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Smoothing Solutions to Initial-Boundary Problems for First-Order Hyperbolic Systems

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Abstract

We consider initial-boundary problems for general linear first-order strictly hyperbolic systems with local or nonlocal nonlinear boundary conditions. While boundary data are supposed to be smooth, initial conditions can contain distributions of any order of singularity. It is known that such problems have a unique continuous solution if the initial data are continuous. In the case of strongly singular initial data we prove the existence of a (unique) delta wave solution. In both cases, we say that a solution is smoothing if it eventually becomes k-times continuously differentiable for each k. Our main result is a criterion allowing us to determine whether or not the solution is smoothing. In particular, we prove a rather general smoothingness result in the case of classical boundary conditions.

1 Introduction

Solutions to hyperbolic PDEs demonstrate a wide spectrum of regularity behavior. The appearance of singularities in nonlinear cases is known as the blow up of a solution [1, 2]. Singularities can appear in a finite time even for small and smooth initial data [3]. In some cases, both linear and nonlinear, a solution with time either preserves the same regularity as it has on the boundary or becomes less or more regular in time. The singularities encountered in the latter case are called anomalous [4, 5]. Criteria for the appearance of anomalous singularities are given in [6, 7, 8, 5, 9]. These papers are devoted to the case of interaction between singularities weaker than the Dirac measure. The singularities resulting from this interaction turn out to be weaker than the incoming singularities. A different effect is observed in [11, 12], where the incoming singularities are derivatives of the Dirac measure. In this case the interaction produces singularities stronger than the initial ones. We will focus on the phenomenon of improving regularity in the case of initial-boundary value problems with nonlinear local and nonlocal boundary conditions for first-order linear strictly hyperbolic systems.

Specifically, in the domain $\Pi = \{(x,t)\,|\,0<x<1, t>0\}$ we address the problem

$$(\partial_t + \Lambda(x,t)\partial_x + A(x,t))u = g(x,t), \qquad (x,t)\in\Pi \tag{1}$$

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$$u(x,0) = \varphi(x), \qquad x \in (0,1) \tag{2}$$

$$\begin{aligned} u_i(1,t) &= h_i(t, v(t)), \quad 1 \le i \le k, \quad t \in (0,\infty) \\ u_i(0,t) &= h_i(t, v(t)), \quad k < i \le n, \quad t \in (0,\infty), \end{aligned} \tag{3}$$

where u, g, and φ are real n-vectors, $A = \{a_{ij}\}_{i,j=1}^n$, $\Lambda = diag(\lambda_1, \dots, \lambda_n)$, and

$$v(t) = (v_1(t), \dots, v_n(t)) = (u_1(0,t), \dots, u_k(0,t), u_{k+1}(1,t), \dots, u_n(1,t)) \,. \tag{4}$$

Note that boundary conditions (3) cover the cases of classical boundary conditions (if h_i do not depend on v) and reflection boundary conditions of local and nonlocal type.

We assume that

$$\lambda_1 < \dots < \lambda_k < 0 < \lambda_{k+1} < \dots < \lambda_n \tag{5}$$

for all $(x,t) \in \overline{\Pi}$. Condition (5) occurs in many applications, where the functions u_j for $j \le k$ (resp. $k+1 \le j \le n$) describe the "species" that travel to the left (resp. to the right) and are reflected in $x = 0$ (resp. $x = 1$) according to the boundary conditions (3).

We will impose the following smoothness assumptions on the initial data: The entries of Λ, A, g, and $h = (h_1, \dots, h_n)$ are smooth in all their arguments in the respective domains, while the entries of φ are allowed to be either continuous functions or strongly singular distributions.

In the case of a continuous φ (considered in Section 2), by a solution to problem (1)–(3) we mean a *continuous solution*, i.e., a continuous vector-function in $\overline{\Pi}$ satisfying an integral system equivalent to (1)–(3). The existence and uniqueness of a continuous solution is proved in [13] (see Theorem 2). In the case of a strongly singular φ (considered in Section 4) by a solution we mean a *delta wave solution*, i.e., a weak limit of solutions to the original problem with regularized initial data that does not depend on a particular regularization. We refer the reader to [10, 9] for a more detailed definition and motivation of delta waves. Theorem 21 in Section 4 establishes the existence of a delta wave solution for a version of (1)–(3).

It is clear that the regularity of initial conditions (2) constraints the regularity of a solution if the latter is considered in the entire domain Π. However, the influence of the initial data can be suppressed if the regularity behavior is considered in dynamics, starting from a point of time T.

Definition 1 A solution u to problem (1)–(3) is called *smoothing* if, whatever $m \in \mathbb{N}$, there exists $T > 0$ such that $u \in C^m\left(\overline{\Pi} \cap \{t \ge T\}\right)^n$.

Our main result is a smoothingness criterion for solutions to problem (1)–(3) in terms of the layout of characteristic curves (Theorems 12 and 22). In the case of classical boundary condition, the criterion implies that the solution is smoothing whenever $\inf |\lambda_i| > 0$ for all $i \le n$. As another consequence, we obtain a class of boundary conditions under which the wave equation has smoothing solutions (see [14] for a special case of this result).

Our analysis of problem (1)–(3) shows a phenomenon usually observed in the situations when solutions to hyperbolic PDEs change their regularity: the smoothness changes jump-like rather than gradually. Another feature of problem (1)–(3) shown in [13] is that, even if we allow non-Lipschitz nonlinearities in (3), this system demonstrates almost linear behavior. The

smoothingness effect in the non-Lipschitz setting contrasts with blowups [1, 2] observed in many nonlinear systems.

In [6, 7], results similar to ours are obtained in some more special cases, namely for homogeneous linear boundary conditions of local type with constant coefficients and continuous initial data. Some restrictions on (1)–(3) are imposed in [6, 7] by technical reasons, as the authors use an approach based on the Laplace transformation and the Green's function method. We extend these results to the case of general linear first-order hyperbolic systems, nonlinear nonlocal boundary conditions, and distributional initial data of the Dirac delta type and derivatives thereof. Note that we use a different approach based on the classical method of characteristics. This method suits well for understanding of the mechanism of the smoothingness effect.

An essential technical difficulty to overcome in demonstrating the regularity self-improvement is caused by the fact that the domain of influence of initial conditions (2) is in general infinite. In other words, the regularity of solutions all the time depends on the regularity of the initial data. However, this dependence is different for the boundary and the integral parts of the equivalent integral form of problem (1)–(3). Since the boundary summands are compositions of boundary data with functions defining characteristic curves and hence are "responsible" for propagation of singularities, the smoothingness effect is encountered whenever the boundary summands have a "bounded memory" or, more rigorously, all characteristics of (1) are bounded and each boundary singularity expands inside Π along a finite number of characteristic curves.[1] Our strategy of obtaining the smoothingness criterion consists in identifying an appropriate class of boundary conditions ensuring the "bounded memory" property and in showing that the integral summands do eventually improve upon the regularity of the initial data. It is important for our analysis that the lower order terms, with the exception of the diagonal ones, contribute into the integral summands transversely to characteristic directions. This is ensured by (5). Therefore, the integral part of the system causes no propagation of singularities and, moreover, suppresses it.

The mathematical motivation of the paper is the scope of the stability theory, the Hopf bifurcation analysis, and the investigation of small periodic forcing of stationary solutions of hyperbolic PDEs. The main reason why those techniques are well established for nonlinear ODEs and parabolic PDEs, but not for nonlinear hyperbolic PDEs, is that, in contrast to parabolic case, hyperbolic operators in general do not improve the regularity of their solutions in time (the question is closely related to propagation of singularities along characteristic curves). This complicates, in particular, proving the Fredholmness property of the linearizations which is crucial for the analysis of solutions to nonlinear problems. We provide a range of boundary conditions ensuring the desired smoothingness effect, which still makes possible to handle the bifurcation analysis of a class of nonlinear problems (this idea goes back to [15]).

The practical motivation is caused by applications to mathematical biology [16], chemical kinetics (describing mass transition in terms of convective diffusion and chemical reaction and analysis of chemical processes in counterflow chemical reactors [17, 18, 19, 20]), and semiconductor laser dynamics (describing the appearance of self-pulsations of lasers and modulation of stationary laser states by time periodic electric pumping [21, 22, 23]).

[1] This sharply contrasts with the case of a Cauchy problem where solutions cannot be smoothing because the boundary part all the time "remembers" the regularity of the initial data.

2 Continuous initial data

Here we consider the case of continuous initial data $\varphi(x)$. We will assume the zero-order compatibility conditions between (2) and (3), namely

$$\begin{array}{rcll} \varphi_i(0) & = & h_i(0, v(0)), & k+1 \le i \le n, \\ \varphi_i(1) & = & h_i(0, v(0)), & 1 \le i \le k, \end{array} \tag{6}$$

where $v(0) = \Big(\varphi_1(0), \ldots, \varphi_k(0), \varphi_{k+1}(1), \ldots, \varphi_n(1)\Big)$. By $\|\cdot\|$ we denote the Euclidian norm in $\mathbb{R}^n$.

Theorem 2 ([13, Thm 3.1]) *Assume that the data λ_i, a_{ij}, g_i, φ_i, and h_i are continuous functions in all their arguments, and the coefficients λ_i are Lipschitz in $x \in [0,1]$ locally in $t \in [0,\infty)$. Suppose that $h_i(t,z)$ are continuously differentiable in $z \in \mathbb{R}^n$ and for each $T > 0$ there exists $C > 0$ such that*

$$\|\nabla_z h(t,z)\| \le C \left(\log\log H(t, \|z\|)\right)^{1/4}, \tag{7}$$

where H is a polynomial in $\|z\|$ with coefficients in $C[0,T]$. If the zero-order compatibility conditions (6) are fulfilled, then problem (1)–(3) has a unique continuous solution in $\overline{\Pi}$ which can be found by the sequential approximation method.

We now introduce the notions of an *Expansion Path* and an *Influence Path*, that will be our main technical tools. Let $\omega_i(\tau; x, t)$ denote the characteristic of the i-th equation of (1) passing through $(x,t) \in \overline{\Pi}$. Let χ be a characteristic of the i-th equation of system (1). Suppose that χ reaches $\partial\Pi$ at two points. Let (x,t) be that of these points having larger ordinate (hence, $x = 0$ or $x = 1$). We say that χ *reflects* at (x,t) if

$$\partial_z h_j(t, v_1(t), \ldots, v_{i-1}(t), z, v_{i+1}(t), \ldots, v_n(t)) \neq 0 \text{ for some } j \le n \text{ and } z \in \mathbb{R}. \tag{8}$$

In this case that of the characteristics $\omega_j(\tau; x, t)$ and $\omega_j(\tau; 1-x, t)$ which lies above the line $\{\tau = t\}$ is called a *reflection* of χ. If the reflection is defined by $\omega_j(\tau; 1-x, t)$, it will be called a *jumping reflection*.

Remark 3 *Note that condition (8) means that the i-th components of the vector $v(t)$ participates in evaluation of $u_j(0,t)$ for $k+1 \le j \le n$ and of $u_j(1,t)$ for $j \le k$. Whenever the continuous solution to problem (1)–(3) is known (see Theorem 2), condition (8) is easily checkable. Otherwise, one can use the following constructive sufficient condition: (8) holds true whenever for every $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \in \mathbb{R}^{n-1}$ there is $j \le n$ such that $\partial_{z_i} h_j(t, z_1, \ldots, z_n) \not\equiv 0$.*

Definition 4 A sequence of characteristics $\chi_1, \chi_2, \ldots$ is called an *Extension Path (EP)* if each χ_{l+1} is a reflection of χ_l.

Definition 5 A sequence of curved segments $B_1, \ldots, B_s$ is called an *Influence Path (IP)* if the following conditions are met.

- Each B_l is a continuous part of a characteristic χ_l of the j_l-th equation for some $j_l \le n$;

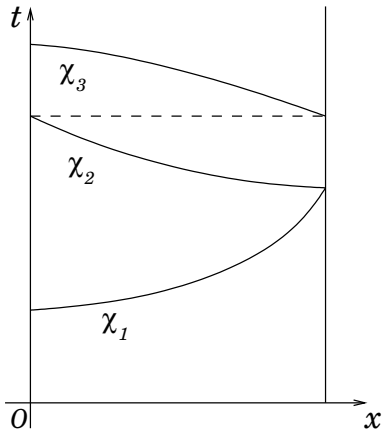

Figure 1: An EP.

- The whole path is monotone in the sense that the coordinate t continuously increases while moving along it;
- The transition from B_l to B_{l+1} can be of three types:
  - B_l and B_{l+1} meet at a point (x,t) such that $a_{j_{l+1}j_l}(x,t) \not\equiv 0$ in any neighborhood of (x,t);
  - B_l and B_{l+1} meet at a point (x,t) with $x=0$ or $x=1$ and in any neighborhood of t a characteristic of the j_l-th equation reflects to a characteristic of the j_{l+1}-th equation;
  - B_l terminates at a point (x,t) with $x=0$ or $x=1$, B_{l+1} starts at the point $(1-x,t)$ on the opposite side of $\partial\Pi$, and in any neighborhood of t a characteristic of the j_l-th equation makes a jumping reflection to a characteristic of the j_{l+1}-th equation.

Roughly speaking, an IP is a piecewise continuous curve with smooth peaces $B_1,\dots,B_s$ lying on characteristic curves such that either B_l and B_{l+1} meet within $\overline{\Pi}$ or B_l terminates at $(x,t)\in\partial\Pi$ and B_{l+1} starts at the point $(1-x,t)$ on the opposite side of $\partial\Pi$.

Remark 6 *Note that any EP is an IP, while the opposite is not necessary true because segments of an IP not necessary lie on reflected characteristics.*

Definition 7 Define a set X_i, called the *domain of influence of the initial data on* u_i, as follows: $(x,t)\in X_i$ if the value $u_i(x,t)$ can be changed by varying a function φ in (2).

Since initial data expand inside Π along characteristic curves according to boundary conditions (3) and the lower order terms of system (1), we have the following characterization.

Lemma 8 *$(x,t)\in X_i$ if and only if there is an IP emanating from the initial axis $t=0$ and going through a part of the characteristic $\omega_i(\tau;x,t)$.*

The sufficiency follows from the proof of the necessity in Theorem 12. The proof of the necessity is based on the constructive description of the domain of dependence of u_i at (x,t) which turns out to be the union of the IPs going through a part of $\omega_i(\tau;x,t)$ below the line $\tau=t$. Under the domain of dependence of u_i at (x_0,t_0) we mean the set of all points $(x',t')\in\overline{\Pi}$ such that if varying the function

$$u(x,t')=\psi(x) \tag{9}$$

in any sufficiently small neighborhood of x', the i-th component of the solution to problem (1), (9), (3) changes at (x_0,t_0).

Definition 9 Define a set X_i° to be the union of IPs emanating from the initial axis $t=0$ and satisfying Definition 5 with additional conditions imposed on the three transition types from B_l to B_{l+1}:

- B_l and B_{l+1} meet at a point (x,t) such that $a_{j_{l+1}j_l}(x,t)\neq 0$;
- B_l and B_{l+1} meet at a point (x,t) with $x=0$ or $x=1$ and at this point χ_l reflects to χ_{l+1};
- B_l terminates at a point (x,t) with $x=0$ or $x=1$, B_{l+1} starts at the point $(1-x,t)$ on the opposite side of $\partial\Pi$, and at (x,t) the characteristic χ_l makes a jumping reflection to χ_{l+1}.

Note that $\overline{X_i^\circ}=X_i$.

We now introduce two conditions that will occur in formulation of our results.

(ι) For every $T>0$ there exists $T'>T$ such that, for all $x\in[0,1]$, every EP passing through (x,T) lies below the line $t=T'$.

($\iota\iota$) For every $T>0$ and $i\le n$ there exists $T'>T$ such that, for all $x\in[0,1]$ with $(x,T)\in X_i^\circ$, every EP containing the characteristic $\omega_i(\tau;x,T)$ lies below the line $t=T'$.

Remark 10 *In many important cases conditions (ι) and ($\iota\iota$) can easily be reformulated and verified in terms of λ, A, and h. Sometimes they can be verified even directly (see examples in Section 3).*

Set $\Pi^T=\{(x,t)\,|\,0<x<1, 0<t<T\}$.

Remark 11 *Let $T>0$. In the domain $\Pi\setminus\overline{\Pi^T}$ let us consider problem (1), (9), (3) with $a_{ij}\equiv 0$ for all $i\neq j$ (i.e., system (1) is decoupled) and with t' replaced by T in (9). Then condition (ι) means that, whatever $T>0$ and $\psi(x)$, the function $\psi(x)$ has a bounded domain of influence on u_i for every $i\le n$. In other words, for any decoupled system (1), if $\psi(x)$ is singular at some point $x\in[0,1]$, then this singularity expands outside $\overline{\Pi^T}$ along a finite number of characteristic curves within $\overline{\Pi^{T'}}$ for some $T'>T$ that does not depend on $x\in[0,1]$. In contrast to (ι), condition ($\iota\iota$) means that, whatever $T>0$ and $\psi_i(x)$, the function $\psi_i(x)$ restricted to $X_i^\circ\cap\{t=T\}$ has a bounded domain of influence on u_i for every $i\le n$.*

Theorem 12 *Assume that the data λ_i, a_{ij}, g_i, and h_i are smooth functions in all their arguments and conditions (5) and (7) are fulfilled.*

- Sufficiency. *Assume that condition (ι) is fulfilled. Then the continuous solution to problem (1)–(3) is smoothing for any $\varphi \in C[0,1]^n$ satisfying equalities (6).*
- Necessity. *Assume that the continuous solution to problem (1)–(3) is smoothing for any $\varphi \in C[0,1]^n$ satisfying equalities (6). Then condition ($\iota\iota$) is fulfilled.*

Remark 13 *One can easily see that the sufficient condition (ι) implies the necessary condition ($\iota\iota$), while the converse is not true. Nevertheless, in a quite general situation when $X_i^\circ = \overline{\Pi}$ for every $i \le n$, it is not difficult to observe that conditions (ι) and ($\iota\iota$) are equivalent. Hence we have the following result.*

Corollary 14 *Assume that the data λ_i, a_{ij}, g_i, and h_i are smooth functions in all their arguments, $X_i^\circ = \overline{\Pi}$ for every $i \le n$, and conditions (5) and (7) are fulfilled. Then the continuous solution to problem (1)–(3) is smoothing for any $\varphi \in C[0,1]^n$ if and only if condition (ι) is fulfilled.*

Proof. *Sufficiency.* Assume that condition (ι) is fulfilled. Define a sequence $T_1, T_2, \dots$ inductively by the following rule. Let T_1 be the infimum of those $\tau > 0$ for which there is x and an EP passing through $(x,0)$ and lying below the line $t = \tau$; let T_j for $j > 1$ be the infimum of those $\tau > 0$ for which there is x and an EP passing through (x, T_{j-1}) and lying below the line $t = T_{j-1}$. Note that T_j is monotone and approaches the infinity. The latter fact is a simple consequence of the smoothness assumptions on Λ. By Theorem 2, problem (1)–(3) has a unique continuous solution u in $\overline{\Pi}$. It suffices to show that $u \in C^j\left(\overline{\Pi} \setminus \Pi^{T_j}\right)^n$.

Consider problem (1)–(3) first in $\overline{\Pi} \setminus \Pi^{T_1}$. The solution satisfies the system of integral equations

$$u_i(x,t) = h_i(t_i(x,t), v(t_i(x,t))) \tag{10}$$
$$+ \int\limits_{t_i(x,t)}^{t} E_i(\tau; x, t) \left[-\sum_{j \ne i} a_{ij}(\xi,\tau) u_j(\xi,\tau) + g_i(\xi,\tau) \right] \Big|_{\xi = \omega_i(\tau;x,t)} d\tau, \quad i \le n,$$

where $E_i(\tau; x, t) = \exp \int_t^\tau a_{ii}(\omega_i(\tau_1; x, t), \tau_1)\, d\tau_1$ and $t_i(x,t)$ denotes the smallest value of $\tau \ge 0$ at which this characteristic reaches $\partial\Pi$. In the sequel, along with the equation $\xi = \omega_i(\tau; x, t)$ we will also use its inverse form $\tau = \tilde{\omega}_i(\xi; x, t)$. Due to the definition of T_1, in the boundary term $h_i(t_i(x,t), v(t_i(x,t)))$ in (10) we can substitute

$$v_j(t_i(x,t)) = h_j(t_j(x_j, t_i(x,t)), v(t_j(x_j, t_i(x,t)))) \tag{11}$$
$$+ \int\limits_{t_j(x_j, t_i(x,t))}^{t_i(x,t)} E_j(\tau; x_j, t_i(x,t)) \left[-\sum_{s \ne j} a_{js}(\xi,\tau) u_s(\xi,\tau) + g_j(\xi,\tau) \right] \Big|_{\xi = \omega_j(\tau; x_j, t_i(x,t))} d\tau,$$

where $x_j = 0$ if $1 \le j \le k$ and $x_j = 1$ if $k+1 \le j \le n$. Continuing in this fashion, the right-hand side of (10) can eventually be brought into a form depending neither on v nor on φ. This version of (10) will be referred to as (10′). We begin with establishing the $C_x^1\left(\overline{\Pi} \setminus \Pi^{T_1}\right)$-smoothness of u. It will be proved once we show that the right-hand side of (10′) has a continuous partial

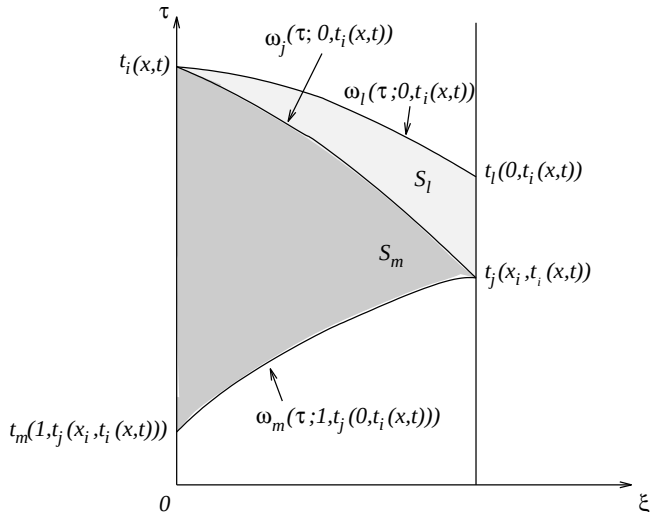

Figure 2: The domain of integration.

derivative in x. The latter can be done by transforming all integrals occurring in $(10')$. The transformation of each integral follows the same scheme, which we illustrate by example of the integral expression

$$I_{ijm}(x,t) = \int\limits_{t_j(x_j,t_i(x,t))}^{t_i(x,t)} E_j(\tau; x_j, t_i(x,t)) a_{jm}(\xi,\tau) u_m(\xi,\tau)\Big|_{\xi=\omega_j(\tau;x_j,t_i(x,t))} d\tau. \tag{12}$$

We will use assumption (5). Suppose, for instance, that $i \le k$, $j \ge k+1$, $m \le k$, and $t_m(1, t_j(x_i, t_i(x,t))) > 0$. This entails, in particular, that $x_i = 0$ and $x_j = 1$. Due to (10), we obtain

$$\begin{aligned} I_{ijm}(x,t) &= \int\limits_{t_j(0,t_i(x,t))}^{t_i(x,t)} E_j(\tau; 0, t_i(x,t)) a_{jm}(\xi,\tau) \Big[h_m(t_m(\xi,\tau), v(t_m(\xi,\tau))) \\ &+ \int\limits_{t_m(\xi,\tau)}^{\tau} E_m(\tau_1;\xi,\tau) \Big\{ -\sum_{p\ne m} (a_{mp}u_p + g_m)(z,\tau_1) \Big\}\Big|_{z=\omega_m(\tau_1;\xi,\tau)} d\tau_1 \Big]\Big|_{\xi=\omega_j(\tau;0,t_i(x,t))} d\tau \\ &= \int\limits_{t_m(1,t_j(0,t_i(x,t)))}^{t_i(x,t)} E_j(\rho; 0, t_i(x,t)) a_{jm}(\omega_j(\rho;0,t_i(x,t)),\rho)\Big|_{\rho=\theta(0,\tau;x,t)} h_m(\tau, v(\tau)) d\tau \end{aligned}$$

$$+\int_{S_m} E_j(\rho;0,t_i(x,t))a_{jm}(\eta,\rho)E_m(\tau;\eta,\rho)\Big|_{\eta=\omega_m(\theta(\xi,\tau;x,t);\xi,\tau),\rho=\theta(\xi,\tau;x,t)}$$
$$\times\left\{-\sum_{p\neq m} a_{mp}(\xi,\tau)u_p(\xi,\tau)+g_m(\xi,\tau)\right\}d\xi\partial\tau, \tag{13}$$

where S_m is the area shown in Fig. 2 and $\theta(\xi,\tau;x,t)$ denotes the t-coordinate of the point where the characteristics $\omega_j(\tau_1;0,t_i(x,t))$ and $\omega_m(\tau_1;\xi,\tau)$ intersect. The other cases are similar. For example, if $m\geq k+1$, then in the formula (13) index m should be replaced by l and the integration over $[t_m(1,t_j(0,t_i(x,t))),t_i(x,t)]$ should be replaced by the integration over $[t_j(0,t_i(x,t)),t_l(0,t_i(x,t))]$.

The desired C^1_x-smoothness of $I_{ijm}(x,t)$ follows from the C^1_x-smoothness of $\theta(\xi,\tau;x,t)$. The latter is a consequence of the smoothness properties of Λ. Indeed, from the equality $\omega_j(\theta(\xi,\tau;x,t);0,t_i(x,t))=\omega_m(\theta(\xi,\tau;x,t);\xi,\tau)$, we conclude that, if $\partial_x\theta$ exists, then it is given by the formula

$$\begin{aligned}
\partial_x\theta(\xi,\tau;x,t) &= \frac{\partial\omega_j(\theta(\xi,\tau;x,t);0,t_i(x,t))}{\partial t_i}\frac{\partial t_i(x,t)}{\partial x}\\
&\times\left(\frac{\partial\omega_m(\theta(\xi,\tau;x,t);\xi,\tau)}{\partial\theta}-\frac{\partial\omega_j(\theta(\xi,\tau;x,t);0,t_i(x,t))}{\partial\theta}\right)^{-1}\\
&= -\lambda_j(0,t_i(x,t))\exp\left\{\int\limits_{t_i(x,t)}^{\theta(\xi,\tau;x,t)}\lambda'_{j\xi}(\xi,\sigma)\Big|_{\xi=\omega_j(\sigma;0,t_i(x,t))}d\sigma\right\}\\
&\times(\lambda_i(0,t_i(x,t)))^{-1}\exp\left\{\int\limits_{t}^{t_i(x,t)}\lambda'_{i\xi}(\xi,\sigma)\Big|_{\xi=\omega_i(\sigma;x,t)}d\sigma\right\}\\
&\times\Big[\lambda_m(\omega_m(\theta(\xi,\tau;x,t);\xi,\tau),\theta(\xi,\tau;x,t))\\
&-\lambda_j(\omega_j(\theta(\xi,\tau;x,t);0,t_i(x,t)),\theta(\xi,\tau;x,t))\Big]^{-1}.
\end{aligned}$$

Thanks to the equality $\omega_m(\theta(\xi,\tau;x,t);\xi,\tau)=\omega_j(\theta(\xi,\tau;x,t);0,t_i(x,t))$ and condition (5), the function $I_{ijm}(x,t)$ is continuously differentiable in x.

Thus, the right-hand side of (10′) is continuously differentiable in x. Therefore, $\partial_x u\in C\left(\overline{\Pi}\setminus\Pi^{T_1}\right)^n$. The membership of u in $C^1\left(\overline{\Pi}\setminus\Pi^{T_1}\right)^n$ now directly follows from system (1).

In the next step, we prove that $u\in C^2\left(\overline{\Pi}\setminus\Pi^{T_2}\right)^n$. For $\partial_x u$ we have equations

$$\begin{aligned}
\partial_x u_i(x,t) &= (R_i u)(x,t)+\int\limits_{t_i(x,t)}E_i(\tau;x,t)\Big[-\sum_{j\neq i}a_{ij}(\xi,\tau)\partial_\xi u_j(\xi,\tau)\\
&-\partial_\xi\lambda_i(\xi,\tau)\partial_\xi u_i(\xi,\tau)+(\partial_\xi f_i)(\xi,\tau,u)\Big]\Big|_{\xi=\omega_i(\tau;x,t)}d\tau,\qquad i\leq n,
\end{aligned}\tag{14}$$

where $f_i(\xi,\tau,u)=g_i(\xi,\tau)-\sum_{j=1}^n a_{ij}(\xi,\tau)u_j$,

$$(R_i u)(x,t)=\lambda_i^{-1}(y_i,\tau)\Big[f_i(y_i,\tau,u)-\nabla_v h_i(\tau,v)\cdot v'(\tau)-(\partial_\tau h_i)(\tau,v)\Big]\Big|_{\tau=t_i(x,t)},\tag{15}$$

$y_i = 0$ for $k+1 \le i \le n$, and $y_i = 1$ for $1 \le i \le k$. Here $w \cdot z$ denotes the scalar product in $\mathbb{R}^n$. In (15) we can represent $v'(t)$ in the form

$$\begin{aligned} v_j'(t) &= f_j(x_j,t,u) - \lambda_j(x_j,t)\partial_x u_j(x_j,t) = f_j(x_j,t,u) - \lambda_j(x_j,t)\Big[(R_j u)(x_j,t) \\ &+ \int\limits_{t_j(x_j,t)}^{t} E_j(\tau;x_j,t)\Big(-\sum_{s\neq j} a_{js}(\xi,\tau)\partial_\xi u_s(\xi,\tau) - \partial_\xi \lambda_j(\xi,\tau)\partial_\xi u_j(\xi,\tau) \\ &+ \left(\partial_\xi f_j\right)(\xi,\tau,u)\Big)\Big|_{\xi=\omega_j(\tau;x_j,t)} d\tau\Big]. \end{aligned}$$

We continue in this fashion up to getting a representation of the boundary term (15) that does not depend on v', what is possible due to the definition of T_2. To show that the right-hand side of the obtained expression for $\partial_x u_i(x,t)$, say (14′), is continuously differentiable in x, we transform all integrals contributing into (14′) similarly to (13). Using the fact that $u \in C^1(\overline{\Pi} \setminus \Pi^{T_1})^n$, one can easily show that $u \in C^{2,1}_{x,t}\left(\overline{\Pi} \setminus \Pi^{T_2}\right)^n$. The desired $C^2\left(\overline{\Pi} \setminus \Pi^{T_2}\right)$-smoothness of the solution is then a direct consequence of system (1) and its differentiations.

We further proceed by induction on k. Assuming that $u \in C^{k-1}\left(\overline{\Pi} \setminus \Pi^{T_{k-1}}\right)^n$ for some $k \ge 2$, we will prove that $u \in C^k\left(\overline{\Pi} \setminus \Pi^{T_k}\right)^n$. We differentiate (1) k times in x, thereby obtaining a system for $\partial_x^k u$. The integral form of this system is similar to (14). Analogously to (14), the definition of T_k makes possible an integral representation of the system for $\partial_x^k u$ which does not depend on $v^{(k)}$ and includes integrals of $\partial_x^{k-1} u$ similar to $I_{ijs}(x,t)$. To show the C^k_x-smoothness of the solution, we transform the integral terms analogously to (13). Finally, the C^k-smoothness of u outside of Π^{T_k} follows from suitable differentiations of system (1). The sufficiency is thereby proved.

Necessity. Suppose that condition (ιu) is not fulfilled and prove that the solution to problem (1)–(3) is not smoothing for some $\varphi \in C[0,1]^n$. Fix $T = t_0 > 0$ and $i \le n$ such that for all $T' > t_0$ there is an EP containing the characteristic $\omega_i(\tau;x,t_0)$ for some $(x,t_0) \in X_i$ and going beyond $\overline{\Pi^{T'}}$.

Since all singularities of solutions expand along EPs, it is sufficient to prove that there exist $m \in \mathbb{N}$ and $\varphi \in C[0,1]^n$ such that, whatever $(x_0,t_0) \in X_i^\circ$, the solution u is not C^m-smooth at (x_0,t_0).

By Lemma 8, for any $(x,t_0) \in X_i$ there is an Influence Path $B_1,\dots,B_s$ emanating from the line $t=0$ and going through a part of $\omega_i(t;x,t_0)$. Denote the smallest possible "length" of such a path by $s(x)$. By the smoothness assumptions on the initial data, if x' is sufficiently close to x and $(x',t_0) \in X_i$, then $s(x') \le s(x)$. Since $X_i \cap \{t = t_0\}$ is closed, the standard compactness argument implies that $s(x)$ is bounded by a constant s_0 uniformly over all x.

Let φ be a continuous nowhere differentiable function. Consider an arbitrary $(x_0,t_0) \in X_i^\circ$. Let us fix a shortest Influence Path $B_1,\dots,B_s$ from some $(z_0,0)$ to (x_0,t_0) with B_s lying on $\omega_i(\tau;x_0,t_0)$ and with the transition from B_l to B_{l+1} as in Definition 9. Using the smoothness assumptions on A and h, we can suppose that $0 < z_0 < 1$. We now intend to prove that the solution u is not C^s-smooth at (x_0,t_0). Since $s \le s_0$, this will give us the necessary part of the theorem. Let (z_l,T_l) denote the starting point of B_{l+1}. Suppose that B_l is a part of a characteristic χ_l of the j_l-th equation. As φ_{j_1} is not continuously differentiable at z_0, the

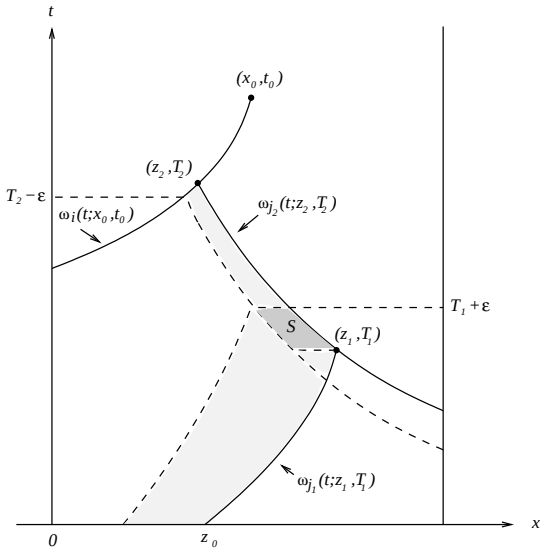

Figure 3: The set $\tilde{\mathcal{B}}$.

function u_{j_1} is not C^1-smooth along χ_1 due to the definition of a characteristic. If χ_2 is a reflection of χ_1, then u_{j_2} is not differentiable at (z_1, T_1) and hence u_{j_2} is not differentiable along χ_2. Otherwise, by Definition 5, we have $a_{j_2 j_1}(z_1, T_1) \neq 0$ and u_{j_2} is not C^2-smooth along χ_2, because the integral of $a_{j_2 j_1} u_{j_1}$ along χ_2 in the integral form of the j_2-th differential equation is not a C^2-function and this nonsmoothness cannot be compensated by any other summands in the integral representation of our problem. It follows that u_{j_2} is not C^2-smooth, in particular, at (z_2, T_2). Continuing in this way, we arrive at the conclusion that u_i is not C^s-smooth along $\omega_i(t; x_0, t_0)$ and hence u is not C^s-smooth at (x_0, t_0).

Going into the details of the above argument, let us represent $u_i(x_0, t_0)$ in an integral form with an integration over a neighborhood of the Influence Path $B_1, \ldots, B_s$. We focus on the case of $s = 3$ where characteristic χ_3 is not a reflection of χ_2 and χ_2 is not a reflection of χ_1 (see Fig. 3). A similar or even simpler argument works as well for other possible cases. Extend each B_l to $B'_l = \chi_l \cap \overline{\Pi^{T_l}}$. Given $\varepsilon > 0$, let $\mathcal{B}$ denote the union of B'_s and one-sided neighborhoods of $B'_1, \ldots, B'_{s-1}$, bounded by the characteristics $\omega_{j_2}(\tau; \omega_i(T_2 - \varepsilon; x_0, t_0), T_2 - \varepsilon)$ and $\omega_{j_1}(\tau; \omega_{j_2}(T_1 + \varepsilon; \omega_i(T_2 - \varepsilon; x_0, t_0), T_2 - \varepsilon), T_1 + \varepsilon)$. Fix an $\varepsilon > 0$ so that $\mathcal{B}$ contains neither $(0, 0)$ nor $(1, 0)$.

Now we write an integral representation of $u_i(x_0, t_0)$ in terms of u over $\mathcal{B}$. We start from the formula (10) for $u_i(x_0, t_0)$ and rewrite a part of the integral in the right-hand side as

$$\int_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau)u_{j_2}(\xi(\tau),\tau)\,d\tau$$
$$= \int_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau)\Bigg[h_{j_2}\left(t_{j_2}(\xi(\tau),\tau),v(t_{j_2}(\xi(\tau),\tau))\right)$$
$$+ \int_{t_{j_2}(\xi(\tau),\tau)}^{\tau} E_{j_2}(\tau_1;\xi(\tau),\tau)\Bigg[\sum_{s\neq j_2} a_{j_2s}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)u_s(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)$$
$$+g_{j_2}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)\Bigg]\,d\tau_1\Bigg]\,d\tau. \tag{16}$$

By construction, $a_{ij_2}(z_2,T_2)\neq 0$. Furthermore, we consider the part of the integral in (16) over the area denoted in Fig. 3 by S and transform it similarly to above as follows:

$$\int_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau)\int_{T_1}^{T_1+\varepsilon} E_{j_2}(\tau_1;\xi(\tau),\tau)$$
$$\times a_{j_2j_1}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)u_{j_1}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)\,d\tau_1\,d\tau$$
$$= \int_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau)\int_{T_1}^{T_1+\varepsilon} E_{j_2}(\tau_1;\xi(\tau),\tau)a_{j_2j_1}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)$$
$$\Bigg[\varphi_{j_1}(\omega_{j_1}(0;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)) + \int_0^{\tau_1} E_{j_1}(\tau_2;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)$$
$$\times\Bigg[\sum_{s\neq j_1} a_{j_1s}(\omega_{j_1}(\tau_2;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1),\tau_2)u_s(\omega_{j_1}(\tau_2;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1),\tau_2)$$
$$+g_{j_1}(\omega_{j_1}(\tau_2;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1),\tau_2)\Bigg]\,d\tau_2\Bigg]\,d\tau_1. \tag{17}$$

Again by construction, $a_{j_2j_1}(z_1,T_1)\neq 0$. Combining (10), (16), and (17), we see that

$$u_i(x_0,t_0) = Q_\varepsilon[u](x_0,t_0) + \Phi_i^\varepsilon(x_0,t_0), \tag{18}$$

where Q_ε is a certain operator and

$$\Phi_i^\varepsilon(x_0,t_0) = \int_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau)\int_{T_1}^{T_1+\varepsilon} E_{j_2}(\tau_1;\xi(\tau),\tau)$$
$$\times a_{j_2j_1}(\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1)\varphi_{j_1}(\omega_{j_1}(0;\omega_{j_2}(\tau_1;\xi(\tau),\tau),\tau_1))\,d\tau_1\,d\tau.$$

A similar representation for $u_i(x,t)$ holds in a sufficiently small neighborhood of (x_0,t_0). From the derivation of (18) it follows that u_i at (x_0,t_0) cannot be more regular than any of the summands in (18). Hence it suffices to show that $\Phi_i^\varepsilon(x_0,t_0)$ is not smooth at (x_0,t_0). Let us make simple transformations:

$$
\begin{aligned}
\Phi_i^\varepsilon(x_0,t_0) &= \int\limits_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau) \\
&\quad\times \int\limits_{\eta(T_1+\varepsilon,\tau)}^{\eta(T_1,\tau)} E_{j_2}(\beta(x,\tau);\xi(\tau),\tau)a_{j_2j_1}(\alpha(x,\tau),\beta(x,\tau))\varphi_{j_1}(x)\,dx\,d\tau \\
&= \int\limits_{T_2-\varepsilon}^{T_2} E_i(\tau;x_0,t_0)a_{ij_2}(\xi(\tau),\tau) \\
&\quad\times\Bigg[\int\limits_{0}^{\eta(T_1,\tau)} E_{j_2}(\beta(x,\tau);\xi(\tau),\tau)a_{j_2j_1}(\alpha(x,\tau),\beta(x,\tau))\varphi_{j_1}(x)\,dx \\
&\quad+\int\limits_{\eta(T_1+\varepsilon,\tau)}^{0} E_{j_2}(\beta(x,\tau);\xi(\tau),\tau)a_{j_2j_1}(\alpha(x,\tau),\beta(x,\tau))\varphi_{j_1}(x)\,dx\Bigg]\,d\tau \\
&= \int\limits_{\eta(T_1,T_2-\varepsilon)}^{\eta(T_1,T_2)} E_i(\rho(y);x_0,t_0)a_{ij_2}(\gamma(y),\rho(y)) \\
&\quad\times\int\limits_{0}^{y} E_{j_2}(\beta(x,\rho(y));\gamma(y),\rho(y))a_{j_2j_1}(\alpha(x,\rho(y)),\beta(x,\rho(y)))\varphi_{j_1}(x)\,dx\,dy \\
&\quad+\int\limits_{\eta(T_1+\varepsilon,T_2-\varepsilon)}^{\eta(T_1+\varepsilon,T_2)} E_i(\zeta(y);x_0,t_0)a_{ij_2}(\sigma(y),\zeta(y)) \\
&\quad\times\int\limits_{y}^{0} E_{j_2}(\beta(x,\zeta(y));\sigma(y),\zeta(y))a_{j_2j_1}(\alpha(x,\zeta(y)),\beta(x,\zeta(y)))\varphi_{j_1}(x)\,dx\,dy,
\end{aligned}
$$

where $\eta(t,\tau)=\omega_{j_1}(0;\omega_{j_2}(t;\xi(\tau),\tau),t)$; $(\alpha(x,\tau),\beta(x,\tau))$ denotes the intersection point of the characteristics $\omega_{j_1}(\tau_1;x,0)$ and $\omega_{j_2}(\tau_1;\xi(\tau),\tau)$; $(\gamma(y),\rho(y))$ denotes the intersection point of the characteristics $\omega_i(\tau;x_0,t_0)$ and $\omega_{j_2}(\tau;\omega_{j_1}(T_1;y,0),T_1)$; $(\sigma(y),\zeta(y))$ denotes the intersection point of the i-th and the j_2-th characteristics passing through the points $\omega_i(\tau;x_0,t_0)$ and $\omega_{j_2}(\tau;\omega_{j_1}(T_1+\varepsilon;y,0),T_1+\varepsilon)$. Note that α, β, γ, ρ, σ, and ζ are (at least) continuous functions, what easily follows from the smoothness assumptions imposed on the initial data. It follows that u_i is not C^3-smooth at (x_0,t_0) (even if α, β, γ, ρ, σ, and ζ are C^∞-functions).

The necessity is proved. □

3 Examples

Here we give some examples to show how the criterion given by Theorem 12 works. Each of these examples is rather general and interesting by its own. Throughout this section it is supposed that all characteristics of system (1) are bounded. This assumption is not restrictive from the practical point of view. It is true, for example, whenever $\inf |\lambda_i| > 0$ for all $i \le n$.

3.1 Classical boundary conditions

As a partial case of problem (1)–(3), consider (1), (2) with classical boundary conditions

$$\begin{array}{ll} u_i(0,t) = h_i(t), & k+1 \le i \le n, \quad t \in (0,\infty) \\ u_i(1,t) = h_i(t), & 1 \le i \le k, \qquad t \in (0,\infty). \end{array} \tag{19}$$

Theorem 15 *Assume that the data λ_i, a_{ij}, g_i, and h_i are smooth in all their arguments. Suppose that condition (5) is fulfilled. Then the continuous solution to problem (1), (2), (19) is smoothing for any $\varphi \in C[0,1]^n$ satisfying (6).*

This result is a straightforward consequence of Theorem 12. Indeed, since no reflection from the boundary is possible, every EP passing through (x,t) consists of a single characteristic. Moreover, due to the smoothness assumptions on Λ and the definition of a characteristic, for any $T > 0$ there is $T' > T$ such that all characteristics passing through the line $t = T$ lei below $t = T'$. Hence the sufficient condition (ι) is fulfilled and the theorem follows.

Remark 16 *Even in the case of smooth classical boundary conditions (19), the domain of influence of the initial data on u_i for every $i \le n$ in general is unbounded (due to the lower order terms in (1)). In spite of this, the influence of the initial data on the regularity of u becomes weaker and weaker in time causing the smoothingness effect.*

3.2 Periodic boundary conditions

Suppose that at least one component of the solution, say the first one, satisfies periodic boundary condition. Specifically, the first equation in (3) is written in the form $u_1(0,t) = u_1(1,t)$. Then the domain of influence of initial data (2) on u_1 is the whole $\overline{\Pi}$. Moreover, for each $(x,t) \in \overline{\Pi}$, there is an unbounded EP passing through (x,t) (at least one constructed by means of characteristics of the first equation). This entails that the necessary condition $(\iota\iota)$ is not fulfilled and hence the solution to problem (1)–(3) is not smoothing.

3.3 Nonseparable linear boundary conditions of local type

Consider now the reflection boundary conditions

$$\begin{array}{ll} u_i(0,t) = \displaystyle\sum_{j=1}^{k} b_{ij} u_j(0,t), & k+1 \le i \le n, \quad t \in (0,\infty) \\ u_i(1,t) = \displaystyle\sum_{j=k+1}^{n} c_{ij} u_j(1,t), & 1 \le i \le k, \qquad t \in (0,\infty), \end{array} \tag{20}$$

where b_{ij} and c_{ij} are constants. Systems like (1), (2), (20) cover linearizations of many mathematical models for chemical kinetics [19, 20].

Our aim is to reformulate sufficient condition (ι) constructively in terms of data (specifically, in terms of boundary data) of our problem. Note first that, if $b_{ij} \neq 0$ for some $k+1 \leq i \leq n$ and $j \leq k$ (resp., $c_{ij} \neq 0$ for some $i \leq k$ and $k+1 \leq j \leq n$), then each characteristic of the j-th equation reflects from the boundary $x = 0$ (resp., from the boundary $x = 1$), the reflections being characteristics of the i-th equation. Since all EPs passing through (x, t) are constructed by means of subsequent reflections, for each EP passing through (x, t) and consisting of more than one smooth piece, there is a unique sequence (finite if the EP passing through (x, t) is bounded and infinite otherwise) either of kind

$$b_{i_2 i_1}, c_{i_3 i_2}, b_{i_4 i_3}, c_{i_5 i_4}, \dots \tag{21}$$

or of kind

$$c_{i_2 i_1}, b_{i_3 i_2}, c_{i_4 i_3}, b_{i_5 i_4}, \dots \tag{22}$$

with nonzero elements. Condition (ι) is satisfied iff for every $T > 0$ all EPs passing through (x, T) are bounded uniformly in $x \in [0, 1]$. The latter is equivalent to the statement that all the sequences (21) and (22) are finite. This means that condition (ι) is expressible here in the algebraic form, namely as a finite number of equalities of kind

$$b_{i_2 i_1} \cdot c_{i_3 i_2} \cdot b_{i_4 i_3} \cdot c_{i_5 i_4} \cdot \dots \cdot b_{i_{n-1} i_{n-2}} \cdot c_{i_n i_{n-1}} = 0 \tag{23}$$

for all possible $i_1, \dots, i_n$ such that the entries of matrices $B_{n-k,k} = \{b_{ij}\}_{k+1 \leq i \leq n, j \leq n}$ and $C_{k,n-k} = \{c_{ij}\}_{i \leq k, k+1 \leq j \leq n}$ appear in (23) no more than once.

Condition (23) can be easily reformulated in the following form (see also [7]). Set

$$R \;=\; \begin{pmatrix} I_{n-k} & B_{n-k,k} \\ C_{k,n-k} & I_k \end{pmatrix},$$

where I_j is a unit matrix of size j. Consider an expansion of the determinant of R along the first $n-k$ rows, namely,

$$\det R = \sum_{i_1 < \dots < i_{n-k}} D_{i_1,\dots,i_{n-k}} F_{i_1,\dots,i_{n-k}}. \tag{24}$$

Here the respective determinants of order $n-k$ are denoted by $D_{i_1,\dots,i_{n-k}}$ and their cofactors by $F_{i_1,\dots,i_{n-k}}$. It turns out that each summand in (24) starting from the second one is a product of kind (23). One can easily show that (23) is fulfilled iff all summands in (24) excepting for the first one, which is obviously equals to 1, vanish. Note that the latter condition is obtained in [7] for problem (1), (2), (20) with coefficients in (1) depending only on x.

3.4 Smoothingness phenomenon for IBVPs for the wave equation

Using Theorems 2 and 12, one can immediately obtain correctly posed IBVPs for the (nonhomogeneous) wave equation

$$(\partial_t^2 - a^2 \partial_x^2) u = f(x, t) \tag{25}$$

and state a smoothingness result for them. For instance, let us consider (25) subjected to initial conditions

$$u(x,0)=\varphi(x),\quad \partial_t u(x,0)=\psi(x) \tag{26}$$

and boundary conditions

$$\begin{array}{rcl} u(0,t) & = & h_1\left(t,u(1,t),(\partial_t u+a\partial_x u)|_{x=0}\right), \\ (\partial_t u+a\partial_x u)|_{x=1} & = & h_2\left(t,u(1,t),(\partial_t u+a\partial_x u)|_{x=0}\right). \end{array} \tag{27}$$

The problem (25)–(27) is equivalent to the following problem for the (2×2)-first-order hyperbolic system

$$(\partial_t+a\partial_x)u=w,\quad (\partial_t-a\partial_x)w=f(x,t) \tag{28}$$

$$u(x,0)=\varphi(x),\quad w(x,0)=\psi(x)+a\frac{d\varphi(x)}{dx} \tag{29}$$

$$\begin{array}{l} u(0,t)=h_1\left(t,u(1,t),w(0,t)\right), \\ w(1,t)=h_2\left(t,u(1,t),w(0,t)\right). \end{array} \tag{30}$$

Under a continuous solution to problem (25)–(27) we mean the continuous solution to problem (28)–(30). In the next theorem conditions (ι) and $(\iota\iota)$ are supposed to be formulated correspondingly to problem (28)–(30). The following smoothingness result for (25)–(27) is a straightforward consequence of Theorems 2 and 12.

Theorem 17 *Assume that the data f and h_i are smooth functions in all their arguments and conditions (5) and (7) are fulfilled.*

- Sufficiency. *If condition (ι) is true, then the continuous solution to problem (25)–(27) is smoothing for any $\varphi\in C^1[0,1]$ and $\psi\in C[0,1]$ satisfying the zero-order compatibility conditions between (29) and (30).*
- Necessity. *Assume that the continuous solution to problem (25)–(27) is smoothing for any $\varphi\in C^1[0,1]$ and $\psi\in C[0,1]$ satisfying the zero-order compatibility conditions between (29) and (30). Then condition $(\iota\iota)$ is fulfilled.*

4 Distributional initial data

Now we address the case of distributional initial data. Specifically, we investigate system (1) with initial conditions

$$\begin{array}{l} u_i(x,0)=b_i(x)\equiv b_{ir}(x)+b_{is}(x)=b_{ir}(x)+\displaystyle\sum_{j=1}^{m_i}c_{ij}\delta^{(l_{ij})}(x-x^*_{ij}), \\ i\le n,\quad 0<x^*_{i1}<\cdots<x^*_{im_i}<1,\quad x\in(0,1) \end{array} \tag{31}$$

and boundary conditions

$$\begin{array}{l} u_i(0,t)=\displaystyle\sum_{j=1}^{n}p_{ij}(t)v^u_j(t)+r_i(t,v^u(t)),\quad k+1\le i\le n,\quad t\in(0,\infty) \\ u_i(1,t)=\displaystyle\sum_{j=1}^{n}p_{ij}(t)v^u_j(t)+r_i(t,v^u(t)),\quad 1\le i\le k,\qquad t\in(0,\infty), \end{array} \tag{32}$$

where the regular part $b_r(x) = (b_{1r}, \dots, b_{nr})$ of the initial data $b(x) = (b_1, \dots, b_n)$ is a continuous vector-function on $[0,1]$, $r_i(t,y) \in C^\infty([0,\infty) \times \mathbb{R}^n)$, r_i and $\nabla_y r_i$ are bounded in $y \in \mathbb{R}^n$ uniformly in t over any compact subset of $\mathbb{R}_+$, and $p_{ij}(t) \in C^\infty[0,\infty)$. Also, $c_{ij} \in \mathbb{R}$, $m_i \in \mathbb{N}$, and $l_{ij} \in \mathbb{N}_0$. Furthermore, $v^u(t)$ just redenotes the vector-function $v(t)$ introduced in Section 1 by (4). We will use such notation below with other superscripts. Without restriction of generality we can suppose that the zero-order compatibility conditions between (31) and (32) are fulfilled, namely,

$$\begin{aligned} b_{ri}(0) &= \sum_{j=1}^n p_{ij}(0) v_j^u(0) + r_i(0, v^u(0)), \quad k+1 \le i \le n \\ b_{ri}(1) &= \sum_{j=1}^n p_{ij}(0) v_j^u(0) + r_i(0, v^u(0)), \quad 1 \le i \le k, \end{aligned} \tag{33}$$

where $v^u(0) = (b_{r1}(0), \dots, b_{rk}(0), b_{r,k+1}(1), \dots, b_{rn}(1))$.

Note that system (1), (31), (32) is unsolvable in the classical sense, since nonlinearities in the boundary conditions and singularities in the initial data entail nonlinear operations in the space of distributions. We, therefore, aim at constructing a delta wave solution to (1), (31), (32). For this purpose we regularize the singular part $b_s(x)$ of initial data (31) by means of the convolution with the so-called model delta nets $(\varphi_\varepsilon)_{\varepsilon>0}$ where $\varphi \in \mathcal{D}(\mathbb{R})$, $\int \varphi \, dx = 1$, $\varphi_\varepsilon(x) = 1/\varepsilon \varphi(x/\varepsilon)$, and $\int |\varphi_\varepsilon| \, dx$ is bounded by a constant independent of ε. The regularized problem for u^ε is this:

$$\begin{aligned} (\partial_t + \Lambda \partial_x + A) u^\varepsilon &= g(x,t) \\ u_i^\varepsilon(x,0) &= b_{ir}(x) + b_{is}^\varepsilon(x) = b_{ir}(x) + \sum_{j=1}^{m_i} c_{ij} \varphi_\varepsilon^{(l_{ij})}(x - x_{ij}^*), \quad 1 \le i \le n \\ u_i^\varepsilon(0,t) &= \sum_{j=1}^n p_{ij}(t) v_j^{u,\varepsilon}(t) + r_i(t, v^{u,\varepsilon}(t)), \qquad k+1 \le i \le n \\ u_i^\varepsilon(1,t) &= \sum_{j=1}^n p_{ij}(t) v_j^{u,\varepsilon}(t) + r_i(t, v^{u,\varepsilon}(t)), \qquad 1 \le i \le k. \end{aligned} \tag{34}$$

Here $v^{u,\varepsilon}(t) = (u_1^\varepsilon(0,t), \dots, u_k^\varepsilon(0,t), u_{k+1}^\varepsilon(1,t), \dots, u_n^\varepsilon(1,t))$. Decompose $A = D + F$ into the diagonal and off-diagonal parts D and F, respectively. Our aim is now to show that $u^\varepsilon \to_{\varepsilon \to 0} z + w$ in $\mathcal{D}'(\Pi)$ where a singular part z and a regular part w are solutions to systems, respectively,

$$\begin{aligned} (\partial_t + \Lambda \partial_x + D) z &= 0 \\ z(x,0) &= b_s(x) \\ z_i(0,t) &= \sum_{j=1}^n p_{ij}(t) v_j^z(t), \quad k+1 \le i \le n \\ z_i(1,t) &= \sum_{j=1}^n p_{ij}(t) v_j^z(t), \quad 1 \le i \le k \end{aligned} \tag{35}$$

and

$$\begin{aligned}(\partial_t + \Lambda\partial_x + A)w + Fz &= g(x,t)\\ w(x,0) &= b_r(x)\\ w_i(0,t) &= \sum_{j=1}^{n} p_{ij}(t)v_j^w(t) + r_i(t,v^w), \quad k+1 \le i \le n\\ w_i(1,t) &= \sum_{j=1}^{n} p_{ij}(t)v_j^w(t) + r_i(t,v^w), \quad 1 \le i \le k\end{aligned} \tag{36}$$

(the two systems are called *nonlinear splitting*). System (35) is linear, corresponds to the singular part of the initial problem, and is responsible for propagation of singularities. System (36) is nonlinear and corresponds to the regular part of our problem. Note that, since F is the off-diagonal part of A, the singular forcing term Fz contributes into the hyperbolic system (36) transversely to the characteristic directions and, therefore, cannot produce strong singularities for w. We now show that problems (35) and (36) are uniquely solvable.

Definition 18 By J_* we denote the union, over $i \le n$ and $j \le m_i$, of all EPs containing the characteristic $\omega_i(\tau; x_{ij}^*, 0)$. Furthermore, J will denote the union, over $i \le n$ and $j \le m_i$, of all EPs emanating from the point $(x_{ij}^*, 0)$.

Note that $J_* \subset J$.

Lemma 19 *System (35) has a unique distributional solution z representable as the sum of singular distributions concentrated on characteristic curves in the set J_*.*

The lemma can be easily proved by the method of characteristics.

Let $C\left(\overline{\Pi} \setminus J\right)$ be the space of piecewise continuous functions on $\overline{\Pi}$ with (possibly) first-order discontinuities on J.

Lemma 20 *System (36) has a unique $C\left(\overline{\Pi} \setminus J\right)^n$-solution w.*

Proof. The uniqueness can be easily proved by considering the problem for the difference of two $C\left(\overline{\Pi} \setminus J\right)^n$-solutions to (36) and applying the method of characteristics.

To prove existence, consider $w = \bar{w} + \tilde{w}$, where $\bar{w}$ is the solution to the linear system with the strongly singular forcing term

$$\begin{aligned}(\partial_t + \Lambda\partial_x + D)\bar{w} + Fz &= g(x,t)\\ \bar{w}(x,0) &= b_r(x)\\ \bar{w}_i(0,t) &= \sum_{j=1}^{n} p_{ij}(t)v_j^{\bar{w}}(t), \quad k+1 \le i \le n\\ \bar{w}_i(1,t) &= \sum_{j=1}^{n} p_{ij}(t)v_j^{\bar{w}}(t), \quad 1 \le i \le k\end{aligned} \tag{37}$$

and $\tilde{w}$ is the solution to the nonlinear system

$$\begin{aligned}
(\partial_t + \Lambda\partial_x + A)\tilde{w} + F\bar{w} &= 0 \\
\tilde{w}(x,0) &= 0 \\
\tilde{w}_i(0,t) &= \sum_{j=1}^{n} p_{ij}(t) v_j^{\tilde{w}}(t) + r_i(t, v^{\bar{w}} + v^{\tilde{w}}), \quad k+1 \le i \le n \\
\tilde{w}_i(1,t) &= \sum_{j=1}^{n} p_{ij}(t) v_j^{\tilde{w}}(t) + r_i(t, v^{\bar{w}} + v^{\tilde{w}}), \quad 1 \le i \le k.
\end{aligned} \tag{38}$$

Note that, if $\bar{w}$ is a solution to (37) and $\tilde{w}$ is a solution to (38), then $\bar{w} + \tilde{w}$ is a solution to (36) indeed.

To show that system (37) has a (unique) solution $\bar{w} \in C\left(\overline{\Pi} \setminus J\right)^n$, we use the method of characteristics again, thereby representing $\bar{w}$ explicitly in the integral form. Since the singularities in the forcing term Fz are strong and transverse to the corresponding characteristics of the hyperbolic system, the Volterra integrals of Fz in the integral representation of $\overline{w}$ are at worst discontinuous functions with the first order discontinuities (if any) on J.

To show that problem (38) has a (unique) solution $\tilde{w} \in C\left(\overline{\Pi} \setminus J\right)^n$, we follow the proof of Theorem [13, Thm 2.1] with minor changes. Fix an arbitrary $T > 0$ and rewrite (38) in Π^T in an equivalent integral form with the boundary term not depending on $v^{\tilde{w}}$ (applying a finite number of integrations along characteristic curves up to reaching the initial axis). Using the assumption that $\nabla_y r_i(t,y)$ is bounded locally in $t \in \mathbb{R}_+$ and globally in $y \in \mathbb{R}^n$, it is not difficult to check that the operator defined by the right-hand side of the integral form of (38) maps $C\left(\overline{\Pi} \setminus J\right)^n$ into itself and is contractible on $\overline{\Pi^{\theta_0}}$ for some $\theta_0 \le T$. The local existence-uniqueness result in $\overline{\Pi^{\theta_0}}$ then follows from the Banach fixed point theorem. If $\theta_0 < T$, we consider problem (38) on $\Pi^T \setminus \overline{\Pi^{\theta_0}}$ with the initial condition replaced by $u(x,\theta_0) = \psi(x) \in C\left([0,1] \setminus J\right)^n$, where the function $\psi(x)$ is determined in the first step. Similarly to the above, the right-hand side of the integral form of this problem defines the operator mapping $C\left(\left(\overline{\Pi} \setminus \Pi^{\theta_0}\right) \setminus J\right)^n$ into itself. Set $\Pi^0 = \emptyset$. Since the contraction property of the operator under consideration does not depend on the initial conditions, one can choose the value of θ_0 from the very beginning so small that the operator is contractible on $\overline{\Pi^{(s+1)\theta_0}} \setminus \Pi^{s\theta_0}$ for any $0 \le s \le \lceil T/\theta_0 \rceil - 1$. Applying the Banach fixed point theorem with respect to the domain $\overline{\Pi^{2\theta_0}} \setminus \Pi^{\theta_0}$, we conclude that problem (38) has a unique $C\left(\left(\overline{\Pi}^{2\theta_0} \setminus \Pi^{\theta_0}\right) \setminus J\right)^n$-solution. The proof of the unique solvability in $\Pi^{(s+1)\theta_0} \setminus \Pi^{s\theta_0}$ for each $s \in \{2, \ldots, \lceil T/\theta_0 \rceil - 1\}$ goes over the same argument. Since $T > 0$ is arbitrary, the unique solvability of (38) in $C\left(\overline{\Pi} \setminus J\right)^n$ follows.

We have thus proved that (36) has a unique solution w in $C\left(\overline{\Pi} \setminus J\right)^n$, which equals to $\bar{w} + \tilde{w}$. □

We are now prepared to state our result about delta waves.

Theorem 21 *Assume that the data λ_i, a_{ij}, g_i, p_{ij}, r_i, and b_{ir} are continuous functions in all their arguments, λ_i are Lipschitz in $x \in [0,1]$ locally in $t \in [0,\infty)$, and both r_i and $\nabla_y r_i$ are in $L^\infty((0,T) \times \mathbb{R}^n)$ for any $T > 0$. Given $\varepsilon > 0$, let u^ε, z^ε, and w^ε be the continuous solutions to problems, respectively (34), (35), and (36) with z^ε, w^ε, and b_s^ε in place of z, w, and b_s, respectively. Then*

1. $u^\varepsilon - z^\varepsilon - w^\varepsilon \to 0$ *in* $L^1_{loc}(\Pi)$ *as* $\varepsilon \to 0$*;*

2. $z^\varepsilon \to z$ *in* $\mathcal{E}'(\Pi)$ *as* $\varepsilon \to 0$*;*

3. $w^\varepsilon \to w$ *in* $C(K)$ *as* $\varepsilon \to 0$ *for any compact subset* $K \subset \overline{\Pi} \setminus J$*;*

4. $u^\varepsilon \to z + w$ *in* $\mathcal{D}'(\Pi)$ *as* $\varepsilon \to 0$*,*

where $z \in \mathcal{E}'(\Pi)^n$ *and* $w \in C\left(\overline{\Pi} \setminus J\right)^n$ *are the solutions to problems (35) and (36), respectively.*

Proof. **1.** Let J^ε_* be the union, over $i \le n$, $j \le m_i$, and $\alpha \in [-\varepsilon; \varepsilon]$, of all EPs containing $\omega_i(\tau; x^*_{ij} + \alpha, 0)$. Note that the set J^ε_* consists of the "tubes" forming the support of z^ε. Set $N^\varepsilon = J^\varepsilon_* \cap (\partial\Pi \setminus \{t = 0\})$.

To show that the splitting of u^ε into a "regular" and a "singular" parts w^ε and z^ε is correct (what is stated in Item 1 of the theorem), we consider the following problem for $\alpha^\varepsilon = u^\varepsilon - z^\varepsilon - w^\varepsilon$:

$$\begin{aligned}
(\partial_t + \Lambda\partial_x + A)\alpha^\varepsilon &= 0 \\
\alpha^\varepsilon(x,0) &= 0 \\
\alpha^\varepsilon_i(0,t) &= \sum_{j=1}^n p_{ij}(t) v^{\alpha,\varepsilon}_j(t) + r_i(t, v^{u,\varepsilon}) - r_i(t, v^{w,\varepsilon}), \quad k+1 \le i \le n \\
\alpha^\varepsilon_i(1,t) &= \sum_{j=1}^n p_{ij}(t) v^{\alpha,\varepsilon}_j(t) + r_i(t, v^{u,\varepsilon}) - r_i(t, v^{w,\varepsilon}), \quad 1 \le i \le k.
\end{aligned} \tag{39}$$

Rewrite

$$\begin{aligned}
&r_i(t, v^{u,\varepsilon}) - r_i(t, v^{w,\varepsilon}) \\
&\quad = \left(r_i(t, v^{\alpha+w+z,\varepsilon}) - r_i(t, v^{w+z,\varepsilon})\right) + \left(r_i(t, v^{w+z,\varepsilon}) - r_i(t, v^{w,\varepsilon})\right) \\
&\quad = \int_0^1 \left(\nabla_y r_i\right)\left(t, \sigma v^{\alpha+w+z,\varepsilon} + (1-\sigma) v^{w+z,\varepsilon}\right) d\sigma \cdot v^{\alpha,\varepsilon} + \left(r_i(t, v^{w+z,\varepsilon}) - r_i(t, v^{w,\varepsilon})\right) \\
&\quad \equiv \sum_{j=1}^n K^\varepsilon_{ij}(t) v^{\alpha,\varepsilon}_j + L^\varepsilon_i(t),
\end{aligned} \tag{40}$$

where the symbol $\equiv$ introduces the short-hand notation K^ε_{ij} and L^ε_i. By assumption, $r_i(t,y)$ and $\nabla_y r_i(t,y)$ are in $\mathrm{L}^\infty\left((0,T) \times \mathbb{R}^n\right)$ for any $T > 0$. Hence $K^\varepsilon_{ij}(t)$ and $L^\varepsilon_i(t)$ are bounded uniformly in $t \in [0,T]$ and $\varepsilon > 0$ for an arbitrarily fixed $T > 0$.

To prove the desired statement, we will use dominated convergence. It is sufficient to show that, given $T > 0$, the sequence α^ε is bounded on $\overline{\Pi}^T$ uniformly in $\varepsilon > 0$ and $\alpha^\varepsilon(x,t) \to 0$ as $\varepsilon \to 0$ pointwise off J_*.

Fix an arbitrary $T > 0$ and rewrite (39) in $\overline{\Pi}^T$ in the integral form

$$\alpha^\varepsilon_i(x,t) = (R_i \alpha^\varepsilon)(x,t) - \int_{t_i(x,t)}^t \sum_{j=1}^n \left(a_{ij}\alpha^\varepsilon_j\right)\left(\omega_i(\tau; x, t), \tau\right) d\tau, \quad i \le n, \tag{41}$$

where $(R_i\alpha^\varepsilon)(x,t)=0$ if $t_i(x,t)=0$ and

$$(R_i\alpha^\varepsilon)(x,t)=\sum_{j=1}^{n}\left(p_{ij}+K_{ij}^\varepsilon\right)(t_i(x,t))\,v_j^{\alpha,\varepsilon}(t_i(x,t))+L_i^\varepsilon(t_i(x,t)) \tag{42}$$

otherwise. We will use a modification of (41) where the boundary term $(R_i\alpha^\varepsilon)(x,t)$ does not depend on $v^{\alpha,\varepsilon}$. More specifically, using (41), we express $v_j^{\alpha,\varepsilon}(t_i(x,t))$ contributing into (42) in an integral form. If the resulting expression for $\alpha^\varepsilon(x,t)$ still depends on $v^{\alpha,\varepsilon}$, we apply the same procedure again. In a finite number of steps we reach the initial axis and thereby obtain the desired representation for $\alpha_j^\varepsilon(x,t)$. It is the sum of a continuous function (of K_{ij}^ε, L_i^ε, and p_{ij}) and a finite number of integrals over characteristic curves in the EPs passing through $\omega_i(\tau;x,t)$ and restricted to $\overline{\Pi^t}$. Let N be the maximal number of such integrals where the range of maximization is $i\le n$ and $(x,t)\in\overline{\Pi^T}$. Applying the sequential approximation method, we easily derive the apriori estimate in $\overline{\Pi^T}$:

$$\sup_{\varepsilon,i,x,t}|\alpha_i^\varepsilon|\le\sup_{i,t,\varepsilon}|L_i^\varepsilon|\sum_{s=0}^{N}n^s\left(\sup_{i,j,t}|p_{ij}|+\sup_{\varepsilon,i,j,t}|K_{ij}^\varepsilon|\right)^s$$
$$\times\exp\left\{nT\sup_{i,j,x,t}|a_{ij}|\sum_{s=0}^{N}n^s\left(\sup_{i,j,t}|p_{ij}|+\sup_{\varepsilon,i,j,t}|K_{ij}^\varepsilon|\right)^s\right\}.$$

This implies that $\alpha^\varepsilon(x,t)$ is bounded uniformly in $(x,t)\in\overline{\Pi}^T$ and $\varepsilon>0$, as desired.

It remains to prove that $\alpha^\varepsilon(x,t)\to 0$ as $\varepsilon\to 0$ pointwise off J_*. Note that, due to the support properties of z^ε, we have $v^{w+z,\varepsilon}(t)\equiv v^{w,\varepsilon}(t)$ and hence $L_i^\varepsilon(t)\equiv 0$ on $\partial\Pi\setminus(N^\varepsilon\cup\{t=0\})$. As $N^{\varepsilon_1}\subset N^{\varepsilon_2}$ for all $\varepsilon_1<\varepsilon_2$, we have the identity $L_i^\varepsilon(t)\equiv 0$ on $\partial\Pi\setminus(N^{\varepsilon_0}\cup\{t=0\})$ for an arbitrarily fixed $\varepsilon_0>0$ and all $\varepsilon\le\varepsilon_0$. Fix $(x,t)\in\overline{\Pi^T}\setminus J_*$ and $\varepsilon_0>0$ so that $(x,t)\in\overline{\Pi^T}\setminus J_*^{\varepsilon_0}$. Then $\alpha^\varepsilon(x,t)$ is representable in the form (41) with $L_i^\varepsilon(t_i(x,t))=0$. Note that $t_i(x,t)\in\partial\Pi\setminus N^{\varepsilon_0}$; hence $v_j^{\alpha,\varepsilon}(t_i(x,t))$ is expressible in the form (41) again with $L_j^\varepsilon=0$. Continuing in this way, similarly to the above we arrive at the integral representation of α^ε where the boundary term depends neither on L^ε nor on $v^{\alpha,\varepsilon}$. Roughly speaking, each $\alpha_i^\varepsilon(x,t)$ is a finite sum of integrals (of kind as in (41)) over characteristics in the EPs passing through $\omega_i(\tau;x,t)$ and restricted to $\overline{\Pi^t}$. We split each of the integrals into two parts, the splitting procedure being illustrated by example of the integral in (41):

$$\int\limits_{t_i(x,t)}^{t}d\tau=\int\limits_{[t_i(x,t),t]\setminus\triangle_{\varepsilon_0}(x,t)}d\tau+\int\limits_{\triangle_{\varepsilon_0}(x,t)}d\tau,$$

where $\triangle_{\varepsilon_0}(x,t)=\{\tau\in[t_i(x,t),t]\,:\,(\omega_i(\tau;x,t),\tau)\in J_*^{\varepsilon_0}\}$. Since α^ε is bounded on $\overline{\Pi}^T$ uniformly in $\varepsilon>0$, there is $C>0$ (depending on T) such that the absolute value of the second integral is bounded from above by $C\varepsilon_0$. This gives us the following apriori estimate in $\overline{\Pi^T}$:

$$\sup_{\varepsilon\le\varepsilon_0,i,x,t}|\alpha_i^\varepsilon|\le C\varepsilon_0\sum_{s=0}^{N}n^s\left(\sup_{i,j,t}|p_{ij}|+\sup_{\varepsilon>0i,j,t}|K_{ij}^\varepsilon|\right)^s$$
$$\times\exp\left\{nT\sup_{i,j,x,t}|a_{ij}|\sum_{s=0}^{N}n^s\left(\sup_{i,j,t}|p_{ij}|+\sup_{\varepsilon>0,i,j,t}|K_{ij}^\varepsilon|\right)^s\right\}. \tag{43}$$

Note that (43) is true with ε_1 in place of ε_0 for an arbitrarily fixed $\varepsilon_1 \le \varepsilon_0$. The desired convergence is thereby proved. The first statement of the theorem now follows by dominated convergence.

2. This part easily follows from diagonality of matrix D by the method of characteristics.

3. Given ε, denote by $\bar{w}^\varepsilon$ the continuous solution to (37) with z^ε in place of z and let $\bar{w}$ denote the $C\left(\overline{\Pi} \setminus J\right)^n$-solution to (37). Let K be an arbitrary compact and connected subset of $\overline{\Pi} \setminus J$.

Claim 1. $\bar{w}^\varepsilon \to \bar{w}$ in $C(K)$ as $\varepsilon \to 0$.

Proof of Claim. The difference $\bar{w} - \bar{w}^\varepsilon$ satisfies system

$$\begin{aligned}
\left(\partial_t + \Lambda\partial_x + D\right)\left(\bar{w}^\varepsilon - \bar{w}\right) + F(z^\varepsilon - z) &= 0 \\
\bar{w}^\varepsilon(x,0) - \bar{w}(x,0) &= 0 \\
\bar{w}_i^\varepsilon(0,t) - \bar{w}_i(0,t) &= \sum_{j=1}^{n} p_{ij}(t)\left(v_j^{\bar{w},\varepsilon}(t) - v_j^{\bar{w}}(t)\right), \quad k+1 \le i \le n \\
\bar{w}_i^\varepsilon(1,t) - \bar{w}_i(1,t) &= \sum_{j=1}^{n} p_{ij}(t)\left(v_j^{\bar{w},\varepsilon}(t) - v_j^{\bar{w}}(t)\right), \quad 1 \le i \le k.
\end{aligned} \tag{44}$$

Since $z^\varepsilon \to z$ in $\mathcal{E}'(\Pi)$ as $\varepsilon \to 0$, we obtain the convergence $\bar{w}^\varepsilon \to \bar{w}$ in $\mathcal{E}'(\Pi)$ as $\varepsilon \to 0$. It remains to prove that $\{\bar{w}^\varepsilon\}_{\varepsilon>0}$ converges in $C\left(K\right)$. Note first that the solution to problem (37) is the sum of the solution to the version of (37) with $b_r(x) \equiv 0$ and $g(x,t) \equiv 0$ and the solution to the version of (37) with $F(x,t) \equiv 0$. Since the solution to the latter problem is continuous in $\overline{\Pi}$, our task reduces to proving the convergence of $\{\bar{w}^\varepsilon\}_{\varepsilon>0}$ in $C\left(K\right)$ in the case when $b_r(x) \equiv 0$ and $g(x,t) \equiv 0$.

Let J^ε be the union, over $i \le n$, $j \le m_i$, and $\alpha \in [-\varepsilon;\varepsilon]$, of all EPs emanating from $(x_{ij}^* + \alpha, 0)$. Fix an arbitrary $i \le n$ and $\varepsilon_0 > 0$ such that $K \subset \overline{\Pi} \setminus J^{\varepsilon_0}$. It is easily seen that, given $(x,t) \in K$, $\bar{w}_i^\varepsilon(x,t)$ for each $\varepsilon \le \varepsilon_0$ is a finite sum of integrals over the characteristics in the EPs passing through $\omega_i(\tau;x,t)$ and restricted to $\overline{\Pi^t} \cap J^\varepsilon$ (see Fig. 4). Denote these by $C_1^\varepsilon, \ldots, C_l^\varepsilon$. Moreover, suppose that C_j^ε is given by equation $x = c_j(t)$.

Let $f_{q,s}$ denote the entries of the matrix F. Moreover, $q(j)$ is the index of the characteristic curve passing through C_j^ε and $s(j)$ is the index of the characteristic curves bounding the "tube" containing C_j^ε. An explicit calculation of $\bar{w}_i^\varepsilon(x,t)$ gives us equality

$$\bar{w}_i^\varepsilon(x,t) = \sum_{j=1}^{l} P_{ij}(x,t) \int_{t_j^{\varepsilon-}}^{t_j^{\varepsilon+}} E_{q(j)}(\tau; c_j(t_j^{\varepsilon+}), t_j^{\varepsilon+}) f_{q(j),s(j)}(c_j(\tau),\tau) z_{s(j)}^\varepsilon(c_j(\tau),\tau)\, d\tau, \tag{45}$$

where $[t_j^{\varepsilon-}, t_j^{\varepsilon+}]$ is the projection of C_j^ε onto t-axis and $P_{ij}(x,t)$ is a smooth function of p_{ij}.

Furthermore, for any $j \le l$ there are $\alpha(j) \le n$, $\beta(j) \le m_\alpha$, $i_1, \ldots, i_\gamma \in \mathbb{N}$, and smooth mappings $T_{i_1} : t \longrightarrow t_{i_1}(c_j(t),t)$, $T_{i_p} : t \longrightarrow t_{i_p}(x_{i_p},t)$ for $1 < p < \gamma$, and $T_{i_\gamma} : t \longrightarrow \omega_{i_\gamma}(0; x_{i_\gamma}, t)$, where $x_i = 0$ for $i \le k$ and $x_i = 1$ for $k+1 \le i \le n$. Geometrically speaking, for each $j \le l$ there is $x^*_{\alpha(j),\beta(j)}$ and a continuous path in J_*^ε containing both the curve C_j^ε and the point $(x^*_{\alpha(j),\beta(j)}, 0)$, whatsoever $\varepsilon \le \varepsilon_0$. Moreover, the value of γ equals to the number of the "tubes" creating this path.

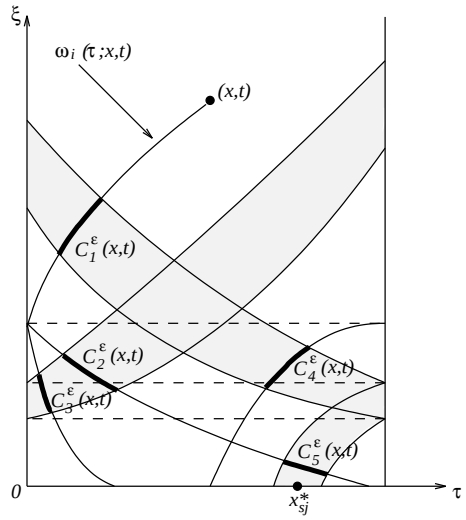

Figure 4: Construction of the set J_*^ε.

Note that l as a function of (x,t) is constant on any connected component of $\overline{\Pi} \setminus J^{\varepsilon_0}$. From now on by l we will denote the value of this function on K. Calculating of $z^\varepsilon_{s(j)}$ and changing variables by means of T_{i_p}, we transform expression (45) for $\bar{w}_i^\varepsilon(x,t)$ on K to a form more convenient for our purpose, namely,

$$\bar{w}_i^\varepsilon(x,t) = \sum_{j=1}^{l} \int\limits_{x^*_{\alpha(j),\beta(j)}-\varepsilon}^{x^*_{\alpha(j),\beta(j)}+\varepsilon} Q_{ij}(\xi;x,t)\varphi_\varepsilon^{(l_{\alpha(j),\beta(j)})}(\xi - x^*_{\alpha(j),\beta(j)})\,d\xi, \tag{46}$$

where the kernels $Q_{ij}(\xi;x,t)$ are certain smooth functions. Therefore,

$$\bar{w}_i^\varepsilon(x,t) \longrightarrow \sum_{j=1}^{l} (-1)^{l_{\alpha(j),\beta(j)}} \partial_\xi^{(l_{\alpha(j),\beta(j)})} Q_{ij}(x^*_{\alpha(j),\beta(j)};x,t) \text{ as } \varepsilon \to 0,$$

where the convergence is uniform for all $(x,t) \in K$. $\square$

Claim 2. $w^\varepsilon \to \bar{w} + \tilde{w}$ as $\varepsilon \to 0$ in $C(K)$, where $\bar{w}$ and $\tilde{w}$ are $C(\overline{\Pi} \setminus J)$-solutions to problems (37) and (38), respectively.

Proof of Claim. On the account of Claim 1, it is sufficient to prove that $w^\varepsilon - \bar{w}^\varepsilon - \tilde{w} \to 0$ as

$\varepsilon \to 0$ in $C(K)$. The difference $w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}$ is a solution to the following problem:

$$\begin{aligned}
\left(\partial_t + \Lambda\partial_x + A\right)\left(w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}\right) &= -F(\bar{w}^\varepsilon - \bar{w}) \\
\left(w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}\right)(x,0) &= 0 \\
\left(w_i^\varepsilon - \bar{w}_i^\varepsilon - \tilde{w}_i\right)(0,t) &= \sum_{j=1}^{n} p_{ij}(t) v_j^{w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}}(0,t) + r_i\left(t, v^{w^\varepsilon}\right) - r_i\left(t, v^{\bar{w}+\tilde{w}}\right), \\
& \qquad k+1 \le i \le n \\
\left(w_i^\varepsilon - \bar{w}_i^\varepsilon - \tilde{w}_i\right)(1,t) &= \sum_{j=1}^{n} p_{ij}(t) v_j^{w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}}(0,t) + r_i\left(t, v^{w^\varepsilon}\right) - r_i\left(t, v^{\bar{w}+\tilde{w}}\right), \\
& \qquad 1 \le i \le k.
\end{aligned} \tag{47}$$

Rewrite

$$\begin{aligned}
& r_i(t, v^{w^\varepsilon}) - r_i(t, v^{\bar{w}+\tilde{w}}) \\
& \quad = \int_0^1 \left(\nabla_y r_i\right)\left(t, \sigma v^{w^\varepsilon} + (1-\sigma) v^{\bar{w}^\varepsilon + \tilde{w}}\right) d\sigma \cdot v^{w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}} + \left(r_i(t, v^{\bar{w}^\varepsilon + \tilde{w}}) - r_i(t, v^{\bar{w}+\tilde{w}})\right).
\end{aligned}$$

Similarly to (39), consider an integral form of (47) on K with the boundary term not depending on $v^{w^\varepsilon - \bar{w}^\varepsilon - \tilde{w}}$. Since $\bar{w}^\varepsilon \to \bar{w}$ as $\varepsilon \to 0$ uniformly on any compact subset of $\overline{\Pi} \setminus J$, the difference $r_i(t, v^{\bar{w}^\varepsilon + \tilde{w}}(t)) - r_i(t, v^{\bar{w}+\tilde{w}}(t))$ tends to zero as $\varepsilon \to 0$ uniformly on any compact subset of $(\partial\Pi \setminus \{t=0\}) \setminus J$.

It remains to show that, for all $i \ne j$, the function

$$P_{ij}^\varepsilon(x,t) = \int_{t_i(x,t)}^{t} E_i(\tau; x, t) f_{ij}(\xi, \tau) \left(\bar{w}_j^\varepsilon - \bar{w}_j\right)(\xi, \tau)|_{\xi = \omega_i(\tau; x, t)} \, d\tau$$

converges to zero as $\varepsilon \to 0$ uniformly on K. Fix an arbitrary $\varepsilon_1 \le \varepsilon_0$ and split up the integral into two parts, namely

$$P_{ij}^\varepsilon(x,t) = \int\limits_{[t_i(x,t),t] - \triangle_{\varepsilon_1}(x,t)} d\tau + \int\limits_{\triangle_{\varepsilon_1}(x,t)} d\tau,$$

where $\triangle_{\varepsilon_1}(x,t) = \{\tau \in [t_i(x,t), t] : (\omega_i(\tau; x, t), \tau) \in J^{\varepsilon_1}\}$. Since $\bar{w}_j^\varepsilon$ is uniformly bounded on each bounded subset of $\overline{\Pi}$ and $\bar{w} \in C(\overline{\Pi} \setminus J)$, the absolute value of the second integral is bounded from above by $C\varepsilon_1$ for all $(x,t) \in K$ and for some $C > 0$ which does not depend on ε_1. Furthermore, there exists $\varepsilon_2 \le \varepsilon_1$ such that for all $\varepsilon \le \varepsilon_2$ the absolute value of the first integral is bounded from above by ε_1, due to the uniform convergence of $\bar{w}_j^\varepsilon$ off J^{ε_1}. We hence conclude that for each $\varepsilon_1 \le \varepsilon_0$ there is $\varepsilon_2 \le \varepsilon_1$ and a constant $C > 0$ such that for all $(x,t) \in K$ and $\varepsilon \le \varepsilon_2$ we have $|P_{ij}^\varepsilon(x,t)| \le C\varepsilon_1$. Therefore, $Q_{ij}^\varepsilon(x,t) \to 0$ as $\varepsilon \to 0$ uniformly on K. $\square$

To finish the proof of this part of the theorem, it remains to recall that problem (36) for w has a unique solution of the form $\bar{w} + \tilde{w}$, where $\bar{w}$ and $\tilde{w}$ are the unique solutions to problems (37) and (38), respectively.

4. This part of the theorem is a direct consequence of the three preceding parts.

$\square$

In Theorem 21 we established the existence of a delta wave solution to the problem under consideration. Now we identify conditions under which this solution is smoothing. In the following theorem conditions (ι) and $(\iota\iota)$ are supposed to be formulated correspondingly to problem (1), (31), (32) .

Theorem 22 *Assume that the data λ_i, a_{ij}, g_i, and p_{ij} are smooth functions in all their arguments and both r_i and $\nabla_y r_i$ are in $L^\infty((0,T)\times\mathbb{R}^n)$ for any $T>0$. Suppose that condition (5) is fulfilled.*

- Sufficiency. *If condition (ι) is true, then the delta wave solution to problem (1), (31), (32) is smoothing for any $b_r \in C[0,1]^n$ satisfying equalities (33).*
- Necessity. *Assume that the delta wave solution to problem (1), (31), (32) is smoothing for any $b_r \in C[0,1]^n$ satisfying equalities (33). Then condition $(\iota\iota)$ is fulfilled.*

Proof. *Sufficiency.* As condition (ι) is true, there exists T_0 such that $J_* \in \overline{\Pi^{T_0}}$ and the restriction of u to $\overline{\Pi}\setminus\Pi^{T_0}$ equals w. The smoothingness of u will follow from Theorem 12 and from the fact that there is $T>T_0$ such that w is continuous on $\overline{\Pi}\setminus\overline{\Pi}^T$. To prove the latter, fix T to be a supremum over $t>0$ such that there is $x\in[0,1]$ and an EP jointing the point (x,t) with the line $t=T_0$. Note that u on $\overline{\Pi}\setminus\overline{\Pi}^T$ fulfills the system of integral equations (10)–(11) where $h_i(t,v)=\sum_{j=1}^n p_{ij}(t)v_j(t)+r_i(t,v)$. By the choice of T, the right hand side of (10) can be rewritten in the form involving integrals of kind $I_{ijs}(x,t)$ defined by (12). This expression will not depend neither on v nor on $u(x,T_0)$. We therefore can use a transformation of $I_{ijs}(x,t)$ similar to that made in (13). Since u is piecewise continuous on $\overline{\Pi}\setminus\overline{\Pi}^T$, the right hand side of (13) is a continuous function on $\overline{\Pi}\setminus\overline{\Pi}^T$, as desired.

Necessity. By Theorem 21, the delta wave solution is given by the sum of a regular part w in $C(\overline{\Pi}\setminus J)^n$ and a singular part $z\in\mathcal{D}'(\Pi)$ which, in its turn, is the sum of strong singularities concentrated on the characteristic curves contributing into the set J_*. By the assumption, the delta wave solution is smoothing for each $b_r\in C[0,1]^n$. This entails the smoothingness of both the regular and the singular parts of the solution. It remains to note that condition $(\iota\iota)$ is necessary for smoothingness of w. This follows by similar argument that was used to prove the necessity in Theorem 12. □

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