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R-matrices and the Yang-Baxter equation on GNS representations of C*-bialgebras (revised)

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Abstract

A new method of construction of R-matrix is given. Let A be a C*-bialgebra with a comultiplication Δ. For two states ω and ψ of A which satisfy certain conditions, we construct a unitary R-matrix $R(\omega, \psi)$ of the C*-bialgebra (A, Δ) on the tensor product of GNS representation spaces associated with ω and ψ. The set $\{R(\omega, \psi) : \omega, \psi\}$ satisfies a kind of Yang-Baxter equation. Furthermore, we show a nontrivial example of such R-matrices.

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Key words. C*-bialgebra, R-matrix, Yang-Baxter equation, GNS representation.

1 Introduction

We have studied C*-bialgebras and tensor products of representations induced from the comultiplication. We proved that a certain C*-bialgebra has no universal R-matrix in [16]. In this paper, we construct (non-universal) R-matrices of the C*-bialgebra, which satisfy the Yang-Baxter equation. For this purpose, we show a theorem for general C*-bialgebras and states. In this section, we show our motivation, definitions of C*-bialgebras and the main theorem.

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1.1 Motivation

In this subsection, we roughly explain our motivation and the background of this study. Explicit mathematical definitions will be shown after § 1.2.

Define the C*-algebra $\mathcal{O}_*$ as the direct sum of all Cuntz algebras except $\mathcal{O}_\infty$:

$$\mathcal{O}_* = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \mathcal{O}_4 \oplus \cdots, \tag{1.1}$$

where $\mathcal{O}_1$ denotes the 1-dimensional C*-algebra $\mathbf{C}$ for convenience. In [13], we constructed a non-cocommutative comultiplication Δ_φ of $\mathcal{O}_*$. Furthermore, we proved that $\mathcal{O}_*$ has no universal R-matrix in [16]. For quasi-cocommutative C*-bialgebras, see [17, 18].

On the other hand, in the theory of quantum groups, R-matrix and the Yang-Baxter equation are fruitful subjects as relations with mathematical physics and topology [4, 6, 7, 10, 19, 23] for quasi-cocommutative bialgebras. They were also considered by Van Daele and Van Keer [22] for Hopf $*$-algebras.

As a construction method of R-matrix, the quantum double is well-known [4, 10, 19]. In this paper, we show a new method. We construct R-matrices from states of a C*-bialgebra under some conditions:

$$\text{states of a C*-bialgebra} \;\Rightarrow\; R\text{-matrices}$$

Furthermore, we show a non-trivial example of this construction.

1.2 Local R-matrix of C*-bialgebra

In this subsection, we recall definitions of C*-bialgebra, and introduce local R-matrix of a C*-bialgebra.

At first, we prepare terminologies about C*-bialgebra according to [20, 21]. For two C*-algebras A and B, let $\mathrm{Hom}(A, B)$ and $A \otimes B$ denote the set of all $*$-homomorphisms from A to B and the minimal C*-tensor product of A and B, respectively. Let $M(A)$ denote the multiplier algebra of a C*-algebra A. We state that $f \in \mathrm{Hom}(A, M(B))$ is *nondegenerate* if $f(A)B$ is dense in a C*-algebra B. A pair (A, Δ) is a *C*-bialgebra* if A is a C*-algebra and $\Delta \in \mathrm{Hom}(A, M(A \otimes A))$ such that Δ is nondegenerate and the following holds:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \tag{1.2}$$

We call Δ the *comultiplication* of A. Remark that A has no unit in general for a C*-bialgebra (A, Δ). Define the *extended flip* $\tilde{\tau}_{A,A}$ from $M(A \otimes A)$ to $M(A \otimes A)$ as $\tilde{\tau}_{A,A}(X)(x \otimes y) \equiv \tau_{A,A}(X(y \otimes x))$ for $X \in M(A \otimes A)$, $x, y \in A$

where $\tau_{A,A}$ denotes the flip of $A \otimes A$. The map Δ^{op} from A to $M(A \otimes A)$ defined as $\Delta^{op} \equiv \tilde{\tau}_{A,A} \circ \Delta$ is called the *opposite comultiplication* of Δ. A C*-bialgebra (A, Δ) is *cocommutative* if $\Delta = \Delta^{op}$.

According to [4, 10, 22], we introduce unitary R-matrix, the quasi-cocommutativity and local R-matrix for C*-bialgebra as follows.

Definition 1.1 (i) *An element R in $M(A \otimes A)$ is called a (unitary) universal R-matrix of (A, Δ) if R is a unitary and*

$$R\Delta(x)R^* = \Delta^{op}(x) \quad (x \in A). \tag{1.3}$$

In this case, we state that (A, Δ) is quasi-cocommutative (or almost cocommutative [2]).

(ii) *Let $(\mathcal{H}_i, \pi_i)$ be a representation of A for $i = 1, 2$. A unitary operator R_{π_1,π_2} on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called a local R-matrix of (A, Δ) on $\mathcal{H}_1 \otimes \mathcal{H}_2$ if it satisfies*

$$R_{\pi_1,\pi_2}(\pi_1 \otimes \pi_2)(\Delta(x))R^*_{\pi_1,\pi_2} = (\pi_1 \otimes \pi_2)(\Delta^{op}(x)) \quad (x \in A). \tag{1.4}$$

In Definition 1.1(ii), if $\pi_1 = \pi_2$, then the flip of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a (trivial) local R-matrix of (A, Δ) on $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\pi_1 \neq \pi_2$, then a local R-matrix does not always exist when (A, Δ) is not cocommutative.

In addition, we introduce a new notion for C*-bialgebra. Let $A \odot B$ denote the algebraic tensor product of C*-algebras A and B.

Definition 1.2 *A C*-bialgebra (A, Δ) is algebraic if there exists a dense $*$-subalgebra A_0 of the C*-algebra A such that $\Delta(A_0) \subset A_0 \odot A_0$. We call A_0 an algebraic part of (A, Δ).*

If a C*-bialgebra (A, Δ) is algebraic with an algebraic part A_0, then $\Delta(x)$ is written as follows for any element $x \in A_0$: There exist $1 \leq m < \infty$ and $x_1^{'}, \ldots, x_m^{'}, x_1^{''}, \ldots, x_m^{''}$ in A_0 such that

$$\Delta(x) = x_1^{'} \otimes x_1^{''} + \cdots + x_m^{'} \otimes x_m^{''}. \tag{1.5}$$

1.3 Main theorem

In this subsection, we show the main theorem. For a C*-algebra A, let $\mathcal{S}(A)$ and Rep$A$ denote the set of all states and the class of all nondegenerate representations of A, respectively.

Definition 1.3 *[5, 20, 21] For $\omega \in \mathcal{S}(A)$ of a C*-algebra A with the Gel'fand-Naĭmark-Segal (=GNS) triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, the map Λ_ω from A to $\mathcal{H}_\omega$ defined by*

$$\Lambda_\omega(x) \equiv \pi_\omega(x)\Omega_\omega \quad (x \in A), \tag{1.6}$$

is called the GNS map associated with ω.

Then our main theorem is stated as follows.

Theorem 1.4 *Let (A, Δ) be an algebraic C*-bialgebra. For $\pi_1, \pi_2 \in \mathrm{Rep}A$ and $\omega_1, \omega_2 \in \mathcal{S}(A)$, we write*

$$\pi_1 \star \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \Delta, \quad \omega_1 \star \omega_2 \equiv (\omega_1 \otimes \omega_2) \circ \Delta \tag{1.7}$$

where $(\omega_1 \otimes \omega_2)(x \otimes y) \equiv \omega_1(x)\omega_2(y)$ for $x, y \in A$ ([9], p.847). Assume that a subset $\mathcal{S}_0$ of $\mathcal{S}(A)$ satisfies the following conditions:

(a) *The set $\mathcal{S}_0$ is closed with respect to the operation $\star$ in (1.7) and the semigroup $(\mathcal{S}_0, \star)$ is abelian.*

(b) *For any $\omega_1, \omega_2 \in \mathcal{S}_0$, $\Omega_{\omega_1} \otimes \Omega_{\omega_2}$ is a cyclic vector for the representation $(\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2},\ \pi_{\omega_1} \star \pi_{\omega_2})$ of A.*

Define the unitary $R(\omega_1, \omega_2)$ from $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ by

$$R(\omega_1, \omega_2)\Lambda_{\omega_1,\omega_2}(\Delta(x)) \equiv \Lambda_{\omega_1,\omega_2}(\Delta^{op}(x)) \quad (x \in A) \tag{1.8}$$

where $\Lambda_{\omega_1,\omega_2} \equiv \Lambda_{\omega_1} \otimes \Lambda_{\omega_2}$. Then $R(\omega_1, \omega_2)$ is well-defined and the following holds:

(i) *For each $\omega_1, \omega_2 \in \mathcal{S}_0$,*

$$R(\omega_1, \omega_2)(\pi_{\omega_1} \otimes \pi_{\omega_2})(\Delta(x))(R(\omega_1, \omega_2))^* = (\pi_{\omega_1} \otimes \pi_{\omega_2})(\Delta^{op}(x)) \tag{1.9}$$

for $x \in A$ on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ and

$$R(\omega_1, \omega_2)\tau_{\omega_2,\omega_1}R(\omega_2, \omega_1)\tau_{\omega_1,\omega_2} = I_{\omega_1} \otimes I_{\omega_2} \tag{1.10}$$

where τ_{ω_1,ω_2} denotes the flip of $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$.

(ii) *$(\pi_{\omega_1} \otimes \pi_{\omega_2}) \circ \Delta$ and $(\pi_{\omega_2} \otimes \pi_{\omega_1}) \circ \Delta$ are unitarily equivalent.*

(iii) *For each $\omega_1, \omega_2, \omega_3 \in \mathcal{S}_0$,*

$$R_{12}(\omega_1, \omega_2)R_{13}(\omega_1, \omega_3)R_{23}(\omega_2, \omega_3) = R_{23}(\omega_2, \omega_3)R_{13}(\omega_1, \omega_3)R_{12}(\omega_1, \omega_2) \tag{1.11}$$

on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2} \otimes \mathcal{H}_{\omega_3}$ where we use the leg numbering notation in [1]. For example, $R_{12}(\omega_1, \omega_2) \equiv R(\omega_1, \omega_2) \otimes I_{\omega_3}$.

From (1.9), $R(\omega_1, \omega_2)$ is a local R-matrix of (A, Δ) on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$. The equation (1.11) is regarded as a kind of Yang-Baxter equation.

Remark 1.5 (i) If $\omega_1 = \omega_2$, then $R(\omega_1, \omega_2) = I_{\omega_1} \otimes I_{\omega_2}$. If not, then this does not hold in general. If $(\mathcal{S}_0, *)$ is not abelian, then (1.9) does not hold in general. We will show these examples in § 3.3.

(ii) We explain conditions in Theorem 1.4. The condition (b) does not always hold even if the condition (a) holds. For $\pi \in \mathrm{Rep}A$, let $[\pi]$ denote the unitary equivalence class of π. For $\pi_1, \pi_2 \in \mathrm{Rep}A$, the new product $[\pi_1] \star [\pi_2] \equiv [\pi_1 \star \pi_2]$ is well-defined. Let $\mathsf{R}(A)$ denote the set of unitary equivalence classes of all representations of A. Then we obtain the map f from $\mathcal{S}_0$ to $\mathsf{R}(A)$ by $f(\omega) \equiv [\pi_\omega]$ for $\omega \in \mathcal{S}_0$. Then the condition in Theorem 1.4 is interpreted as follows:

$$f(\omega \star \psi) = f(\omega) \star f(\psi) \quad (\omega, \psi \in \mathcal{S}_0). \tag{1.12}$$

That is, f is a homomorphism between two semigroups $(\mathcal{S}_0, \star)$ and $(\mathsf{R}(A), \star)$. Such examples will be shown in § 3.2.

(iii) The GNS map in Definition 1.3 was used in constructions of Kac-Takesaki operators [5, 20, 21] and an analogue of multiplicative isometry [15].

In § 2, we will prove Theorem 1.4. In § 3, we will treat $(\mathcal{O}_*, \Delta_\varphi)$ and certain states of $\mathcal{O}_*$ as an example of Theorem 1.4.

2 Proof of main theorem

In this section, we prove Theorem 1.4.

2.1 Lemma for GNS maps

In order to prove Theorem 1.4, we prepare an elementary lemma for GNS maps in (1.6) in this subsection.

Lemma 2.1 *Let Λ_ω be as in (1.6) and let A, B, C be C^*-algebras.*

(i) *For $\phi \in \mathrm{Hom}(A, B)$ and a state ω of B, define*

$$U\Lambda_{\omega\circ\phi}(x) \equiv \Lambda_\omega(\phi(x)) \quad (x \in A). \tag{2.1}$$

Then U is an isometry from $\mathcal{H}_{\omega\circ\phi}$ to $\mathcal{H}_\omega$ such that $U^*\pi_\omega(\phi(x))U = \pi_{\omega\circ\phi}(x)$ for $x \in A$.

(ii) *In addition to (i), if Ω_ω is a cyclic vector for $(\mathcal{H}_\omega, \pi_\omega \circ \phi)$, then U is a unitary.*

(iii) *Let $\phi \in \mathrm{Hom}(A, B \otimes C)$ and let ω_1 and ω_2 be states of B and C, respectively. Let $\omega \equiv (\omega_1 \otimes \omega_2) \circ \phi$. If $\Omega_{\omega_1} \otimes \Omega_{\omega_2}$ is a cyclic vector for $(\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}, (\pi_{\omega_1} \otimes \pi_{\omega_2}) \circ \phi)$, then*

$$U(\omega_1, \omega_2; \phi)\Lambda_\omega(x) \equiv \Lambda_{\omega_1,\omega_2}(\phi(x)) \quad (x \in A) \tag{2.2}$$

defines a unitary $U(\omega_1, \omega_2; \phi)$ from $\mathcal{H}_\omega$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ such that

$$(U(\omega_1, \omega_2; \phi))^*(\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi(x))U(\omega_1, \omega_2; \phi) = \pi_\omega(x) \quad (x \in A). \tag{2.3}$$

(iv) *In addition to the assumption in (iii), assume that $B = C$ and*

$$(\omega_1 \otimes \omega_2) \circ \phi = (\omega_2 \otimes \omega_1) \circ \phi. \tag{2.4}$$

Define the unitary $R(\omega_1, \omega_2; \phi)$ from $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ by

$$R(\omega_1, \omega_2; \phi)\Lambda_{\omega_1,\omega_2}(\phi(x)) \equiv \Lambda_{\omega_1,\omega_2}(\phi^{op}(x)) \quad (x \in A) \tag{2.5}$$

where $\phi^{op} \equiv \tau \circ \phi$ and τ denotes the flip of $B \otimes B$. Then $R(\omega_1, \omega_2; \phi)$ is well-defined and the following holds:

$$R(\omega_1, \omega_2; \phi)(\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi(x))(R(\omega_1, \omega_2; \phi))^* = (\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi^{op}(x)) \tag{2.6}$$

for $x \in A$.

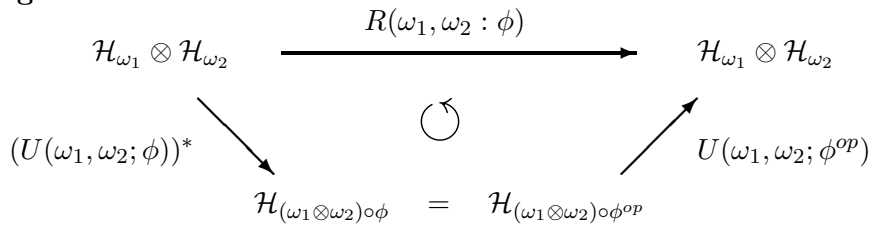
Proof. From the uniqueness of the GNS representation, (i) and (ii) hold. For example, see Proposition 4.5.3 in [8].
(iii) The statement holds from (i) and (ii).
(iv) By (2.4), we obtain $\mathcal{H}_{(\omega_1 \otimes \omega_2) \circ \phi^{op}} = \mathcal{H}_{(\omega_2 \otimes \omega_1) \circ \phi} = \mathcal{H}_{(\omega_1 \otimes \omega_2) \circ \phi}$. From this and the definition of $R(\omega_1, \omega_2; \phi)$, we see that

$$R(\omega_1, \omega_2; \phi) = U(\omega_1, \omega_2; \phi^{op})(U(\omega_1, \omega_2; \phi))^*. \tag{2.7}$$

Hence the statement holds from (iii). ∎

We illustrate Lemma 2.1(iv) in Figure 2.2.

Figure 2.2



Remark 2.3 Lemma 2.1 holds for any pair of a $*$-homomorphism ϕ and a set of states which satisfy assumptions of the cyclicity of the GNS vector and (2.4) without the assumption that ϕ is a comultiplication. Since the logic of the proof is very elementary, the assumption about states is essential.

2.2 Proof of Theorem 1.4

We prove Theorem 1.4 in this subsection. Let (A, Δ) be as in Theorem 1.4. Applying Lemma 2.1(iv) to the case $B = C = A$ and $\phi = \Delta$, we see that $R(\omega_1, \omega_2) = R(\omega_1, \omega_2; \Delta)$. Hence $R(\omega_1, \omega_2)$ is well-defined.
(i) From Lemma 2.1(iv), (1.9) holds. We can verify that (1.10) holds on $\Lambda_{\omega_1,\omega_2}(\Delta^{op}(x))$ for each $x \in A$. Hence (1.10) holds because $\{\Lambda_{\omega_1,\omega_2}(\Delta^{op}(x)) : x \in A\}$ is dense in $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$.
(ii) This follows from (i).
(iii) Define

$$F_R \equiv (id \otimes \Delta) \circ \Delta, \quad F_L \equiv (\Delta \otimes id) \circ \Delta. \tag{2.8}$$

From (1.2), $F_R = F_L$. Let A_0 be an algebraic part of (A, Δ). By assumption, we can write as follows for $x \in A_0$:

$$F_R(x) = \sum_i x_i^{'} \otimes (x_i^{''})^{'} \otimes (x_i^{''})^{''}, \tag{2.9}$$

$$F_L(x) = \sum_j (y_j^{'})^{'} \otimes (y_j^{'})^{''} \otimes y_j^{''} \tag{2.10}$$

where R.H.S.s of (2.9) and (2.10) are finite sums.

Here we write elements in $\mathcal{S}_0$ as $a, b, c, \ldots$ for simplicity of description. From (2.3) and the assumption for $\Omega_a \otimes \Omega_b$, $\pi_a \star \pi_b$ is unitarily equivalent to $\pi_{a \star b}$ for each $a, b \in \mathcal{S}_0$. From this, $(\pi_a \star \pi_b) \star \pi_c$ is unitarily equivalent to $\pi_{a\star b} \star \pi_c$. By assumption, $\Omega_{a\star b} \otimes \Omega_c$ is a cyclic vector for $(\mathcal{H}_{a\star b} \otimes \mathcal{H}_b, \pi_{a\star b} \star \pi_c)$. Hence $\Omega_a \otimes \Omega_b \otimes \Omega_c$ is a also cyclic vector for $(\pi_a \star \pi_b) \star \pi_c$. From this, $\Omega_a \otimes \Omega_b \otimes \Omega_c$ is a cyclic vector for $(\pi_a \otimes \pi_b \otimes \pi_c) \circ F_R$ $(= (\pi_a \otimes \pi_b \otimes \pi_c) \circ F_L)$ on $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$. Therefore $\{\Lambda_{a,b,c}(F_R(x)) : x \in A_0\}$

is dense in $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ where $\Lambda_{a,b,c} \equiv \Lambda_a \otimes \Lambda_b \otimes \Lambda_c$. Hence it is sufficient to show (1.11) on $\{\Lambda_{a,b,c}(F_R(x)) : x \in A_0\}$. The following holds:

$$
\begin{aligned}
& R_{12}(a,b)R_{13}(a,c)R_{23}(b,c)\Lambda_{a,b,c}(F_R(x)) \\
&= R_{13}(a,c)R_{23}(b,c)\Lambda_{a,b,c}((id \otimes \Delta^{op})(\Delta(x))) \\
&= \sum R_{12}(a,b)R_{13}(a,c)\Lambda_{a,b,c}(x_i^{'} \otimes (x_i^{''})^{''} \otimes (x_i^{'})^{'}) \\
&= \sum R_{12}(a,b)R_{13}(a,c)\Lambda_{a,b,c}((y_j^{'})^{'} \otimes y_j^{''} \otimes (y_j^{'})^{''})) \quad (\text{by } F_R(x) = F_L(x)) \\
&= \sum R_{12}(a,b)\Lambda_{a,b,c}((y_j^{'})^{''} \otimes y_j^{''} \otimes (y_j^{'})^{'})) \\
&= \sum R_{12}(a,b)\Lambda_{a,b,c}((x_i^{'})^{'} \otimes (x_i^{''})^{''} \otimes x_i^{'}) \quad (\text{by } F_R(x) = F_L(x)) \\
&= \sum \Lambda_{a,b,c}((x_i^{'})^{''} \otimes (x_i^{'})^{''} \otimes x_i^{'}) \\
&= \Lambda_{a,b,c}((\Delta^{op} \otimes id)(\Delta^{op}(x))).
\end{aligned}
$$

Remark that every "Σ" is a finite sum. As the same token,

$$R_{23}(b,c)R_{13}(a,c)R_{12}(a,b)\Lambda_{a,b,c}(F_L(x)) = \Lambda_{a,b,c}((id \otimes \Delta^{op})(\Delta^{op}(x))). \quad (2.11)$$

Hence the statement holds from the coassociativity of Δ^{op}. ∎

3 C*-bialgebra defined as the direct sum of Cuntz algebras

We show a set of states of the C*-algebra $\mathcal{O}_*$ in (1.1) which satisfies assumptions in Theorem 1.4 in this section.

3.1 C*-bialgebra $(\mathcal{O}_*, \Delta_\varphi)$

In this subsection, we recall the C*-bialgebra $(\mathcal{O}_*, \Delta_\varphi)$ in [13]. For $n \geq 2$, consider the *Cuntz algebra* $\mathcal{O}_n$ [3], that is, a C*-algebra which is universally generated by generators $s_1, \ldots, s_n$ satisfying $s_i^* s_j = \delta_{ij} I$ for $i, j = 1, \ldots, n$ and $\sum_{i=1}^n s_i s_i^* = I$ where I denotes the unit of $\mathcal{O}_n$. Since $\mathcal{O}_n$ is simple, that is, there is no non-trivial closed two-sided ideal, any $*$-homomorphism

from $\mathcal{O}_n$ to a C*-algebra is injective. If $t_1, \ldots, t_n$ are elements of a unital C*-algebra A such that $t_1, \ldots, t_n$ satisfy the relations of canonical generators of $\mathcal{O}_n$, then the correspondence $s_i \mapsto t_i$ for $i = 1, \ldots, n$ is uniquely extended to a $*$-embedding of $\mathcal{O}_n$ into A from the uniqueness of $\mathcal{O}_n$. Therefore we simply call such a correspondence among generators by an embedding of $\mathcal{O}_n$ into A.

Redefine the C*-algebra $\mathcal{O}_*$ in (1.1) as the direct sum of the set $\{\mathcal{O}_n : n \in \mathbf{N}\}$ of Cuntz algebras:

$$\mathcal{O}_* \equiv \bigoplus_{n \in \mathbf{N}} \mathcal{O}_n = \{(x_n) : \|(x_n)\| \to 0 \text{ as } n \to \infty\} \tag{3.1}$$

where $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathcal{O}_1$ denotes the 1-dimensional C*-algebra for convenience. For $n \in \mathbf{N}$, let I_n denote the unit of $\mathcal{O}_n$ and let $s_1^{(n)}, \ldots, s_n^{(n)}$ denote canonical generators of $\mathcal{O}_n$ where $s_1^{(1)} \equiv I_1$. For $n, m \in \mathbf{N}$, define $\varphi_{n,m} \in \mathrm{Hom}(\mathcal{O}_{nm}, \mathcal{O}_n \otimes \mathcal{O}_m)$ by

$$\varphi_{n,m}(s_{m(i-1)+j}^{(nm)}) \equiv s_i^{(n)} \otimes s_j^{(m)} \quad (i = 1, \ldots, n,\ j = 1, \ldots, m). \tag{3.2}$$

Theorem 3.1 *([13], Theorem 1.1) For the set $\varphi \equiv \{\varphi_{n,m} : n, m \in \mathbf{N}\}$ in (3.2), define the $*$-homomorphism Δ_φ from $\mathcal{O}_*$ to $\mathcal{O}_* \otimes \mathcal{O}_*$ by*

$$\Delta_\varphi \equiv \oplus\{\Delta_\varphi^{(n)} : n \in \mathbf{N}\}, \tag{3.3}$$

$$\Delta_\varphi^{(n)}(x) \equiv \sum_{(m,l) \in \mathbf{N}^2,\, ml = n} \varphi_{m,l}(x) \quad (x \in \mathcal{O}_n,\ n \in \mathbf{N}). \tag{3.4}$$

Then $(\mathcal{O}_*, \Delta_\varphi)$ is an algebraic C-bialgebra such that $\Delta_\varphi(\mathcal{O}_*) \subset \mathcal{O}_* \otimes \mathcal{O}_*$.*

About properties of $(\mathcal{O}_*, \Delta_\varphi)$, see [13, 15]. About a generalization of $(\mathcal{O}_*, \Delta_\varphi)$, see [14].

Let $\mathrm{Rep}\mathcal{O}_n$ denote the class of all $*$-representations of $\mathcal{O}_n$. For $\pi_1, \pi_2 \in \mathrm{Rep}\mathcal{O}_n$, we define the relation $\pi_1 \sim \pi_2$ if π_1 and π_2 are unitarily equivalent. Then the following holds.

Lemma 3.2 *([12], Lemma 1.2) For $\varphi_{n,m}$ in (3.2), $\pi_1 \in \mathrm{Rep}\mathcal{O}_n$ and $\pi_2 \in \mathrm{Rep}\mathcal{O}_m$, define $\pi_1 \otimes_\varphi \pi_2 \in \mathrm{Rep}\mathcal{O}_{nm}$ by*

$$\pi_1 \otimes_\varphi \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \varphi_{n,m}. \tag{3.5}$$

Then the following holds for $\pi_1, \pi_1' \in \mathrm{Rep}\mathcal{O}_n$, $\pi_2, \pi_2' \in \mathrm{Rep}\mathcal{O}_m$ and $\pi_3 \in \mathrm{Rep}\mathcal{O}_l$:

(i) *If* $\pi_1 \sim \pi_1^{'}$ *and* $\pi_2 \sim \pi_2^{'}$, *then* $\pi_1 \otimes_\varphi \pi_2 \sim \pi_1^{'} \otimes_\varphi \pi_2^{'}$.

(ii) $\pi_1 \otimes_\varphi (\pi_2 \oplus \pi_2^{'}) = \pi_1 \otimes_\varphi \pi_2 \oplus \pi_1 \otimes_\varphi \pi_2^{'}$.

(iii) $\pi_1 \otimes_\varphi (\pi_2 \otimes_\varphi \pi_3) = (\pi_1 \otimes_\varphi \pi_2) \otimes_\varphi \pi_3$.

From Lemma 3.2(i), we can define $[\pi_1] \otimes_\varphi [\pi_2] \equiv [\pi_1 \otimes_\varphi \pi_2]$ where $[\pi]$ denotes the unitary equivalence class of π.

Let $\mathcal{S}_n$ denote the set of all states of $\mathcal{O}_n$. For $(\omega, \omega^{'}) \in \mathcal{S}_n \times \mathcal{S}_m$, define

$$\omega \otimes_\varphi \omega^{'} \equiv (\omega \otimes \omega^{'}) \circ \varphi_{n,m} \tag{3.6}$$

where $(\omega \otimes \omega^{'})(x \otimes y) \equiv \omega(x)\omega^{'}(y)$ for $x \in \mathcal{O}_n$ and $y \in \mathcal{O}_m$. Then we see that $\omega \otimes_\varphi (\omega^{'} \otimes_\varphi \omega^{''}) = (\omega \otimes_\varphi \omega^{'}) \otimes_\varphi \omega^{''}$.

By identifying $\mathcal{O}_n$ with a C*-subalgebra of $\mathcal{O}_*$, any state and representation of $\mathcal{O}_n$ are naturally identified with those of $\mathcal{O}_*$. From this and the definition of Δ_φ, the following holds:

$$\pi_1 \otimes_\varphi \pi_2 = (\pi_1 \otimes \pi_2) \circ \Delta_\varphi, \quad \omega_1 \otimes_\varphi \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta_\varphi. \tag{3.7}$$

3.2 Pure states of Cuntz algebras parametrized by unit vectors

In this subsection, we show examples of set of states which satisfies assumptions in Theorem 1.4. We recall states in [11] and show tensor product formulae among them. Let $S(\mathbf{C}^n)$ denote the set of all unit vectors in $\mathbf{C}^n$ for $n \geq 2$.

Definition 3.3 *([11], Proposition 3.1) For* $n \geq 2$, *let* $s_1, \ldots, s_n$ *denote canonical generators of* $\mathcal{O}_n$. *For* $z = (z_1, \ldots, z_n) \in S(\mathbf{C}^n)$, *define the state* ϱ_z *of* $\mathcal{O}_n$ *by*

$$\varrho_z(s_{j_1} \cdots s_{j_a} s_{k_b}^* \cdots s_{k_1}^*) \equiv \overline{z}_{j_1} \cdots \overline{z}_{j_a} z_{k_b} \cdots z_{k_1} \tag{3.8}$$

for each $j_1, \ldots, j_a, k_1, \ldots, k_b \in \{1, \ldots, n\}$ *and* $a, b \geq 1$.

The following results for ϱ_z are known: For any z, ϱ_z is pure when $n \geq 2$. Define the state ϱ_1 of $\mathcal{O}_1 = \mathbf{C}$ by $\varrho_1(x) = x$ for each $x \in \mathcal{O}_1$. We define $S(\mathbf{C}^1) \equiv \{1\} \subset \mathbf{C}$ for convenience. Then ϱ_z in (3.8) makes sense for each $z \in \bigcup_{n \geq 1} S(\mathbf{C}^n)$. If $z, y \in S(\mathbf{C}^n)$ and $z \neq y$, then GNS representations associated with ϱ_z and ϱ_y are not unitarily equivalent.

Let $GP(z)$ denote the unitary equivalence class of the GNS representation associated with ϱ_z. If π_1 and π_2 are representatives of $GP(z)$ and $GP(y)$ for $z \in S(\mathbf{C}^n)$ and $y \in S(\mathbf{C}^m)$, respectively, then we write $GP(z) \otimes_\varphi GP(y)$ as $[\pi_1] \otimes_\varphi [\pi_2]$ for simplicity of description.

Theorem 3.4 *([15], Theorem 3.2) For $\otimes_\varphi$ in (3.5), the following holds for each $z \in S(\mathbf{C}^n)$ and $y \in S(\mathbf{C}^m)$:*

(i) $\varrho_z \otimes_\varphi \varrho_y = \varrho_{z \boxtimes y}$,

(ii) $GP(z) \otimes_\varphi GP(y) = GP(z \boxtimes y)$

where $z \boxtimes y \in S(\mathbf{C}^{nm})$ is defined as

$$(z \boxtimes y)_{m(i-1)+j} \equiv z_i y_j \quad (i = 1, \dots, n,\ j = 1, \dots, m). \tag{3.9}$$

For $z \in S(\mathbf{C}^n)$, let $(\mathcal{H}_z, \pi_z, \Omega_z)$ denote the GNS triple associated with ϱ_z. From Theorem 3.4(ii), $(\mathcal{H}_z \otimes \mathcal{H}_y, \pi_z \otimes_\varphi \pi_y)$ is irreducible for each $z, y \in \bigcup_{n \geq 1} S(\mathbf{C}^n)$. From this, $\Omega_z \otimes \Omega_y$ is a cyclic vector for $(\mathcal{H}_z \otimes \mathcal{H}_y, \pi_z \otimes_\varphi \pi_y)$. In consequence,

$$\mathcal{S}_0 \equiv \bigcup_{n \geq 1} \{\varrho_z : z \in S(\mathbf{C}^n)\} \tag{3.10}$$

satisfies all assumptions in Theorem 1.4 with respect to $(\mathcal{O}_*, \Delta_\varphi)$. Furthermore, if $\mathcal{M}$ is a subsemigroup of the semigroup $(\mathcal{S}_0, \boxtimes)$, then $\{\varrho_z : z \in \mathcal{M}\}$ also satisfies all assumptions in Theorem 1.4. Such subsemigroups are shown in the last paragraph of § 3 in [15].

3.3 Examples

In this subsection, we show two examples of Theorem 1.4.

(i) We show a nontrivial example of Theorem 1.4. Let $\{s_i^{(n)}\}_{i=1}^n$ denote the canonical generators of $\mathcal{O}_n$ and let $l_2(\mathbf{N})$ denote the Hilbert space with the orthonormal basis $\{e_k : k \in \mathbf{N}\}$ where $\mathbf{N} = \{1, 2, 3, \dots\}$. Define the representation π_n of $\mathcal{O}_n$ on $l_2(\mathbf{N})$ by

$$\pi_n(s_i^{(n)}) e_k \equiv e_{n(k-1)+i} \quad (i = 1, \dots, n,\ k \in \mathbf{N}). \tag{3.11}$$

Let ω_n denote the vector state of $\mathcal{O}_n$ associated with the unit vector e_1. Then $\omega_n \star \omega_m = \omega_{nm}$ for each $n, m \geq 2$. We identify π_n with the GNS representation of $\mathcal{O}_n$ by ω_n. Here we consider cases $n = 2, 3$. Let $R \equiv R(\omega_2, \omega_3)$ be as in (1.8). Remark that

$$e_1 \otimes e_3 = (\pi_2 \otimes \pi_3)(\Delta_\varphi(s_3^{(6)}))(e_1 \otimes e_1) = (\pi_2 \otimes \pi_3)(\varphi_{2,3}(s_3^{(6)}))(e_1 \otimes e_1). \tag{3.12}$$

Then

$$\begin{aligned} R(e_1 \otimes e_3) = &\ (\pi_2 \otimes \pi_3)(\varphi_{3,2}^{op}(s_3^{(6)}))(e_1 \otimes e_1) \\ = &\ \pi_2(s_1^{(2)})e_1 \otimes \pi_3(s_2^{(3)})e_1 \\ = &\ e_1 \otimes e_2. \end{aligned}$$

In this way, R is not the identity operator on $l_2(\mathbf{N}) \otimes l_2(\mathbf{N})$.

(ii) We show an example which does not satisfy (1.9) because of the non-commutativity of states. Let ω_2 and π_2 be as in the previous example. Define $\overline{\omega}_2 \equiv \omega_2 \circ \alpha$ where α denotes the flip automorphism of the canonical generators of $\mathcal{O}_2$ and let $\overline{\pi}_2$ denote the GNS representation of $\mathcal{O}_2$ by $\overline{\omega}_2$ which is identified with $\pi_2 \circ \alpha$. Then $\omega_2 \star \overline{\omega}_2 \neq \overline{\omega}_2 \star \omega_2$. Let $v \equiv e_1 \otimes e_1$. Then $(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi(s_2^{(4)}))v = v$. Define the operator R on $l_2(\mathbf{N}) \otimes l_2(\mathbf{N})$ by

$$R(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi(x))v \equiv (\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi^{op}(x))v \quad (x \in \mathcal{O}_4). \tag{3.13}$$

Then

$$R(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi(s_2^{(4)}))R^*v = R(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi(s_2^{(4)}))v = Rv = v. \tag{3.14}$$

Since $(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi^{op}(s_2^{(4)})) = \pi_2(s_2^{(2)}) \otimes \overline{\pi}_2(s_1^{(2)})$ and $\langle v|(\pi_2(s_2^{(2)}) \otimes \overline{\pi}_2(s_1^{(2)}))v\rangle = 0$, we see that $v \neq (\pi_2(s_2^{(2)}) \otimes \overline{\pi}_2(s_1^{(2)}))v$. Hence $R(\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi(s_2^{(4)}))R^* \neq (\pi_2 \otimes \overline{\pi}_2)(\Delta_\varphi^{op}(s_2^{(4)}))$.

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