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Conditionally monotone independence

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Abstract

We define the notion of conditionally monotone product as a part of conditionally free product, which naturally includes monotone and Boolean products. Then we define conditionally monotone cumulants which are useful to calculate the limit distributions in central limit theorem and Poisson's law of small numbers. Moreover, we introduce deformed convolutions arising from the conditionally monotone convolution of probability measures and compute the limit distributions. In order to understand the validity of cumulants, we discuss what are cumulants of a given convolution product in general.

1 Introduction

Non-commutative notions of independence are connected deeply with the products of states on the free product of algebras (with or without the identification of units). There are two "universal products" (classical and free) defined on the free product of algebras with the identification of units [2, 22] and there are five "natural products" (classical, free, Boolean, monotone and anti-monotone) defined on the free product of algebras without the identification of units [17, 18]. These results can also be understood in terms of tensor structures with inclusions [1].

The conditionally free product of states and the conditionally free convolution of probability measures have been introduced as generalizations of the free product and the free convolution [3]. This notion can be seen as a universal product of pairs of states which could be defined similarly to the definition in [22].

Free and Boolean products and their cumulants [23, 24] can be derived from conditionally free product and conditionally free cumulants. In addition, monotone product can also be derived from conditionally free product [10]. This is an interesting fact, which is, however, not easy to manipulate in the following points: it is difficult to repeat the calculation of monotone product in terms of conditionally free product; it is difficult to derive monotone cumulants [8] from conditionally free cumulants.

In this paper, we introduce conditionally monotone product which is a parallel notion to conditionally free product although it can be defined as a part of conditionally free product. In terms of conditionally monotone product, monotone and Boolean products appear naturally. Moreover, monotone and Boolean cumulants can be derived from conditionally monotone cumulants. We also define the notion of conditionally monotone independence as an abstraction of the conditionally monotone product.

In connection to conditionally free convolution, the t-transformation has been introduced in [4], which is later generalized in [12]. It is important that the t-deformation of free convolution gives an interpolation between free and Boolean convolutions [4, 5]. Moreover, several other deformations of convolutions arising from conditionally free convolution have been found in [11, 19, 20]. (In [14] another type of interpolation has been obtained.) Motivated by these works, we define deformed convolutions arising from conditionally monotone convolution associated with maps from the set of probability measures into itself. These kinds of convolutions naturally include monotone and Boolean convolutions. Then

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we show that the generalized t-transformation, the V_a-transformation and a transformation related to a $\triangleright$-infinitely divisible distribution give associative convolutions in the present situation. In particular, the t-transformation gives an interpolation between monotone and Boolean convolutions. Thus, many transformations found in [4, 5, 11, 12, 19, 20] also give associative convolutions also in the present situation.

Moreover, we show necessary and sufficient conditions under which the deformed convolution preserves the set $\{\mu; \operatorname{supp}\mu \subset [0,\infty)\}$ and the set $\{\mu; \mu \text{ is symmetric}\}$. These results generalize the results in Boolean and monotone cases [7].

We introduce cumulants of the deformed convolution by a map T. Then we calculate limit distributions for some associative convolutions. In order to confirm the validity of the cumulants of the deformed convolution, we consider the uniqueness and the existence of cumulants of a general convolution product in Section 4.

2 Preliminaries

2.1 Reciprocal Cauchy transform

The Cauchy transform of a probability measure μ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x), \ z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.1}$$

The reciprocal Cauchy transform of a probability measure μ is defined by

$$H_\mu(z) = \frac{1}{G_\mu(z)}, \ z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.2}$$

which is an analytic function from the upper halfplane into itself. As a result, H_μ has the following form:

$$H_\mu(z) = b + z + \int_{\mathbb{R}} \frac{1+xz}{x-z} d\eta(x), \tag{2.3}$$

where $b \in \mathbb{R}$ and η is a positive finite measure. Conversely, any function of the form of the right hand side of (2.3) is a reciprocal Cauchy transform of a probability measure.

2.2 Monotone independence

Muraki has defined the notion of monotone independence in [16]. The following definition is slightly different from the original one. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and let I be a linearly ordered set. A family of subalgebras $\{\mathcal{A}_i\}_{i\in I}$ is said to be monotone independent if the equality

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_j)\varphi(a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n) \tag{2.4}$$

holds for $a_k \in \mathcal{A}_{i_k}$ with $i_1, i_2, \cdots, i_n \in I$, $i_{j-1} < i_j > i_{j+1}$ and $1 \le j \le n$. When $j = 1$ (resp. $j = n$), the condition $i_{j-1} < i_j > i_{j+1}$ becomes $i_1 > i_2$ (resp. $i_{n-1} < i_n$). He has also defined the monotone convolution $\mu \triangleright \nu$ of two probability measures μ, ν and proved that monotone convolution is characterized by Muraki's formula

$$H_{\mu\triangleright\nu} = H_\mu \circ H_\nu. \tag{2.5}$$

Monotone convolution is non-commutative and associative.

2.3 Conditionally free independence

Let I be an index set. Let $\mathcal{A}_i$ be a unital $*$-algebra and let φ_i, ψ_i be states on $\mathcal{A}_i$ for $i \in I$. The conditionally free product of triples $(\mathcal{A}_i, \varphi_i, \psi_i)_{i\in I}$ has been introduced in the paper [3]. We set $(\mathcal{A}, \varphi, \psi) = *_{i\in I}(\mathcal{A}_i, \varphi_i, \psi_i)$ by $\mathcal{A} := *_{i\in I}\mathcal{A}_i$ (the free product with the identification of units) and $\psi := *_{i\in I}\psi_i$ (the free product of states). φ is defined by the following condition: the equality

$$\varphi(a_1 \cdots a_n) = \prod_{k=1}^{n} \varphi_{i_k}(a_k) \tag{2.6}$$

holds if $a_k \in \mathcal{A}_{i_k}$ with $i_1 \neq \cdots \neq i_n$ and $\psi_{i_k}(a_k) = 0$ for all $1 \leq k \leq n$.

The conditionally free convolution of pairs of probability measures (μ_1, ν_1) and (μ_2, ν_2) also has been introduced in the paper [3]. Define the R-transform and the conditionally free R-transform by

$$\frac{1}{G_{\nu_i}(z)} = z - R_{\nu_i}(G_{\nu_i}(z)), \tag{2.7}$$

$$\frac{1}{G_{\mu_i}(z)} = z - R_{(\mu_i,\nu_i)}(G_{\nu_i}(z)), \tag{2.8}$$

for $i = 1, 2$. We expand $R_{(\mu,\nu)}(z) = \sum_{n=1}^{\infty} R_n(\mu, \nu)z^{n-1}$ as a formal power series. $R_N(\mu, \nu)$ are called conditionally free cumulants. Similarly, we expand $R_\nu(z) = \sum_{n=1}^{\infty} R_n(\nu)z^{n-1}$ and $R_n(\nu)$ are called free cumulants. In this paper, $H_\mu(z)$ is more useful than $G_\mu(z)$ and correspondingly we use $\phi_{(\mu,\nu)}(z) := R_{(\mu,\nu)}(\frac{1}{z})$ and $\phi_\nu(z) := R_\nu(\frac{1}{z})$. Then (2.7) and (2.8) can be written as

$$H_{\nu_i}(z) = z - \phi_{\nu_i}(H_{\nu_i}(z)), \tag{2.9}$$

$$H_{\mu_i}(z) = z - \phi_{(\mu_i,\nu_i)}(H_{\nu_i}(z)), \tag{2.10}$$

The conditionally free convolution of (μ_1, ν_1) and (μ_2, ν_2) is the pair $(\mu, \nu) = (\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$, where μ and ν are characterized by

$$\phi_\nu(z) = \phi_{\nu_1}(z) + \phi_{\nu_2}(z), \tag{2.11}$$

$$\phi_{(\mu,\nu)}(z) = \phi_{(\mu_1,\nu_1)}(z) + \phi_{(\mu_2,\nu_2)}(z). \tag{2.12}$$

We denote by $(\mu_1\nu_1 \boxplus_{\nu_2} \mu_2, \nu_1 \boxplus \nu_2)$ the conditionally free convolution of (μ_1, ν_1) and (μ_2, ν_2).

2.4 Technical facts

We summarize the notation and several lemmata which will be used in this paper. Let $\mathcal{P}$, $\mathcal{P}^2$, $\mathcal{P}_{\mathrm{m}}$, $\mathcal{P}_{\mathrm{c}}$, $\mathcal{P}_+$ and $\mathcal{P}_{\mathrm{sym}}$ be the set of probability measures, the set of probability measures with finite variance, the set of probability measures with finite moments of all orders, the set of probability measures with compact support, the set of probability measures whose supports are contained in $[0, \infty)$ and the set of symmetric probability measures, respectively. The following lemma is a slight generalization of Proposition 2.2 in [15].

Lemma 2.1. *A probability measure μ belongs to $\mathcal{P}^2$ if and only if H_μ has the representation*

$$H_\mu(z) = a + z + \int_{\mathbb{R}} \frac{1}{x - z} d\rho(x), \tag{2.13}$$

where $a \in \mathbb{R}$, ρ a positive finite measure. a and ρ are determined uniquely. Furthermore, we have $\rho(\mathbb{R}) = \sigma^2(\mu)$ and $a = -m(\mu)$, where $m(\mu)$ denotes the mean of μ and $\sigma^2(\mu)$ denotes the variance of μ.

We define $a(\mu) := \inf\{x \in \mathrm{supp}\mu\}$ and $b(\mu) := \sup\{x \in \mathrm{supp}\mu\}$. We note that $-\infty \leq a(\mu) < \infty$ and $-\infty < b(\mu) \leq \infty$ hold. The following lemmata 2.2-2.5 have been proved in [7].

Lemma 2.2. *Let ν and μ be probability measures. Then the following inequalities hold:*
(1) *If $\operatorname{supp}\nu \cap (-\infty, 0] \neq \emptyset$ and $\operatorname{supp}\nu \cap [0, \infty) \neq \emptyset$, then $a(\mu) \geq a(\nu \rhd \mu)$ and $b(\mu) \leq b(\nu \rhd \mu)$.*
(2) *If $\operatorname{supp}\nu \subset (-\infty, 0]$, then $a(\mu) \geq a(\nu \rhd \mu)$ and $b(\nu) + b(\mu) \leq b(\nu \rhd \mu)$.*
(3) *If $\operatorname{supp}\nu \subset [0, \infty)$, then $a(\nu) + a(\mu) \geq a(\nu \rhd \mu)$ and $b(\mu) \leq b(\nu \rhd \mu)$.*

Lemma 2.3. *We use the notation in (2.3). For $\mu \in \mathcal{P}$, the condition $\mu \in \mathcal{P}_+$ is equivalent to the following conditions.*

(1) $\operatorname{supp}\eta \subset [0, \infty)$;

(2) $H_\mu(-0) \leq 0$.

The conditions (2) is equivalent to the conditions $\eta(\{0\}) = 0$, $\int_0^\infty \frac{1}{x} d\eta(x) < \infty$ and $b + \int_0^\infty \frac{1}{x} d\eta(x) \leq 0$.

Lemma 2.4. *Let $\{\mu_t\}_{t \geq 0}$ be a weakly continuous $\rhd$-convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) *there exists $t_0 > 0$ such that $\operatorname{supp}\mu_{t_0} \subset [0, \infty)$;*

(2) *$\operatorname{supp}\mu_t \subset [0, \infty)$ for all $0 \leq t < \infty$;*

Lemma 2.5. *We assume that the support of each μ_t is compact (this is a time-independent property). Then the following statements are equivalent.*

(1) *There exists $t_0 > 0$ such that μ_{t_0} is symmetric.*

(2) *μ_t is symmetric for all $t > 0$.*

Lemma 2.6. *$\mathcal{P}_+$ and $\mathcal{P}_{\mathrm{sym}}$ are closed subsets of $\mathcal{P}$ under the weak topology.*

Proof. Let $\{\mu_n\} \subset \mathcal{P}_+$ be a sequence converging to $\mu \in \mathcal{P}$. The weak convergence implies that $\mu((-\infty, 0)) \leq \liminf \mu_n((-\infty, 0)) = 0$; therefore, $\mu \in \mathcal{P}_+$.

For a probability measure ν, $\nu \in \mathcal{P}_{\mathrm{sym}}$ is equivalent to the condition

$$\int_{\mathbb{R}} g(x) d\nu(x) = 0 \text{ for all } g \in C_b(\mathbb{R}),\ g(-x) = -g(x), \tag{2.14}$$

where $C_b(\mathbb{R})$ is the set of bounded continuous functions on $\mathbb{R}$. This equivalence can be proved by a simple approximation argument. Then the conclusion is not difficult. □

3 Conditionally monotone independence

It is known that free, Boolean and monotone product of states (denoted by $*, \diamond$ and $\rhd$, respectively) can be derived from conditionally free product [3, 10], which we explain briefly. We consider triples of algebras and states $(\mathcal{A}_1, \varphi_1, \psi_1)$ and $(\mathcal{A}_2, \varphi_2, \psi_2)$. We assume that $\mathcal{A}_i$ has a decomposition

$$\mathcal{A}_i = \mathbb{C}\,1 \oplus \mathcal{A}_i^0 \tag{3.1}$$

with a subalgebra $\mathcal{A}_i^0$ $(i = 1, 2)$. Then the delta state δ_i $(i = 1, 2)$ is defined by $\delta_i(\lambda 1 + a^0) = \lambda$ for $\lambda \in \mathbb{C}$ and $a^0 \in \mathcal{A}_i^0$. δ_i is a homomorphism from $\mathcal{A}_i$ to $\mathbb{C}$. Conversely, if there exists a homomorphism from $\mathcal{A}_i$ to $\mathbb{C}$, then $\mathcal{A}_i^0$ is given by its kernel. Then we have the following relations.

$$(\varphi, \varphi) * (\psi, \psi) = (\varphi * \psi, \varphi * \psi) \text{ on } \mathcal{A}_1 * \mathcal{A}_2, \tag{3.2}$$

$$(\varphi, \delta_1) * (\psi, \delta_2) = (\varphi \diamond \psi, \delta_1 * \delta_2) \text{ on } \mathcal{A}_1^0 * \mathcal{A}_2^0, \tag{3.3}$$

$$(\varphi, \delta_1) * (\psi, \psi) = (\varphi \rhd \psi, \psi) \text{ on } \mathcal{A}_1^0 * \mathcal{A}_2. \tag{3.4}$$

In terms of (compactly supported) probability measures, the equalities (3.2)-(3.4) can be written as

$$(\mu,\mu)\boxplus(\nu,\nu)=(\mu\boxplus\nu,\mu\boxplus\nu), \tag{3.5}$$
$$(\mu,\delta_0)\boxplus(\nu,\delta_0)=(\mu\uplus\nu,\delta_0), \tag{3.6}$$
$$(\mu,\delta_0)\boxplus(\nu,\nu)=(\mu\rhd\nu,\nu). \tag{3.7}$$

We can understand the associativity of Boolean convolution by (3.6) since conditionally free convolution is associative. The associativity of monotone convolution is, however, not easy to understand by (3.7) since we cannot repeat the calculation of (3.7). We show that the monotone convolution of the second components of pairs of probability measures is suitable to repeat the calculation.

Definition 3.1. (1) Let $(\mathcal{A}_1,\varphi_1,\psi_1)$ and $(\mathcal{A}_2,\varphi_2,\psi_2)$ be triples consisting of algebras and two linear functionals and let $\mathcal{A}_1*_{nu}\mathcal{A}_2$ be the free product without the identification of units. We define conditionally monotone product

$$(\mathcal{A}_1,\varphi_1,\psi_1)\rhd(\mathcal{A}_2,\varphi_2,\psi_2):=(\mathcal{A}_1*_{nu}\mathcal{A}_2,\varphi_1\rhd_{\psi_2}\varphi_2,\psi_1\rhd\psi_2), \tag{3.8}$$

where $\varphi_1\rhd_{\psi_2}\varphi_2$ is defined as follows. We denote the unitization of each $\mathcal{A}_i$ by $\widetilde{\mathcal{A}_i}:=\mathbb{C}1_{\widetilde{\mathcal{A}_i}}\oplus\mathcal{A}_i$ (see [1]). (When $\mathcal{A}_i$ contains a unit $1_{\mathcal{A}_i}$, $1_{\mathcal{A}_i}\neq 1_{\widetilde{\mathcal{A}_i}}$.) Then we have $\widetilde{\mathcal{A}_1*_{nu}\mathcal{A}_2}\cong\widetilde{\mathcal{A}_1}*\widetilde{\mathcal{A}_2}$ naturally. The linear functionals φ_i and ψ_i are naturally extended to $\widetilde{\varphi_i}$ and $\widetilde{\psi_i}$ on $\widetilde{\mathcal{A}_i}$ by $\widetilde{\psi_i}(1_{\widetilde{\mathcal{A}_i}})=1$ and $\widetilde{\psi_i}(1_{\widetilde{\mathcal{A}_i}})=1$. We denote by $\widetilde{\delta_1}$ the delta state of the decomposition $\widetilde{\mathcal{A}_1}=\mathbb{C}1\oplus\mathcal{A}_1$. We define $\varphi_1\rhd_{\psi_2}\varphi_2$ by the restriction of $\widetilde{\varphi_1}_{\widetilde{\delta_1}}*_{\widetilde{\psi_2}}\widetilde{\varphi_2}$ on $\mathcal{A}_1*_{nu}\mathcal{A}_2$.
(2) For pairs of probability measures (μ_1,ν_1) and (μ_2,ν_2), we define the conditionally monotone convolution $(\mu_1,\nu_1)\rhd(\mu_2,\nu_2)$ by $(\mu_1,\nu_1)\rhd(\mu_2,\nu_2):=(\mu_1\rhd_{\nu_2}\mu_2,\nu_1\rhd\nu_2)$, where $\mu_1\rhd_{\nu_2}\mu_2$ is defined by $\mu_1\rhd_{\nu_2}\mu_2:=\mu_{1\delta_0}\boxplus_{\nu_2}\mu_2$.

Remark 3.2. Since conditionally monotone product is defined on the free product without the identification of units, the definition has been given for linear functionals. Conditionally monotone product, however, preserves the positivity of linear functionals since it is defined by the restriction of conditionally free product.

Proposition 3.3. *$\mu_1\rhd_{\nu_2}\mu_2$ is characterized by*

$$H_{\mu_1\rhd_{\nu_2}\mu_2}=H_{\mu_1}\circ H_{\nu_2}+H_{\mu_2}-H_{\nu_2}. \tag{3.9}$$

Proof. We immediately obtain the equality $H_{\mu_1}\circ H_{\nu_2}(z)=H_{\nu_2}(z)-\phi_{(\mu_1,\delta_0)}(H_{\nu_2}(z))$ by (2.10). Since conditionally free product is characterized by the sum of $\phi_{(,)}$, we have $H_{\mu_1}\circ H_{\nu_2}(z)-H_{\nu_2}(z)+H_{\mu_2}(z)=z-\phi_{(\mu_1\rhd_{\nu_2}\mu_2,\nu_2)}\circ H_{\nu_2}(z)$. □

We can easily check by Proposition 3.3 that the conditionally monotone convolution of probability measures is associative, i.e., $\big((\mu_1,\nu_1)\rhd(\mu_2,\nu_2)\big)\rhd(\mu_3,\nu_3)=(\mu_1,\nu_1)\rhd\big((\mu_2,\nu_2)\rhd(\mu_3,\nu_3)\big)$. Then it is expected to be the case that the conditionally monotone product of pairs of linear functionals is also associative, which will be proved below. We use the following notation: for a linearly ordered index set $I=\{i_1,i_2,\cdots,i_n\}$ with $i_1<i_2<\cdots<i_n$, we set

$$\overrightarrow{\prod_{i\in I}}a_i:=a_{i_1}a_{i_2}\cdots a_{i_n}.$$

Proposition 3.4. *Let $(\mathcal{A}_1,\varphi_1,\psi_1)$ and $(\mathcal{A}_2,\varphi_2,\psi_2)$ be triples consisting of algebras and two linear functionals. We have the following formula.*

$$\varphi_1\rhd_{\psi_2}\varphi_2(a_1b_1a_2\cdots b_{n-1}a_n)=\sum_{S\subset\{1,\cdots,n-1\}}\Big(\prod_{j\notin S}\psi_2(b_j)\Big)\Big(\prod_{j\in S}(\varphi_2(b_j)-\psi_2(b_j))\Big)\prod_j\varphi_1(a_{S_j}) \tag{3.10}$$

for $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$, where $a_{S_j} := \overrightarrow{\prod}_{k\in S_j} a_k$. $\{S_j\}$ is a partition of $\{1,\cdots,n\}$ determined by S as follows: if $S = \{k_1,\cdots,k_m\}$ with $1 \le k_1 < \cdots < k_m \le n-1$, then $S_j := \{k_{j-1}+1,\cdots,k_j\}$ for $1 \le j \le m+1$, where $k_0 := 0$ and $k_{m+1} := n$. If $S = \emptyset$, then $S_1 = \{1,\cdots,n\}$. $\prod_{j\in\emptyset}\psi_2(b_j)$ is defined to be 1.

Proof. Let x_j and y_k be (non-commutative) variables in an algebra over $\mathbb{C}$ with unit 1 and let $p_j \in \mathbb{C}$. Then the following identity holds:

$$x_1y_1x_2\cdots y_{n-1}x_n = \sum_{S\subset\{1,\cdots,n-1\}} \Big(\prod_{j\notin S} p_j\Big)\Big(x_{S_1}(y_{k_1}-p_{k_1}1)x_{S_2}(y_{k_2}-p_{k_2}1)\cdots(y_{k_m}-p_{k_m}1)x_{S_{m+1}}\Big) \quad (3.11)$$

where we have used the same notation as explained below (3.10). (3.11) is proved by induction about the number n easily. We put $x_j = a_j \in \mathcal{A}_1$, $y_j = b_j \in \mathcal{A}_2$, $p_j = \psi_2(b_j)$ and we use the unit $1_{\widetilde{\mathcal{A}_1 *_{nu} \mathcal{A}_2}}$. Taking the expectation of (3.11) w.r.t. $\widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}$, we obtain the result by the definition of the conditionally free product. □

Theorem 3.5. *Let $(\mathcal{A}_1,\varphi_1,\psi_1)$ and $(\mathcal{A}_2,\varphi_2,\psi_2)$ be triples consisting of algebras and two linear functionals. The calculation rules for the conditionally monotone product are what follows.*

(1) $\varphi_1 \rhd_{\psi_2} \varphi_2(b_1a_1b_2a_2\cdots a_nb_{n+1}) = \varphi_2(b_1)\varphi_2(b_{n+1})\varphi \rhd_{\psi_2} \varphi_2(a_1b_2a_2\cdots b_na_n)$
for $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$, $n \in \mathbb{N}$.

(2) $\varphi_1 \rhd_{\psi_2} \varphi_2(a_1b_1\cdots b_{n-1}a_n) = (\varphi_2(b_j)-\psi_2(b_j))\varphi_1 \rhd_{\psi_2} \varphi_2(a_1b_1a_2\cdots b_{j-1}a_j)\varphi_1 \rhd_{\psi_2} \varphi_2(a_{j+1}b_{j+1}\cdots b_{n-1}a_n)$
$+\ \psi_2(b_j)\varphi_1 \rhd_{\psi_2} \varphi_2(a_1b_1a_2\cdots b_{j-1}a_ja_{j+1}b_{j+1}\cdots b_{n-1}a_n)$
for $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$, $1 \le j \le n-1$, $n \in \mathbb{N}$.

Proof. Since (1) can be proved similarly, we only prove (2). Moreover, we assume that $j = 1$ since the proof for general j is essentially the same. We denote by 1 the unit $1_{\widetilde{\mathcal{A}_1 *_{nu} \mathcal{A}_2}}$ for simplicity. First we obtain

$$\begin{aligned}\widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}(a_1b_1a_2\cdots b_{n-1}a_n) &= \widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}(a_1(b_1-\psi_2(b_1)1)a_2\cdots b_{n-1}a_n)\\ &\quad + \psi_2(b_1)\widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}(a_1a_2\cdots b_{n-1}a_n).\end{aligned} \quad (3.12)$$

Using the identity (3.11) and the definition of conditionally free product, we have

$$\begin{aligned}&\widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}(a_1(b_1-\psi_2(b_1)1)a_2\cdots b_{n-1}a_n)\\ &= \widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}\Big(a_1(b_1-\psi_2(b_1)1)\sum_{S\subset\{2,\cdots,n-1\}}\Big(\prod_{j\notin S}\psi_2(b_j)\Big)\Big(a_{S_1}(b_{k_1}-\psi_2(b_{k_1})1)\cdots a_{S_{m+1}}\Big)\Big)\\ &= (\varphi_2(b_1)-\psi_2(b_1))\sum_{S\subset\{2,\cdots,n-1\}}\Big(\prod_{j\notin S}\psi_2(b_j)\Big)\Big(\prod_{j\in S}(\varphi_2(b_j)-\psi_2(b_j))\Big)\varphi_1(a_1)\prod_{j=1}^{|S|+1}\varphi_1(a_{S_j})\\ &= \varphi_1(a_1)(\varphi_2(b_1)-\psi_2(b_1))\widetilde{\varphi_1}_{\widetilde{\delta}_1} *_{\widetilde{\psi_2}} \widetilde{\varphi_2}(a_2b_2\cdots b_{n-1}a_n).\end{aligned}$$

□

Theorem 3.6. *Conditionally monotone product is associative.*

Proof. Let $(\mathcal{A}_i,\varphi_i,\psi_i)$ $(1 \le i \le 3)$ be triples consisting of algebras and two linear functionals. We identify $(\mathcal{A}_1 *_{nu} \mathcal{A}_2) *_{nu} \mathcal{A}_3$ with $\mathcal{A}_1 *_{nu} (\mathcal{A}_2 *_{nu} \mathcal{A}_3)$ naturally and denote them by $\mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$. What should be proved is the equality $\varphi_1 \rhd_{\psi_2\rhd\psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3) = (\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3$ on $\mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$. We call $x \in \mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$ an elementary element if x is of such a form as $x = x_1x_2\cdots x_N$, $x_j \in \mathcal{A}_{i_j}$, $i_1 \ne \cdots \ne i_n$. We give a proof by induction about the number of elements of $\mathcal{A}_3$ in elementary elements. In this proof we use the notation a, a' or a_j for elements in $\mathcal{A}_1$, b, b' or b_j for elements in $\mathcal{A}_2$ and c or c_j for elements in $\mathcal{A}_3$ without any references.

When $x \in \mathcal{A}_2 *_{nu} \mathcal{A}_3 \subset \mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$, we can easily prove that $\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(x) = (\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(x)$. We assume that the statement is the case for elementary elements containing not more than n elements in $\mathcal{A}_3$. Let x be an elementary element containing $n+1$ elements in $\mathcal{A}_3$. Let c be the $(n+1)$-th element in $\mathcal{A}_3$ contained in x. Except for some cases explained below, x can be written as $x = uavcw$, where each component satisfies the following property:
(i) u is an elementary element in $\mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$ containing not more than n elements in $\mathcal{A}_3$;
(ii) v is an elementary element in $\mathcal{A}_2 *_{nu} \mathcal{A}_3$ containing not more than n elements in $\mathcal{A}_3$. The last element in v belongs to $\mathcal{A}_2$, i.e., $v = v'b_1$;
(iii) w is an elementary element in $\mathcal{A}_1 *_{nu} \mathcal{A}_2$.
When x is of such a form as $x = uacw$, the following proof is applicable if v is understood to play the role of a unit. By the property (1) in Theorem 3.5, the proof is easy when x is of such a form as $x = uavc$. When a does not appear, again by the property (1) in Theorem 3.5 the proof becomes easy. Therefore, we only prove for x of the form $uavcw$. We consider the following two cases: (1) The first element in w belongs to $\mathcal{A}_1$, i.e., $w = a'w'$; (2) The first element in w belongs to $\mathcal{A}_2$, i.e., $w = b_2w'$.

Case (1): We obtain

$$\begin{aligned}
&(\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(uavcw) \\
&\quad = (\varphi_3(c) - \psi_3(c))(\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(uav)(\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(w) \\
&\qquad + \psi_3(c)(\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(uavw) \\
&\quad = (\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uav)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uavw) \\
&\quad = (\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uav)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)(\varphi_2 \rhd_{\psi_3} \varphi_3(v) - \psi_2 \rhd \psi_3(v))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(v)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw),
\end{aligned} \tag{3.13}$$

where we have used the assumption of induction and the property (2) in Theorem 3.5,. On the other hand, we obtain

$$\begin{aligned}
&\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uavcw) \\
&\quad = (\varphi_2 \rhd_{\psi_3} \varphi_3(vc) - \psi_2 \rhd \psi_3(vc))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(w) \\
&\qquad + \psi_2 \rhd \psi_3(vc)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw) \\
&\quad = \varphi_2 \rhd_{\psi_3} \varphi_3(v)\varphi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(w) \\
&\qquad - \psi_2 \rhd \psi_3(v)\psi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(w) \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(v)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw) \\
&\quad = (\varphi_3(c) - \psi_3(c) + \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uav)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad - \psi_2 \rhd \psi_3(v)\psi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(v)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw) \\
&\quad = (\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uav)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\Big(\varphi_2 \rhd_{\psi_3} \varphi_3(v) - \psi_2 \rhd \psi_3(v)\Big)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(v)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw).
\end{aligned} \tag{3.14}$$

Therefore, they coincide with each other.

Case (2): By the similar calculation, we have

$$\begin{aligned}
&(\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(uavcw) \\
&\quad = (\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uav)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uavw) \\
&\quad = (\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_2 \rhd_{\psi_3} \varphi_3(v)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)(\varphi_2 \rhd_{\psi_3} \varphi_3(vb_2) - \psi_2 \rhd \psi_3(vb_2))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(vb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw').
\end{aligned} \tag{3.15}$$

On the other hand we have

$$\begin{aligned}
&\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uavcw) \\
&\quad = (\varphi_2 \rhd_{\psi_3} \varphi_3(vcb_2) - \psi_2 \rhd \psi_3(vcb_2))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad + \psi_2 \rhd \psi_3(vcb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw') \\
&\quad = \varphi_2 \rhd_{\psi_3} \varphi_3(v)(\varphi_3(c) - \psi_3(c))\varphi_2(b_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad + \psi_3(c)\varphi_2 \rhd_{\psi_3} \varphi_3(vb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad - \psi_2 \rhd \psi_3(vcb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad + \psi_2 \rhd \psi_3(vcb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw') \\
&\quad = \varphi_2 \rhd_{\psi_3} \varphi_3(v)(\varphi_3(c) - \psi_3(c))\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w) \\
&\qquad + \psi_3(c)\varphi_2 \rhd_{\psi_3} \varphi_3(vb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad - \psi_3(c)\psi_2 \rhd \psi_3(vb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(ua)\varphi_1 \rhd_{\psi_2} \varphi_2(w') \\
&\qquad + \psi_3(c)\psi_2 \rhd \psi_3(vb_2)\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(uaw').
\end{aligned} \tag{3.16}$$

Then $\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(x) = (\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(x)$ for x containing $n+1$ elements in $\mathcal{A}_3$. By induction, $\varphi_1 \rhd_{\psi_2 \rhd \psi_3} (\varphi_2 \rhd_{\psi_3} \varphi_3)(x) = (\varphi_1 \rhd_{\psi_2} \varphi_2) \rhd_{\psi_3} \varphi_3(x)$ holds for all elementary elements x in $\mathcal{A}_1 *_{nu} \mathcal{A}_2 *_{nu} \mathcal{A}_3$. □

By the associativity of conditionally monotone product, we can define the product $\rhd_{i \in I}(\mathcal{A}_i, \varphi_i, \psi_i)$ for a linearly ordered set I.

Now we define the notion of of conditionally monotone independence.

Definition 3.7. Let $(\mathcal{A}, \varphi, \psi)$ be an algebraic probability space equipped with two states. Now we assume that $\mathcal{A}$ is unital. Let I be a linearly ordered set. A family of subalgebras $\{\mathcal{A}_i\}_{i \in I}$ is said to be conditionally monotone independent if the following properties are satisfied:

(1) $\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2 \cdots a_n)$ for $a_k \in \mathcal{A}_{i_k}$ with $i_1, i_2, \cdots, i_n \in I$ and $i_1 > i_2$;

(2) $\varphi(a_1 a_2 \cdots a_n) = \varphi(a_n)\varphi(a_1 \cdots a_{n-1})$ for $a_k \in \mathcal{A}_{i_k}$ with $i_1, i_2, \cdots, i_n \in I$ and $i_n > i_{n-1}$;

(3) $\varphi(a_1 a_2 \cdots a_n) = (\varphi(a_j) - \psi(a_j))\varphi(a_1 \cdots a_{j-1})\varphi(a_{j+1} \cdots a_n) + \psi(a_j)\varphi(a_1 a_2 \cdots a_{j-1} a_{j+1} \cdots a_n)$ for $a_k \in \mathcal{A}_{i_k}$ with $i_1, i_2, \cdots, i_n \in I$ and $i_{j-1} < i_j > i_{j+1}$ with $2 \le j \le n-1$.

Remark 3.8. This definition includes monotone (resp. Boolean) independence when $\varphi = \psi$ (resp. $\psi = 0$) on $\mathcal{A}_i$ for all $i \in I$.

We can prove that $\mathcal{A}_i$ $(i \in I)$ are conditionally monotone independent in $(\mathcal{A}, \varphi, \psi) = \rhd_{i \in I}(\mathcal{A}_i, \varphi_i, \psi_i)$.

4 Cumulants of a convolution

To define conditionally monotone cumulants and the related cumulants in the sections 5 and 10, we consider what are cumulants of a convolution product. We do not consider cumulants of a convolution of pairs of probability measures, which can be also possible with a slight generalization.

For a given convolution $\square$ defined on $\mathcal{P}_{\rm m}$, we consider what are cumulants of $\square$. We shall treat convolutions which are not necessarily commutative for the later applications. All results in this section hold for both $\mathcal{P}_{\rm m}$ and $\mathcal{P}_{\rm c}$, except for Theorem 4.9. Then we use the set $\mathcal{P}_{\rm m}$ mainly.

Definition 4.1. (1) We define recursively $\nu^{\square n} := \nu\square\nu^{\square n-1}$ for $\nu \in \mathcal{P}_{\rm m}$. $\square$ is said to be power associative if $\nu^{\square(n+m)} = \nu^{\square n}\square\nu^{\square m}$ for all m, $n \geq 0$.

Let $m_n(\mu)$ be the n-th moment of $\mu \in \mathcal{P}_{\rm m}$. We put the following assumptions.

(M1) There exists a polynomial P_n of $2n-2$ variables for each $n \geq 1$ such that

$$m_n(\mu\square\nu) = m_n(\mu) + m_n(\nu) + P_n(m_1(\mu), \cdots, m_{n-1}(\mu), m_1(\nu), \cdots, m_{n-1}(\nu)). \tag{4.1}$$

(M2) The polynomial P_n contains no constants for any $n \geq 1$.

Remark 4.2. The condition (M2) is equal to the condition $\delta_0\square\delta_0 = \delta_0$.

Let $r_n(\mu)$ be a polynomial of $\{m_k(\mu)\}_{k\geq 1}$ for any $n \geq 1$. We consider the following properties.

(C1) Power additivity: for any n, $N \geq 1$,

$$r_n(\mu^{\square N}) = Nr_n(\mu). \tag{4.2}$$

(C2) There exists a polynomial Q_n of $n-1$ variables such that

$$r_n(\mu) = m_n(\mu) + Q_n(m_1(\mu), \cdots, m_{n-1}(\mu)). \tag{4.3}$$

(C2') In addition to the condition (C2), the polynomial Q_n never contains linear terms $m_k(\mu)$, $1 \leq k \leq n-1$ for any n.

If a sequence $\{r_n\}$ satisfies (C2), we can write m_n by r_n as

$$m_n(\mu) = r_n(\mu) + R_n(r_1(\mu), \cdots, r_{n-1}(\mu)), \tag{4.4}$$

where R_n is a polynomial of $n-1$ variables.

We note that in many important examples the condition of homogeneity

$$r_n(D_\lambda\mu) = \lambda^n r_n(\mu) \tag{4.5}$$

holds, where D_λ is the dilation operator defined by $(D_\lambda\mu)(B) = \mu(\lambda^{-1}B)$ (see [13]). Indeed, this condition holds for classical, free, Boolean, monotone convolutions (see [13] for Boolean and free convolutions and see [8] for monotone convolution). Clearly (C2) and (4.5) imply (C2'). We do not assume this condition since the uniqueness of cumulants follows from only (C1) and (C2') (see Theorem 4.4). Moreover, there are examples which satisfy (C2') but do not satisfy (4.5) such as V_a-transformation in [11].

If there exists a sequence $\{r_n\}$ satisfying (C1) and (C2), we consider a transformation of the form

$$r_n \mapsto r'_n := r_n + \sum_{k=1}^{n-1} a_{n,k} r_k \tag{4.6}$$

for real numbers $a_{n,k}$, $1 \leq k \leq n-1$, $1 \leq n < \infty$. This transformation clearly preserves the properties (C1) and (C2). Moreover, we obtain the following property (1).

Proposition 4.3. *We assume (M1) and (M2) and assume that $\square$ is power associative.*
(1) If there are two sequences $\{r_n\}$ and $\{r'_n\}$ satisfying (C1) and (C2), there exists a unique transformation of the form (4.6) which maps $\{r_n\}$ to $\{r'_n\}$.
(2) The polynomial Q_n in (C2) never contains a constant term.

Proof. (1) There exists a polynomial A_n of variables $n-1$ for each $n \geq 1$ such that $r'_n = r_n + A_n(r_1, \cdots, r_{n-1})$ by (4.3) and (4.4). Replacing μ by $\mu^{\boxplus N}$, we obtain $Nr'_n = Nr_n + A_n(Nr_1, \cdots, Nr_{n-1})$ for any N. This is an equality of polynomials of N, and hence, A_n is of the form $A_n(r_1, \cdots, r_{n-1}) = \sum_{k=1}^{n-1} a_{n,k} r_k$.
(2) We show the fact inductively. For $n = 1$, there exists $b_1 \in \mathbb{R}$ such that $r_1 = m_1 + b_1$. Since P_n does not contain a constant term in (4.1), we have $m_1(\mu \boxplus \mu) = 2m_1(\mu)$, which implies $2r_1(\mu) - b_1 = 2r_1(\mu) - 2b_1$. Therefore, $b_1 = 0$. We assume that Q_n does not contain a constant term for $n \leq k$. By a similar argument, we can prove that Q_{k+1} does not contain a constant term. □

Proposition 4.4. *We assume (M1) and (M2) and assume that $\boxplus$ is power associative. The following statements are equivalent:*

(a) There exists a sequence $\{r_n\}_{n\geq 1}$ satisfying (C1) and (C2);

(b) There exists a sequence $\{r_n\}_{n\geq 1}$ satisfying (C1) and (C2');

(c) $m_n(\mu^{\boxplus N})$ is a polynomial of $m_1(\mu), \cdots, m_n(\mu)$ and N for any n.

Moreover, the sequence $\{r_n\}$ in (b) is unique and is given by

$$r_n(\mu) = \frac{\partial}{\partial N} m_n(\mu^{\boxplus N})\Big|_{N=0}. \tag{4.7}$$

Remark 4.5. We can see from (4.7) that cumulants are deeply connected with convolution semigroups $\{\mu_t\}_{t\geq 0}$ with $\mu_0 = \delta_0$ and the notion of infinite divisibility.

Proof. (1) (a) $\Rightarrow$ (c): If there exists a sequence $\{r_n\}_{n\geq 1}$ satisfying (C1) and (C2), we have

$$\begin{aligned} m_n(\mu^{\boxplus N}) &= r_n(\mu^{\boxplus N}) + R_n(r_1(\mu^{\boxplus N}, \cdots, r_{n-1}(\mu^{\boxplus N})) \\ &= Nr_n(\mu) + R_n(Nr_1(\mu), \cdots, Nr_{n-1}(\mu)) \\ &= Nm_n(\mu) + NQ_n(m_1(\mu), \cdots, m_{n-1}(\mu)) \\ &\quad + R_n(Nm_1(\mu), \cdots, Nm_{n-1}(\mu) + NQ_{n-1}(m_1(\mu), \cdots, m_{n-2}(\mu))). \end{aligned} \tag{4.8}$$

Therefore, $m_n(\mu^{\boxplus N})$ is a polynomial of N and $m_k(\mu)$.
(c) $\Rightarrow$ (a): By (M1), (M2) and the assumption (c), $m_n(\mu^{\boxplus N})$ has such a form as

$$m_n(\mu^{\boxplus N}) = Nm_n(\mu) + \sum_{l=0}^{L} N^l S_l(m_1(\mu), \cdots, m_{n-1}(\mu)). \tag{4.9}$$

We define

$$\begin{aligned} r_n(\mu) &:= \frac{\partial}{\partial N} m_n(\mu^{\boxplus N})\Big|_{N=0} \\ &= m_n(\mu) + S_1(m_1(\mu), \cdots, m_{n-1}(\mu)). \end{aligned} \tag{4.10}$$

By the power associativity of $\boxplus$ we obtain (C1). (C2) follows from (4.10).
(a) $\Rightarrow$ (b): For a sequence $\{r_n\}$ satisfying (C1) and (C2), we can write r_n in the form $r_n = m_n + \sum_{k=1}^{n-1} b_{n,k} m_k + T_n(m_1, \cdots, m_{n-1})$, where T_n is a polynomial which does not contain linear terms m_k, $1 \leq k \leq n-1$. We define a new sequence $\{r'_n\}$ inductively as follows: $r'_1 := r_1$, $r'_2 = r_2 - b_{2,1} r_1$, $r'_n = r_n - \sum_{k=1}^{n-1} a_{n,k} r'_k$ for $n \geq 2$. Then r'_n do not contain linear terms m_k.

We note that Q_n does not contain a constant term by Proposition 4.3 (2). If there exists a sequence $\{r_n\}$ satisfying (C1) and (C2'), the corresponding polynomial R_n in (4.4) also does not contain linear terms m_k, $1 \leq k \leq n-1$ and a constant term. Therefore, the equality $m_n(\mu^{\boxplus N}) = Nr_n(\mu) + R_n(Nr_1(\mu), \cdots, Nr_{n-1}(\mu))$ implies that $r_n = \frac{\partial}{\partial N} m_n(\mu^{\boxplus N})\Big|_{N=0}$. □

Definition 4.6. Let $\square$ be a power associative convolution defined on $\mathcal{P}_{\mathrm{m}}$ satisfying (M1) and (M2). Then the polynomials r_n satisfying (C1) and (C2') are called the cumulants of the convolution $\square$. Cumulants are unique.

Remark 4.7. This definition is a generalization of the cumulants in classical, free, Boolean and monotone probability theories.

We can prove the existence of cumulants.

Theorem 4.8. *We assume the conditions (M1) and (M2) for a power associative convolution $\square$. Then cumulants of $\square$ exist.*

Proof. It is sufficient to prove that $m_n(\mu^{\square N})$ is a polynomial of N by Proposition 4.4. Then the proof is the same one as given in [8], which we omit here. □

We discuss when the additivity of cumulants holds. In the proof of the following theorem, we assume the convolution is defined on $\mathcal{P}_{\mathrm{c}}$ to assure that moments determine a unique probability measure.

Theorem 4.9. *Let $\square$ be a power associative convolution defined on $\mathcal{P}_{\mathrm{c}}$ satisfying (M1) and (M2). Let r_n be the cumulants. Then the following conditions are equivalent.*

(1) $r_n(\mu\square\nu) = r_n(\mu) + r_n(\nu)$ for all n and μ, $\nu \in \mathcal{P}_{\mathrm{c}}$.

(2) $\square$ is associative, commutative and P_n in (4.1) does not contain linear terms $m_k(\mu)$ and $m_k(\nu)$, $1 \le k \le n-1$ for any n.

Proof. (1) ⇒ (2): The associativity and commutativity follow immediately since probability measures with compact supports are determined by the cumulants. By (M1) and (4.4) we obtain the identity

$$\begin{aligned} &P_n(m_1(\mu), \cdots, m_{n-1}(\mu), m_1(\nu), \cdots, m_{n-1}(\nu)) \\ &\quad = Q_n(m_1(\mu), \cdots, m_{n-1}(\mu)) + Q_n(m_1(\nu), \cdots, m_{n-1}(\nu)) \\ &\qquad + R_n(r_1(\mu) + r_1(\nu), \cdots, r_{n-1}(\mu) + r_{n-1}(\nu)). \end{aligned}$$

By (C2'), Q_n and R_n do not contain linear terms.
(2) ⇒ (1): By (M1), (C2') and (4.4)

$$\begin{aligned} r_n(\mu\square\nu) &= m_n(\mu\square\nu) + Q_n(m_1(\mu\square\nu), \cdots, m_{n-1}(\mu\square\nu)) \\ &= r_n(\mu) + r_n(\nu) + P_n(m_1(\mu), \cdots, m_{n-1}(\mu), m_1(\nu), \cdots, m_{n-1}(\nu)) \\ &\quad + R_n(r_1(\mu), \cdots, r_{n-1}(\mu)) + R_n(r_1(\nu), \cdots, r_{n-1}(\nu)) \\ &\quad + Q_n(m_1(\mu\square\nu), \cdots, m_{n-1}(\mu\square\nu)). \end{aligned}$$

Therefore, there exists a polynomial U_n which does not contain linear terms such that $r_n(\mu\square\nu) = r_n(\mu) + r_n(\nu) + U_n(r_1(\mu), \cdots, r_{n-1}(\mu), r_1(\nu), \cdots, r_{n-1}(\nu))$. We replace μ and ν by $\mu^{\square N}$ and $\nu^{\square N}$, respectively. We note that by the associativity and commutativity, it holds that $r_n(\mu^{\square N}\square\nu^{\square N}) = r_n((\mu\square\nu)^{\square N}) = Nr_n(\mu\square\nu)$. Then we obtain $Nr_n(\mu\square\nu) = Nr_n(\mu) + Nr_n(\nu) + U_n(Nr_1(\mu), \cdots, Nr_{n-1}(\mu), Nr_1(\nu), \cdots, Nr_{n-1}(\nu))$. This can be seen as an identity of polynomials of N; therefore, we have $U_n = 0$. □

Theorem 4.10. *Let $\square$ be a power associative convolution defined on $\mathcal{P}_{\mathrm{m}}$ satisfying (M1) and (M2). Let r_n be the cumulants.*
(1) (central limit theorem) For $\mu \in \mathcal{P}_{\mathrm{m}}$ with $m_1(\mu) = 0$ and $m_2(\mu) = 1$, we define $\mu_N := (D_{\frac{1}{\sqrt{N}}}\mu)^{\square N}$. Then $r_1(\mu_N) \to 0$, $r_2(\mu_N) \to 1$ and $r_n(\mu_N) \to 0$ as $N \to \infty$ for any $n \ge 3$.
(2) (Poisson's law of small numbers) Let $\{\mu^{(N)}\}$ be a sequence such that for any $n \ge 1$ $Nm_n(\mu^{(N)}) \to \lambda > 0$ as $N \to \infty$. We define $\mu_N := (\mu^{(N)})^{\square N}$. Then $r_n(\mu_N) \to \lambda$ as $N \to \infty$ for any $n \ge 1$.

5 Conditionally monotone cumulants

We discuss the cumulants of conditionally monotone convolution based on Definition 4.6. Although conditionally monotone convolution is defined as a part of conditionally free convolution, conditionally free cumulants seem to be not suitable to look for the cumulants. For the second component of a pair of probability measures, we use monotone cumulants defined in [8]. We define conditionally monotone cumulants for the first component.

Lemma 5.1. *Let $\mu \in \mathcal{P}_{\mathrm{m}}$. We define $b_n = b_n(\mu)$ by $H_\mu(z) = z\Big(1+\sum_{n=1}^{\infty}\frac{b_n}{z^n}\Big)$ in the sense of asymptotic expansion. Then b_n is of the form*

$$b_n = -m_n + W_n(m_1, \cdots, m_{n-1}), \tag{5.1}$$

where W_n is a polynomial. Moreover, $b_n(D_\lambda\mu) = \lambda^n b_n(\mu)$ for $\lambda > 0$. In particular, W_n does not contain a constant term and linear terms.

Proof. We have

$$\begin{aligned} H_\mu(z) &= z\frac{1}{1+\sum_{k=1}^{\infty}\frac{m_k}{z^k}} \\ &= z\Big(1 - \sum_{k=1}^{\infty}\frac{m_k}{z^k} + (\sum_{k=1}^{\infty}\frac{m_k}{z^k})^2 - \cdots\Big). \end{aligned} \tag{5.2}$$

Then we obtain the polynomial W_n. The last statement is obtained by the relation $H_{D_\lambda\mu}(z) = \lambda H_\mu(\lambda^{-1}z)$. □

Lemma 5.2. *For any $n \geq 3$, there exists a polynomial Y_n of $2n-3$ variables such that $b_n(\mu_1 \rhd_{\nu_2} \mu_2) = b_n(\mu_1) + b_n(\mu_2) + Y_n(b_1(\mu_1), \cdots, b_{n-1}(\mu_1), b_1(\nu_2), \cdots, b_{n-2}(\nu_2))$. In addition, Y_n does not contain a constant term and linear terms. For $n = 1$ and 2, we have $b_1(\mu_1 \rhd_{\nu_2} \mu_2) = b_1(\mu_1) + b_1(\mu_2)$ and $b_2(\mu_1 \rhd_{\nu_2} \mu_2) = b_2(\mu_1) + b_2(\mu_2)$.*

Proof. The additivity of b_1 and b_2 has been proved in Proposition 7.3. We have the equality

$$\begin{aligned} H_{\mu_1\rhd_{\nu_2}\mu_2}(z) &= H_{\mu_1}(H_{\nu_2}(z)) + H_{\mu_2}(z) - H_{\nu_2}(z) \\ &= b_1(\mu_1) + \sum_{n=1}^{\infty} b_{n+1}(\mu_1)G_{\nu_2}(z)^n + z + b_1(\mu_2) + \sum_{n=1}^{\infty}\frac{b_{n+1}(\mu_2)}{z^n} \\ &= b_1(\mu_1) + \sum_{n=1}^{\infty}\frac{b_{n+1}(\mu_1)}{z^n}\Big(1+\sum_{k=1}^{\infty}\frac{m_k(\nu_2)}{z^k}\Big)^n + z + b_1(\mu_2) + \sum_{n=1}^{\infty}\frac{b_{n+1}(\mu_2)}{z^n}. \end{aligned} \tag{5.3}$$

By Lemma 5.1, $m_k(\nu_2)$ has such a form as $m_k(\nu_2) = -b_k(\nu_2) + X_k(b_1(\nu_2), \cdots, b_{k-1}(\nu_2))$, where X_k is a polynomial without a constant term and linear terms. Therefore, we have the conclusion. □

We now consider cumulants. For cumulants of the second component, we use monotone cumulants $r_n(\nu)$ and its generating function $A_\nu(z) = -\sum_{n=1}^{\infty}\frac{r_n(\nu)}{z^{n-1}}$. In order to define conditionally monotone cumulants, we first give a formal discussion based on the formula (4.7). Let $\{(\mu_t, \nu_t)\}_{t\geq 0}$ be a conditionally monotone convolution semigroup with $(\mu_0, \nu_0) = (\delta_0, \delta_0)$. Then $\{\nu_t\}_{t\geq 0}$ is a monotone convolution semigroup with $\nu_0 = \delta_0$. We define $\mu := \mu_1$ and $\nu = \nu_1$. By (7.7) we have $H_{\mu_{s+t}} = H_{\mu_s} \circ H_{\nu_t} + H_{\mu_t} - H_{\nu_t}$. We assume that $m_n(\mu_t)$ is a differentiable function of t. Differentiating the equality by s formally and putting $s = 0$, we obtain

$$\frac{d}{dt}H_{\mu_t}(z) = A_{(\mu,\nu)}(H_{\nu_t}(z)), \tag{5.4}$$

where $A_{(\mu,\nu)}(z)$ is defined by $A_{(\mu,\nu)}(z) = \frac{d}{dt}H_{\mu_t}(z)|_{t=0}$. We expand $A_{(\mu,\nu)}(z)$ as a formal power series:

$$A_{(\mu,\nu)}(z) = -\sum_{n=1}^{\infty} \frac{r_n(\mu,\nu)}{z^{n-1}}. \tag{5.5}$$

(5.4) is equivalent to

$$\frac{d}{dt}G_{\mu_t}(z) = \sum_{n=1}^{\infty} r_n(\mu,\nu)G_{\mu_t}(z)^2 G_{\nu_t}(z)^{n-1} \tag{5.6}$$

as a formal power series. There exists a polynomial M_n for any $n \geq 3$ such that

$$\begin{aligned}\frac{d}{dt}m_n(\mu_t) = r_n(\mu,\nu) + M_n(r_1(\mu,\nu),\cdots,r_{n-1}(\mu,\nu), m_1(\mu_t),\cdots,m_{n-1}(\mu_t),\\ m_1(\nu_t),\cdots,m_{n-2}(\nu_t)).\end{aligned} \tag{5.7}$$

It is easy to prove that $M_1 = 0$ and M_2 only depends on $r_1(\mu,\nu)$ and $m_1(\mu_t)$. Since $m_k(\nu_t)$ can be expressed by the polynomial of $r_1(\nu),\cdots,r_k(\nu)$, we can prove inductively that $m_n(\mu_t)$ is of the form

$$m_n(\mu_t) = r_n(\mu,\nu)t + t^2 C_n(t, r_1(\mu,\nu),\cdots,r_{n-1}(\mu,\nu), r_1(\nu),\cdots,r_{n-2}(\nu)), \tag{5.8}$$

where C_n is a polynomial for any $n \geq 3$ and $r_k(\mu)$ is the monotone k-th cumulant of μ. It is easy to show that $r_1(\mu,\nu) = m_1(\mu)$ and $r_2(\mu,\nu) = \sigma^2(\mu) = m_2(\mu) - m_1(\mu)^2$. Therefore, $r_n(\mu,\nu)$ are expressed by polynomials of $m_k(\mu)$ and $m_k(\nu)$ $(1 \leq k \leq n)$. These relations can be generalized to any probability measures $\mu,\ \nu \in \mathcal{P}_{\mathrm{m}}$.

Definition 5.3. For probability measures $\mu,\ \nu \in \mathcal{P}_{\mathrm{m}}$, we define the conditionally monotone cumulants $r_n(\mu,\nu)$ $(n \geq 1)$ by the equations $m_n(\mu) = r_n(\mu,\nu) + C_n(1, r_1(\mu,\nu),\cdots,r_{n-1}(\mu,\nu), r_1(\nu),\cdots,r_{n-2}(\nu))$ for $n \geq 3$. For $n = 1, 2$, we define $r_1(\mu,\nu) := m_1(\mu)$ and $r_2(\mu,\nu) := m_2(\mu) - m_1(\mu)^2$.

Theorem 5.4. *$r_n((\mu,\nu)^{\triangleright N}) = N r_n(\mu,\nu)$ holds for $\mu,\ \nu \in \mathcal{P}_{\mathrm{m}}$ and $N \in \mathbb{N}$.*

Proof. We define μ_N by $(\mu_N, \nu^{\triangleright N}) = (\mu,\nu)^{\triangleright N}$. We define

$$m_n(\nu,t) := \sum_{k=1}^{n} \sum_{1=i_0<i_1<\cdots<i_{k-1}<i_k=n+1} \frac{t^k}{k!} \prod_{l=1}^{k} i_{l-1} r_{i_l - i_{l-1}}(\nu) \tag{5.9}$$

and

$$m_n(\mu,\nu,t) := r_n(\mu,\nu)t + t^2 C_n(t, r_1(\mu,\nu),\cdots,r_{n-1}(\mu,\nu), r_1(\nu),\cdots,r_{n-2}(\nu)), \tag{5.10}$$

which may not be moments of a probability measure for general t. It is noted that $m_n(\mu,\nu,1) = m_n(\mu)$. We shall show that $m_n(\mu,\nu,N) = m_n(\mu_N, \nu^{\triangleright N}, 1)(= m_n(\mu_N))$ for any $n,\ N \geq 1$. We define formal power series $A_{(\mu,\nu)}(z)$ by (5.5) and $H_{(\mu,\nu)}(t,z) := \left(\sum_{n=0}^{\infty} \frac{m_n(\mu,\nu,t)}{z^{n+1}}\right)^{-1}$ and $H_\nu(t,z) := \left(\sum_{n=0}^{\infty} \frac{m_n(\nu,t)}{z^{n+1}}\right)^{-1}$. We note that the equality $H_\nu(t+s,z) = H_\nu(t, H_\nu(s,z))$ holds as formal power series (see [8] for the proof). Now we show that

$$H_{(\mu,\nu)}(t+s,z) = H_{(\mu,\nu)}(t, H_\nu(s,z)) - H_\nu(s,z) + H_{(\mu,\nu)}(s,z). \tag{5.11}$$

We define $K_s(t,z) := H_{(\mu,\nu)}(t, H_\nu(s,z)) - H_\nu(s,z) + H_{(\mu,\nu)}(s,z)$. By the arguments given before Definition 5.3, we have

$$\frac{d}{dt}H_{(\mu,\nu)}(t,z) = A_{(\mu,\nu)}(H_\nu(t,z)). \tag{5.12}$$

It is clear that $H_{(\mu,\nu)}(t+s,z)$ satisfies the differential equation $\frac{d}{dt}H_{(\mu,\nu)}(t+s,z) = A_{(\mu,\nu)}(H_\nu(t+s,z)$ for fixed s with the initial value $H_{(\mu,\nu)}(s,z)$. On the other hand, by (5.12) we have $\frac{d}{dt}K_s(t,z) = A_{(\mu,\nu)}(H_\nu(t, H_\nu(s,z))) = A_{(\mu,\nu)}(H_\nu(t+s,z))$. Moreover, we have $K_s(0,z) = H_{(\mu,\nu)}(s,z)$. Therefore,

$H_{(\mu,\nu)}(t+s,z) = K_s(t,z)$. Since the coefficients of z^k in the equality (5.11) are polynomials of t and s, we can substitute real numbers. Setting $t = s = 1$, we obtain $m_n(\mu,\nu,2) = m_n(\mu_2, \nu \rhd \nu, 1) = m_n(\mu_2)$. Inductively, we can show that $m_n(\mu,\nu,N) = m_n(\mu_N)$. By the (power) associativity of conditionally monotone convolution, we also obtain $m_n(\mu,\nu,MN) = m_n(\mu_N, \nu^{\rhd N}, M)$ for $M, N \in \mathbb{N}$. Then we have

$$\begin{aligned} &MNr_n(\mu,\nu) + M^2N^2C_n(MN, r_1(\mu,\nu),\cdots,r_{n-1}(\mu,\nu), r_1(\nu),\cdots,r_{n-2}(\nu)) \\ &= Mr_n(\mu_N,\nu^{\rhd N}) + M^2C_n(M, r_1(\mu_N,\nu^{\rhd N}),\cdots,r_{n-1}(\mu_N,\nu^{\rhd N}), r_1(\nu^{\rhd N}),\cdots,r_{n-2}(\nu^{\rhd N})). \end{aligned} \tag{5.13}$$

Since the coefficients of M coincide, it holds that $r_n(\mu_N,\nu^{\rhd N}) = Nr_n(\mu,\nu)$. □

Before closing this section, we summarize the basic relations of generating functions:

$$\frac{d}{dt}H_{(\mu,\nu)}(t,z) = A_{(\mu,\nu)}(H_\nu(t,z)),\ H_{(\mu,\nu)}(1,z) = H_\mu(z),\ H_{(\mu,\nu)}(0,z) = z, \tag{5.14}$$

$$\frac{d}{dt}H_\nu(t,z) = A_\nu(H_\nu(t,z)),\ H_\nu(1,z) = H_\nu(z),\ H_\nu(0,z) = z. \tag{5.15}$$

These relations are parallel to

$$H_\mu(z) = z - \phi_{(\mu,\nu)}(H_\nu(z)), \tag{5.16}$$

$$H_\nu(z) = z - \phi_\nu(H_\nu(z)), \tag{5.17}$$

which are basic relations in conditionally free convolution. We note that monotone cumulants and Boolean cumulants are obtained as special cases of conditionally monotone cumulants: $A_{(\mu,\mu)}$ is a generating function of monotone cumulants and $A_{(\mu,\delta_0)}$ is a generating function of Boolean cumulants.

6 Limit theorems of conditionally monotone convolution

We consider the central limit theorem and Poisson's law of small numbers of conditionally monotone convolution. Let $\mu,\nu \in \mathcal{P}_{\mathrm{m}}$ such that $m_1(\mu) = m_1(\nu) = 0$ and $m_2(\mu) = \alpha^2$ and $m_2(\nu) = \beta^2$ with $\alpha,\beta > 0$. We define $(\mu_N,\nu_N) := (D_{\frac{1}{\sqrt{N}}}\mu, D_{\frac{1}{\sqrt{N}}}\nu)^{\rhd N}$. By a simple calculation, we can show that $r_1(\mu_N,\nu_N) = 0$, $r_2(\mu_N,\nu_N) = \alpha^2$ and $r_n(\mu_N,\nu_N) \to 0$ as $N \to \infty$. The second component is the usual monotone convolution. Therefore, there exist limit distributions $(\nu_{\alpha,\beta},\nu_\beta)$ at least in the sense of moments. The limit distributions are characterized by

$$\frac{d}{dt}H_{(\nu_{\alpha,\beta},\nu_\beta)}(t,z) = -\frac{\alpha^2}{H_{\nu_\beta}(t,z)},\ H_{(\nu_{\alpha,\beta},\nu_\beta)}(1,z) = H_{\nu_{\alpha,\beta}}(z), \tag{6.1}$$

$$\frac{d}{dt}H_{\nu_\beta}(t,z) = -\frac{\beta^2}{H_{\nu_\beta}(t,z)},\ H_{\nu_\beta}(1,z) = H_{\nu_\beta}(z). \tag{6.2}$$

We obtain $H_{\nu_\beta}(t,z) = \sqrt{z^2 - 2\beta^2 t}$ from (6.2). Setting $t = 1$, we obtain the limit measure $\nu_\beta(dx) = \frac{1}{\pi\sqrt{2\beta^2 - x^2}}dx$, $x \in (-\sqrt{2}\beta, \sqrt{2}\beta)$. For the first component, we obtain $H_{(\nu_{\alpha,\beta},\nu_\beta)}(1,z) = (1 - \frac{\alpha^2}{\beta^2})z + \frac{\alpha^2}{\beta^2}\sqrt{z^2 - 2\beta^2}$, and hence, we have

$$G_{\nu_{\alpha,\beta}}(z) = \frac{\alpha^2\sqrt{z^2 - 2\beta^2} + (\alpha^2 - \beta^2)z}{(2\alpha^2 - \beta^2)z^2 - 2\alpha^4}. \tag{6.3}$$

This is the Stieltjes transform of the Kesten distribution. If we use the t-transformation, the limit distribution can be written by $\mathcal{U}_{\alpha^2/\beta^2}(\nu_\beta)$. The limit distribution is compactly supported, and then the convergence holds in the sense of weak convergence. We note that $\nu_{\beta,\beta} = \nu_\beta$.

Next we consider Poisson's law of small numbers in the setting of triangular arrays. Let $\mu^{(N)},\nu^{(N)} \in \mathcal{P}_{\mathrm{m}}$ such that $Nm_n(\mu^{(N)}) \to \lambda > 0$ and $Nm_n(\nu^{(N)}) \to \rho > 0$ as $N \to \infty$ for any $n \geq 1$. We define

$(\mu_N, \nu_N) := (\mu^{(N)}, \nu^{(N)})^{\triangleright N}$. It is not difficult to prove that $r_n(\mu_N, \nu_N) \to \lambda$ as $N \to \infty$ for any $n \geq 1$. It is known that ν_N converges weakly to the monotone Poisson distribution p_ρ with parameter ρ, which is characterized by the differential equation

$$\frac{d}{dt} H_{p_\rho}(t,z) = \frac{\rho H_{p_\rho}(t,z)}{1 - H_{p_\rho}(t,z)}, \ H_{p_\rho}(1,z) = H_{p_\rho}(z), \tag{6.4}$$

where $H_{p_\rho}(1,z) = H_{p_\rho}(z)$. We denote the limit distribution $\lim_{N\to\infty} \mu_N$ by $p_{\lambda,\rho}$ in the sense of moments. Then $p_{\lambda,\rho}$ is characterized by the differential equation

$$\frac{d}{dt} H_{(p_{\lambda,\rho}, p_\rho)}(t,z) = \frac{\lambda H_{p_\rho}(t,z)}{1 - H_{p_\rho}(t,z)}, \ H_{(p_{\lambda,\rho}, p_\rho)}(1,z) = H_{p_{\lambda,\rho}}(z). \tag{6.5}$$

By (6.4) and (6.5) we have $\frac{d}{dt} H_{(p_{\lambda,\rho}, p_\rho)}(t,z) = \frac{\lambda}{\rho} \frac{d}{dt} H_{p_\rho}(t,z)$, which implies

$$H_{p_{\lambda,\rho}}(z) = \Big(1 - \frac{\lambda}{\rho}\Big) z + \frac{\lambda}{\rho} H_{p_\rho}(z). \tag{6.6}$$

It is not difficult to see that $p_{\lambda,\rho}$ has a compact support. Therefore, μ_N converges weakly to $p_{\lambda,\rho}$. Also in this case, $p_{\lambda,\rho}$ can be written as $p_{\lambda,\rho} = \mathcal{U}_{\lambda/\rho}(p_\rho)$.

7 Convolutions arising from conditionally monotone convolution

Conditionally monotone convolution generates monotone convolution and Boolean convolution in the left component:

$$(\mu, \mu) \triangleright (\nu, \nu) = (\mu \triangleright \nu, \mu \triangleright \nu), \tag{7.1}$$
$$(\mu, \delta_0) \triangleright (\nu, \delta_0) = (\mu \uplus \nu, \delta_0), \tag{7.2}$$

which are very parallel relations to (3.5) and (3.6). For a map $T\colon \mathcal{P} \to \mathcal{P}$, one can define a new convolution $\boxplus_T$ by

$$(\mu \boxplus_T \nu, T\mu \boxplus T\nu) = (\mu, T\mu) \boxplus (\nu, T\nu). \tag{7.3}$$

A problem of Bożejko is to find all maps $T\colon \mathcal{P} \to \mathcal{P}$ such that

$$T(\mu \boxplus_T \nu) = T\mu \boxplus T\nu. \tag{7.4}$$

If such T is found, then the new convolution is associative and commutative. Moreover, this is a generalization of free and Boolean convolutions. This problem has been considered in [4, 11, 12, 20], in which several such convolutions have been found. Motivated by these works, we consider the following type of convolution:

$$(\mu \triangleright_{\mathrm{T}} \nu, T\nu) = (\mu, \delta_0) \boxplus (\nu, T\nu). \tag{7.5}$$

This relation is parallel to (7.3) in terms of conditionally monotone convolution:

$$(\mu \triangleright_{\mathrm{T}} \nu, T\mu \triangleright T\nu) = (\mu, T\mu) \triangleright (\nu, T\nu). \tag{7.6}$$

Clearly, this convolution includes Boolean and monotone convolutions if we take T as $T\mu = \delta_0$ for all μ and $T = \mathrm{Id}$, respectively.

The convolution may not be associative. Now we give a characterization when the convolution $\triangleright_{\mathrm{T}}$ becomes associative.

Proposition 7.1. (1) $\rhd_{\mathrm{T}}$ *is characterized by the equality*

$$H_{\mu \rhd_{\mathrm{T}} \nu} = H_\mu \circ H_{T\nu} + H_\nu - H_{T\nu}. \tag{7.7}$$

(2) *The convolution* $\rhd_{\mathrm{T}}$ *becomes associative if and only if*

$$T(\mu \rhd_{\mathrm{T}} \nu) = T\mu \rhd T\nu \tag{7.8}$$

for all μ *and* ν.

Remark 7.2. (1) In many cases T is only defined in a subset of $\mathcal{P}$ such as $\mathcal{P}_{\mathrm{m}}$. In such a case, the above Proposition holds if the subset is closed under the convolution $\rhd_{\mathrm{T}}$. Hereafter, we often state results about the convolution $\rhd_{\mathrm{T}}$ only for $T\colon \mathcal{P} \to \mathcal{P}$ if such a generalization of the domain of T is trivial.
(2) It seems to be not known whether the condition (7.4) is necessary condition for the associativity of $\boxplus_T$.

Proof. (1) The proof is easy.
(2) First we calculate $H_{(\mu \rhd_{\mathrm{T}} \nu) \rhd_{\mathrm{T}} \lambda}$.

$$\begin{aligned} H_{(\mu \rhd_{\mathrm{T}} \nu) \rhd_{\mathrm{T}} \lambda} &= H_{\mu \rhd_{\mathrm{T}} \nu} \circ H_{T\lambda} + H_\lambda - H_{T\lambda} \\ &= H_\mu \circ H_{T\nu} \circ H_{T\lambda} + H_\nu \circ H_{T\lambda} - H_{T\nu} \circ H_{T\lambda} + H_\lambda - H_{T\lambda}. \end{aligned} \tag{7.9}$$

$H_{\mu \rhd_{\mathrm{T}} (\nu \rhd_{\mathrm{T}} \lambda)}$ is calculated as follows.

$$\begin{aligned} H_{\mu \rhd_{\mathrm{T}} (\nu \rhd_{\mathrm{T}} \lambda)} &= H_\mu \circ H_{T(\nu \rhd_{\mathrm{T}} \lambda)} + H_{\nu \rhd_{\mathrm{T}} \lambda} - H_{T(\nu \rhd_{\mathrm{T}} \lambda)} \\ &= H_\mu \circ H_{T(\nu \rhd_{\mathrm{T}} \lambda)} + H_\nu \circ H_{T\lambda} + H_\lambda - H_{T\lambda} - H_{T(\nu \rhd_{\mathrm{T}} \lambda)}. \end{aligned} \tag{7.10}$$

Then the associativity of the convolution implies that $H_\mu \circ H_{T\nu} \circ H_{T\lambda} - H_\mu \circ H_{T(\nu \rhd_{\mathrm{T}} \lambda)}$ only depends on ν and λ. If $H_{T\nu} \circ H_{T\lambda}$ were not equal to $H_{T(\nu \rhd_{\mathrm{T}} \lambda)}$ for some ν and λ, then there would exist $w \in \mathbb{C} \setminus \mathbb{R}$ such that $H_{T\nu} \circ H_{T\lambda}(w) \neq H_{T(\nu \rhd_{\mathrm{T}} \lambda)}(w)$. In this case, $H_\mu \circ H_{T\nu} \circ H_{T\lambda}(w) - H_\mu \circ H_{T(\nu \rhd_{\mathrm{T}} \lambda)}(w)$ clearly depends on μ, which is a contradiction. Therefore, we conclude that

$$H_{T\nu} \circ H_{T\lambda} = H_{T(\nu \rhd_{\mathrm{T}} \lambda)} \tag{7.11}$$

for all $\nu, \lambda \in \mathcal{P}$. Conversely, if (7.11) holds, it is not difficult to see that the convolution is associative. □

We show the additivity of mean and variance, which will be used in the proof of Theorem 8.2.

Proposition 7.3. *Let* $T\colon \mathcal{P} \to \mathcal{P}$ *be an arbitrary map. Then we have the following properties.*

(1) $\mathcal{P}^2$ *is closed under the convolution* $\rhd_{\mathrm{T}}$.

(2) $m(\mu)$ *and* $\sigma^2(\mu)$ *are additive w.r.t. the convolution* $\rhd_{\mathrm{T}}$ *considered in* $\mathcal{P}^2$*:*

$$m(\mu \rhd_{\mathrm{T}} \nu) = m(\mu) + m(\nu), \tag{7.12}$$

$$\sigma^2(\mu \rhd_{\mathrm{T}} \nu) = \sigma^2(\mu) + \sigma^2(\nu). \tag{7.13}$$

Proof. We use the notation

$$H_\mu(z) = -m(\mu) + z + \int \frac{1}{x - z} d\rho_\mu(x). \tag{7.14}$$

Then we have

$$\begin{aligned} H_{\mu \rhd_{\mathrm{T}} \nu}(z) &= H_\mu \circ H_{T\nu}(z) + H_\nu(z) - H_{T\nu}(z) \\ &= -m(\mu) + H_{T\nu}(z) + \int_{\mathbb{R}} \frac{1}{x - H_{T\nu}(z)} d\rho_\mu(x) + H_\nu(z) - H_{T\nu}(z) \\ &= -m(\mu) + H_\nu(z) + \int_{\mathbb{R}} \frac{1}{x - z} d\Big(\int_{\mathbb{R}} (T\nu)_y(x) d\rho_\mu(y) \Big) \\ &= -m(\mu) - m(\nu) + z + \int_{\mathbb{R}} \frac{1}{x - z} d\Big(\int_{\mathbb{R}} (T\nu)_y(x) d\rho_\mu(y) + \rho_\nu(x) \Big), \end{aligned} \tag{7.15}$$

where $\lambda_y \in \mathcal{P}$ is defined by $H_{\lambda_y} = H_\lambda - y$ for $\lambda \in \mathcal{P}$. By Lemma 2.13, $\mathcal{P}^2$ is closed under the convolution; moreover, the mean and the variance are additive. □

Thus, $\mathcal{P}^2$ is closed under the convolution $\rhd_{\mathrm{T}}$ for arbitrary map T. On the contrary, the closedness of $\mathcal{P}_+$ and $\mathcal{P}_{\mathrm{sym}}$ under the convolution $\rhd_{\mathrm{T}}$ does not hold without any assumptions. We show the necessary and sufficient conditions.

Proposition 7.4. (1) *The following two conditions are equivalent.*

(1*a*) $T(\mathcal{P}_+) \subset \mathcal{P}_+$,

(1*b*) $\mu \rhd_{\mathrm{T}} \nu \in \mathcal{P}_+$ *for all* μ, $\nu \in \mathcal{P}_+$.

(2) *The following two conditions are equivalent.*

(2*a*) $T(\mathcal{P}_{\mathrm{sym}}) \subset \mathcal{P}_{\mathrm{sym}}$,

(2*b*) $\mu \rhd_{\mathrm{T}} \nu \in \mathcal{P}_{\mathrm{sym}}$ *for all* μ, $\nu \in \mathcal{P}_{\mathrm{sym}}$.

Proof. (1) We assume (1*a*). By Lemma 2.3, H_λ is analytic in $\mathbb{C}\setminus[0,\infty)$ and $H_\lambda < 0$ in $(-\infty, 0)$ for $\lambda = \mu, \nu, T\nu$. Then the composition $H_\mu \circ H_{T\nu}$ is analytic in $\mathbb{C}\setminus[0,\infty)$, and hence, $H_{\mu \rhd_{\mathrm{T}} \nu}$ is also analytic in the same region. This implies the condition (1) in Lemma 2.3. We use the notation

$$H_\mu(z) = b_\mu + z + \int_{\mathbb{R}} \frac{1+xz}{x-z} d\eta_\mu(x) \tag{7.16}$$

and the similar notation for H_ν. We put $g_\mu(z) := \int_{\mathbb{R}} \frac{1+xz}{x-z} d\eta_\mu(x)$. By Proposition 7.1, we have

$$\begin{aligned} H_{\mu \rhd_{\mathrm{T}} \nu}(z) &= H_\mu \circ H_{T\nu}(z) + H_\nu(z) - H_{T\nu}(z) \\ &= b_\mu + g_\mu(H_{T\nu}(z)) + H_\nu(z). \end{aligned} \tag{7.17}$$

Since g_μ and $H_{T\nu}$ are nondecreasing, $g_\mu \circ H_{T\nu}$ is also nondecreasing. Then $b_\mu + g_\mu \circ H_{T\nu}(-0) \le b_\mu + g_\mu(-0) = H_\mu(-0) \le 0$. Therefore, we obtain $H_{\mu \rhd_{\mathrm{T}} \nu}(-0) \le 0$.

Next we assume (1*b*). We shall prove the fact by *reductio ad absurdum*; we assume that there exists $\nu \in \mathcal{P}_+$ such that $T\nu \notin \mathcal{P}_+$. In the notation (7.15), we have

$$\begin{aligned} H_{\mu \rhd_{\mathrm{T}} \nu}(z) &= H_\mu \circ H_{T\nu}(z) + H_\nu(z) - H_{T\nu}(z) \\ &= -m(\mu) + H_\nu(z) + \int_{\mathbb{R}} \frac{1}{x-z} d\Big(\int_{\mathbb{R}} (T\nu)_y(x) d\rho_\mu(y) \Big) \\ &= -m(\mu) + H_\nu(z) - G_{\rho_\mu \rhd T\nu}(z) \end{aligned} \tag{7.18}$$

for all $\mu \in \mathcal{P}^2$, where we defined $\rho_\mu \rhd T\nu$ by the affinity in the left component of monotone convolution. We can construct $\mu \in \mathcal{P}_+$ such that $a(\rho_\mu) = 0$. Then from (3) of Lemma 2.2 we have $a(\rho_\mu \rhd_{\mathrm{T}} T\nu) \le a(T\nu) < 0$, which means that $G_{\rho_\mu \rhd T\nu}$ is not analytic in $\mathbb{C}\setminus[0,\infty)$. By assumption, both $H_{\mu \rhd_{\mathrm{T}} \nu}(z)$ and $H_\nu(z)$ are analytic in $\mathbb{C}\setminus[0,\infty)$, which is a contradiction to the equality (7.18) (we note that Lemma 2.2 is applicable for any positive finite measure).

(2) We assume (2*a*). Since symmetric probability measures are characterized by $H_\mu(-z) = -H_\mu(z)$ for $z \in \mathbb{C}\setminus\mathbb{R}$, the proof is not difficult.

Conversely, we assume (2*b*). We take μ to be the arcsine law with mean 0 and variance 1. Clearly, $\mu \in \mathcal{P}_{\mathrm{sym}}$. By assumption, $H_{\mu \rhd_{\mathrm{T}} \nu}(-z) = -H_{\mu \rhd_{\mathrm{T}} \nu}(z)$ for all $\nu \in \mathcal{P}_{\mathrm{sym}}$. Then (7.7) implies that

$$\sqrt{H_{T\nu}(z)^2 - 2} - \sqrt{H_{T\nu}(-z)^2 - 2} = H_{T\nu}(z) + H_{T\nu}(-z) \tag{7.19}$$

for $z \in \mathbb{C}_+$. After some calculations we obtain $H_{T\nu}(-z) = -H_{T\nu}(z)$, which means $T\nu \in \mathcal{P}_{\mathrm{sym}}$. □

Remark 7.5. The above property is a generalization of the Boolean and monotone cases. In the cases of Boolean and monotone, moreover, we can show that $\nu^{\square n} \in \mathcal{P}_+$ implies $\nu \in \mathcal{P}_+$ ($\square$ is Boolean or monotone convolution).

Sometimes the limit distribution of Poisson's law of small numbers concerning a deformed convolution does not belong to $\mathcal{P}_+$ [11]. We can give a sufficient condition under which the limit distribution belongs to $\mathcal{P}_+$ in the case of the convolution $\rhd_{\mathrm{T}}$.

Corollary 7.6. (1) *We assume that $T(\mathcal{P}_+) \subset \mathcal{P}_+$. If Poisson's law of small numbers holds, the limit distribution belongs to $\mathcal{P}_+$.*
(2) *We assume that $T(\mathcal{P}_{\mathrm{sym}}) \subset \mathcal{P}_{\mathrm{sym}}$. If the central limit theorem holds, the limit distribution belongs to $\mathcal{P}_{\mathrm{sym}}$.*

Remark 7.7. Poisson's law of small numbers and the central limit theorem mean the statements as given in Theorem 11.6.

Proof. If we take $\mu^{(N)} := (1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_1 \in \mathcal{P}_+$, the limit distribution $\lim_{N\to\infty}(\mu^{(N)})^{\rhd_{\mathrm{T}} N} \in \mathcal{P}_+$ by Lemma 2.6. For the central limit theorem, we take $\mu := \frac{1}{2}(\delta_{-1} + \delta_1)$. Then $(D_{\frac{1}{\sqrt{N}}}\mu)^{\rhd_{\mathrm{T}} N} \in \mathcal{P}_{\mathrm{sym}}$ and the limit distribution also belongs to $\mathcal{P}_{\mathrm{sym}}$ by Lemma 2.6. □

8 $V_{t,u,a}$-transformation

We introduce $V_{t,u,a}$-transformation and prove that the transformation satisfies the condition (7.8). Moreover, this family of transformations includes the generalized t-transformation [12] and the V_a-transformation [11].

Let $f\colon \mathcal{Q} \to \mathbb{R}$, where $\mathcal{Q}$ is a subset of $\mathcal{P}$. Typically $\mathcal{Q}$ is chosen to be $\mathcal{P}^2$, $\mathcal{P}_{\mathrm{m}}$ or $\mathcal{P}_{\mathrm{c}}$. Motivated by the generalized t-transformation in [12] and V_a-transformation in [11], we look for a transform $V_{t,f}$ of the form

$$\mu \mapsto V_{t,f}\mu, \quad H_{V_{t,f}\mu}(z) = tH_\mu(z) + (1-t)z + f(\mu). \tag{8.1}$$

If $f(\mu) = (t-u)m(\mu)$, this is the same as the generalized t-transformation. If $t = 1$ and $f(\mu) = ar_\mu(2)$, this is the same as V_a-transformation ($r_\mu(2)$ is the second free cumulant of μ, which is also equal to the variance).

Lemma 8.1. *Assume that $\mathcal{Q}$ is closed under the convolution $\rhd_{V_{t,f}}$. $V_{t,f}$ satisfies the associativity condition (7.8) considered in $\mathcal{Q}$ if and only if $f(\mu \rhd_{V_{t,f}} \nu) = f(\mu) + f(\nu)$ for all $\mu \in \mathcal{Q}$.*

Proof. We denote $\rhd_{V_{t,f}}$ by $\rhd_{t,f}$ for simplicity. By (7.7) we obtain

$$\begin{aligned} H_{V_{t,f}(\mu\rhd_{t,f}\nu)}(z) &= tH_{\mu\rhd_{t,f}\nu}(z) + (1-t)z + f(\mu \rhd_{t,f} \nu) \\ &= tH_\mu \circ H_{V_{t,f}\nu}(z) + tH_\nu(z) - tH_{V_{t,f}\nu}(z) + (1-t)z + f(\mu \rhd_{t,f} \nu) \\ &= tH_\mu \circ H_{V_{t,f}\nu}(z) + tH_\nu(z) + (1-t)z + f(\nu) - tH_{V_{t,f}\nu}(z) + f(\mu \rhd_{t,f} \nu) - f(\nu) \\ &= tH_\mu \circ H_{V_{t,f}\nu}(z) + (1-t)H_{V_{t,f}\nu}(z) + f(\mu \rhd_{t,f} \nu) - f(\nu). \end{aligned} \tag{8.2}$$

On the other hand, we have

$$\begin{aligned} H_{V_{t,f}\mu\rhd V_{t,f}\nu}(z) &= H_{V_{t,f}\mu} \circ H_{V_{t,f}\nu}(z) \\ &= tH_\mu \circ H_{V_{t,f}\nu}(z) + (1-t)H_{V_{t,f}\nu}(z) + f(\mu). \end{aligned} \tag{8.3}$$

Therefore, the associativity condition (7.8) is equivalent to $f(\mu \rhd_{t,f} \nu) = f(\mu) + f(\nu)$. □

We define $V_{t,u,a}$-transformation by letting

$$f(\mu) = f_{t,u,a}(\mu) = (t-u)m(\mu) + a\sigma^2(\mu). \tag{8.4}$$

More clearly, we define

$$H_{V_{t,u,a}\mu}(z) = tH_\mu(z) + (1-t)z + (t-u)m(\mu) + a\sigma^2(\mu). \tag{8.5}$$

We shall prove that $V_{t,u,a}$-transformation gives an associative convolution defined on $\mathcal{P}^2$. It may be interesting to consider f including higher order moments, which we do not treat now. We use the notation $\rhd_{t,u,a}$ for the convolution defined by $V_{t,u,a}$.

Theorem 8.2. *The convolution $\rhd_{t,u,a}$ defined on $\mathcal{P}^2$ is associative.*

Proof. This fact follows from Lemma 8.1 and Proposition 7.3. □

In order to calculate the inverse transformation of $V_{t,u,a}$, we show the following facts.

Lemma 8.3. *We have the following equalities.*

$$m(V_{t,u,a}\mu) = um(\mu) - a\sigma^2(\mu), \tag{8.6}$$

$$\sigma^2(V_{t,u,a}\mu) = t\sigma^2(\mu). \tag{8.7}$$

Proof. We use the same notation as in Theorem 8.2. We have

$$H_{V_{t,u,a}\mu}(z) = -um(\mu) + a\sigma^2(\mu) + z + t\int \frac{1}{x-z} d\rho_\mu(x), \tag{8.8}$$

which implies the conclusion by Lemma 2.13. □

Proposition 8.4. *We have the following equality:*

$$V_{t',u',a'}V_{t,u,a} = V_{t't,u'u,u'a+a't} \text{ for } t \geq 0,\ u,\ a \in \mathbb{R}\,. \tag{8.9}$$

In particular, we have

$$V^{-1}_{t,u,a,} = V_{t^{-1},u^{-1},-\frac{a}{tu}} \text{ for } t > 0,\ u \neq 0,\ a \in \mathbb{R}\,. \tag{8.10}$$

Proposition 8.5. (1) *$\mathcal{P}_+ \cap \mathcal{P}^2$ is closed under the convolution $\rhd_{t,u,a}$ if and only if $u \geq t$ and $a = 0$.*
(2) *$\mathcal{P}_{\mathrm{sym}} \cap \mathcal{P}^2$ is closed under the convolution $\rhd_{t,u,a}$ if and only if $a = 0$.*

Proof. These facts are easy consequences of Proposition 7.4. □

Example 8.6. (1) Oravecz has introduced Fermi convolution $\bullet$ in [19]. Oravecz has mentioned the relation between Fermi convolution and conditionally free convolution:

$$(\mu \bullet \nu, \delta_{m(\mu)} \boxplus \delta_{m(\nu)}) = (\mu, \delta_{m(\mu)}) \boxplus (\nu, \delta_{m(\nu)}), \tag{8.11}$$

where $m(\mu)$ denotes the mean of μ. We can easily extend the Fermi convolution to the convolution coming from the map F_u defined by $F_u\mu = \delta_{um(\mu)}$. We have $V_{0,u,0} = F_u$; therefore, we obtain an associative convolution $\rhd_{F_u}$ related to Fermi convolution defined by

$$(\mu \rhd_{F_u} \nu, \delta_{um(\mu)+um(\nu)}) = (\mu, \delta_{um(\mu)}) \rhd (\nu, \delta_{um(\nu)}). \tag{8.12}$$

(2) $\mathcal{U}_t := V_{t,t,0}$ is called the t-transformation. We obtain an associative convolution $\rhd_t$ arising from $\mathcal{U}_t$ defined by

$$(\mu \rhd_t \nu, \mathcal{U}_t(\mu) \rhd \mathcal{U}_t(\nu)) = (\mu, \mathcal{U}_t(\mu)) \rhd (\nu, \mathcal{U}_t(\nu)). \tag{8.13}$$

We note that the t-transformation gives an interpolation between Boolean and monotone convolutions in the present situation: monotone convolution is obtained when $t = 1$ and Boolean convolution is obtained when we take the limit $t \to 0$.

(3) V_a-transformation is equal to $V_{1,1,a}$.

In the following we discuss the meaning of these results. It is known that the t-transformation $\mathcal{U}_t$ ($t > 0$) satisfies the condition (7.4), which gives a family of new convolutions $\boxplus_{\mathcal{U}_t}$ [4]. These convolutions can also be written as

$$\mu \boxplus_{\mathcal{U}_t} \nu = \mathcal{U}_{1/t}(\mathcal{U}_t(\mu) \boxplus \mathcal{U}_t(\nu)). \tag{8.14}$$

Apart from the context of conditionally free convolution, it seems interesting to study the deformations of Boolean and classical convolutions defined by the right hand side of (8.14), with $\boxplus$ replaced by $\uplus$ and $*$, respectively. The new convolutions have been studied in [5]. By definition, the deformed convolutions are associative and commutative.

We can also define the same deformations in the monotone case. The results in this section show that the deformation has a natural meaning in terms of conditionally monotone convolution as in the case of free convolution (cf. (7.3) and (7.6)). Of course the above discussion is meaningful for any T which is invertible such as some class of **t**-transformations.

9 Deformations related to monotone infinitely divisible distributions

Krystek and Wojakowski have introduced a deformation connected to a $\boxplus$-infinitely divisible distribution in [11], which we explain now. For a $\boxplus$-infinitely divisible distribution φ with compact support, there corresponds a unique weakly continuous $\boxplus$-convolution semigroup $\{\varphi_t\}_{t\geq 0}$ with $\varphi_0 = \delta_0$ and $\varphi_1 = \varphi$. Define a transformation Φ_t^{φ} by

$$\Phi_t^{\varphi}\mu = \varphi_{\sigma^2(\mu)t}. \tag{9.1}$$

This map satisfies the condition (7.4).

We introduce the monotone analog of Φ_t^{φ}.

Definition 9.1. For a $\rhd$-infinitely divisible distribution $\xi \in \mathcal{P}^2$, let $\{\xi_t\}_{t\geq 0}$ be the corresponding weakly continuous $\rhd$-convolution semigroup with $\xi_0 = \delta_0$ and $\xi_1 = \xi$. Let $f\colon \mathcal{P}^2 \to \mathbb{R}$. We define a transformation Ξ_f^{ξ} by

$$\Xi_f^{\xi}\mu := \xi_{f(\mu)}. \tag{9.2}$$

Lemma 9.2. *Ξ_f^{ξ} satisfies the associativity condition (7.8) in $\mathcal{P}^2$ if and only if $f(\mu \rhd_{\Xi_f^{\xi}} \nu) = f(\mu) + f(\nu)$ for all μ, $\nu \in \mathcal{P}^2$.*

Proof. This fact follows from the equality $\Xi_f^{\xi}\mu \rhd \Xi_f^{\xi}\nu = \xi_{f(\mu)+f(\nu)}$. □

Theorem 9.3. *The map Ξ_t^{ξ} ($t \geq 0$) defined by $f(\mu) = t\sigma^2(\mu)$ satisfies the condition (7.8).*

Proof. The fact follows from Proposition 7.3. □

Remark 9.4. (1) If $\xi = \delta_a$, the map $f_{s,t}(\mu) = -sm(\mu) + t\sigma^2(\mu)$ ($s, t \in \mathbb{R}$) is also possible. In this case we have $\Xi_{f_{u,a}}^{\delta_1} = V_{0,u,a}$.
(2) It may be interesting to consider higher moments than two, which we do not consider now.

Proposition 9.5. (1) *$\mathcal{P}_+ \cap \mathcal{P}^2$ is closed under the convolution $\rhd_{\Xi_t^{\xi}}$ if and only if $\xi \in \mathcal{P}_+ \cap \mathcal{P}^2$.*
(2) *We assume that $\xi \in \mathcal{P}_{\mathrm{c}}$. Then $\mathcal{P}_{\mathrm{sym}} \cap \mathcal{P}_{\mathrm{c}}$ is closed under the convolution $\rhd_{\Xi_t^{\xi}}$ if and only if $\xi \in \mathcal{P}_{\mathrm{sym}} \cap \mathcal{P}_{\mathrm{c}}$.*

Proof. This is an immediate consequence of Lemma 2.4, Lemma 2.5 and Proposition 7.4. □

10 Cumulants of deformed convolutions

We define $r_n^T(\mu) := r_n(\mu, T\mu)$. We show that $r_n^T(\mu)$ is suitable for the cumulants of the convolution $\rhd_{\mathrm{T}}$.

Proposition 10.1. *We assume that there exists a polynomial V_n of $n+1$ variables, which does not contain a constant term, such that*

$$m_n(T\mu) = V_n(m_1(\mu), \cdots, m_{n+1}(\mu)) \tag{10.1}$$

for any $n \geq 1$. Then the conditions (M1) and (M2) hold for the convolution $\rhd_{\mathrm{T}}$.

Proof. By (5.1) and (10.1), we obtain (M1). (M2) follows from the facts that both W_n and Y_n do not contain constant terms and linear terms. □

Theorem 10.2. *Let $T\colon \mathcal{P}_{\mathrm{m}} \to \mathcal{P}_{\mathrm{m}}$ be a map satisfying (7.8) and (10.1). Then $r_n(\mu, T\mu)$ satisfy the conditions (C1) and (C2') w.r.t. the convolution $\rhd_{\mathrm{T}}$.*

Proof. (C2') follows from the definition of conditionally monotone cumulants and (10.1). (C1) is showed as follows:
$r^T(\mu^{\rhd_{\mathrm{T}} N}) = r_n(\mu^{\rhd_{\mathrm{T}} N}, T(\mu^{\rhd_{\mathrm{T}} N})) = r_n((\mu, T\mu)^{\rhd N}) = N r_n(\mu, T\mu)$. □

We shall show that the conditionally free cumulants $R_n(\mu, T\mu)$ satisfy the conditions (C1) and (C2') under the same restriction. This result will be helpful to improve the understandings of cumulants.

Proposition 10.3. *Let $T\colon \mathcal{P}_{\mathrm{m}} \to \mathcal{P}_{\mathrm{m}}$ be a map satisfying the condition (7.4). We assume that the n-th moment of $T\mu$ is of the form*

$$m_n(T\mu) = V_n(m_1(\mu), \cdots, m_{n+1}(\mu)) \tag{10.2}$$

for any $n \geq 1$, where V_n is a polynomial which does not contain a constant term. Then the convolution $\boxplus_T$ satisfies the conditions (M1) and (M2), and $R_n(\mu, T\mu)$ satisfy the conditions (C1) and (C2').

Proof. (C1) is obtained by (2.12), (7.3) and (7.4). It is not difficult to see that $R_n(\mu, \nu)$ is of the form

$$\begin{aligned}
&R_1(\mu, \nu) = m_1(\nu),\\
&R_2(\mu, \nu) = m_2(\mu) - m_1(\mu)^2,\\
&R_n(\mu, \nu) = m_n(\mu) + W_n(m_1(\mu), \cdots, m_{n-1}(\mu), m_1(\nu), \cdots, m_{n-2}(\nu)) \text{ for } n \geq 3,
\end{aligned}$$

where W_n is a polynomial. We can show that W_n does not contain a constant term and linear terms. Then we obtain (C2'). We obtain (M1) by the equalities $R_n(\mu \boxplus_T \nu, T(\mu \boxplus_T \nu)) = R_n(\mu, T\mu) + R_n(\nu, T\nu)$ and (4.4). (M2) follows from the absence of a constant term in W_n which is a consequence of $R_{(\delta_0, \nu)}(z) = 0$. □

Remark 10.4. (1) The condition (10.2) is satisfied in all convolutions studied in [4, 5, 11, 12, 19].
(2) It is interesting that the condition (10.1) is the same as (10.2).

11 Limit theorems of deformed convolutions

Central limit theorem and Poisson's law of small numbers of the convolution $\rhd_{\mathrm{T}}$ hold by Theorem 4.10 in the sense of the convergence of moments under the conditions (7.8) and (10.1). We summarize the statements combining Theorem 4.10 and Theorem 10.2.

Theorem 11.1. *Let $T\colon \mathcal{P}_{\mathrm{m}} \to \mathcal{P}_{\mathrm{m}}$ be a map which satisfies (7.8) and (10.1).*
(1) (central limit theorem) Let μ be a probability measure in $\mathcal{P}_{\mathrm{m}}$ with mean 0 and variance 1. We define $\mu_N := \left(D_{\frac{1}{\sqrt{N}}}\mu\right)^{\rhd_{\mathrm{T}} N}$. Then $m_n(\mu_N)$ converges to $m_n(\nu_1^{(T)})$, where $m_n(\nu_t^{(T)})$ are characterized by

$$\frac{\partial}{\partial t} H_{\nu_t^{(T)}}(z) = -\frac{1}{H_{T\nu_t^{(T)}}(z)}. \tag{11.1}$$

(2) (Poisson's law of small numbers) Let $\{\mu^{(N)}\}_{N=1}^{\infty}$ be a sequence of probability measures in $\mathcal{P}_{\mathrm{m}}$ such that $Nm_n(\mu^{(N)}) \to \lambda > 0$ as $N \to \infty$ for all $n \geq 1$. We define $\mu_N := (\mu^{(N)})^{\triangleright_{\mathrm{T}} N}$. Then $m_n(\mu_N)$ converges to $m_n(p_\lambda^{(T)})$, where $m_n(p_\lambda^{(T)})$ are characterized by

$$\frac{\partial}{\partial\lambda} H_{p_\lambda^{(T)}}(z) = \frac{H_{Tp_\lambda^{(T)}}(z)}{1 - H_{Tp_\lambda^{(T)}}(z)}. \tag{11.2}$$

In this section we calculate the limit distributions for T constructed in the sections 8 and 9. If T is invertible, we can use monotone cumulants to calculate the limit distributions since $\mu \triangleright_{\mathrm{T}} \nu = T^{-1}(T\mu \triangleright T\nu)$. Cumulants introduced in Section 10, however, enable us to calculate the limit distributions for even non-invertible T. In this section, we always use cumulants introduced in Section 10.

11.1 $V_{t,u,a}$-transformation

We now calculate the limit distribution in central limit theorem of the convolution $\triangleright_{t,u,a}$ for $t > 0$, $u \in \mathbb{R}$ and $a \in \mathbb{R}$. We only calculate the cases $a = 0$ and $t = 0$. We denote $r_n^{V_{t,u,a}}(\mu)$ by $r_n^{(t,u,a)}(\mu)$. In this section we use two logarithms $\log_{[1]}$ and $\log_{[2]}$: $\log_{[1]}(z)$ is defined by $\log_{[1]}(z) := \log|z| + i\arg(z)$, $\arg(z) \in (-\pi, \pi)$, $z \in \mathbb{C}\setminus(-\infty, 0]$; $\log_{[2]}$ is defined by $\log|z| + i\arg(z)$, $\arg(z) \in (0, 2\pi)$, $z \in \mathbb{C}\setminus[0, \infty)$. We define $\sqrt{z}$ by $\exp(\frac{1}{2}\log_{[2]} z)$ for $z \in \mathbb{C}\setminus[0, \infty)$. Then the Cauchy transform of the normalized arcsine law is given by $\frac{1}{\sqrt{z^2-2}}$ for $z \in \mathbb{C}_+$.

Theorem 11.2. *(1) Let μ be a probability measure in $\mathcal{P}_{\mathrm{m}}$ with mean 0 and variance 1. Then $\mu_N := (D_{\frac{1}{\sqrt{N}}}\mu)^{\triangleright_{t,u,0} N}$ converges weakly to the Kesten distribution $\nu^{(t,0)}$. For $t \geq 1$, the absolutely continuous part is $\frac{1}{2\pi}\frac{\sqrt{2t-x^2}}{1-(1-\frac{t}{2})x^2}dx$ and is supported on $[-\sqrt{2t}, \sqrt{2t}]$. For $t < 1$, $\nu^{(t,0)}$ contains atoms at $x = \pm\frac{1}{\sqrt{1-\frac{t}{2}}}$.*
(2) Let μ be a probability measure in $\mathcal{P}_{\mathrm{m}}$ with mean 0 and variance 1. Then $\mu_N := (D_{\frac{1}{\sqrt{N}}}\mu)^{\triangleright_{0,0,a} N}$ converges weakly to a probability measure $\nu^{(0,a)}$. The absolutely continuous part of $\nu^{(0,a)}$ is given by

$$\nu^{(0,a)}|_{ac} = \begin{cases} \frac{a}{(\log|1+\frac{a}{x}|-ax)^2+\pi^2}dx, & x \in [-a, 0], \quad a > 0, \\ \frac{|a|}{(\log|1+\frac{a}{x}|-ax)^2+\pi^2}dx, & x \in [0, |a|], \quad a < 0. \end{cases} \tag{11.3}$$

$\nu^{(0,a)}$ contains two atoms: one in $(-\infty, -a)$ and one in $(0, \infty)$ for $a > 0$; one in $(-\infty, 0)$ and one in $(|a|, \infty)$ for $a < 0$.

Remark 11.3. (1) The Kesten distribution also appears in the central limit theorem of $\boxplus_{\mathcal{U}_t}$ [4, 5] with the parameter t replaced by $2t$.
(2) The limit distribution of (2) is symmetric only in the case of $a = 0$ where the convolution is Boolean convolution (cf. Proposition 8.5).

Proof. (1) We denote $r_n^{V_{t,u,a}}(\nu)$ by $r_n^{t,u,a}(\nu)$. Let $\{\nu_s^{(t,0)}\}_{s\geq 0}$ be a (formal) convolution semigroup which is a solution of (11.1) for $T = V_{t,u,0}$. (The word "formal" means that the limit moments might not be deterministic. Therefore, we consider $\nu_s^{(t,0)}$ as a sequence of moments.) We note that $m_1(\nu_s^{(t,0)}) = sr_1^{(t,u,0)}(\nu_1^{(t,0)}) = 0$. Then $H_{V_{t,u,a}\nu_s^{(t,a)}}(z) = tH_{\nu_s^{(t,a)}}(z) + (1-t)z$. We denote $H_{\nu_s^{(t,0)}}(z)$ by $H_s(z)$. (11.1) can be integrated and we obtain $\frac{t}{2}H_s(z)^2 + (1-t)zH_s(z) = -s + (1-\frac{t}{2})z^2$, which implies

$$G_s(z) = \frac{(\frac{1}{2}-\frac{t}{2}) + \frac{1}{2}\sqrt{z^2-2st}}{(1-\frac{t}{2})z^2 - 2s}. \tag{11.4}$$

G_1 is the Cauchy transform of the Kesten distribution (see [5]), whose support is compact. Then the weak convergence holds (see Theorem 4.5.5 in [6]).
(2) Let $\{\nu_s^{(0,a)}\}_{s\geq 0}$ be a (formal) convolution semigroup which is a solution of (11.1) for $T = V_{0,0,a}$. We

note that $m_1(\nu_s^{(0,a)})=0$ and $\sigma^2(\nu_s^{(0,a)})=r_2^{(0,0,a)}(\nu_s^{(0,a)})=s$. Then $H_{V_{0,0,a}\nu_s^{(0,a)}}(z)=z+as$. We denote $H_{\nu_s^{(0,a)}}(z)$ by $H_s(z)$ for simplicity. We have

$$\begin{aligned} H_s(z) &= -\int_0^s \frac{1}{z+ar}dr+z \\ &= z-\frac{1}{a}\log_{[1]}\Big(1+\frac{as}{z}\Big). \end{aligned} \tag{11.5}$$

case $a>0$: The absolutely continuous part of the limit distribution is given by $\frac{a}{(\log|1+\frac{a}{x}|-a\pi)^2+\pi^2}dx$ which is supported on the set $\{x\in\mathbb{R}; G_1(x+i0)<0\}=[-a,0]$. We can show that the limit distribution contains one atom in $(0,\infty)$ and one atom in $(-\infty,-a)$.
case $a<0$: The absolutely continuous part is given by $\frac{|a|}{(\log|1+\frac{a}{x}|-a\pi)^2+\pi^2}dx$ which is supported on the set $[0,|a|]$. We can show that the limit distribution contains one atom in $(|a|,\infty)$ and one atom in $(-\infty,0)$.

We note that the case $a=0$ corresponds to the Boolean convolution, and hence, the limit distribution is $\frac{1}{2}(\delta_{-1}+\delta_1)$. □

We calculate the limit distribution of Poisson's law of small numbers. We consider only the case $T=V_{0,u,a}$; otherwise, the explicit form is difficult to obtain.

Theorem 11.4. *Let $\{\mu^{(N)}\}_{N=1}^{\infty}$ be a sequence of probability measures in $\mathcal{P}_{\mathrm{m}}$ such that $Nm_n(\mu^{(N)})\to\lambda>0$ as $N\to\infty$ for all $n\geq 1$. Then $\mu_N:=(\mu^{(N)})^{\triangleright_{0,u,a}N}$ converges weakly to a compactly supported distribution $p_\lambda^{(u,a)}$. The absolutely continuous part of $p_\lambda^{(u,a)}$ is given by*

$$p_\lambda^{(u,a)}|_{ac}=\begin{cases} \dfrac{a-u}{\Big(\log|1+\frac{(a-u)\lambda}{x-1}|-(a-u)(x-\lambda)\Big)^2+\pi^2}dx, & x\in[1-(a-u)\lambda,1],\quad a>u, \\ \dfrac{|a-u|}{\Big(\log|1+\frac{(a-u)\lambda}{x-1}|-(a-u)(x-\lambda)\Big)^2+\pi^2}dx, & x\in[1,1+(u-a)\lambda],\quad a<u. \end{cases} \tag{11.6}$$

$p_\lambda^{(u,a)}$ contains two atoms: one in $(-\infty,1-(a-u)\lambda)$ and one in $(1,\infty)$ for $a>u$; one in $(-\infty,1)$ and one in $(1+(u-a)\lambda,\infty)$ for $a<u$.

Remark 11.5. One can see that $p_\lambda^{(u,a)}$ is in $\mathcal{P}_+$ if and only if $u\geq a$ (cf. Proposition 8.5).

Proof. We note that $m_1(p_\lambda^{(u,a)})=r_1^{(0,u,a)}(p_\lambda^{(u,a)})=\lambda$ and $\sigma^2(p_\lambda^{(u,a)})=\lambda$. Then we obtain the differential equation

$$\frac{\partial}{\partial\lambda}H_{p_\lambda^{(u,a)}}(z)=-1-\frac{1}{z-1+(a-u)\lambda}. \tag{11.7}$$

The remaining arguments are similar to Theorem 11.2 and we omit the proof. □

11.2 Deformations related to $\triangleright$-infinitely divisible distributions

For a compactly supported $\triangleright$-infinitely divisible distribution ξ, let $\{\xi_t\}_{t\geq 0}$ be the corresponding weakly continuous $\triangleright$-convolution semigroup with $\xi_0=\delta_0$ and $\xi_1=\xi$. Then ξ_t is compactly supported for each $t>0$ [16]. Let $\{\nu_s^{[\xi,t]}\}_{s\geq 0}$ and $\{p_s^{[\xi,t]}\}_{s\geq 0}$ be the (formal) convolution semigroups defined by (11.1) and (11.2), respectively. We denote $r_n^{\Xi_t^\xi}(\mu)$ by $r_n^{[\xi,t]}(\mu)$. Since $r_2^{[\xi,t]}(\nu_s^{[\xi,t]})=\sigma^2(\nu_s^{[\xi,t]})=s$ we obtain $\Xi_t^\xi(\nu_s^{[\xi,t]})=\xi_{st}$. Similarly, we obtain $\Xi_t^\xi(p_\lambda^{[\xi,t]})=\xi_{\lambda t}$. Therefore, (11.1) and (11.2) become

$$\frac{\partial}{\partial s}H_{\nu_s^{[\xi,t]}}(z)=-\frac{1}{H_{\xi_{st}}(z)}, \tag{11.8}$$

$$\frac{\partial}{\partial\lambda}H_{p_\lambda^{[\xi,t]}}(z)=\frac{H_{\xi_{t\lambda}}(z)}{1-H_{\xi_{t\lambda}}(z)}. \tag{11.9}$$

These equations have been defined in the sense of formal power series of moments. Moreover, in the present case, we can show that any one of (11.8) and (11.9) gives a sequence of moments of a compactly supported probability measure for each $s > 0$ since the right hand sides of (11.8) and (11.9) are analytic outside a ball. Therefore, $\nu_s^{[\xi,t]}$ and $p_\lambda^{[\xi,t]}$ make sense as uniquely determined probability measures. Since the limit distribution is compactly supported, the convergence of moments in Theorem 11.1 becomes the weak convergence. We summarize the above arguments. We denote $\rhd_{\Xi_t^\xi}$ by $\rhd_{[\xi,t]}$.

Theorem 11.6. *Let ξ be a $\rhd$-infinitely divisible distribution in $\mathcal{P}_{\mathrm{c}}$.*
(1) (central limit theorem) Let μ be a probability measure in $\mathcal{P}_{\mathrm{m}}$ with mean 0 and variance 1. Then $\mu_N := \left(D_{\frac{1}{\sqrt{N}}}\mu\right)^{\rhd_{[\xi,t]}N}$ converges to $\nu_1^{[\xi,t]}$ weakly.
(2) (Poisson's law of small numbers) Let $\{\mu^{(N)}\}_{N=1}^\infty$ be a sequence of probability measures in $\mathcal{P}_{\mathrm{m}}$ such that $Nm_n(\mu^{(N)}) \to \lambda > 0$ as $N \to \infty$ for all $n \geq 1$. Then $\mu_N := (\mu^{(N)})^{\rhd_{[\xi,t]}N}$ converges to $p_\lambda^{[\xi,t]}$ weakly.

We calculate the limit distributions explicitly when ξ is the normalized arcsine law.

Theorem 11.7. *Let η be the normalized arcsine law.*
(1) The limit distribution $\nu_1^{[\eta,t]}$ is given by the Kesten distribution $\nu^{(t,0)}$.
(2) The absolutely continuous part of $p_\lambda^{[\eta,t]}$ is supported on $[-\sqrt{2\lambda t}, \sqrt{2\lambda t}] \cup [1, \sqrt{2\lambda t + 1}]$. The singular part consists of three parts: one atom exists in $(\sqrt{2\lambda t + 1}, \infty)$; one atom exists in $(\sqrt{2\lambda t}, 1)$ if $0 < t < \frac{1}{2\lambda}$ and $(1-\frac{1}{t})\sqrt{2\lambda t} - \frac{1}{t}\log(1-\sqrt{2\lambda t}) - \lambda < 0$; one atom exists in $(-\infty, -\sqrt{2\lambda t})$ if $(\frac{1}{t}-1)\sqrt{2\lambda t} - \frac{1}{t}\log(\sqrt{2\lambda t}+1) > 0$.

Remark 11.8. It is remarkable that the limit distribution in (1) also appears in Theorem 10 in [11] with the parameter t replaced by $2t$.

Proof. (1) The differential equation (11.8) becomes

$$\frac{\partial}{\partial s} H_{\nu_s^{[\eta,t]}}(z) = -\frac{1}{\sqrt{z^2 - 2ts}}, \tag{11.10}$$

which implies $H_{\nu_s^{[\eta,t]}}(z) = (1 - \frac{1}{t})z + \frac{1}{t}\sqrt{z^2 - 2ts}$. Therefore, the limit distribution is the Kesten distribution.
(2) The differential equation (11.9) becomes

$$\frac{\partial}{\partial \lambda} H_{p_\lambda^{[\xi,t]}}(z) = \frac{\sqrt{z^2 - 2\lambda t}}{1 - \sqrt{z^2 - 2\lambda t}}. \tag{11.11}$$

We can solve this and obtain

$$H_{p_\lambda^{[\xi,t]}}(z) = \left(1 - \frac{1}{t}\right)z + \frac{1}{t}\sqrt{z^2 - 2\lambda t} + \frac{1}{t}\log_{[1]}\left(\frac{\sqrt{z^2 - 2\lambda t} - 1}{z - 1}\right) - \lambda. \tag{11.12}$$

One can see that $\lim_{\mathrm{Im}\, z \searrow 0} H_{p_\lambda^{[\xi,t]}}(z) > 0$ if and only if $\mathrm{Re}\, z \in (-\sqrt{2\lambda t}, \sqrt{2\lambda t}) \cup [1, \sqrt{2\lambda t + 1}]$. We remark that $H_{p_\lambda^{[\xi,t]}}(x)$ is strictly increasing in the intervals $(-\infty, -\sqrt{2\lambda t})$, $(\sqrt{2\lambda t}, 1)$ and $(\sqrt{2\lambda t + 1}, \infty)$. Then it is not difficult to show the existence of atoms. □

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