For the first equality, note that $G^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^\infty \lambda^{\alpha-1}e^{-\lambda G}\,d\lambda$. The second identity uses well-known facts about the Γ function.

Proof of Lemma A.2. By (A.4),

$$\lim_{t\to\infty} \mathbb{P}\big(|\sigma \cap [0,t]| \geq at^\alpha\big) = \lim_{t\to\infty} \mathbb{P}\Big(\frac{\sigma_{\lfloor at^\alpha\rfloor}}{\lfloor at^\alpha\rfloor^{1/\alpha}} \leq \frac{t}{\lfloor at^\alpha\rfloor^{1/\alpha}}\Big) = \mathbb{P}(G \leq \frac{1}{a^{1/\alpha}}),$$

since the distribution of G contains no atoms. ∎

We will need the following uniform one-sided large deviation estimate.

Lemma A.3 *We have for any sequence $c_n \to \infty$*

$$\lim_{n\to\infty} \sup_{t \geq c_n n^{\frac{1}{\alpha}}} \Big|\frac{K^{*n}(t)}{nK(t)} - 1\Big| = 0. \tag{A.6}$$

Proof. This follows from [Z99, Thm. 1] by specialising to the one-dimensional asymmetric case. Note that Zaigraev [Z99] attributes the result in the present case (one-dimensional situation, K in the normal domain of attraction of a stable law) to Tkačuk [T73], which the authors unfortunately could not access. ∎

Proof of Lemma A.1. Our proof follows more or less the scheme of [D97, Thm. 3], with [D97, Thm. 2] replaced by Lemma A.3. Even though we use Lemma A.1 in this paper only for $\alpha = 1/2$, the proof is the same for all $\alpha \in (0,1)$.

By a local limit theorem for sums of random variables in the domain of attraction of a stable law, e.g. [IL71, Thm. 4.3.1], we have

$$\sup_{t\in\mathbb{R}_+} \big|n^{\frac{1}{\alpha}}K^{*n}(n^{\frac{1}{\alpha}}t) - g(t)\big| \to 0 \quad \text{as } n \to \infty, \tag{A.7}$$

where g is the density of the one-sided stable random variable appearing as the limit in (A.4). Thus, we can find a continuous, strictly decreasing function $\rho : [0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty}\rho(t) = 0$ such that

$$\sup_{t\in\mathbb{R}_+} \big|n^{\frac{1}{\alpha}}K^{*n}(n^{\frac{1}{\alpha}}t) - g(t)\big| \leq \rho(n) \quad \text{for } n \in \mathbb{N}. \tag{A.8}$$

Obviously, $\rho^{-1} : (0, \rho(0)] \to [0, \infty)$ is continuous and strictly decreasing with $\lim_{y\to 0+} \rho^{-1}(y) = \infty$. Note that the function $\psi : (0, \rho(0)^{1/(2-\alpha)}] \to [0, \infty)$ with $\psi(y) = (\rho^{-1}(y^{2-\alpha}))^{1/\alpha}/y$ is strictly decreasing, and $\lim_{y\to 0+} \psi(y) = \infty$. Define $\delta(t) := \psi^{-1}(t)$ for $t \geq 0$. Observe that then $t \mapsto \delta(t)$ is strictly decreasing and satisfies $\lim_{t\to\infty} \delta(t) = 0$. Furthermore,

$$\rho\big((\delta(t)t)^\alpha\big) = \rho\big(\big(\delta(t)\psi(\delta(t))\big)^\alpha\big) = \rho\big(\rho^{-1}\big(\delta(t)^{2-\alpha}\big)\big) = \delta(t)^{2-\alpha}, \tag{A.9}$$

proving that $t\delta(t) \to \infty$ as $t \to \infty$, and

$$\frac{\rho\big((\delta(t)t)^\alpha\big)}{\delta(t)^{1-\alpha}} = \delta(t) \to 0 \quad \text{as } t \to \infty. \tag{A.10}$$

Decompose

$$t^{1-\alpha} \sum_{n\geq 1} K^{*n}(t) = t^{1-\alpha} \sum_{n>(\delta(t)t)^\alpha} K^{*n}(t) + t^{1-\alpha} \sum_{n=1}^{[(\delta(t)t)^\alpha]} K^{*n}(t) =: S_1 + S_2. \tag{A.11}$$

We have

$$S_1 = t^{1-\alpha} \sum_{n>(\delta(t)t)^\alpha} \frac{1}{n^{\frac{1}{\alpha}}} g\big(\frac{t}{n^{\frac{1}{\alpha}}}\big) + t^{1-\alpha} \sum_{n>(\delta(t)t)^\alpha} \frac{1}{n^{\frac{1}{\alpha}}} \Big(n^{\frac{1}{\alpha}} K^{*n}(n^{\frac{1}{\alpha}} \frac{t}{n^{\frac{1}{\alpha}}}) - g\big(\frac{t}{n^{\frac{1}{\alpha}}}\big)\Big) =: S_1' + R_1,$$

where

$$|R_1| \leq \rho\big((\delta(t)t)^\alpha\big)\, t^{1-\alpha} \sum_{n>(\delta(t)t)^\alpha} \frac{1}{n^{\frac{1}{\alpha}}}.$$

Since $\sum_{n>(\delta(t)t)^\alpha} \frac{1}{n^{\frac{1}{\alpha}}} \sim \int_{(\delta(t)t)^\alpha}^\infty x^{-1/\alpha}\, dx \sim \frac{\alpha}{1-\alpha}(\delta(t)t)^{\alpha-1}$, we obtain from (A.10) that $R_1 \to 0$ as $t \to \infty$.

Put $x_n^{(t)} := t/n^{\frac{1}{\alpha}}$, then we have

$$\frac{t}{n^{\frac{1}{\alpha}}} \sim \alpha n\big(x_n^{(t)} - x_{n+1}^{(t)}\big) = \alpha (t/x_n^{(t)})^\alpha \big(x_n^{(t)} - x_{n+1}^{(t)}\big)$$

since $\alpha n n^{\frac{1}{\alpha}}\big(x_n^{(t)} - x_{n+1}^{(t)}\big)/t = \alpha n\big(1 - \frac{n^{\frac{1}{\alpha}}}{(n+1)^{\frac{1}{\alpha}}}\big) \sim \alpha n\big(1 - \big(1 - \frac{1}{n+1}\big)^{1/\alpha}\big) \to 1$, and hence

$$S_1' \sim \alpha \sum_{n>(\delta(t)t)^\alpha} \frac{(t/x_n^{(t)})^\alpha}{t^\alpha} \big(x_n^{(t)} - x_{n+1}^{(t)}\big) g(x_n^{(t)}) \sim \alpha \sum_{n\,:\, n^{\frac{1}{\alpha}} > \delta(t)t} \big(x_n^{(t)} - x_{n+1}^{(t)}\big) \big(x_n^{(t)}\big)^{-\alpha} g(x_n^{(t)}).$$

The term on the right is an approximating Riemann sum and $n^{\frac{1}{\alpha}} > \delta(t)t$ means $x_n^{(t)} < 1/\delta(t)$, which tends to ∞ as $t \to \infty$. Thus, recalling (A.5) and the discussion above it, we have

$$S_1 \to \alpha \int_0^\infty x^{-\alpha} g(x)\, dx = \frac{\alpha \sin(\alpha\pi)}{c_K \pi} \quad \text{as } t \to \infty.$$

To bound S_2, note that $n \leq (\delta(t)t)^\alpha$ implies $t \geq n^{1/\alpha}/\delta(t) \geq n^{1/\alpha}$ for t sufficiently large. In particular, for such t and n, $\delta(t) \leq \delta(n^{1/\alpha})$, so $t \geq (\delta(n^{1/\alpha}))^{-1} n^{1/\alpha}$. Applying Lemma A.3 with $c_n := 1/\delta(n^{1/\alpha}) \to \infty$, we see that there exists $n_0 \in \mathbb{N}$, $t_0 < \infty$ and $C < \infty$ such that

$$K^{*n}(t) \leq CnK(t) \quad \text{for all } n \geq n_0,\ t \geq \frac{n^{1/\alpha}}{\delta(t)} \vee t_0. \tag{A.12}$$

Note that

$$K^{*n}(t) \le 2nc_K(t/n)^{-1-\alpha} \le 4n^{2+\alpha}K(t) \quad \text{for } t \text{ sufficiently large,} \tag{A.13}$$

which follows from (A.1) and the observation that $K^{*n}(t)$ is bounded from above by

$$\begin{aligned}
&\sum_{j=1}^{n} \int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t} 1_{\{\sigma_j-\sigma_{j-1}\ge t/n\}} \prod_{i=1}^{m} K(\sigma_i-\sigma_{i-1}) \prod_{i=1}^{m-1} d\sigma_i \\
&\le\ 2c_K\Big(\frac{t}{n}\Big)^{-1-\alpha} \sum_{j=1}^{n} \int\cdots\int_{\sigma_0=0<\sigma_1<\cdots<\sigma_m=t} \prod_{i=1,\, i\ne j}^{m} K(\sigma_i-\sigma_{i-1}) \prod_{i=1}^{m-1} d\sigma_i \\
&\le\ 2nc_K\Big(\frac{t}{n}\Big)^{-1-\alpha} \Big(\int_0^\infty K(\sigma)\,d\sigma\Big)^{n-1}.
\end{aligned}$$

Therefore if the constant C appearing in (A.12) is suitably increased, (A.12) holds for all $n \in \mathbb{N}$. Thus for t sufficiently large, we have

$$S_2 \ =\ t^{1-\alpha} \sum_{n=1}^{[(\delta(t)t)^\alpha]} K^{*n}(t) \le 2Cc_K t^{-2\alpha} \sum_{n=1}^{[(\delta(t)t)^\alpha]} n \le 2Cc_K t^{-2\alpha}(\delta(t)t)^{2\alpha} = 2Cc_K\delta(t)^{2\alpha},$$

which converges to 0 as $t \to \infty$. ∎

B Random walk estimates

Lemma B.1 [Local central limit theorem] *Let $(X_t)_{t\ge0}$ with $X_0 = 0$ be a continuous time random walk on $\mathbb{Z}^3$ with jump rate 1 and jump probability kernel $(q(x))_{x\in\mathbb{Z}^3}$, which is irreducible and symmetric with finite covariance matrix $Q_{ij} = \sum_{x\in\mathbb{Z}^3} x_i x_j q(x)$, $1 \le i, j \le 3$. Let $p_t(\cdot)$ denote the transition probability kernel of X at time t. Then*

$$p_t(x) \le p_t(0) \qquad \text{for all } x \in \mathbb{Z}^3 \text{ and } t \ge 0, \tag{B.1}$$

and

$$\lim_{t\to\infty} (2\pi t)^{\frac{3}{2}} \sqrt{\det Q}\, p_t(0) = 1. \tag{B.2}$$

Proof. Since $\hat{p}(k) := \sum_{x\in\mathbb{Z}^3} e^{i\langle k,x\rangle} p_t(x) = e^{-t(1-\phi(k))}$, where $\phi(k) = \sum_{x\in\mathbb{Z}^3} e^{i\langle k,x\rangle} q(x)$ is real by the symmetry of q, by inverse Fourier transform,

$$p_t(x) = \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} e^{-i\langle k,x\rangle} e^{-t(1-\phi(k))} dk \le \frac{1}{(2\pi)^3} \int_{[-\pi,\pi]^3} e^{-t(1-\phi(k))} dk = p_t(0).$$

For (B.2), see e.g. [S76, Prop. 7.9, Chap. II] where a discrete time version was proved. The proof for the continuous time version is identical. ∎

Lemma B.2 *Let X, $q(\cdot)$, and $p_t(\cdot)$ be as in Lemma B.1 without the symmetry assumption on q. Then for any $a, b, c > 0$, there exists some $C > 0$ depending only on q such that*

$$\sum_{x\in\mathbb{Z}^d} p_a(x) p_b(x) p_c(x) \le \frac{C}{(1+ab+bc+ca)^{\frac{d}{2}}}. \tag{B.3}$$

Proof. Without loss of generality, assume that $a \geq b \geq c$. By the local central limit theorem, there exists $C_1 > 0$ such that uniformly in $t > 0$ and $x \in \mathbb{Z}^d$, we have $p_t(x) \leq \frac{C_1}{(1+t)^{\frac{d}{2}}}$. Then

$$\sum_{x\in\mathbb{Z}^d} p_a(x)p_b(x)p_c(x) \leq \frac{C_1^2}{(1+ab)^{\frac{d}{2}}} \sum_{x\in\mathbb{Z}^d} p_c(x) = \frac{C_1^2}{(1+ab)^{\frac{d}{2}}} \leq \frac{C}{(1+ab+bc+ca)^{\frac{d}{2}}}.$$

■

Lemma B.3 *Let X, $q(\cdot)$, Q, and $p_t(\cdot)$ be as in Lemma B.1 so that q is symmetric. Then there exist $C_1, C_2 > 0$ depending on q, such that for all $t > 1$ and $r > 0$,*

$$\frac{C_1 r}{t^{\frac{d}{2}}(t+r)} \leq p_t(0) - p_{t+r}(0) \leq \frac{C_2 r}{t^{\frac{d}{2}}(t+r)}. \tag{B.4}$$

Proof. By the symmetry of q, $\phi(k) := \sum_x e^{i\langle k,x\rangle} q(x) \in [-1,1]$, and $\mathbb{E}[e^{i\langle k, X_t\rangle}] = e^{-t(1-\phi(k))}$. Therefore,

$$p_t(0) - p_{t+r}(0) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \big(e^{-t(1-\phi(k))} - e^{-(t+r)(1-\phi(k))}\big) dk.$$

By irreducibility of $q(\cdot)$, $\phi(k) = 1$ only at $k = 0$, and hence $c := \inf_{|k|\geq\epsilon, k\in[-\pi,\pi]^d}(1-\phi(k)) > 0$ for any $\epsilon > 0$. By Taylor expansion, if $\epsilon > 0$ is sufficiently small, then

$$\frac{1}{4}\langle k, Qk\rangle \leq (1-\phi(k)) \leq \langle k, Qk\rangle \qquad \forall\, |k| < \epsilon.$$

Therefore

$$\begin{aligned}
(2\pi)^d(p_t(0) - p_{t+r}(0)) &= \int_{[-\pi,\pi]^d} e^{-t(1-\phi(k))}(1 - e^{-r(1-\phi(k))}) dk \\
&\leq r \int_{[-\pi,\pi]^d} (1-\phi(k)) e^{-t(1-\phi(k))} dk \\
&\leq 2r \int_{|k|>\epsilon, k\in[-\pi,\pi]^d} e^{-t(1-\phi(k))} dk + r \int_{|k|\leq\epsilon} \langle k, Qk\rangle e^{-\frac{t\langle k,Qk\rangle}{4}} dk \\
&\leq 2(2\pi)^d e^{-ct} r + \frac{r}{t^{\frac{d}{2}+1}} \int_{\mathbb{R}^d} \langle k, Qk\rangle e^{-\frac{\langle k,Qk\rangle}{4}} dk \\
&\leq \frac{Cr}{t^{\frac{d}{2}+1}},
\end{aligned} \tag{B.5}$$

which implies that $p_t(0) - p_{t+r}(0) \leq \frac{C_2 r}{t^{\frac{d}{2}}(t+r)}$ for $r < t$. When $r \geq t$, the same bound follows from the local central limit theorem.

Similarly,

$$\begin{aligned}
(2\pi)^d(p_t(0) - p_{t+r}(0)) &= \int_{[-\pi,\pi]^d} e^{-(t+r)(1-\phi(k))}(e^{r(1-\phi(k))} - 1) dk \\
&\geq r \int_{|k|\leq\epsilon, k\in[-\pi,\pi]^d} (1-\phi(k)) e^{-(t+r)(1-\phi(k))} dk \geq r \int_{|k|\leq\epsilon, k\in[-\pi,\pi]^d} \frac{\langle k, Qk\rangle}{4} e^{-(t+r)\langle k,Qk\rangle} dk \\
&\geq \frac{Cr}{(t+r)^{\frac{d}{2}+1}},
\end{aligned}$$

which follows by a change of variable for k and the fact that $t+r > 1$. This implies $p_t(0) - p_{t+r}(0) \geq \frac{C_1 r}{t^{\frac{d}{2}}(t+r)}$ for $r < t$. When $r > t$, the same bound follows from the local central limit theorem. ■

Lemma B.4 *Let X, $q(\cdot)$ and $p_t(\cdot)$ be as in Lemma B.1 so that q is symmetric. Then there exist $C > 0$ depending only on q such that, for any $a, b > 0$ and $t > 1$,*

$$|p_t(0)p_{t+a+b}(0) - p_{t+a}(0)p_{t+b}(0)| \le \frac{Cab}{t^d(t+a)(t+b)}. \tag{B.6}$$

Proof. Clearly (B.6) holds for suitable choice of c and C when $t < 1$. Therefore we need only consider $t > 1$. Note that

$$\begin{aligned}
&p_t(0)p_{t+a+b}(0) - p_{t+a}(0)p_{t+b}(0)\\
&= p_{t+a+b}(0)(p_t(0) - p_{t+a}(0)) - p_{t+a}(0)(p_{t+b}(0) - p_{t+a+b}(0))\\
&= \big(p_{t+a+b}(0) - p_{t+a}(0)\big)\big(p_t(0) - p_{t+a}(0)\big) + p_{t+a}(0)(p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0)).
\end{aligned} \tag{B.7}$$

By Lemma B.3, the first term in (B.7) is bounded in absolute value by

$$\frac{Cb}{(t+a)^{\frac{d}{2}}(t+a+b)} \cdot \frac{Ca}{t^{\frac{d}{2}}(t+a)},$$

which is clearly bounded by the RHS of (B.6).

For the second term in (B.7), we claim that

$$0 \le p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Cab}{t^{\frac{d}{2}}(t+a)(t+b)}, \tag{B.8}$$

which together with the fact that $p_{t+a}(0) \le Ct^{-\frac{d}{2}}$ imply (B.6). Note that

$$\begin{aligned}
(2\pi)^d\big(p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0)\big) &= \int_{[-\pi,\pi]^d} e^{-t(1-\phi(k))}(1 - e^{-a(1-\phi(k))})(1 - e^{-b(1-\phi(k))})dk\\
&\le ab\int_{[-\pi,\pi]^d} (1-\phi(k))^2 e^{-t(1-\phi(k))}dk.
\end{aligned}$$

Clearly $p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \ge 0$. For the upper bound, exactly as in (B.5), we can Taylor expand $\phi(k)$ around $k = 0$ for $|k| \le \epsilon$ and bound $|\phi(k)|$ uniformly for $|k| > \epsilon$, which gives

$$p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Cab}{t^{\frac{d}{2}+2}}.$$

When $a, b < t$, this implies (B.8). If $b > t$, then (B.8) follows from the bound

$$p_t(0) - p_{t+a}(0) - p_{t+b}(0) + p_{t+a+b}(0) \le \frac{Ca}{t^{\frac{d}{2}}(t+a)} + \frac{Ca}{(t+b)^{\frac{d}{2}}(t+a+b)}$$

by Lemma B.3. The same argument applies when $a > t$. ∎

Lemma B.5 [Comparison of return probabilities] *Let X, $q(\cdot)$ and $p_t(\cdot)$ be as in Lemma B.1 so that q is symmetric. For $1 \le i \le n$, let $a_i, b_i > 0$, and let Z_i be an independent random variable distributed as X_{a_i} conditioned on $X_{a_i+b_i} = 0$. Then*

$$\mathbb{P}(Z_1 + \cdots + Z_n = 0) > \mathbb{P}(X_{a_1+\cdots+a_n} = 0). \tag{B.9}$$

Proof. Let $\phi(k) = \sum_x e^{i\langle k,x\rangle}q(x)$ and $\psi_i(k) = \mathbb{E}[e^{i\langle k,Z_i\rangle}]$. Since $\mathbb{E}[e^{i\langle k,X_t\rangle}] = e^{-t(1-\phi(k))}$, by Fourier transform, (B.9) is equivalent to

$$\int_{[-\pi,\pi]^d} \psi_1(k)\cdots\psi_n(k)\mathrm{d}k > \int_{[-\pi,\pi]^d} e^{-\sum_{i=1}^n a_i(1-\phi(k))}\mathrm{d}k. \tag{B.10}$$

By symmetry of q, $\phi(k) \in [-1,1]$ and $e^{-a_i(1-\phi(k))} \in (0,1]$. Therefore to verify (B.10), it suffices to show that for each $1 \le i \le n$,

$$\psi_i(k) \ge e^{-a_i(1-\phi(k))} \tag{B.11}$$

for all $k \in [-\pi,\pi]^d$, with strict inequality for some $k \in [-\pi,\pi]^d$.

Note that $\hat{p}_s(k) := \sum_x e^{i\langle k,x\rangle} p_s(x) = e^{-s(1-\phi(k))}$. By definition, $\mathbb{P}(Z_i = x) = \frac{p_{a_i}(x)p_{b_i}(x)}{p_{a_i+b_i}(0)}$, and hence

$$\psi_i(k) = \frac{(\hat{p}_{a_i} * \hat{p}_{b_i})(k)}{p_{a_i+b_i}(0)} = \frac{\int_{[-\pi,\pi]^d} e^{-a_i(1-\phi(k-u))-b_i(1-\phi(u))}\mathrm{d}u}{\int_{[-\pi,\pi]^d} e^{-(a_i+b_i)(1-\phi(u))}\mathrm{d}u}.$$

By symmetry, $\psi_i(k) = \psi_i(-k)$, and hence

$$\begin{aligned}
\psi_i(k) &= \frac{\int_{[-\pi,\pi]^d} e^{-b_i(1-\phi(u))} \frac{e^{-a_i(1-\phi(k-u))}+e^{-a_i(1-\phi(-k-u))}}{2}\mathrm{d}u}{\int_{[-\pi,\pi]^d} e^{-(a_i+b_i)(1-\phi(u))}\mathrm{d}u} \\
&\ge \frac{\int_{[-\pi,\pi]^d} e^{-b_i(1-\phi(u))} e^{-a_i(1-\frac{\phi(k-u)+\phi(-k-u)}{2})}\mathrm{d}u}{\int_{[-\pi,\pi]^d} e^{-(a_i+b_i)(1-\phi(u))}\mathrm{d}u},
\end{aligned} \tag{B.12}$$

where we applied Jensen's inequality. Note that since $\phi(x)$ is not identically equal to 1, for some choice of k and u, we have $\phi(k-u) \ne \phi(-k-u)$ so that there is strict inequality in (B.12) for some k. By symmetry,

$$\begin{aligned}
\phi(k-u)+\phi(-k-u) &= \sum_x q(x)(e^{i\langle k-u,x\rangle} + e^{i\langle -k-u,x\rangle}) \\
&= \sum_x q(x)\big(\cos\langle k-u,x\rangle + \cos\langle -k-u,x\rangle\big) \\
&= 2\sum_x q(x)\cos\langle k,x\rangle\cos\langle u,x\rangle \\
&\ge 2\sum_x q(x)\big(\cos\langle k,x\rangle + \cos\langle u,x\rangle - 1\big) \\
&= 2(\phi(k)+\phi(u)-1),
\end{aligned} \tag{B.13}$$

where we used $(1-\cos\alpha)(1-\cos\beta) \ge 0$. Plugging this bound into (B.12) then yields (B.11). ∎

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