An Efficient Game Form for Unicast Service Provisioning

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Abstract

We consider the decentralized bandwidth/rate allocation problem in unicast service provisioning with strategic users. We present a mechanism/game form which possesses the following properties when the users' utilities are concave: (1) It implements in Nash equilibria the solution of the corresponding centralized rate allocation problem in unicast service provisioning. (2) It is individually rational. (3) It is budget-balanced at all Nash equilibria of the game induced by the mechanism/game form as well as off equilibrium. When the users' utilities are quasi-concave the mechanism possesses properties (2) and (3) stated above. Moreover, every Nash equilibrium of the game induced by the proposed mechanism results in a Walrasian equilibrium.

I. Introduction

A. Motivation and Challenges

Most of today's networks, called integrated services networks support the delivery of a variety of services to their users each with its own quality of service (QoS) requirements (e.g. delay, percentage of packet loss, jitter, etc). As the number of services offered by the network and the demand for the services increase, the need for efficient network operation increases. One of the key factors that contributes to efficient network operation is the efficient utilization of the network's resources. The design of resource allocation mechanisms which guarantee the delivery of different services, each with its own QoS requirements, and maximize some network-wide performance criterion (e.g. the network's utility to its users) is an important and challenging task.

The challenge comes from: (1) the fact that the network is an informationally decentralized system; (2) the network's users may behave strategically (i.e. they may behave selfishly). Networks are informationally decentralized systems. Each user's utility is its own private information. Users are unaware of each others' utilities and of the

resources (e.g. bandwidth, buffers, spectrum) available to the network. The network (network manager) knows the network's topology and its resources but is unaware of the users' utilities. If information were centralized, the resource allocation problem could be formulated and solved as a mathematical programming problem or as a dynamic programming problem. Since information is not centralized such formulations are not possible. The challenge is: (1) to determine a message exchange process among the network and users and an allocation rule (based on the outcome of the message exchange process) that eventually lead to a resource allocation that is optimal for the centralized problem, (2) To take into account, in the determination of the allocation mechanism, the possible strategic (selfish) behavior of the network's users.

The topic of resource allocation for informationally decentralized systems has been explored in great detail by mathematical economists in the context of mechanism design. Decentralized resource allocation problems arising in networks have recently attracted significant attention among engineers. Below we present a brief survey of the existing literature on decentralized network resource allocation and briefly refer to texts and survey articles written by mathematical economists on mechanism design. A more detailed discussion of the existing literature on network resource allocation, and a comparison of the results of our paper with this literature will be presented in section VIII of the paper.

Within the context of communication networks most of existing literature (e.g. [4], [7], [8], [22], [25], [26], [27], [28], [29], [32], [33], [34], [37], [43], [44], [50], [51], [53], [57], [62], [63] and [65]) has approached the design of decentralized resource allocation mechanisms under the assumption that the network (network manager) and its users are cooperative, that is, they obey the rules of the proposed decentralized resource allocation mechanism.

Decentralized resource allocation problems with strategic users are solved through the development of game forms/mechanisms which together with the users' utilities give rise to games. Depending on the information available to the users, the resulting games are either ones with complete information or incomplete information ([10]); the nature of the game dictates the behavioral/equilibrium concept (e.g. Nash equilibrium (NE), subgame perfect equilibrium, Bayesian Nash equilibrium, sequential equilibrium) that is appropriate/suitable for the solution of the resource allocation problem. The allocation mechanisms specify; (1) the allocations made at all equilibrium and non-equilibrium points of the game; and (2) the *tax* (positive or negative) each user pays as a result of his/her participation in the game. Desirable properties of resource allocation mechanisms with strategic users are: (1) the allocations corresponding to all equilibria of the resulting game must be globally optimal; (2) the users must voluntarily participate

in the allocation process; (3) the budget must be balanced, that is the sum of taxes paid by the users must be zero (there should be no money left to be thrown away at the end of the allocation process). A brief discussion of game forms and precise specifications of their desirable properties are presented in section II of the paper.

Recently, within the context of communication networks, researchers have investigated decentralized resource allocation problems under the assumption that users behave strategically (i.e. they do not necessarily obey the rules of the mechanism but have to be induced to follow the rules). Within the context of wired networks, decentralized resource allocation mechanisms have been proposed and analyzed in [14], [23], [24], [27], [29], [30], [31], [35], [36], [60] and [68]. A more detailed discussion of these references and comparison with the results of this paper will be presented in section VIII. Decentralized power allocation mechanisms for wireless networks with strategic users where every user's transmission creates interference to every other user or to a subset of the network's users were proposed and analyzed in [56], [57].

Resource allocation for informationally decentralized systems has been explored in great detail by mathematical economists in the context of mechanism design. There is an enormous economics literature for the situation where users are cooperative (e.g [17], [67] and references therein) as well as the situation where users are strategic (e.g [17], [39], [67] and references therein). For a survey on the relationship of mechanism design and decentralized resource allocation in communication networks we refer the reader to [61].

B. Contribution of the paper

We investigate the unicast service provisioning problem in wired networks with arbitrary topology and strategic users. *The main contribution of this paper is the discovery of a decentralized rate allocation mechanism for unicast service provisioning in networks with arbitrary/general topology and strategic users, which possesses the following properties.*
When each user's utility is concave, then:

- *(**P1**) The mechanism implements the solution of the centralized unicast service provisioning problem in Nash equilibria. That is, the allocation corresponding to each NE of the game induced by the mechanism is a globally optimal solution of the corresponding centralized resource allocation problem.*
- *(**P2**) The mechanism is individually rational, that is, the network users voluntarily participate in the rate allocation process.*
- *(**P3**) The mechanism is budget-balanced*[1] *at all feasible allocations, that is, at all the*

[1]In a budget-balanced mechanism the sum of the taxes paid by a subset of users is equal to the sum of subsidies received by the rest of the users.

allocations that correspond to NE messages/strategies as well as at all the allocations that correspond to off-equilibrium messages/strategies.

When each user's utility is quasi-concave but differentiable, then:

- *The mechanism possesses properties (**P2**) and (**P3**).*
- *(**P4**) Every NE of the game induced by the mechanism results in a Walrasian equilibrium ([37] Ch. 15) , consequently, a Pareto optimal allocation.*

*To the best of our knowledge, none of the decentralized resource allocation mechanisms proposed so far for the unicast service provisioning problem in communication networks possesses simultaneously all three properties (**P1**)-(**P3**) when the network's topology is general/arbitrary, the users are strategic and their utilities are concave. Furthermore, we are not aware of the existence of any publications in unicast service provisioning containing the analysis of a decentralized rate allocation mechanism when the users are strategic and their utilities are quasi-concave.*

We compare our contributions with the existing literature in section VIII of this paper, after we present and prove our results.

C. Organization of the paper

The rest of the paper is organized as follows. In section III we formulate the unicast service provisioning problem with strategic users. In section IV we describe the allocation mechanism/game form we propose for the solution of the unicast service provisioning problem. In section V we analyze the properties of the proposed mechanism. In section VI we discuss how the game form/mechanism presented in this paper can be implemented in a network. In section VII we investigate the properties of the game form proposed in this paper when the users' utilities $\mathbf{U}_i, i \in \mathcal{N}$, are quasi-concave. In section VIII we compare the results of this paper with the existing literature. We conclude in section IX.

II. Mechanism Design/Implementation Theory

Formally, Mechanism design provides a systematic methodology for the design of decentralized resource allocation mechanisms for informationally decentralized systems that achieve optimal allocations of the corresponding centralized systems. In the mechanism design framework, a centralized resource allocation problem is described by the triple $(\mathbf{E}, \mathcal{A}, \pi)$: the environment space $\mathbf{E}$, the action/allocation space $\mathcal{A}$ and the goal correspondence π. Below, we briefly describe each component separately.

`Environment Space` ($\mathbf{E}$): We define the environment space $\mathbf{E}$ of an allocation problem to be the set of individual preferences (or the set of utilities), endowments and the

technology taken together. The environment $\mathbf{E}$ is the set of circumstances that can not be changed either by the designer of the allocation mechanism or by the agents/users that participate in the allocation mechanism.

For instance, in the problem considered in this paper, the environment $\mathbf{E}_i$ of user $i, i \in \mathcal{N}$, consists of the set $\mathcal{U}_i$ of her utility functions, the set Ω_i of routes for user i together with the set of link capacities along each route $\mathcal{R}_i \in \Omega_i$. A realization $\mathbf{e}_i \in \mathbf{E}_i$ of user i's environment consists of a utility function $\mathbf{U}_i \in \mathcal{U}_i$, a route $\mathcal{R}_i \in \Omega_i$ and the capacities $\mathbf{c}^l$ of the links l that belong to $\mathcal{R}_i$.

The environment $\mathbf{E}$ space is the cartesian products of users' individual environment spaces $\mathbf{E}_i$, i.e., $\mathbf{E} := \mathbf{E}_1 \times \mathbf{E}_2 \cdots \times \mathbf{E}_N$. A realization $\mathbf{e} \in \mathbf{E}$ of the environment is a collection of the users' individual realizations $\mathbf{e}_i$, $i = 1, 2, \cdots, N$, that is, $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N)$.

`Action Space`($\mathcal{A}$): We define the action space $\mathcal{A}$ of a resource allocation problem to be the set of all possible actions/resource allocations.

For instance, in the problem we investigate in this paper, $\mathcal{A}$ is the set of all tax and rate/bandwidth allocation to the users.

`Goal Correspondence`(π): Goal correspondence is a map from $\mathbf{E}$ to $\mathcal{A}$ which assigns to every environment $\mathbf{e} \in \mathbf{E}$ the set of actions/allocations which are solutions to the centralized resource allocation problem associated with/corresponding to the decentralized resource allocation problem under consideration. That is,

$$\pi : \mathbf{E} \to \mathcal{A}.$$

The setting described above corresponds to the case where one of the agents (e.g. a network manager) has enough information about the environment so as to determine the action according to the goal correspondence π. In general, information is decentralized; for every instance $\mathbf{e} \in \mathbf{E}$ of the allocation problem, agent i, $i = 1, 2, \cdots, N$, knows only $\mathbf{e}_i \in \mathbf{E}_i$. Therefore, we wish to devise: (i) a message exchange process among the agents participating in the allocation process; and (ii) a rule f that determines an allocation (or a set of allocations) based on the outcome/result of the message exchange process, such that for every $\mathbf{e} \in \mathbf{E}$ we achieve $\pi(\mathbf{e})$. The design of the message exchange process and the rule f must take into account the following: (1)The fact that the participating agents may be strategic(selfish), thus, they may have an incentive to misrepresent (during the message exchange process) their private information so as to shift the result of the

allocation to their favor. (2) The fact that agents, may choose not to participate in the allocation process; hence, their participation must be voluntary. The formal development of decentralized resource allocation processes/mechanisms that take into account the aforementioned facts is the subject of implementation theory, a branch of mechanism design (see [17], [67]).

In implementation theory a decentralized resource allocation process is described by a N-user/agent game form $(\mathcal{M}, f)$, where $\mathcal{M} = \prod_{i=1}^{N} \mathcal{M}_i$ is the message space, specifying for each user i, $i \in \mathcal{N}$, the set of messages $\mathcal{M}_i$ that user i can communicate to other users, and f is an outcome function that describes the actions that are taken for every $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_N) \in \mathcal{M}$; that is

$$f : \mathcal{M} \to \mathcal{A}.$$

The game form $(\mathcal{M}, f)$ is common knowledge ([**?**], [66]) among all the N agents/users. A game form is different from a game, as the consequence of a profile $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_N)$ of messages is an allocation (or a set of allocations if f is a correspondence) rather than a vector of utility payoffs. Once a realization $\mathbf{e} := (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N)$, $\mathbf{e}_i \in \mathbf{E}_i$, $i \in \mathcal{N}$ of the environment is specified, a game form induces a game. A solution concept (or equilibrium concept) specifies the strategic behavior of the agents/users faced with a game $(\mathcal{M}, f, \mathbf{e})$ induced by the game form $(\mathcal{M}, f)$. Consequently, a solution concept is a correspondence Λ that identifies a subset of $\mathcal{M}$ for any given specification $(\mathcal{M}, f, \mathbf{e})$. We define

$$\Theta_\Lambda := \{a \in \mathcal{A} : \exists \mathbf{m} \in \Lambda(\mathcal{M}, f, \mathbf{e}) : f(\mathbf{m}) = a\} \tag{1}$$

as the set of outcomes associated with the solution concept Λ.

The solution concept (equilibrium concept) appropriate for a decentralized resource allocation problem depends on the information that is available to the agents/users about the environment. For example, if agent i, $i = 1, 2, \cdots, N$, knows $\mathbf{e}_i \in \mathbf{E}_i$ and has a probability mass function (on $\mathbf{E}_{-i} = \prod_{\substack{j=1 \\ j \neq i}}^{N} \mathbf{E}_j$), then an appropriate solution concept is a Baysian Nash equilibrium (BNE), [49]. On the other hand, if agent i, $i \in \mathcal{N}$ knows $\mathbf{e}_i \in \mathbf{E}_i$, and $\mathbf{E}_j$, for all $j \neq i$, then an appropriate solution concept is a Nash equilibrium (NE) [38] or a subgame perfect NE or a sequential NE ([10]), etc.

In the problem investigated in this paper we consider NE as the solution/equilibrium concept (see the discussion at the end of section 3.2). For this reason, we use NE as the solution concept to define implementation of a goal correspondence π by a game form

$(\mathcal{M}, f)$ via a solution/equilibrium concept. For any $(\mathcal{M}, f, \mathbf{e})$, a pure NE is a message $\mathbf{m}^* := (\mathbf{m}_1^*, \mathbf{m}_2^*, \cdots, \mathbf{m}_N^*) \in \mathcal{M}$ such that

$$\mathbf{e}_i\big(f(\mathbf{m}^*)\big) \geq \mathbf{e}_i\big(f(\mathbf{m}_{-i}^*, \mathbf{m}_i)\big) \qquad \forall i \in \mathcal{N} \quad \& \quad \forall \mathbf{m}_i \in \mathcal{M}_i \tag{2}$$

where $\mathbf{m}_{-i} := (\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_{i-1}, \mathbf{m}_{i+1}, \cdots, \mathbf{m}_N)$. Denote the messages satisfying (2) by $NE(\mathcal{M}, f, \mathbf{e})$. Then, the set of associated outcomes is

$$\Theta_{NE}(\mathcal{M}, f, \mathbf{e}) := \Big\{a \in \mathcal{A} | \exists \mathbf{m} \in NE(\mathcal{M}, f, \mathbf{e}) \ s.t. \ f(\mathbf{m}) = a\Big\}. \tag{3}$$

Based on the above description we can now specify how a goal correspondence π is implicitly enforced via a game form in NE.

Definition *1:* A game form $(\mathcal{M}, f)$ is said to implement (respectively, fully implement) in NE the goal correspondence $\pi : \mathbf{E} \to \mathcal{A}$ if for all $\mathbf{e} \in \mathbf{E}$ ($\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N)$)

$$\Theta_{NE}(\mathcal{M}, f, \mathbf{e}) \subseteq \pi(\mathbf{e}) \tag{4}$$

(respectively, $\Theta_{NE}(\mathcal{M}, f, \mathbf{e}) = \pi(\mathbf{e})$.)

Definition *2:* A goal correspondence $\pi : \mathbf{E} \to \mathcal{A}$ is said to be implemented in NE if there exists a game form $(\mathcal{M}, f)$ that implements it.

Within the context of implementation theory there have been significant developments in the characterization of goal correspondences that can be implemented in the following solution concepts: dominant strategies [6], [11], Nash equilibria [39], [40], [55], [65], refined Nash equilibria such as subgame perfect equilibria [**?**], [42], undominated Nash equilibria [**?**], [19], [21], [47], trembling hand perfect Nash equilibria [59], Bayesian Nash equilibria [18], [46], [48], [52]. Excellent survey articles on implementation theory are [20], [40], [45]. These articles summarize the state of the art on implementation theory up to the time of their publication.

Definitions 1 above implies that a game form $(\mathcal{M}, f)$ that implements in NE a goal correspondence π takes into account the agents'/users' strategic behavior and obtains centralized solutions. An implicit assumption in the above definitions is that the users participate in the message exchange process specified by the game form. In order for the users to voluntarily participate in a mechanism/decentralized allocation process specified by a game form $(\mathcal{M}, f)$, the game form must satisfy an additional property defined as follows. Let the initial endowment of a user be defined as the amount of resources that the user has before participating in a game form/decentralized resource allocation process. (For example, in the problem we considered in this paper the initial

endowment of user i, $i \in \mathcal{N}$, is the tax or subsidy and rate/bandwidth before the rate allocation mechanism is run, and this tax and rate/bandwidth are both zero). Then we have the following.

Definition *3:* INDIVIDUAL RATIONALITY: A game form $(\mathcal{M}, f)$ is said to be individually rational if for all users i, $i \in \mathcal{N}$, $\mathbf{e}_i(f(\mathbf{m})) \geq \mathbf{e}_i(0,0)$ for all $\mathbf{m} \in NE(\mathcal{M}, f, \mathbf{e})$, $\forall\ \mathbf{e} \in \mathbf{E}$.

Individual rationality 3 asserts that at any NE (or at any solution/equilibrium concept that is appropriate for the allocation process) the utility of each user is at least as much as its utility before participating in the allocation process.

Definitions 1 and 3 imply that a game form that is individually rational and implements in NE (or the appropriate solution/equilibrium concept) a goal correspondence, obtains optimal allocations of the corresponding centralized system by having users voluntarily participate in the allocation process.

We wish to have a mechanism, which, in addition to possessing the above properties, is budget-balanced at all equilibria as well as off equilibria. Budget balance means that the sum of taxes paid by some of the users/agents is equal to the sum of the money (subsidies) received by the rest of the users participating in the allocation process. Budget balance implies that there is no money left unallocated at the end of the allocation.

The above presentation highlights the properties that are desirable in any mechanism/game form designed for the problem considered in this paper.

In summary, any mechanism/game form for our decentralized allocation problem must achieve:

1) Implementation in NE of the goal correspondence π defined by the solution of the corresponding centralized problem.
2) Individual rationality.
3) Budget balance.

After this brief introduction to the key concepts and ideas of implementation theory, we proceed to formulate the unicast service provisioning problem.

III. THE UNICAST PROBLEM WITH STRATEGIC NETWORK USERS, PROBLEM FORMULATION

In this section we present the formulation of the unicast problem in wired communication networks with strategic users. We proceed as follows, In section III-A we formulate

the centralized unicast problem the solution of which we want to implement in Nash equilibria. In section III-B we formulate the decentralized unicast problem with strategic network users, we state our assumptions, our objective and provide an interpretation of the equilibrium concept (Nash equilibrium) in which we want to implement the solution of the centralized problem of section III-A.

A. The centralized problem

We consider a wired network with N, $N > 2$, users. The set of these users is denoted by $\mathcal{N}$, i.e. $\mathcal{N} = \{1, 2, \cdots, N\}$. The network topology, the capacity of the network links, and the routes assigned to users' services are fixed and given. The users' utility functions have the form

$$\mathcal{V}_i(x_i, t_i) = \mathbf{U}_i(x_i) - t_i. \tag{5}$$

The term $\mathbf{U}_i(x_i)$ expresses user i's *satisfaction* from the service it receives. The term t_i represents the *tax* (money) user i pays for the services it receives. We assume that $\mathbf{U}_i$ is a concave and increasing function of the service x_i user i receives, and $t_i \in \mathbb{R}$. When $t_i > 0$ user i pays money for the services it receives; this money is paid to other network users. When $t_i < 0$ user i receives money from other users. Overall, the amount of money paid by some of the network users is equal to the amount of money received by the rest of the users so that $\sum_{i\in\mathcal{N}} t_i = 0$. Denote by $\mathbf{L}$ the set of links of the network, by $\mathbf{c}^l$ the capacity of link l, and by $\mathcal{R}_i$ the set of links l, $l \in \mathbf{L}$, that form the route of user i, $i = 1, 2, \cdots, N$ (as pointed out above each user's route is fixed). We assume that a central authority (the network manager) has access to all of the above information. The objective of this authority is to solve the following centralized optimization problem that we call **Max.0**.

$$\textbf{Max.0} \qquad \max_{x_i, t_i} \quad \sum_{i=1}^{N} \big[\mathbf{U}_i(x_i) - t_i\big] \tag{6}$$

$$subject\ to$$

$$\sum_{i=1}^{N} t_i = 0, \tag{7}$$

$$\sum_{i:l\in\mathcal{R}_i} x_i \le \mathbf{c}^l, \quad \forall\ l \in \mathbf{L}, \tag{8}$$

$$x_i \ge 0, \quad \forall i \in \mathcal{N}, \tag{9}$$

$$t_i \in \mathbb{R}, \quad \forall i \in \mathcal{N}. \tag{10}$$

The inequalities in (8) express the capacity constraints that must be satisfied at each network link. The equality in (7) express the fact that the budget must be balanced, i.e.

the total amount of money paid by some of the users must be equal to the amount of money received by the rest of the users. The inequalities in (9) express the fact that the users' received services $x_i, i \in \mathcal{N}$ must be nonnegative.

Problem **Max.0** is equivalent to problem **Max.1** below,

$$\textbf{Max.1} \qquad \max_{x_i} \quad \sum_{i\in\mathcal{N}} \mathbf{U}_i(x_i) \tag{11}$$

$$subject\ to$$

$$\sum_{i:\ l\in\mathcal{R}_i} x_i \le \mathbf{c}^l, \quad \forall\ l \in \mathbf{L}, \tag{12}$$

$$x_i \ge 0, \quad \forall\ i \in \mathcal{N}, \tag{13}$$

since $t_i = 0$ for every $i \in \mathcal{N}$ is one of maximizing choices of $\mathbf{t} = (t_1, t_2, \cdots, t_N)$ in **Max.0**. Thus, we will refer to **Max.1** as the centralized resource allocation problem.

Let $\mathcal{U}$ denote the set of functions

$$\mathbf{U} : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$$

where $\mathbf{U}$ is concave and increasing. Let $\mathbf{T}$ denote the set of all possible network topologies, network resources and user routes. Consider problem **Max.1** for all possible realizations

$$(\mathbf{U}_1, \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T}$$

of the users' utilities, the network topology, its resources and the users' routes. Then, the solution of **Max.1** for each $(\mathbf{U}_1, \mathbf{U}_2 \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T}$ defines a map

$$\pi : \mathcal{U}^N \times \mathbf{T} \to \mathcal{A}$$

where $\mathcal{A} \in R_+^N$ is the set of all possible rate/bandwidth allocations to the network's users. We call π the solution of the centralized unicast service provisioning problem.

B. The decentralized problem with strategic users

We consider the network model of the previous section with the following assumptions on its information structure.

- (**A1**): Each user knows only his own utility; this utility is his own private information.
- (**A2**): Each user behaves strategically (i.e. each user is selfish, his objective is to maximize his own utility function $\mathbf{U}_i(x_i) - t_i$).

- (**A3**): The network manager knows the topology and resources of the network and is not a profit-maker (i.e. he does not have a utility function).
- (**A4**): The network manager receives requests for service from the network users. Based on these requests, he announces to each user $i, i \in \mathcal{N}$:
  1) The set of links that form user i's route, $\mathcal{R}_i$; that is, the network manager chooses the route for each user and this route remains fixed throughout the user's service.
  2) The capacity of each link in $\mathcal{R}_i$.
  3) The set of other users beside i that use each link in $\mathcal{R}_i$.
- (**A5**) Based on the network manager's announcement, each user competes for resources (bandwidth) at each link of his route with the other users in that link.

From the above description it is clear that the network manager (which is not profit maker) acts like an accountant who sets up the users' routes, specifies the users competing for resources/bandwidth at each link, collects the money from the users i that pay tax (i.e. $t_i > 0$) and distributes it to those users j that receive money (i.e. $t_i < 0$).

As a consequence of assumptions (**A1**)-(**A5**) we have at each link of the network a decentralized resource allocation problem which can be studied/analyzed within the context of implementation theory [17]. These decentralized resource allocation problems are not completely independent/decoupled, as the rate that each user receives at any link of its own route must be the same. This constraint is dictated by the nature of the unicast service provisioning problem and has a direct implication on the nature of the mechanism/game form we present in section IV.

Under the above assumptions the objective is to determine a game form/mechanism which has the following properties,

- (**P1**). For each realization $(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T}$, the Nash equilibria of the game induced by the mechanism result in allocations that are an optimal solution of the corresponding centralized problem **Max.0**.
- (**P2**). The mechanism is individually rational, that is, for every realization

$$(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T},$$

  the network users voluntarily participate in the corresponding game.
- (**P3**). For every realization $(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T}$ we have a balanced budget at every equilibrium point of the corresponding game form as well as off equilibrium.

Before proceeding with the specification of our mechanism, we comment on the appropriateness of Nash equilibrium as a solution concept for the decentralized problem

under consideration. Nash equilibria describe strategic behavior in games of complete information. Since in our model, the network users do not know each others' utilities, for any realization $(\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_N, T) \in \mathcal{U}^N \times \mathbf{T}$ the resulting game is not a game of complete information. We can create a game of complete information by increasing the strategy space of the game following Maskin's approach [39] . Such an approach, however, would result in an infinite dimensional strategy space for the corresponding game. We do not follow Maskin's approach but we adopt the philosophy presented by Reichelstein and Reiter in [54] and Groves and Ledyard in [13]. Specifically:

> *"We interpret our analysis as applying to an unspecified (message exchange) process in which users grope their way to a stationary message and in which the Nash property is a necessary condition for stationarity"*, Reichelstein and Reiter ([54] pg. 664).
>
> *" We do not suggest that each agent knows* $\boldsymbol{e}$[2] *when he computes* $\boldsymbol{m}_i$[3]*, ... We do suggest, however, that the 'complete information' Nash equilibrium game-theoretic equilibrium messages may be the possible equilibrium of the iterative process–that is, the stationary messages–just as the demand-equal-supply price is thought of the equilibrium of some unspecified market dynamic process."*, Groves and Ledyard ([13] pp. 69-70).

A philosophy similar to ours has also been adopted by Stoenescu and Ledyard in [60].

In the rest of paper we present a mechanism/game form for the problem formulated in this section which possess properties $\mathtt{P1} - \mathtt{P3}$ stated above.

IV. A Mechanism for Rate Allocation

In section IV-A, we specify a mechanism/game form for the decentralized rate allocation problem formulated in section III. In section IV-B, we discuss and interpret the components of the mechanism.

A. Specification of the mechanism

A game form/mechanism ([17]) consists of two components $\mathcal{M}, f$. The component $\mathcal{M}$ denotes the users' *message/strategy space*. The component f is the *outcome function*; f defines for every message/strategy profile, the bandwidth/rate allocated to each user and the tax (subsidy) each user pays (or receives).

[2]In our mechanism $\mathbf{e} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_N, T)$, that is, a realization of the users' utilities as well as of the topology and resources of the network.

[3]In our problem $\mathbf{m}_i$ is the strategy of user i, $i \in \mathcal{N}$.

For the decentralized resource allocation problem formulated in section III we propose a game form/mechanism the components of which we describe below.

Message space: The message/strategy space for user i, $i = 1, 2, ..., N$, is given by $\mathcal{M}_i \subset R_+^{|\mathcal{R}_i|+1}$. Specifically a message of user i is of the form

$$\mathbf{m}_i = (x_i, p_i^{l_{i_1}}, p_i^{l_{i_2}}, \cdots, p_i^{l_{i_{|\mathcal{R}_i|}}})$$

where $0 \leq x_i \leq \min_{l \in \mathcal{R}_i} \mathbf{c}^l$ and $0 \leq p_i^{l_{i_k}} \leq M, k = 1, 2, , \cdots, |\mathcal{R}_i|$, $0 < M < \infty$, and $|\mathcal{R}_i|$ denotes the number of links along route $\mathcal{R}_i, i \in \mathcal{N}$. The component x_i denotes the bandwidth/rate user i requests at all the links of his route. The component $p_i^{l_{i_j}}, j = 1, 2, \cdots, |\mathcal{R}_i|$, denotes the price per unit of bandwidth user i is willing to pay at link l_{i_j}.

As noted in section III-B, the nature of the unicast service provisioning problem dictates/requires that the bandwidth/rate allocated to any user i, $i \in \mathcal{N}$, must be the same at all links of its route. Thus, the nature of message $\mathbf{m}_i$ is a consequence of the above requirement.

Outcome Function: The outcome function f is given by

$$f : \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_N \to (\mathbb{R}_+^N \times \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R})$$

and is defined as follows. For any $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_N) \in \mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_N$,

$$f(\mathbf{m}) = f(\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_N) = (x_1, x_2, \cdots, x_N, t_1, t_2, \cdots, t_N)$$

where $x_i, i \in \mathcal{N}$, is the amount of bandwidth/rate allocated to user i (this is equal to the amount of bandwidth user i, $i \in \mathcal{N}$, requests), and $t_i, i \in \mathcal{N}$, is determined by t_i^l, the tax (subsidy) user i pays (receives) for link $l \in \mathcal{R}_i$, and by other additional subsidies $\mathcal{Q}^i$ that user i may receive. We proceed now to specify $t_i^l, l \in \mathcal{R}_i$, and $\mathcal{Q}^i$ for every user $i \in \mathcal{N}$.

The tax $t_i^{l_{i_j}}, j = 1, 2, \cdots, |\mathcal{R}_i|$, $i \in \mathcal{N}$, is defined according to the number of users using link l. Let $\mathcal{G}^l$ denotes the set of users using link l and let $|\mathcal{G}^l|$ denote the cardinality of $\mathcal{G}^l$. We consider three cases[4]

- CASE 1 , $\left|\mathcal{G}^l\right| = 2$
  Let $i,\ j \in \mathcal{G}^l$. Then,

$$t_i^l = p_j^l x_i + \frac{(p_i^l - p_j^l)^2}{\alpha} - 2p_j^l(p_i^l - p_j^l)\Big(\frac{x_i + x_j - \mathbf{c}^l}{\gamma}\Big) + \frac{1\{x_i > 0\}1\{x_i + x_j - \mathbf{c}^l > 0\}}{1 - 1\{x_i > 0\}1\{x_i + x_j - \mathbf{c}^l > 0\}}$$

$$t_j^l = p_i^l x_j^l + \frac{(p_j^l - p_i^l)^2}{\alpha} - 2p_i^l(p_j^l - p_i^l)\Big(\frac{x_i + x_j - \mathbf{c}^l}{\gamma}\Big) + \frac{1\{x_j > 0\}1\{x_i + x_j - \mathbf{c}^l > 0\}}{1 - 1\{x_j > 0\}1\{x_i + x_j - \mathbf{c}^l > 0\}}$$

[4]We consider only the cases where $|\mathcal{G}^l| \geq 2$. If $|\mathcal{G}^l| = 1$ and $i \in \mathcal{G}^l$, then $t_i^l = 0 \cdot 1\{x_i \leq \mathbf{c}^l\} + \frac{1\{x_i > \mathbf{c}^l\}}{1 - 1\{x_i > \mathbf{c}^l\}}$.

where α and γ are positive constants that are sufficiently large and, the function $1\{A\}$, used throughout the paper, is defined as follows

$$1\{A\} = \begin{cases} 1-\varepsilon & \text{if } A \text{ holds;} \\ 0 & \text{otherwise.} \end{cases}$$

where ε is bigger than zero and sufficiently small[5]; ε is chosen by the mechanism designer.

- CASE 2, $\left|\mathcal{G}^l\right| = 3$
  Let $i,\ j$ and $k \in \mathcal{G}^l$. Then

$$\begin{aligned} t_i^l &= \left(P_{-i}^l + \frac{p_k^l(p_j^l - p_k^l)}{\gamma}\right) x_i + (p_i^l - P_{-i}^l)^2 - 2P_{-i}^l(p_i^l - P_{-i}^l)\Big(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma}\Big) \\ &\quad + \frac{1\{x_i > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}}{1 - 1\{x_i > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}} + \Omega_i^l \qquad (14) \\ t_j^l &= \left(P_{-j}^l + \frac{p_i^l(p_k^l - p_i^l)}{\gamma}\right) x_j + (p_j^l - P_{-j}^l)^2 - 2P_{-j}^l(p_j^l - P_{-j}^l)\Big(\frac{\mathcal{E}_{-j}^l + x_j}{\gamma}\Big) \\ &\quad + \frac{1\{x_j > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}}{1 - 1\{x_j > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}} + \Omega_j^l \qquad (15) \\ t_k^l &= \left(P_{-k}^l + \frac{p_j^l(p_i^l - p_j^l)}{\gamma}\right) x_k + (p_k^l - P_{-k}^l)^2 - 2P_{-k}^l(p_k^l - P_{-k}^l)\Big(\frac{\mathcal{E}_{-k}^l + x_k}{\gamma}\Big) \\ &\quad + \frac{1\{x_k > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}}{1 - 1\{x_k > 0\}1\{x_i + x_j + x_k - \mathbf{c}^l > 0\}} + \Omega_k^l \qquad (16) \end{aligned}$$

  where,

$$\begin{aligned} &P_{-i}^l = \frac{p_j^l + p_k^l}{2}, P_{-j}^l = \frac{p_k^l + p_i^l}{2}, P_{-k}^l = \frac{p_j^l + p_i^l}{2}, \\ &\mathcal{E}_{-i}^l = x_j + x_k - \mathbf{c}^l, \mathcal{E}_{-j}^l = x_i + x_k - \mathbf{c}^l, \mathcal{E}_{-k}^l = x_i + x_j - \mathbf{c}^l \\ &\mathcal{E}_i^l = 2x_i - \mathbf{c}^l, \mathcal{E}_j^l = 2x_j - \mathbf{c}^l, \mathcal{E}_k^l = 2x_i - \mathbf{c}^l, \qquad (17) \end{aligned}$$

[5]Therefore, when A and B (both) hold, then $\frac{1\{A\}1\{B\}}{1-1\{A\}1\{B\}} \approx \frac{1}{0^+}$ is well defined and it becomes a large number.

and Ω_i^l is defined as

$$\begin{aligned}\Omega_i^l &= \frac{{p_j^l}^2 x_k}{\gamma} + \frac{\sum_{\substack{r\in\mathcal{G}^l\\ r\neq i}}\sum_{\substack{s\in\mathcal{G}^l\\ s\neq i,r}}\Big(2p_r^l p_s^l(1+\frac{x_r}{\gamma}) - x_r p_s^l\Big)}{(|\mathcal{G}^l|-1)(|\mathcal{G}^l|-2)} + \frac{\sum_{\substack{r\in\mathcal{G}^l\\ r\neq i}}\sum_{\substack{s\in\mathcal{G}^l\\ s\neq i,j}} 2p_s^l(p_r^l\mathcal{E}_s^l - x_r p_s^l)}{\gamma(|\mathcal{G}^l|-1)^2(|\mathcal{G}^l|-2)} \\ &\quad - \frac{\sum_{\substack{j\in\mathcal{G}^l\\ r\neq i}} {p_r^l}^2}{|\mathcal{G}^l|-1} - {P_{-i}^l}^2 - 2\frac{\mathcal{E}_{-i}^l {P_{-i}^l}^2}{\gamma}. \end{aligned} \tag{18}$$

The terms Ω_j^l and Ω_k^l are defined in a way similar to Ω_i^l.

- CASE 3, $|\mathcal{G}^l| > 3$
  Let $i \in \mathcal{G}^l \subseteq \mathcal{N}$. Then,

$$\begin{aligned} t_i^l &= P_{-i}^l x_i + (p_i^l - P_{-i}^l)^2 - 2P_{-i}^l(p_i^l - P_{-i}^l)\Big(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma}\Big) + \frac{1\{x_i>0\}1\{\mathcal{E}_{-i}^l + x_i > 0\}}{1 - 1\{x_i>0\}1\{\mathcal{E}_{-i}^l + x_i>0\}} \\ &\quad + \Phi_i^l \end{aligned} \tag{19}$$

where,

$$P_{-i}^l = \frac{\sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}} p_j^l}{|\mathcal{G}^l|-1}, \quad \mathcal{E}_{-i}^l = \sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}} x_j - \mathbf{c}^l, \quad \mathcal{E}_i^l = (|\mathcal{G}^l|-1)x_i - \mathbf{c}^l,$$

and

$$\begin{aligned}\Phi_i^l &= \frac{\sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}}\sum_{\substack{k\in\mathcal{G}^l\\ k\neq i,j}}\Big(2p_j^l p_k^l(1+\frac{x_j}{\gamma}) - x_j p_k^l\Big)}{(|\mathcal{G}^l|-1)(|\mathcal{G}^l|-2)} + \frac{\sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}}\sum_{\substack{k\in\mathcal{G}^l\\ k\neq i,j}}\sum_{\substack{r\in\mathcal{G}^l\\ r\neq i,j,k}} 2p_k^l(p_j^l\mathcal{E}_r^l - x_j p_r^l)}{\gamma(|\mathcal{G}^l|-1)^2(|\mathcal{G}^l|-3)} \\ &\quad + \frac{\sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}}\sum_{\substack{k\in\mathcal{G}^l\\ k\neq i,j}} 2p_k^l(p_j^l\mathcal{E}_k^l - x_j p_k^l)}{\gamma(|\mathcal{G}^l|-1)^2(|\mathcal{G}^l|-2)} - \frac{\sum_{\substack{j\in\mathcal{G}^l\\ j\neq i}} {p_j^l}^2}{|\mathcal{G}^l|-1} - {P_{-i}^l}^2 - 2\frac{\mathcal{E}_{-i}^l {P_{-i}^l}^2}{\gamma}. \end{aligned} \tag{20}$$

Next we specify additional subsidies $\mathcal{Q}^i$ that user $i, i \in \mathcal{N}$, may receive. For that matter we consider all links $l \in \mathbf{L}$ such that $|\mathcal{G}^l| = 2$. For each such link l, we define the quantity

$$\begin{aligned}\mathcal{Q}^l &:= -2\frac{(p_i^l - p_j^l)^2}{\alpha} - p_j^l x_i - p_i^l x_j\left[2p_j^l(p_i^l - p_j^l) + 2p_i^l(p_j^l - p_i^l)\right]\left(\frac{x_i + x_j - \mathbf{c}^l}{\gamma}\right) \\ &= o(1) - p_j^l x_i - p_i^l x_j. \end{aligned} \tag{21}$$

Furthermore for each such $l \in \mathbf{L}$ where $|\mathcal{G}^l| = 2$ the network manager chooses at random a user $k_l \notin \mathcal{G}^l$ and assigns the subsidy $\mathcal{Q}^l$ to user k_l. Let $l_1, l_2, \cdots, l_r$ be the set of links such that $|\mathcal{G}^{l_i}| = 2, i = 1, 2, \cdots, r$, and let k_{l_i} be the corresponding users that receive $\mathcal{Q}^{l_i}$.

Based on the above, the tax (subsidy) paid (received) by user $j, j \in \mathcal{N}$, is the following. If $j \neq k_{l_1}, k_{l_2}, \cdots k_{l_r}$ then

$$t_j = \sum_{l \in \mathcal{R}_j} t_j^l, \tag{22}$$

where for each $l \in \mathcal{R}_j$, t_j^l is determined according to the cardinality of $\mathcal{G}^l$. If $j = k_{l_i}, i = 1, 2, \cdots, r$, then

$$t_{k_{l_i}} = \sum_{l \in \mathcal{R}_{k_{l_i}}} t_{k_{l_i}}^l + \mathcal{Q}^{l_i}. \tag{23}$$

where $\mathcal{Q}^{l_i}$ is defined by (21).

Note that $\mathcal{Q}^{l_i}$ is not controlled by user k_{l_i}, that is, $\mathcal{Q}^{l_i}$ does not depend on user k_{l_i}'s message/strategy. Thus, the presence (or absence) of $\mathcal{Q}^{l_i}$ does not influence the strategic behavior of user k_{l_i}. We have assumed here that the users $k_{l_1}, k_{l_2}, \cdots, k_{l_r}$, are distinct. Expressions similar to the above hold when the users $k_{l_1}, k_{l_2}, \cdots, k_{l_r}$ are not distinct.

B. Discussion/Interpretation of the Mechanism

As pointed out in section III-B, the design of a decentralized resource allocation mechanism has to achieve the following goals. (1) It must induce strategic users to voluntarily participate in the allocation process. (2) It must induce strategic users to follow its operational rules. (3) It must result in optimal allocations at all equilibria of the induced game. (4) It must result in a balanced budget at all equilibria and off equilibrium.

Since the designer of the mechanism can not alter the users' utility functions, $\mathbf{U}_i, i \in \mathcal{N}$, the only way it can achieve the aforementioned objectives is through the use of appropriate *tax incentives/tax functions*. At each link l, the tax incentive of our mechanism for user i consists of three components $\Delta_1^l(i)$, $\Delta_2^l(i)$ and $\Delta_3^l(i)$. We specify and interpret these components for *Case 3* (Eq. (19)). Similar interpretations hold for *Case 1* and *Case 2*.

For *Case 3* we have,

$$t_i^l := \Delta_1^l(i) + \Delta_2^l(i) + \Delta_3^l(i) \tag{24}$$

where

$$\Delta_1^l(i) := P_{-i}^l x_i \tag{25}$$

$$\Delta_2^l(i) := (p_i^l - P_{-i}^l)^2 - 2P_{-i}^l(p_i^l - P_{-i}^l)\Big(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma}\Big) + \frac{1\{x_i > 0\}1\{\mathcal{E}_{-i}^l + x_i > 0\}}{1 - 1\{x_i > 0\}1\{\mathcal{E}_{-i}^l + x_i > 0\}} \tag{26}$$

$$\Delta_3^l(i) := \Phi_i^l \tag{27}$$

- $\Delta_1^l(i)$ specifies the amount user i has to pay for the bandwidth it gets at link l. It is important to note that the price per unit of bandwidth that a user pays is determined by the message/proposal of the other users using the same link. Thus, a user does not control the price it pays per unit of the service it receives.
- $\Delta_2^l(i)$ provides the following incentives to the users of a link: (1) To bid/propose the same price per unit of bandwidth at that link (2) To collectively request a total bandwidth that does not exceed the capacity of the link. The incentive provided to all users to bid the same price per unit of bandwidth is described by the term $(p_i^l - P_{-i}^l)^2$. The incentive provided to all users to collectively request a total bandwidth that does not exceed the link's capacity is captured by the term

$$\frac{1\{x_i > 0\}1\{\mathcal{E}_{-i}^l + x_i > 0\}}{1 - 1\{x_i > 0\}1\{\mathcal{E}_{-i}^l + x_i > 0\}}. \tag{28}$$

  Note that a user is very heavily penalized if it requests a nonzero bandwidth, and, collectively, all the users of the link request a total bandwidth that exceeds the link's capacity. A joint incentive provided to all users to bid the same price per unit of bandwidth and to utilize the total capacity of the link is captured by the term

$$2P_{-i}(p_i^l - P_{-i}^l)\Big(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma}\Big) \tag{29}$$

- $\Delta_3^l(i)$, The goal of this component is to lead to a balanced budget. That is,

$$\sum_{i \in \mathcal{G}^l} [\Delta_1^l(i) + \Delta_2^l(i)] \neq 0, \tag{30}$$

  but,

$$\sum_{i \in \mathcal{G}^l} [\Delta_1^l(i) + \Delta_2^l(i) + \Delta_3^l(i)] = 0. \tag{31}$$

  Note that, $\Delta_3^l(i)$ is not controlled by user i's messages (simply because there is no term in $\Delta_3^l(i)$ under the control of user i), so $\Delta_3^l(i)$ does not have any influence on the strategic behavior of the user.

As indicated in (31), when the number of users at link $l \in \mathbf{L}$ is larger than or equal to

three, i.e. $|\mathcal{G}^l| \geq 3$, the mechanism is budget-balanced at that link, that is $\sum_{i\in\mathcal{G}^l} t_i^l = 0$. When $|\mathcal{G}^l| = 2$ the mechanism is not budget balanced at link l. The amount $\mathcal{Q}^l = -\sum_{\substack{i\in\mathcal{G}^l \\ |\mathcal{G}^l|=2}} t_i^l$, is given as subsidy to a randomly chosen user, say j, that does not compete for resources at link l. Such money transfers results in an overall balanced budget, and are always possible whenever $N > 2$. Furthermore, the money transfered to user j does not alter j's strategic behavior since $\mathcal{Q}^l$ does not depend on user j's strategy. The existence of the term $\mathcal{Q}_{l_j}$ in the tax function couples the games that are taking place at various links of the network. The presence of $\mathcal{Q}^{l_j}$ implies that the designer of the mechanism must not consider links individually; for the allocation of resources at certain links (specially those links l with $|\mathcal{G}^l| = 2$) the design must consider network users that do not compete for resources in those links.

V. Properties of the Mechanism

We prove that the mechanism proposed in section IV has the following properties: **(P1)** It implements the solution of problem **Max.0** in Nash equilibria. **(P2)** It is individually rational. **(P3)** It is budget-balanced at every feasible allocation, that is the mechanism is budget-balanced at allocations corresponding to all NE messages as well as those corresponding to off-equilibrium messages. We also prove the existence of NE in the game induced by the mechanism and characterized all of them.

We establish the above properties by proceeding as follows. First we prove that all Nash equilibria of the game induced by the game form/mechanism of section IV result in feasible solutions of the centralized problem **Max.1**, (Theorem 1). Then, we show that network users voluntarily participate in the allocation process. We do this by showing that the allocations they receive at all Nash equilibria of the game induced by the game form of section IV are weakly preferred to the $(0, 0)$ allocation they receive when they do not participate in the allocation process (Theorem 5). Afterwards, we establish that the mechanism is budget-balanced at all Nash equilibria; we also prove that the mechanism is budget-balanced off equilibrium (Lemma 2). Finally, we show that the mechanism implements in Nash equilibria the solution of the centralized allocation problem **Max.0** (Theorem 6).

We present the proofs of the following theorems and lemmas in Appendix.

Theorem *1:* (Feasibility): *If $\boldsymbol{m}^* = (\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a NE point of the game induced by the game form and the users' utility(outcome) functions presented in section IV then the allocation $\boldsymbol{x}^*$ is a feasible solution of problem **Max.1**.*

The following lemma presents some key properties of NE prices and rates.

Lemma *2: Let $\boldsymbol{m}^* = (\boldsymbol{x}^*, \boldsymbol{p}^*)$ be a NE. Then for every $l \in \boldsymbol{L}$ and $i \in \mathcal{G}^l$, we have,*

$$p_i^{*l} = p_j^{*l} = P_{-i}^{*l} := p^{*l}, \tag{32}$$

$$p^{*l}\Big(\frac{\mathcal{E}^{*l}}{\gamma}\Big) = 0, \tag{33}$$

$$\frac{\partial t_i^l}{\partial x_i}\bigg|_{\mathbf{m}=\mathbf{m}^*} = p^{*l}, \tag{34}$$

where $\mathcal{E}^{*l} = \sum_{i\in\mathcal{G}^l} x_i^* - \mathbf{c}^l$.

An immediate consequence of Lemma 2 is the following.

Corollary *3:* At every NE point $\mathbf{m}^*$ of the mechanism the tax function has the following form,

$$t_i^l(\mathbf{m}^*) = \begin{cases} p^{*l}x_i^* & \text{if } |\mathcal{G}^l| = 2; \\ p^{*l}\Big[x_i^* - \frac{x_j^*+x_k^*}{2}\Big] + \frac{p^{*l^2}x_k^*}{\gamma} & \text{if } |\mathcal{G}^l| = 3; \\ p^{*l}(x_i^* - x_{-i}^*) & \text{if } |\mathcal{G}^l| > 3. \end{cases} \tag{35}$$

Thus,

$$t_i(\mathbf{m}^*) = \sum_{l\in\mathcal{R}_i} t_i^l(\mathbf{m}^*), \tag{36}$$

for $i \neq k_{l_1}, k_{l_2}, \cdots, k_{l_r}$, (cf section IV), and for $i = k_{l_j}, j = 1, 2, \cdots, r$,

$$t_{k_{l_j}}(\mathbf{m}^*) = p^{*l_j}(x_{i_1}^* + x_{i_2}^*) + \sum_{l\in\mathcal{R}_{k_{l_j}}} t_{k_{l_j}}^l(\mathbf{m}^*) \tag{37}$$

where $\{i_1, i_2\} = \mathcal{G}^{l_j}$.

Proof: The result follows directly from Lemma 2, and specification of the tax for each user, designed by Eq. (22) and (23). ■

In the following lemma, we prove that the proposed mechanism is always budget balanced.

Lemma *4:* The proposed mechanism/game form is always budget balanced at every feasible allocation. That is, the mechanism is budget-balanced at all allocations corresponding to NE messages as well as messages that are off equilibrium.

The next result asserts that the mechanism/game form proposed in section IV is individually rational.

Theorem *5:* (INDIVIDUAL RATIONALITY): *The game form specified in section IV is individually rational, i.e. at every NE point the corresponding allocation $(\boldsymbol{x}^*, \boldsymbol{t}^*)$ is weakly preferred by all users to the initial allocation $(0, 0)$.*

Finally, we prove that the mechanism of section IV impements in NE the correspondence π defined by the solution of problem **Max.0**.

Theorem *6:* (NASH IMPLEMENTATION): Consider any NE $\mathbf{m}^*$ of the game induced by the mechanism of section IV. Then, the allocation $(\mathbf{x}^*, \mathbf{t}^*)$ corresponding to $\mathbf{m}^*$ is an optimal solution of the centralized problem **Max.0**.

EXISTENCE AND CHARACTERIZATION OF THE NASH EQUILIBRIA:
So far, we have assumed the existence of NE of the game induced by the proposed game form/mechanism. In the following theorem, we prove that NE exist (recall the interpretation of NE we have given at the end of section III) and characterize all of them.

Theorem *7:* Let $(x_1^*, x_2^*, \cdots, x_N^*)$ be an optimal solution of problem **Max.1** and λ^{*l}, $l \in \mathbf{L}$, be the corresponding Lagrange multipliers of the *KKT* conditions. Then

$$\mathbf{m}^* := (x_1^*, x_2^*, \cdots, x_N^*, p^{*l_1}, p^{*l_2}, \cdots, p^{*l_L})$$

with $p^{*l} = \lambda^{*l}, l \in \mathbf{L}$ is a NE of the game induced by the proposed game form.

VI. IMPLEMENTATION OF THE DECENRALIZED MECHANISM

First, we discuss how the mechanism specified by the game form of section IV can be implemented at equilibrium. Then, we address the computation of the NE of the game induced by the game form of this paper.

We present one way of implementing the proposed mechanism at equilibrium. Consider an arbitrary link l of the network. The users of that link communicate their equilibrium messages to one another and to the network manager. The network manager determines the rate and tax (or subsidy) of each user and announces this information to the user. The users $i, i \in \mathcal{N}$ with tax $t_i^l > 0$ pay the amount t_i^l to the network manager; the network manager redistributes the amount of money it receives to the users $j \in \mathcal{N}$ with $t_j^l < 0$. In the situation where the number of users in the link is equal to two the network manager chooses randomly a third user to whom it gives the subsidy $\mathcal{Q}^l$ defined by (21). The above described process is repeated/takes place at every network link. This process implements the mechanism described in the paper at equilibrium.

Even though for the specific form of the tax we have provided a complete characterization of the NE of the game induced by the game form proposed in the paper, currently we do not have an algorithm for the computation of these equilibria. Based on preliminary investigation, we believe that best response algorithms do not, in general, guarantee convergence to NE equilibria, because the game induced by the game form proposed in this paper is not supermodular (due to the capacity constraint present at

each link). Thus, the algorithmic computation of the NE of the game induced by the game form proposed in the paper remains as an open problem.

VII. An Extension

So far we required that the users' utility functions be concave. We now weaken this requirement; we assume that the users' utilities are quasi-concave. We consider the game form proposed in section IV. By repeating the arguments of Theorem 1, Lemma 2, Corollary 3, Lemma 4 and Theorem 5 we can show that: every NE of the game induced by the game form is feasible; the game form/mechanism is individually rational and budget-balanced at all feasible allocations, i.e. at every NE and off equilibrium. In the following theorem we prove that every NE of the game induced by the proposed game form results in a Walrasian Equilibrium (WE) [37].

Theorem *8:* Consider the game $(\mathcal{M}, f, \mathcal{V}_i, i = 1, 2, \cdots, N)$, induced by the game form of section IV, with quasi-concave utilities $\mathbf{U}_i, i \in \mathcal{N}$. Then, every NE $\mathbf{m}^*$ of this game results in a Walrasian equilibrium, hence a pareto optimal allocation $(\mathbf{x}^*, \mathbf{t}^*)$.

VIII. Discussion and related work

The game form/mechanism we proposed and analyzed in this paper has the following properties.

1) When the users' utilities are concave the mechanism implements in Nash equilibria the solution of the centralized unicast service provisioning problem, it is individually rational, and budget-balanced at all NE and off-equilibrium.
2) When the users' utilities are quasi-concave the mechanism is individually rational and budget-balanced, and the allocations corresponding to any NE are Pareto optimal.

We now explain why the proposed mechanism and the above results are distinctly different from all game forms/mechanisms proposed so far for the unicast service provisioning problem with strategic users.

Most of the previous work on the unicast service provisioning problem in networks with general topology is based on Vickrey-Clark-Groves(VCG)-type mechanisms, [9], [68], [24], [30], [31], [64], [3], [12]. The game forms/mechanisms proposed in [68] and [24] induce games that establish the existence of a unique Nash equilibrium at which the allocation is globally optimal under some conditions; but these mechanisms are not budget-balanced even at equilibrium. The mechanisms/game forms proposed in [9], [30], [31] induce games that have multiple NE; these mechanisms are not budget-balanced even at equilibrium, and the allocations corresponding to the Nash equilibria are not always globally optimal (that is these mechanisms do not implements in Nash

equilibria the solution of the centralized unicast service provisioning problem). Our mechanism is not of the VCG-type, thus, it is philosophically different from those of [9], [68], [24], [30], [31].

The work in [35], [36] and [69] deals with single link networks. For these single-link networks the authors of [36] proposed a class of efficient (optimal) allocation mechanisms, called ESPA, for the allocation of a single divisible good, ESPA mechanisms were further developed in [35]. It is not currently known whether ESPA mechanisms implement in Nash equilibria the optimal solution of the unicast service provisioning problem in networks with arbitrary/general topology. The network model considered in this paper has arbitrary/general topology.

In [14], [23] the authors show that when the resource allocation mechanism proposed in [27] is considered the users are strategic and NE is the equilibrium concept, the allocations corresponding to any NE are different from any allocations that are optimal solutions of the corresponding centralized unicast service provisioning problem; that is, the allocation corresponding to any NE suffer from a certain efficiency loss. Particularly, in [23] it is shown that there exists a lower bound on the efficiency loss. The mechanism we propose in this paper is distinctly different from those of [14], [23] and results in the same performance as the optimal centralized allocations (that is, the allocations corresponding to any NE are efficient).

Philosophically, our work is most closely related to [60], but it is distinctly different from [60] for the following reasons: (1) the game form proposed in our paper is distinctly different from that of [60]. (2) The mechanism of [60] is not balanced off equilibrium. (3) In the mechanism of [60] there is no coupling among the games that are being played at different links. In our mechanism such a coupling exists (see section IV), and results in a balanced-budget off equilibrium.

Finally, we are not aware of any publication, other than this paper, containing the analysis of a decentralized rate allocation mechanism when the users are strategic and their utilities are quasi-concave.

IX. Conclusion

We have proposed a mechanism for rate/bandwidth allocation in unicast service provisioning and performed an equilibrium analysis of the mechanism. We discovered that when the users' utilities are concave, the mechanism possesses the following properties: (i) It implements in Nash equilibria the solution of the corresponding centralized rate allocation problem. (ii) It is individually rational. (iii) It is budget-balanced at all feasible allocations, i.e. at all Nash equilibria of the game induced by the mechanism/game

form as well as off equilibrium. When the users' utilities are quasi-concave the proposed mechanism possesses properties (ii) and (iii) stated above. Moreover, every Nash equilibrium of the game induced by the proposed mechanism results in a Walrasian equilibrium, hence a Pareto optimal allocation.

The development of algorithms that guarantee convergence to Nash equilibria of the game corresponding to the mechanism of this paper is an important problem. We have not studied this problem in detail. Preliminary investigation indicates that best response algorithms do not guarantee convergence to Nash equilibria, since the game induced by the game form developed in this paper is not supermodular (due to the capacity constraint present in the game played at each link).

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APPENDIX

Proof of Theorem 1: By the construction of the mechanism $x_i^* \geq 0$ for all $i \in \mathcal{N}$. Suppose that $\mathbf{x}^* = (x_1^*, \cdots, x_N^*)$ is such that the capacity constraint is violated at some link l and $x_j^* > 0$ (i.e. user j will be heavily charged because $\frac{1\{x_j^*>0\}1\{\mathcal{E}_{-j}^{*l}+x_j^*>0\}}{1-1\{x_j^*>0\}1\{\mathcal{E}_{-j}^{*l}+x_j^*>0\}} \approx \frac{1}{0^+}$ which is a large number). Now, Consider x_j such that: (i) either $x_j > 0$ or $x_j = 0$ and $\sum_{\substack{k\in\mathcal{G}^l \\ k\neq j}} x_k^* + x_j \leq \mathbf{c}^l$; or (ii) $x_j = 0$. Hence, $\frac{1\{x_j>0\}1\{\mathcal{E}_{-j}^{*l}+x_j>0\}}{1-1\{x_j>0\}1\{\mathcal{E}_{-j}^{*l}+x_j>0\}} = 0$, therefore,

$$\mathcal{V}_j(\mathbf{m}_j, \mathbf{m}_{-j}^*) > \mathcal{V}_j(\mathbf{m}_j^*, \mathbf{m}_{-j}^*), \tag{38}$$

this contradicts the fact that $\mathbf{m}^* = (\mathbf{m}_j^*, \mathbf{m}_{-j}^*)$ is a NE. Consequently, $\mathbf{x}^*$ is a feasible allocation of problem **Max.1**. ■

Proof of Lemma 2: We prove this lemma by considering the three cases.

- CASE 1, $\left|\mathcal{G}^l\right| = 2$ Assume that link l is used by two users $i, j \in \mathcal{N}$. From (5), defined taxes for Case 1, and Theorem 1, we get,

$$\frac{\partial \mathcal{V}_i(x_i, \mathbf{t}_i)}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} = \frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} = -2p_j^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) + 2\frac{p_i^{*l} - p_j^{*l}}{\alpha} = 0 \tag{39}$$

$$\frac{\partial \mathcal{V}_j(x_j, \mathbf{t}_j)}{\partial p_j^l}|_{\mathbf{m}=\mathbf{m}^*} = \frac{\partial t_j^l}{\partial p_j^l}|_{\mathbf{m}=\mathbf{m}^*} = -2p_i^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) + 2\frac{p_j^{*l} - p_i^{*l}}{\alpha} = 0 \tag{40}$$

therefore,

$$\frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} + \frac{\partial t_j^l}{\partial p_j^l}|_{\mathbf{m}=\mathbf{m}^*} = -2p_i^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) - 2p_j^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) = 0. \tag{41}$$

Then, the positivity of prices and Theorem 1 imply that,

$$p_i^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) = p_j^{*l}\Big(\frac{x_i^* + x_j^* - \mathbf{c}^l}{\gamma}\Big) = 0. \tag{42}$$

Eq. (42) combined with (39) and (40) gives

$$p_i^{*l} = p_j^{*l} = p^{*l} \tag{43}$$

Furthermore, using equations (14), (14), (42) and (43) we conclude that,

$$\frac{\partial t_i^l}{\partial x_i}|_{\mathbf{m}=\mathbf{m}^*} = p_j^{*l} = p^{*l} \tag{44}$$

$$\frac{\partial t_j^l}{\partial x_j}|_{\mathbf{m}=\mathbf{m}^*} = p_i^{*l} = p^{*l} \tag{45}$$

- CASE 2, $\left|\mathcal{G}^l\right| = 3$

Assume that link l is used by three users i, j and $k \in \mathcal{N}$. Note that Ω_i^l defined by (18), is not controlled by user i (i.e. Ω_i^l does not depend on x_i and p_i^l). Similarly Ω_j^l (resp. Ω_k^l) is not controlled by user j (resp. user k). Therefore,

$$\frac{\partial \Omega_i^l}{\partial x_i} = \frac{\partial \Omega_i^l}{\partial p_i^l} = \frac{\partial \Omega_j^l}{\partial x_j} = \frac{\partial \Omega_j^l}{\partial p_j^l} = \frac{\partial \Omega_k^l}{\partial x_k} = \frac{\partial \Omega_k^l}{\partial p_k^l} = 0. \tag{46}$$

By (14)-(16) anf Theorem 1 and (46) we obtain,

$$\frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} = 2\left[(p_i^{*l} - P_{-i}^{*l}) - P_{-i}^{*l}\Big(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\Big)\right] = 0, \tag{47}$$

$$\frac{\partial t_j^l}{\partial p_j^l}|_{\mathbf{m}=\mathbf{m}^*} = 2\left[(p_j^{*l} - P_{-j}^{*l}) - P_{-j}^{*l}\Big(\frac{\mathcal{E}_{-j}^{*l} + x_j^*}{\gamma}\Big)\right] = 0, \tag{48}$$

$$\frac{\partial t_k^l}{\partial p_k^l}|_{\mathbf{m}=\mathbf{m}^*} = 2\left[(p_k^{*l} - P_{-k}^{*l}) - P_{-k}^{*l}\Big(\frac{\mathcal{E}_{-k}^{*l} + x_k^*}{\gamma}\Big)\right] = 0. \tag{49}$$

Therefore,

$$\begin{aligned}\frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} \;+\; \frac{\partial t_j^l}{\partial p_j^l}|_{\mathbf{m}=\mathbf{m}^*} + \frac{\partial t_k^l}{\partial p_k^l}|_{\mathbf{m}=\mathbf{m}^*} &= -2P_{-i}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) \\ -\; 2P_{-j}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) - 2P_{-k}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) &= 0. \end{aligned} \tag{50}$$

The positivity of prices, Theorem 1, and (50) imply

$$P_{-i}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) = P_{-j}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) = P_{-k}^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) = 0. \tag{51}$$

Eq. (51) combined with (47)-(49) results in

$$p_i^{*l} = p_j^{*l} = p_k^{*l} = p^{*l} \tag{52}$$

$$p^{*l}\Big(\frac{x_i^* + x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\Big) = 0 \tag{53}$$

Moreover, because of (14)-(16), (51), (52) and (53),

$$\frac{\partial t_i^l}{\partial x_i}|_{\mathbf{m}=\mathbf{m}^*} = \frac{\partial t_j^l}{\partial x_j}|_{\mathbf{m}=\mathbf{m}^*} = \frac{\partial t_k^l}{\partial x_k}|_{\mathbf{m}=\mathbf{m}^*} = p^{*l} \tag{54}$$

Consider user $i \in \mathcal{G}^l$ ($|\mathcal{G}^l| > 3$). Since user i does not control Φ_i^l defined by (20), (i.e. Φ_i^l does not depend on x_i and p_i^l),

$$\frac{\partial \Phi_i^l}{\partial x_i} = \frac{\partial \Phi_i^l}{\partial p_i^l} = 0. \tag{55}$$

Eq. (19) along with (55) imply

$$\frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} = 2\left[(p_i^{*l} - P_{-i}^{*l}) - P_{-i}^{*l}\Big(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\Big)\right] = 0. \tag{56}$$

Summing Eq. (56) over all $i \in \mathcal{G}^l$, we get,

$$\sum_{i\in\mathcal{G}^l} \frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\mathbf{m}^*} = \sum_{i\in\mathcal{G}^l}\left[(p_i^{*l} - P_{-i}^{*l}) - P_{-i}^{*l}\Big(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\Big)\right] = \sum_{i\in\mathcal{G}^l} -P_{-i}^{*l}\Big(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\Big) = 0, \tag{57}$$

which, because of Theorem 1 and positivity of prices, implies

$$-P_{-i}^{*l}\Big(\frac{\mathcal{E}_{-i}^{*l} + x_i^*}{\gamma}\Big) = 0. \tag{58}$$

for every $i \in \mathcal{G}^l$. Then Eq. (58) gives

$$p_i^{*l} = P_{-i}^{*l}. \tag{59}$$

for all $i \in \mathcal{G}^l$. From Eqs. (58) and (59) it follows that,

$$p^{*l}\Big(\frac{\mathcal{E}^{*l}}{\gamma}\Big) = 0, \tag{60}$$

$$p_i^{*l} = p_j^{*l} = P_{-i}^{*l} = p^{*l}. \tag{61}$$

Eqs. (60) and (61) along with (19) give

$$\frac{\partial t_i^l}{\partial x_i}|_{\mathbf{m}=\mathbf{m}^*} = p^{*l}. \tag{62}$$

By (61), (60) and (62)[6] the proof is complete. ■

Proof of Lemma 4: Equation (35) together with (36) and (37) imply that $\sum_{l\in\mathbf{L}}\sum_{i\in\mathcal{G}^l} t_i^{*l} = 0$. Now, we prove that the proposed mechanism is also budget balanced off equilibrium.

[6]Note that, since the derivative of an indicator function is a Dirac delta function ([1], p. 94), then to have a well defined derivative subject to x_i at the boundary, i.e., when $\sum_{i\in\mathcal{G}^l} x_i = \mathbf{c}^l$, the derivation is from left. This observation holds throughout the proofs appearing in this Appendix.

First we show that, for every $l \in \mathbf{L}$ where $|\mathcal{G}^l| \geq 3$ then

$$\sum_{i \in \mathcal{G}^l, |\mathcal{G}^l| \geq 3} t_i^l = 0. \tag{63}$$

We prove (63) in detail when $|\mathcal{G}^l| > 3$. By a little algebra we can show the following equalities,

$$\sum_{i \in \mathcal{G}^l} {p_i^l}^2 = \sum_{i \in \mathcal{G}^l} \left[\frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} {p_j^l}^2}{|\mathcal{G}^l| - 1} \right], \tag{64}$$

$$\sum_{i \in \mathcal{G}^l} \left(2p_i^l P_{-i}^l + 2P_{-i}^l p_i^l \frac{x_i}{\gamma} - P_{-i}^l x_i \right) = \sum_{i \in \mathcal{G}^l} \left[\frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} \sum_{\substack{k \in \mathcal{G}^l \\ k \neq i,j}} \left(2p_j^l p_k^l (1 + \frac{x_j}{\gamma}) - x_j p_k^l \right)}{(|\mathcal{G}^l| - 1)(|\mathcal{G}^l| - 2)} \right], \tag{65}$$

$$\sum_{i \in \mathcal{G}^l} P_{-i}^l p_i^l \frac{\mathcal{E}_{-i}^l}{\gamma} = \sum_{i \in \mathcal{G}^l} \left[\frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} \sum_{\substack{k \in \mathcal{G}^l \\ k \neq i,j}} \sum_{\substack{r \in \mathcal{G}^l \\ r \neq i,j,k}} 2p_k^l p_j^l \mathcal{E}_r^l}{\gamma(|\mathcal{G}^l| - 1)^2(|\mathcal{G}^l| - 3)} + \frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} \sum_{\substack{k \in \mathcal{G}^l \\ k \neq i,j}} 2p_k^l p_j^l \mathcal{E}_k^l}{\gamma(|\mathcal{G}^l| - 1)^2(|\mathcal{G}^l| - 2)} \right], \tag{66}$$

$$\sum_{i \in \mathcal{G}^l} {P_{-i}^l}^2 \frac{x_i}{\gamma} = \sum_{i \in \mathcal{G}^l} \left[\frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} \sum_{\substack{k \in \mathcal{G}^l \\ k \neq i,j}} \sum_{\substack{r \in \mathcal{G}^l \\ r \neq i,j,k}} x_j p_r^l}{\gamma(|\mathcal{G}^l| - 1)^2(|\mathcal{G}^l| - 3)} + \frac{\sum_{\substack{j \in \mathcal{G}^l \\ j \neq i}} \sum_{\substack{k \in \mathcal{G}^l \\ k \neq i,j}} x_j p_k^l}{\gamma(|\mathcal{G}^l| - 1)^2(|\mathcal{G}^l| - 2)} \right]. \tag{67}$$

where From (64)-(67) we conclude that

$$\sum_{i \in \mathcal{G}^l} \left[P_{-i}^l x_i + (p_i^l - P_{-i}^l)^2 - 2P_{-i}^l (p_i^l - P_{-i}^l) \left(\frac{\mathcal{E}_{-i}^l + x_i}{\gamma} \right) \right] = -\sum_{i \in \mathcal{G}^l} \Phi_i^l \tag{68}$$

Eq. (68) along with Eq. (19) imply that $\sum_{i \in \mathcal{G}^l, |\mathcal{G}^l| > 3} t_i^l = 0$.
By arguments similar to the above we can prove that whenever $|\mathcal{G}^l| = 3, \sum_{|\mathcal{G}^l| = 3}^{i \in \mathcal{G}^l} t_i^l = 0$.

Next consider all links $l \in \mathbf{L}$ where $|\mathcal{G}^l| = 2$. In accordance with the notation in section IV, let these links be $l_1, l_2, \cdots, l_r$. Then, by the specification of the tax function (cf. section IV) we obtain,

$$\sum_{j=1}^{r} \left[t_{i_{l_{j,1}}}^{l_j} + t_{i_{l_{j,2}}}^{l_j} \right] + \sum_{j=1}^{r} \mathcal{Q}^{l_j} = 0, \tag{69}$$

where $\{i_{l_{j,1}}, i_{l_{j,2}}\} = \mathcal{G}^{l_j}, j = 1, 2, \cdots, r$.
Finally note that,

$$\sum_{i=1}^{N} t_i = \sum_{l \in \mathbf{L}: |\mathcal{G}^l| = 2} \sum_{i \in \mathcal{G}^l} t_i^l + \sum_{l \in \mathbf{L}: |\mathcal{G}^l| = 3} \sum_{i \in \mathcal{G}^l} t_i^l + \sum_{l \in \mathbf{L}: |\mathcal{G}^l| > 3} \sum_{i \in \mathcal{G}^l} t_i^l + \sum_{j=1}^{r} \mathcal{Q}^{l_j} = 0. \tag{70}$$

■

Proof of Theorem 5: We need to show that $\mathcal{V}_i(\mathbf{x}^*, \mathbf{t}_i^*) \geq \mathcal{V}_i(0,0) = 0$ for every $i \in \mathcal{N}$. By the property of NE it follows that

$$\mathcal{V}_i(\mathbf{x}^*, \mathbf{t}^*) \geq \mathcal{V}_i(\mathbf{x}^*_{-i}, x_i, \mathbf{t}_i, \mathbf{t}^*_{-i}) \qquad \forall\ (x_i, \mathbf{t}_i). \tag{71}$$

So, it is enough to find $(x_i, \mathbf{p}_i) \in \mathcal{M}_i$ such that

$$\mathcal{V}_i(\mathbf{x}^*_{-i}, x_i, \mathbf{p}_i, \mathbf{p}^*_i) \geq 0. \tag{72}$$

We set $x_i = 0$ and examine the cases $|\mathcal{G}^l| = 2$, $|\mathcal{G}^l| = 3$ and $|\mathcal{G}^l| > 3$, separately.

- CASE 1, $\left|\mathcal{G}^l\right| = 2$
  With $x_i = 0, p_j^l = p^{*l}$ and $x_j = x_j^*$, Eq. (14) defines the following function $\mathcal{F}_2(p_i^l)$:

$$\mathcal{F}_2(p_i^l) \ := \ \frac{(p_i^l - p^{*l})^2}{\alpha} - 2^{*l}(p_i^l - p^{*l})\left(\frac{x_j^* - \mathbf{c}^l}{\gamma}\right)$$

  Clearly, at

$$p_i^l = p^{*l} \tag{73}$$

  $\mathcal{F}_2(p^{*l}) = 0$. Then, by Eq. (14) it follows that

$$t_i^l(\mathbf{x}^*_{-i}, 0, \mathbf{p}^*_{-i}, p^{*l}) = 0. \tag{74}$$

- CASE 2, $\left|\mathcal{G}^l\right| = 3$
  Denote by i, j, k the users of link l. With $x_i = 0, x_j = x_j^*, x_k = x_k^*$ and $p_j^l = p_k^l = p^{*l}$, Eq. (14) defines the following function $\mathcal{F}_3(p_i^l)$:

$$\begin{aligned}\mathcal{F}_3(p_i^l) \ &:= \ (p_i^l - p^{*l})^2 - 2p^{*l}(p_i^l - p^{*l})\left(\frac{x_j^* + x_k^* - \mathbf{c}^l}{\gamma}\right) + \frac{p^{*l^2} x_k^*}{\gamma} + \Omega_i^{*l} \\ &= \ p_i^{l^2} - 2p_i^l p^{*l}\Big(1 + \frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) + p^{*l^2}\Big(1 + 2\frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) - x_{-i}^* p^{*l} + \frac{p^{*l^2} x_k^*}{\gamma}\end{aligned} \tag{75}$$

  $\mathcal{F}_3(p_i^l)$ is a quadratic polynomial in p_i^l. Setting $\mathcal{F}_3(p_i^l) = 0$ we obtain the root

$$\wp_{i,3}^l \ = \ \sqrt{\Big(p^{*l}\frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big)^2 + x_{-i}^* p^{*l} - \frac{p^{*l^2} x_k^*}{\gamma}} + p^{*l}\Big(1 + \frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) \tag{76}$$

  Since by its definition γ is sufficiently large, it follows from Eq. (76) that $\wp_{i,3}^l > 0$, i.e. $\wp_{i,3}^l$ is a feasible price. Therefore, from Eq. (14) we obtain

$$t_i^l(\mathbf{x}_{-i}^*, 0, \mathbf{p}_{-i}^*, \wp_{i,3}^l) = 0. \tag{77}$$

- CASE 3, $\left|\mathcal{G}^l\right| > 3$
  With $x_i = 0, x_j = x_j^*\ \forall j \neq i, j \in \mathcal{G}^l$, $p_j^l = p^{*l}$, $j \in \mathcal{G}^l$, Eqs. (19) and (20) define (after a little algebra) the following function $\mathcal{F}_{>3}(p_i^l)$,

$$\begin{aligned}\mathcal{F}_{>3}(p_i^l) &:= (p_i^l - p^{*l})^2 - 2p^{*l}(p_i^l - p^{*l})\Big(\frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) + \Phi_i^{*l} \\ &= {p_i^l}^2 - 2p_i^l p^{*l}\Big(1 + \frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) + {p^{*l}}^2\Big(1 + 2\frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) - x_{-i}^* p^{*l}\end{aligned} \tag{78}$$

  $\mathcal{F}_{>3}(p_i^l)$ is a quadratic polynomial in p_i^l. Setting $\mathcal{F}_{>3}(p_i^l) = 0$ we obtain the root

$$\wp_{i,>3}^l = p^{*l}\Big(1 + \frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big) + \sqrt{\Big(p^{*l}\frac{\mathcal{E}_{-i}^{*l}}{\gamma}\Big)^2 + x_{-i}^* p^{*l}} \tag{79}$$

  where

$$x_{-i}^* := \frac{\sum_{j\neq i} x_j^*}{|\mathcal{G}^l| - 1}. \tag{80}$$

  Since by its definition γ is sufficiently large, it follows from Eq. (79) that $\wp_{i,>3}^l > 0$ (i.e. $\wp_{i,>3}^l$ is a feasible price). Therefore, from Eq. (19) we get

$$t_i^l(\mathbf{x}_{-i}^*, 0, \mathbf{p}_{-i}^*, \wp_{i,>3}^l) = 0. \tag{81}$$

  Consequently, at $\mathbf{m}_i = (x_i, \mathtt{p}_i) = (0, p_i^{l_{i_1}}, p_i^{l_{i_2}}, \cdots, p_i^{l_{i_{|\mathcal{R}_i|}}})$, (where, $p_i^{l_{i_k}}, k = 1, 2, \cdots, |\mathcal{R}_i|$, are defined either by (73),(76) or (79), depending on the cardinality $\mathcal{G}^{l_{i_k}}, k = 1, 2, \cdots, |\mathcal{R}_i|$), we obtain

$$\mathcal{V}_i(x, \mathbf{t})\Big|_{\mathbf{m}=(\mathbf{m}_i, \mathbf{m}_{-i}^*)} = \mathbf{U}_i(0) - \sum_{k=0}^{|\mathcal{R}_i|} t_i^{l_{i_k}}(\mathbf{x}_{-i}^*, 0, \mathbf{p}_{-i}^{*l_{i_k}}, p_i^{l_{i_k}}) = \mathbf{U}_i(0) = 0. \tag{82}$$

  when $i \neq k_{l_1}, k_{l_2}, \cdots, k_{l_r}$.

When $i = k_{l_j}, j = 1, 2, \cdots, r$,

$$\begin{aligned}\mathcal{V}_i(x, \mathbf{t})|_{\mathbf{m}=(\mathbf{m}_i, \mathbf{m}^*_{-i})} &= \mathbf{U}_i(0) - \sum_{k=0}^{|\mathcal{R}_i|} t_i^{l_{i_k}}(\mathbf{x}^*_{-i}, 0, \mathbf{p}^{*l_{i_k}}_{-i}, p_i^{l_{i_k}}) - \mathcal{Q}^{*l_j} \\ &= -\mathcal{Q}^{*l_j} = p^{*l_j}(x^*_{i_{l_{j,1}}} + x^*_{i_{l_{j,2}}}) \\ &\geq 0, \end{aligned} \tag{83}$$

where $\{i_{l_{j,1}}, i_{l_{j,2}}\} = \mathcal{G}^{l_j}$. Combining (71), (82) and (83) we obtain

$$\mathcal{V}_i(x_i^*, \mathbf{t}^*) \geq \mathcal{V}_i(x, \mathbf{t})\Big|_{\mathbf{m}=(\mathbf{m}_i, \mathbf{m}_{-i})} \geq 0 \tag{84}$$

and this establishes (72) and completes the proof. ■

Proof of Theorem 6: Let $(\mathbf{x}^*, \mathbf{p}^*)$ be an arbitrary NE of the game $(\mathcal{M}, f, \mathcal{V})$ induced by the proposed game form. Then by the properties of NE, we must have that for every user $i \in \mathcal{N}$,

$$\frac{\partial \mathcal{V}_i(\mathbf{m})}{\partial x_i}|_{\mathbf{m}=\mathbf{m}^*} = \left[\frac{\partial \mathbf{U}_i(x_i)}{\partial x_i} - \frac{\partial t_i(\mathbf{m})}{\partial x_i}\right]|_{\mathbf{m}=\mathbf{m}^*} = 0. \tag{85}$$

By Lemma 2, Eq. (85) is equivalent to

$$\frac{\partial \mathbf{U}_i(x_i)}{\partial x_i} - \sum_{l \in \mathcal{R}_i} p^{*l} = 0. \tag{86}$$

Furthermore, by Lemma 2 we have $p^{*l}\mathcal{E}^{*l}/\gamma = 0$ and since $\gamma > 0$

$$p^{*l}\mathcal{E}^{*l} = p^{*l}\Big[\sum_{k \in \mathcal{G}^l} x_k^* - \mathbf{c}^l\Big] = 0 \tag{87}$$

Equations (85) and (87) hold for every user $i \in \mathcal{N}$.

Consider now the centralized problem **Max.1**. Since the functions $\mathbf{U}_i, i \in \mathcal{N}$ are concave and differentiable and the constraints are linear, Slater's condition ([2]) is satisfied, the duality gap is equal to zero, and the Karush Kuhn Tucker (*KKT*) conditions are necessary and sufficient to guarantee the optimality of any allocation $\mathbf{x} := (x_1, x_2, \cdots, x_N)$ that satisfies them. Let λ^l be the Lagrange multiplier corresponding to the capacity constraint for link l and ν_i be the Lagrange multiplier corresponding to the demand constraint. The Lagrangian for problem **Max.1** is

$$\mathcal{L}(x, \lambda, \nu) = \sum_{i \in \mathcal{N}} \mathbf{U}_i(x_i) - \sum_{l \in \mathbf{L}} \lambda^l \Big(\sum_{i \in \mathcal{G}_l} x_i - \mathbf{c}^l\Big) + \sum_{i \in \mathcal{N}} \nu_i x_i \tag{88}$$

and the KKT conditions are:

$$\frac{\partial \mathcal{L}(x^*, \lambda^*, \nu^*)}{\partial x_i} = \frac{\partial \mathbf{U}_i(x_i^*)}{\partial x_i} - \sum_{l \in \mathcal{R}_i} \lambda^{*l} + \nu_i^* = 0 \tag{89}$$

$$\lambda^{*l}\Big(\sum_{i \in \mathcal{G}^l} x_i^* - \mathbf{c}^l\Big) = 0 \quad \forall\, l \in \mathbf{L} \tag{90}$$

$$\nu_i^* x_i^* = 0 \quad \forall i \in \mathcal{N} \tag{91}$$

Since the KKT conditions are necessary and sufficient to guarantee the optimality of any allocation $\mathbf{x} = (x_1, x_2, \cdots, x_N)$ that satisfies them, it is enough to find ν_i^* and λ^{l*}, $l \in \mathbf{L}$, such that Eqs. (89), (90) and (91) are satisfied.

Set ν_i^*, $i \in \mathcal{N}$, equal to zero and $\lambda^{l*} = p^{*l}$, $l \in \mathbf{L}$. Then (91) is satisfied and (85) and (87) become,

$$\frac{\partial \mathbf{U}_i(x_i^*)}{\partial x_i} - \sum_{l \in \mathcal{R}_i} \lambda^{*l} = 0 \tag{92}$$

$$\lambda^{*l}\Big(\sum_{i \in \mathcal{G}^l} x_i - \mathbf{c}^l\Big) = 0 \quad \forall\, l \in \mathbf{L} \tag{93}$$

Equations (92) and (93) are identical to to (85) and (87), respectively. Thus, they are satisfied by the allocation $f(\mathbf{m}^*)$ corresponding to the NE $\mathbf{m}^*$. Furthermore, by the construction of the game form $\sum_{i \in \mathcal{N}} t_i^*$ is equal to zero. Consequently, the NE $\mathbf{m}^*$ results in an optimal solution of problem **Max.0**. Since the NE $\mathbf{m}^*$ was arbitrarily chosen, every NE $\mathbf{m}^*$ of the game induced by the game form proposed in section IV results in an optimal solution of problem **Max.0**. ■

Proof of Theorem 7: First we note that an optimal solution $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_N^*)$ of problem **Max.1** exists. This follows from the fact that each $\mathbf{U}_i,\ \forall i \in \mathcal{N}$, is concave and the space of the constraints described by Eqs. (12) and (13) is convex. The KKT conditions for problem **Max.1** result in the following equations,

$$\frac{\partial \mathbf{U}_i(x_i^*)}{\partial x_i^*} - \sum_{l \in \mathcal{R}_i} \lambda^{*l} + \nu_i^* = 0 \quad (N\ equations) \tag{94}$$

$$\lambda^{*l}(\sum_{i \in \mathcal{G}^l} x_i^* - \mathbf{c}^l) = 0 \quad (L\ equations) \tag{95}$$

$$\nu_i^* x_i^* = 0 \quad (N\ equations) \tag{96}$$

We have $N + L + N$ equations in $L + N$ unknowns, $\lambda^{*l}, l \in \mathbf{L}$ and $\nu_i^*, i \in \mathcal{N}$. In general we have multiple solutions.

We want to show that for every solution $(\lambda^{*l}, \nu_i^*, l = 1, 2, \cdots, L, i = 1, 2, \cdots, N)$ of Eqs. (94), (95) and (96) the message $\bar{\mathbf{m}} = (\bar{\mathbf{m}}_1, \bar{\mathbf{m}}_2, \cdots, \bar{\mathbf{m}}_N)$, $\bar{\mathbf{m}}_i = (\bar{x}_i, \bar{p}_i^l : l \in \mathcal{R}_i)$ with $\bar{x}_i = x_i^*$

and $\bar{p}_i^l = \lambda^{*l}$ for all $i \in \mathcal{N}$ and $l \in \mathcal{R}_i$, is a Nash equilibrium of the game induced by the proposed game form.

For that matter we note that by the selection of $\bar{\mathbf{m}}$ we have

$$p_i^{*l} = p_j^{*l} = \lambda^{*l} = p^{*l} \tag{97}$$

for every i and $j \in \mathcal{G}^l$. By (95) and (97)

$$p^{*l}\Big(\sum_{j\in\mathcal{G}^l} x_j^* - \mathbf{c}^l\Big) = \lambda^{*l}\Big(\sum_{j\in\mathcal{G}^l} x_j^* - \mathbf{c}^l\Big) = 0 \tag{98}$$

and by (35) we obtain

$$\frac{\partial t_i^{*l}}{\partial x_i^*} = p^{*l} = \lambda^{*l} \tag{99}$$

for every $l \in \mathcal{R}_i$ and every $i \in \mathcal{N}$. Therefore, the message $\bar{\mathbf{m}}$ satisfies all the conditions of Lemma 2.

Next we show that for every $i \in \mathcal{N}$, $\bar{\mathbf{m}}_i$ is a solution of the problem,

$$\begin{aligned} &\max_{\mathbf{m}_i\in\mathcal{M}_i} \Big\{\mathbf{U}_i(x_i) - \sum_{l\in\mathcal{R}_i} t_i^l(\bar{\mathbf{m}}_{-i}, \mathbf{m}_i)\Big\} \\ &\textit{subject to} \\ &\qquad x_i \geq 0,\ p_i^l \geq 0 \qquad\qquad \forall l \in \mathcal{R}_i. \end{aligned} \tag{100}$$

Any maximizing solution of (100) must satisfy

$$\frac{\partial \mathbf{U}_i(x_i)}{\partial x_i} - \sum_{l\in\mathcal{R}_i} \frac{\partial t_i^l(\bar{\mathbf{m}}_{-i}, \mathbf{m}_i)}{\partial x_i} + r_i = 0 \tag{101}$$

$$\frac{\partial \mathbf{U}_i(x_i)}{\partial p_i^l} - \sum_{l\in\mathcal{R}_i} \frac{\partial t_i^l(\bar{\mathbf{m}}_{-i}, \mathbf{m}_i)}{\partial p_i^l} + q_i^l = 0 \tag{102}$$

$\forall\ l \in \mathcal{R}_i$, where r_i and q_i^l are the Lagrange multipliers associated with the constraints $x_i \geq 0$, and $p_i^l \geq 0$, $l \in \mathcal{R}_i$, respectively. We set $r_i = \nu_i^*$ and $q_i^l = 0$ for every $l \in \mathcal{R}_i$. At $\mathbf{m}_i = \bar{\mathbf{m}}_i$, Eq. (101) is satisfied because of Eq. (94) and Eq. (99).

Furthermore at $\mathbf{m}_i = \bar{\mathbf{m}}_i$ Eq. (102) is satisfied since

$$\frac{\partial \mathcal{V}_i}{\partial p_i^l}|_{\mathbf{m}=\bar{\mathbf{m}}} = -\sum_{l\in\mathcal{R}_i} \frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\bar{\mathbf{m}}} \tag{103}$$

and

$$\frac{\partial t_i^l}{\partial p_i^l}|_{\mathbf{m}=\bar{\mathbf{m}}} = \begin{cases} 0 & \text{if } |\mathcal{G}^l| = 2; \\ -2p^{*l}\left[\frac{x_i^*+x_j^*+x_k^*-\mathbf{c}^l}{\gamma}\right] = 0 & \text{if } |\mathcal{G}^l| = 3; \\ -2P_{-i}^{*l}\left[\frac{\mathcal{E}_{-i}^{*l}+x_i^*}{\gamma}\right] = 0 & \text{if } |\mathcal{G}^l| > 3, \end{cases} \tag{104}$$

for any $l \in \mathbf{L}$ because of (98). Hence, $(x_1^*, x_2^*, \cdots, x_N^*, \lambda^{*l_1}, \lambda^{*l_2}, \cdots, \lambda^{*l_L})$ is a NE point of the game induced by the game form proposed in section IV. ■

Proof of Theorem 8: The users' utilities $\mathcal{V}_i(x_i, \mathbf{t}_i) = \mathbf{U}_i(x_i) - \mathbf{t}_i$, $i = 1, 2, \cdots, N$, (where $\mathbf{t}_i$ is specified by the game form of section IV) are quasi-concave in $\mathbf{m}_i = (x_i, \mathbf{p}_i)$ and continuous in $\mathbf{m} = (\mathbf{m}_1, \cdots, \mathbf{m}_N) = ((x_1, \mathbf{p}_1), (x_2, \mathbf{p}_2), \cdots, (x_N, \mathbf{p}_N))$. Furthermore, the message/strategy spaces $\mathcal{M}_i$, specified in the game form in section IV, are compact, convex and non-empty. Therefore, by Glicksberg's theorem [10], there exists a pure NE of the game $(\mathcal{M}, f, \mathcal{V}_i, i = 1, 2, \cdots, N)$ induced by the game form of section IV.

Let $\mathbf{m}^*$ be a NE of this game. Then, for every user $i \in \mathcal{N}$,

$$\mathcal{V}_i(\mathbf{m}^*) \geq \mathcal{V}_i(\mathbf{m}_{-i}^*, \mathbf{m}_i) \quad \text{for every } \mathbf{m}_i \in \mathcal{M}_i. \tag{105}$$

That is,

$$\mathbf{U}_i(x_i^*) - \sum_{l\in\mathcal{R}_i} t_i^{*l}(\mathbf{m}^*) \geq \mathbf{U}_i(x_i) - \sum_{l\in\mathcal{R}_i} t_i^l(\mathbf{m}_{-i}^*, \mathbf{m}_i) \quad \forall\ \mathbf{m}_i \in \mathcal{M}_i, \tag{106}$$

where

$$\sum_{l\in\mathcal{R}_i} t_i^{*l}(\mathbf{m}^*) = \sum_{l\in\mathcal{R}_i} p^{*l}x_i^* + \sum_{\substack{l\in\mathcal{R}_i \\ |\mathcal{G}^l|=3}} \Omega_i^{*l} + \sum_{\substack{l\in\mathcal{R}_i \\ |\mathcal{G}^l|>3}} \Phi_i^{*l} + \sum_{j=1}^{r} \mathcal{Q}^{*l_j} 1\{i = k_{l_j}\} \tag{107}$$

and

$$\begin{aligned} \sum_{l\in\mathcal{R}_i} t_i^l(\mathbf{m}_{-i}^*, \mathbf{m}_i) = {} & \sum_{\substack{l\in\mathcal{R}_i \\ |\mathcal{G}^l|=2}} \Pi_2(\mathbf{m}_{-i}^*, \mathbf{m}_i) + \sum_{\substack{l\in\mathcal{R}_i \\ |\mathcal{G}^l|=3}} \Pi_3(\mathbf{m}_{-i}^*, \mathbf{m}_i) + \sum_{\substack{l\in\mathcal{R}_i \\ |\mathcal{G}^l|>3}} \Pi_{>3}(\mathbf{m}_{-i}^*, \mathbf{m}_i) \\ & + \sum_{j=1}^{r} \mathcal{Q}^{*l_j} 1\{i = k_{l_j}\}, \end{aligned} \tag{108}$$

where

$$\Pi_2(\mathbf{m}^*_{-i}, \mathbf{m}_i) := p^{*l}x_i + \frac{(p^l_i - p^{*l})^2}{\alpha} - 2p^{*l}(p^l_i - p^{*l})\left(\frac{x_i + x^*_j - \mathbf{c}^l}{\gamma}\right),$$

$$\Pi_3(\mathbf{m}^*_{-i}, \mathbf{m}_i) := p^{*l}x_i + (p^l_i - p^{*l})^2 + \Omega^{*l}_i - 2p^{*l}(p^l_i - p^{*l})\left(\frac{x_i + x^*_j + x^*_k - \mathbf{c}^l}{\gamma}\right),$$

$$\Pi_{>3}(\mathbf{m}^*_{-i}, \mathbf{m}_i) := p^{*l}x_i + (p^l_i - p^{*l})^2 + \Phi^{*l}_i - 2p^{*l}(p^l_i - p^{*l})\left(\frac{x_i + \mathcal{E}^{*l}_{-i}}{\gamma}\right). \tag{109}$$

Since (106) holds for every feasible $(x_i, \mathbf{P}_i)$, setting $p^l_i = p^{*l}$ for every $l \in \mathcal{R}_i$ we obtain,

$$\mathcal{V}_i(x^*_i, \mathbf{P}^*) = \mathbf{U}_i(x^*_i) - \sum_{l\in\mathcal{R}_i} p^{*l}x^*_i \geq \mathcal{V}_i(x_i, \mathbf{P}^*) = \mathbf{U}_i(x_i) - \sum_{l\in\mathcal{R}_i} p^{*l}x_i \tag{110}$$

for every feasible x_i. Therefore, for every $i = 1, 2, \cdots, N$,

$$x^*_i = \arg\max_{x_i\in\mathbf{D}^*_{-i}} \left\{\mathbf{U}_i(x_i) - \sum_{l\in\mathcal{R}_i} p^{*l}x_i\right\} \tag{111}$$

where $\mathbf{D}^*_{-i} := \left\{x_i : 0 \leq x_i \leq \min_{l\in\mathcal{R}_i}\{\mathbf{c}^l - \sum_{\substack{j\in\mathcal{G}^l \\ j\neq i}} x^*_j\}\right\}$. Consequently, $(\mathbf{x}^*, \mathbf{p}^*)$ is a Walrasian equilibrium, therefore $(\mathbf{x}^*, \mathbf{t}^*)$ is Pareto optimal ([37] Chapter 15). ■