arXiv:0810.4074v2 [math.GR] 5 Feb 2009

ON FINITE THURSTON-TYPE ORDERINGS OF BRAID GROUPS

TETSUYA ITO

Abstract. We give an combinatorial description of a finite Thurston type ordering $<_T$ of the positive braid monoid B_n^+ by introducing the $\mathcal{C}$-normal form. The $\mathcal{C}$-normal form can be seen as a generalization of Burckel's normal form and Dehornoy's Φ-normal form. Using this description, we determine the order type of well-ordered set $(B_n, <_T)$ for all finite Thurston type orderings $<_T$.

1. Introduction

The braid group B_n is a group defined by the presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle$$

The monoid generated by the positive generators $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ is called *the positive braid monoid* and denoted by B_n^+. An element in B_n^+ is called a *positive braid*. The braid group B_n, first introduced by Artin, appears many fields of mathematics and have been studied from many points of view.

In 1990's the new feature of braid groups, the left-invariant total orderings called *the Dehornoy ordering* was discovered by Dehornoy [D1]. After the discovery of the orderings, many studies have been done from both an algebraic and geometrical prospective and the subject of braid orderings and related topics have been developed rapidly. Many other equivalent definitions of the Dehornoy ordering are found (see [DDRW1],[DDRW2] to study various aspects of Dehornoy's ordering) and consequently it is found that the Dehornoy orderings is a quite natural structure of the braid groups.

In [SW], infinite families of left-invariant total orderings of the braid groups, called *Thurston type orderings* are constructed via the hyperbolic geometry. This family of orderings contains the Dehornoy ordering as a special one, hence it is an extension of the Dehornoy ordering.

2000 *Mathematics Subject Classification.* Primary 20F36, Secondary 20F60.

Key words and phrases. Thurston-type ordering, $\mathcal{C}$-normal form, Braid groups, order-type.

It is known that Thurston type orderings have a property called Property S, and consequently they define the well-ordering when they are restricted to the positive braid monoid B_n^+.

One of the problem of braid orderings is to determine the order type of the well-ordered set $(B_n^+, <)$ for a Thurston type ordering $<$ and to construct a method to compute ordinals for each element in B_n^+. For the Dehornoy ordering case, Burckel solved this problem by using a correspondence between positive braid words and rooted labeled trees ([B1],[B2]). He introduced a normal form of a positive braid called *Burckel's normal form* which makes it possible to compare the Dehornoy ordering and compute ordinals. In [D3], Dehornoy gives another description of Burckel's normal form by introducing the Φ-normal forms of positive braids. He utilize the alternate decomposition, which use the Garside structure of braid groups. He shows for the positive braid monoid, the Φ-normal form coincide with Burckel's normal form. However, the connection between the Φ-normal forms and the Dehornoy ordering are indirect. It depends on Burckel's result, so there still lies some gap to understand Dehornoy ordering on positive braid monoid directly.

In this paper, we give a combinatorial description for special kinds of Thurston type orderings, called *finite type orderings*. The Dehornoy ordering is a typical example of finite type orderings so our results are extensions of results of Dehornoy and Burckel.

We introduce a notion of the code of positive braid words and a new normal form of positive braids, named the $\mathcal{C}$*-normal form*. The set of codes has the two lexicographical orderings $<_{left}$ and $<_{right}$. The $\mathcal{C}$-normal form of a positive braid β is defined as a positive word representative of β whose code is maximal with respect to the ordering $<_{right}$. The code of the $\mathcal{C}$-normal form of positive braid β is denoted by $\mathcal{C}(\beta; <)$.

Using these notions, first we give a new description of finite Thurston type orderings.

Theorem 1. *Let $<$ be a finite Thurston type ordering on B_n. Then for $\alpha, \beta \in B_n^+$, $\alpha < \beta$ holds if and only if $\mathcal{C}(\alpha; <) <_{left} \mathcal{C}(\beta; <)$ holds.*

This $\mathcal{C}$-normal form description of finite Thurston type orderings makes it possible to compare braids by the purely algebraic and combinatorial ways. Using this description, next we determines the order types of well-ordered set $(B_n^+, <)$ for all finite Thurston type orderings $<$. This gives an answer of the question listed in [DDRW1] and [DDRW2].

Theorem 2 (The order types of finite Thurston type orderings)**.** *Let $<$ be a finite Thurston type ordering on the braid group B_n. Then the order type of $(B_n^+, <)$ is $\omega^{\omega^{n-2}}$.*

Our $\mathcal{C}$-normal form can be seen as an extension of Burckel's normal form and Dehornoy's Φ-normal form. We define the tail-twisted Φ-normal form of positive braids. This is an extension of Φ-normal form defined by a purely algebraic way and we show the tail-twisted Φ-normal forms coincide with the $\mathcal{C}$-normal forms. Using this algebraic formulation of $\mathcal{C}$-normal form, we finally give a computational complexity of $\mathcal{C}$-normal forms and finite Thurston type orderings.

Theorem 3. *Let $<$ be an arbitrary finite Thurston type ordering of the braid group B_n.*

1. *The $\mathcal{C}$-Normal form $W(\beta; <)$ of a positive braid $\beta \in B_n^+$ is computed by the time $O(l^2 n \log n)$, where l is the word length of β.*
2. *For a braid (not necessarily positive) $\beta \in B_n$, whether $\beta > 1$ holds or not can be decided by the time $O(l^2 n^3 \log n)$ where l is the word length of β.*

Thus, our $\mathcal{C}$-normal form argument provides an efficient algorithm to compare finite Thurston type orderings.

The plan of the paper is following. In section 2, we summarize the definitions and some basic facts about Thurston-type orderings of the braid groups which will be used later. In section 3 we introduce a notion of the code and define the $\mathcal{C}$-normal form, and prove theorem 1 and theorem 2. Section 4 is devoted to the study of the relationships between our $\mathcal{C}$-normal form and Dehornoy's Φ-normal form. We define the tail-twisted Φ-normal form and show that it coincide with the $\mathcal{C}$-normal form, and prove theorem 3.

Unfortunately, our method to obtain a combinatorial descriptions requires the condition "finite type". However, intuitively speaking, an infinite type ordering is a "limit" of finite type orderings, the following conjecture seems to be plausible.

Conjecture 1. For *all* Thurston-type orderings $<$ on the braid group B_n, the order type of $(B_n^+, <)$ is $\omega^{\omega^{n-2}}$.

The main point is that on the positive braid monoids, we can approximate the restriction of an infinite Thurston type ordering by a sequence of finite Thurston type orderings. That is, for $l > 0$ and an arbitrary infinite Thurston type ordering $<_\infty$, we can choose a finite type ordering $<_l$ which completely coincides with the set of positive

braids with the length less than l. In this sense, an infinite type ordering is indeed a limit of finite type orderings: we can say $\lim_{l\to\infty} <_l = <_\infty$. The author does not know this implies the order type of $(B_n^+, <_\infty)$ is $\omega^{\omega^{n-2}}$.

Finally we mention another aspect of our code constructions. The braid group B_n has another well-known monoid B_n^{+*} called *the dual braid monoid*, which has similar properties of the positive braid monoids B_n^+. Indeed, the dual braid monoid also defines the Garside group structure on the braid group B_n, and two monoids B_n^+ and B_n^{+*} are closely related ([BKL],[Be]). Fromentin studies the Dehornoy orderings via the dual braid monoids using the rotating normal forms, which corresponds to Dehornoy's Φ-normal form ([F1],[F2],[F3]). Our code constructions and the $\mathcal{C}$-normal forms equally work for the dual braid monoids with an appropriate modification. We will describe this in the subsequent paper [I3].

Acknowledgements. The author would like to express his gratitude to Professor Toshitake Kohno for his encouragement and helpful suggestions. He also thanks to Bert Wiest for pointing out many notational errors and mistakes of early version of the paper.

2. Thurston type ordering

2.1. **Construction via hyperbolic geometry.** First we briefly review an construction of Thurston type orderings. For details, see [SW].

Let D_n be the n-punctured disc. The braid group B_n is naturally identified with the relative mapping class group $MCG(D_n, \partial D_n)$, which is a group of the isotopy classes of homeomorphisms of D_n whose restrictions on ∂D_n are the identity maps. Under this identification, the generator σ_i corresponds to the isotopy class of the positive half Dehn-twist along the straight line connecting the i-th and the $(i+1)$-th puncture points.

Choose an complete hyperbolic structure on D_n which makes ∂D_n a geodesic, and a base point $* \in \partial D_n$ (there are many of choices of a hyperbolic metric and a base point, but constructed families of orderings are independent of these choices). Then the universal covering $\pi : \widetilde{D_n} \to D_n$ can be isometrically embedded in the hyperbolic plane $\mathbb{H}^2$. By attaching the points of infinity S^1_∞ to $\widetilde{D_n}$, we obtain a topological disc (by abuse of notation, we still denote this disc by the same notation $\widetilde{D_n}$). Take a lift of the base point $\widetilde{*}$ and let C be the component of $\pi^{-1}(\partial D_n)$ which contains $\widetilde{*}$.

For a point $x \in \partial\widetilde{D_n}\backslash C$, we define the relation $<_x$ on the braid group $B_n = MCG(D_n, \partial D_n)$ as follows. For two elements of $[f], [g] \in MCG(D_n, \partial D_n) = B_n$, choose their representative homeomorphisms $f, g : D_n \to D_n$ and take their lifts $\widetilde{f}, \widetilde{g} : \widetilde{D_n} \to \widetilde{D_n}$ so that they fix $\widetilde{*}$. We say $[f] <_x [g]$ holds if and only if $\widetilde{f}(x) < \widetilde{g}(x)$ holds under an orientation-preserving identification of $\partial\widetilde{D_n}\backslash C$ with the real line $\mathbb{R}$. Although there are many choices of representative homeomorphisms f and g, the restrictions of their lifts $\widetilde{f}$, $\widetilde{g}$ on $\partial\widetilde{D_n}\backslash C$ are independent of these choices, so $<_x$ is well-defined. The relation $<_x$ defines a left-invariant partial ordering on B_n. It is known that if we choose a point x in a nice way, $<_x$ becomes a total ordering. We call left-invariant total orderings constructed the above way *Thurston type orderings*. As we will see in the next subsection, Thurston type orderings contain the Dehornoy ordering so it is an extension of the Dehornoy ordering. The above construction of orderings can be generalized to other relative mapping class groups of hyperbolic surfaces with non-empty boundaries.

It is known that Thurston type orderings fall into two types: *finite type* and *infinite type*. In next subsection, we formulate a finite type ordering as an ordering which can be described by curve diagrams.

2.2. **Construction of finite Thurston type ordering via curve diagram.** In this subsection we give an alternative description of special kind of Thurston type orderings using a curve diagram, which gives more accessible meanings and closely related to our combinatorial description.

Let Γ be a diagram in D_n, which consists of an union of oriented, embedded arcs $\Gamma_1, \Gamma_2, \cdots, \Gamma_{n-1}$ satisfying the following properties.

- Interiors of Γ_i are disjoint each other.
- The initial point of Γ_1 lies at ∂D_n.
- The initial point of Γ_i lies at some point of $(\bigcup_{j=1}^{i-1} \Gamma_j) \cup (\partial D_n)$ and the end point of Γ_i lies at some puncture points or $(\bigcup_{j=1}^{i-1} \Gamma_j) \cup (\partial D_n) \cup$ (Interior of γ_i).
- Each component of $D_n \backslash \Gamma$ is a disc or one-punctured disc.

We say such a diagram *curve diagram*. We identify two isotopic curve diagrams as the same curve diagrams.

For two curve diagrams Γ, Γ', we say Γ and Γ' are tight if they forms no bigon. A bigon is a disc whose boundary consists of two subarcs $\gamma \subset \Gamma_i$, $\gamma' \subset \Gamma'_j$, as depicted in figure 1. We remark that if we equip a complete hyperbolic metric on D_n and realize both Γ and Γ' as geodesics, then Γ and Γ' become tight. Thus, we can always isotope

given two curve diagrams so that they are tight. For a curve diagram Γ on D_n and $\alpha, \beta \in B_n$, we say $\alpha <_\Gamma \beta$ holds if and only if when we put $\alpha(\Gamma)$ and $\beta(\Gamma)$ tight, then $\beta(\Gamma)$ moves more left than $\alpha(\Gamma)$.

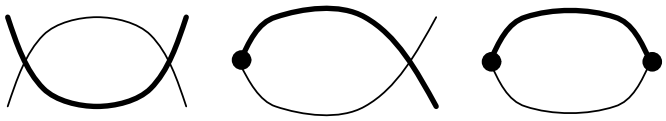


FIGURE 1. Bigons

The relation $<_\Gamma$ defines a left-invariant total ordering on B_n. We say a Thurston-type ordering $<$ is *finite type* if it coincides with the ordering $<_\Gamma$ for some curve diagram Γ. Although this definition of finite type ordering is different from that in [SW], [DDRW1],[DDRW2], but this definition agrees with usual definition. Conversely, orderings constructed via curve diagram are always Thurston-type orderings.

We remark that the notion of curve diagram and its defining ordering are also generalized to other mapping class groups for arbitrary surface with non-empty boundaries. See [RW] for such a construction of left-invariant total orderings on mapping class groups.

We say a curve diagram Γ is *normal* if all of arcs $\{\Gamma_i\}$ are vertical (that means, we can write Γ_i by the equation $x = const$ by using $x - y$ coordinate, regarding D_n as the unit disc removed puncture points lying in the x-axis) and they are orented upwards.

For a normal curve diagram Γ, we define the integer $k(i)$ so that Γ_i lies between the $k(i)$-th and the $(k(i)+1)$-th punctures. Then the normal curve diagrams Γ are represented by the permutation of $n-1$ integers $\mathbf{k} = \{k(1), k(2), \cdots, k(n-1)\}$. A finite Thurston type ordering $<$ is called *normal* if $<=<_\Gamma$ for some normal curve diagram Γ.

In our paper, we mainly consider a normal finite Thurston-type ordering. This is justified by the following facts.

Two left invariant total orderings $<, <'$ on a group G are called *conjugate* if there exists an element $g \in G$ such that $f < h$ is equivalent to $fg <' hg$ for all $f, h \in G$. We call an element g *conjugating element* between $<$ and $<'$. In [SW], it is shown that the number of conjugacy classes of finite Thurston type orderings on B_n is finite and each conjugacy class contains at least one normal finite type ordering. As this definition suggest, the description of conjugate ordering is easily obtained from original ordering. For the braid group B_n and the positive

braid monoid B_n^+, there are one more good property which makes more easier to handle conjugate orderings.

Lemma 1. *Let $<,<'$ be two left-invariant total orderings on the braid group B_n. If $<$ and $<'$ are conjugate, then we can choose a conjugating element α as a positive braid.*

Proof. Let α' be a conjugating element between $<$ and $<'$. Let $\Delta = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\sigma_2\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)$ be the Garside fundamental braid. Since Δ^2 is central in B_n and $\Delta^{2p}\alpha'$ is a positive braid for sufficiently large $p > 0$ (See [Bi], for example), so $\Delta^{2p}\alpha$ is a positive conjugating element. □

We give some examples of normal finite Thurston type orderings.

Example 1 (The Dehornoy ordering and the reverse of the Dehornoy ordering)**.** Let Γ_D be a curve diagrams as shown in the left figure of figure 2, which corresponds to the trivial permutation $\{1, 2, \cdots, n-1\}$. It is known that the ordering $<_{\Gamma_D}$ defined by curve diagram Γ coincide with the well-known Dehornoy ordering $<_D$.

Algebraically, the Dehornoy ordering $<_D$ on B_n is defined as follows. For $\alpha, \beta \in B_n$, $\alpha <_D \beta$ holds if and only if $\alpha^{-1}\beta$ admit a word representative which contains no $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \cdots \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$ but contains at least one σ_i for some i.

Next we consider another curve diagram Γ' in the middle figure of figure 2, which corresponds to the permutation $\{n-1, n-2, \cdots, 1\}$. The ordering $<_{\Gamma_{D'}}$ is called the reverse of the Dehornoy ordering. Algebraically, $<_{D'}$ is defined by $\alpha <_D \beta$ if and only if $\alpha^{-1}\beta$ admit a word representative which contains no $\sigma_{n-1}^{\pm 1}, \sigma_{n-2}^{\pm 1}, \cdots \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}$ but contains at least one σ_i for some i.

We remark that in some papers (for example, [D2]) the term "the Dehornoy ordering" is used to represent the reverse of Dehornoy ordering. This is not a serious difference because these two orderings are conjugate by the Garside fundamental braid Δ. That is, $\alpha <_{D'} \beta$ is equivalent to $\alpha\Delta <_D \beta\Delta$.

Example 2 (The Most simple example of finite Thurston type ordering which is not conjugate to the Dehornoy ordering)**.** Let Γ be the curve diagram of D_4, as shown in the right figure of figure 2, corresponding to the permutation $(2, 1, 3)$. It is known that this ordering $<_\Gamma$ is not conjugate to the Dehornoy ordering $<_D$. (See [DDRW1]).

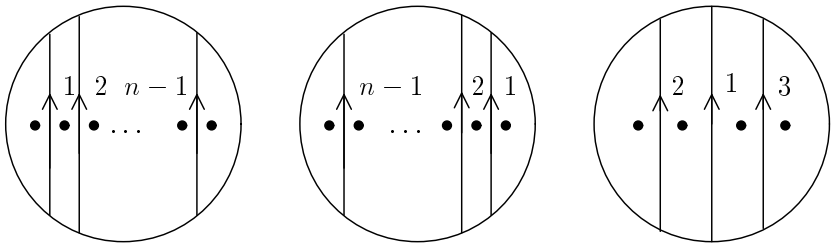


FIGURE 2. Examples of curve diagrams

2.3. **Property S of Thurston type orderings.** Thurston type orderings are generalization of the Dehornoy ordering, and most of important properties of the Dehornoy orderings remains holds. The most important property we use in the paper is property S.

Propotition 1 (Property S)**.** *Let $<$ be a Thurston type ordering on B_n. Then*

$$\alpha\sigma_i\beta > \alpha\beta > \alpha\sigma_i^{-1}\beta$$

holds for all $\alpha, \beta \in B_n$ and $1 \leq i \leq n-1$.

Roughly speaking, property S means we can freely insert positive generators σ_i at arbitrary points without decreasing the ordering. We remark that the property S is confirmed from the geometric definition of Thurston type orderings. This property will be heavily used to prove our description of orderings.

Using property S, we can prove the following corollary by the same argument of the Dehornoy ordering case. (see [DDRW1] Chapter 1, for example.)

Corollary 1 (Well-ordering)**.** *For every Thurston type ordering $<$ on the braid group B_n, $(B_n^+, <)$ is a well-ordered set.*

3. Code and the $\mathcal{C}$-normal form

In this section, we give a precise definitions of codes and the $\mathcal{C}$-normal forms. First we consider a normal finite Thurston-type ordering and after we extend results about normal ordering to general one.

Let $<$ be a normal finite Thurston type ordering on B_n. We denote its definitive curve diagram by Γ and the corresponding permutation by $\mathbf{k} = (k(1), k(2), \cdots, k(n-1))$. Throughout in this section we denote $k = k(1)$ and we use the following notations. Let $D : B_n^+ \to B_n^+$ be a homomorphism defined by the conjugate of the Garside fundamental braid Δ. That is, the map D is defined by $D(\alpha) = \Delta^{-1}\alpha\Delta$ or equivalently, $D(\sigma_i) = \sigma_{n-i}$. Similarly, we define the shift map

$Sh : B_n \to B_{n-1}$ by $Sh(\sigma_i) = \sigma_{i-1}$ for $i > 1$ and $Sh(\sigma_1) = 1$. For positive braids $\alpha_1, \alpha_2, \cdots, \alpha_j \in B_n^+$, we denote the submonoid of B_n^+ generated by $\{\alpha_1, \alpha_2, \cdots, \alpha_j\}$ by $\langle \alpha_1, \alpha_2, \cdots, \alpha_j \rangle$.

3.1. **Definitions of code and the $\mathcal{C}$-normal form.** In this subsection, we define the *code* of a positive n-braid word W with respect to a finite Thurston type ordering $<$, denoted by $\mathcal{C}(W; <)$ (or simply $\mathcal{C}(W)$) and the $\mathcal{C}$-normal form. The code plays a central role in our description of finite Thurston type orderings.

First we define the code of a positive braid word (with additional structure, called a subword decomposition) with respect to a normal finite Thurston type ordering $<$ by the following inductive way.

Our starting point is the positive 2-braid monoids B_2^+. There are only one normal finite Thurston type ordering $<_D$ on B_2^+, namely, the Dehornoy ordering on B_2. Of course this ordering $<_D$ is defined by $\sigma_1^a <_D \sigma_1^b$ if $a < b$. For a positive 2-braid word $W = \sigma_1^a$, we simply define the code of W with respect to the ordering $<$ by the positive integer (a). That is, we define $\mathcal{C}(\sigma_1^a; <_D) = a$.

Assume that we have already defined the codes of a positive i-braid word with respect to arbitrary normal finite Thurston type orderings of B_i for all $i < n$. Now we define the code of n-braid word W with respect to the normal finite Thurston type ordering $<$ on B_n.

For a positive n-braid word W, first we decompose the word W into the products of subwords as

$$W = A_m A_{m-1} \cdots A_0 A_{-1}$$

where

$$\begin{cases} A_{-1} \in \langle \sigma_1 \sigma_2, \cdots \sigma_{k-1}, \sigma_{k+1}, \cdots, \sigma_{n-1} \rangle \\ A_i \in \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-2} \rangle \ i \geq 0 : odd \\ A_i \in \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle \ i \geq 0 : even \end{cases} .$$

Given such a decomposition of the word W, the code of W is defined as a sequence of codes with fewer strands

$$\begin{aligned} \mathcal{C}(W; <) \ = \ & (\cdots, \mathcal{C}(D^{m-1}(A_m); <_D), \mathcal{C}(D^{m-2}(A_{m-1}); <_D), \\ & \cdots, \mathcal{C}((D^0(A_1); <_D), \mathcal{C}(A_0; <_{res}), \mathcal{C}(A_{-1}; <)) \end{aligned}$$

where $<_D$ denotes the Dehornoy ordering of B_{n-1}. We remark that since $D^{i-1}(A_i) \in B_{n-1}^*$, the term $\mathcal{C}(D^{i-1}(A_i); <_D)$ is well-defined from our inductive hypothesis. The codes $\mathcal{C}(A_0; <_{res})$ and $\mathcal{C}(A_{-1}; <)$ are defined by the following manner.

Let $<_{res}$ be the restriction of the ordering $<$ on the submonoid $\langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle \subset B_n^+$. Under the identification of the shift map $Sh : \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle \to B_{n-1}^+$, the ordering $<_{res}$ also can be seen as

a normal finite Thurston type ordering on B^+_{n-1}. We remark that the identification using the shift map Sh is different from the identification using the map D. We define $\mathcal{C}(A_0; <_{res})$ as a code of A_0 with respect to the ordering $<_{res}$ under this identification Sh.

Next we define the last term $\mathcal{C}(A_{-1}; <)$. For $1 \leq j \leq n-2$, let

$$\begin{cases} m_j = \max\{1, k(i)+1 \mid i = 1, 2, \cdots, j,\ k(i) < k(j+1)\} \\ M_j = \min\{n, k(i) \mid i = 1, 2, \cdots, j,\ k(i) > k(j+1)\}. \end{cases}$$

Now decompose the word A_{-1} as the following form.

$$A_{-1} = X_1 X_2 \cdots X_{n-2}$$

where $X_j \in \langle \sigma_{m_j}, \sigma_{m_j+1}, \cdots, \sigma_{M_j-1} \rangle$. We remark that for general word A_{-1}, such a form of decomposition might be impossible. We choose the first decomposition $W = A_m \cdots A_{-1}$ so that such a decomposition is possible.

Let $<_j$ be the restriction of the ordering $<$ to $\langle \sigma_{m_j}, \sigma_{m_j+1}, \cdots, \sigma_{M_j-1} \rangle$. By identifying $\langle \sigma_{m_j}, \sigma_{m_j+1}, \cdots, \sigma_{M_j-1} \rangle$ with $B^+_{M_j-m_j}$ by the m_j-fold shift map Sh^{m_j}, we can consider $<_i$ as a normal finite Thurston type ordering on $B^+_{M_j-m_j}$. Let $\mathcal{C}(X_j; <_j)$ be the code of X_j with respect to the ordering $<_j$. From our inductive hypothesis, we have already defined these codes. Then, the code of A_{-1} is defined as a sequence of codes

$$\mathcal{C}(A_{-1}; <) = (\mathcal{C}(X_1; <_1), \mathcal{C}(X_2; <_2), \cdots, \mathcal{C}(X_{n-2}; <_{n-2})).$$

This completes the definition of codes.

We remark that to define the code, it is not sufficient to choose a positive braid word: we need to indicate the way to decompose W into subwords. To indicate a subword decomposition of W, we sometimes use the symbol $|$. For example, To express the decomposition of $W = \sigma_1^{p+q}$ with $A_{-1} = \sigma_1^q$ and $A_0 = \sigma_1^p$, we use a notation $W = \sigma_1^p | \sigma_1^q$.

Let $Code(n, <)$ be the set of all codes of positive n-braid word with respect to $<$. Now we define two natural lexicographical orderings $<_{left}$ and $<_{right}$ on the set $Code(n, <)$ by inductive way. To this end, we regard the code of a positive n-braid word as a left-infinite sequences of the code of subwords $(\cdots, a_i, a_{i-1}, \cdots, a_1)$, with only finite a_i is non-trivial. This means, for example, we regard a code (2) of a 2-braid word σ_1^2 as a left infinite sequence of integers $(\cdots, 0, 0, 2)$.

Definition 1 (Lexicographical ordering $<_{left}$ and $<_{right}$)**.** Let $<$ be a normal finite Thurston type ordering. The *Lexicographical ordering from left* $<_{left}$ and the *Lexicographical ordering from right* $<_{right}$ are total orderings on $Code(n, <)$ define by the following inductive way.

For two codes (a) and (b) in $Code(2,<)$, we define $(a) <_{left} (b)$ if and only if $a < b$, and $(a) <_{right} (b)$ if and only if $a < b$.

Assume that we have already defined $<_{left}$ and $<_{right}$ for any normal finite Thurston type ordering $<$ on B_i for all $i < n$.

Let $\mathcal{C} = (\cdots, \mathcal{C}(A_m), \cdots, \mathcal{C}(A_{-1}))$ and $\mathcal{C}' = (\cdots, \mathcal{C}(A'_m), \cdots, \mathcal{C}(A'_{-1}))$ be the code of positive n-braid word (with decomposition) W and W' with respect to the normal finite Thurston type ordering $<$. From definition of $\mathcal{C}(A_{-1})$ (and $\mathcal{C}(A'_{-1})$), these terms are written by

$$\begin{cases} \mathcal{C}(A_{-1};<) = (\mathcal{C}(X_1), \mathcal{C}(X_2), \cdots, \mathcal{C}(X_{n-2})) \\ \mathcal{C}(A'_{-1};<) = (\mathcal{C}(X'_1), \mathcal{C}(X'_2), \cdots, \mathcal{C}(X'_{n-2})) \end{cases}$$

respectively. From our inductive hypothesis, the ordering $<_{left}$ and $<_{right}$ are already defined on such codes. We define $\mathcal{C} <_{left} \mathcal{C}'$ if and only if

$$\begin{cases} \mathcal{C}(A_i) = \mathcal{C}(A'_i) \text{ for } i > l > -1 \text{ and } \mathcal{C}(A_l) <_{left} \mathcal{C}(A'_l) \\ or \\ \mathcal{C}(A_i) = \mathcal{C}(A'_i) \text{ for } i > -1,\ \mathcal{C}(X_j) = \mathcal{C}(X'_j) \text{ for } j < l \\ \text{ and } \mathcal{C}(X_l) <_{left} \mathcal{C}(X'_l) \text{ for some } l < n-2 \end{cases}$$

Similarly, we define $\mathcal{C} <_{right} \mathcal{C}'$ if and only if

$$\begin{cases} \mathcal{C}(X_j) = \mathcal{C}(X'_j) \text{ for } j > l \text{ and } \mathcal{C}(X_l) <_{right} \mathcal{C}(X'_l) \text{ for some } l < n-2 \\ or \\ \mathcal{C}(A_i) = \mathcal{C}(A'_i \text{ for } i < l \text{ and } \mathcal{C}(A_l) <_{right} \mathcal{C}_{A'_l} \text{ for some } l > -1 \end{cases}$$

We are now ready to define our $\mathcal{C}$-*normal form* of a positive braid $\beta \in B_n^+$ for normal finite Thurston type orderings.

Let $<$ be a normal finite Thurston type ordering on B_n. For a positive braid $\beta \in B_n^+$, the $\mathcal{C}$-*normal form* of β with respect to the ordering $<$ is defined to be the unique positive word representative of β whose code is maximal among the set of codes of positive word representative of β with respect to the ordering $<_{right}$. Since the number of positive braid word representative of β is finite, $\mathcal{C}$-normal forms of positive braids are always uniquely determined. We denote the $\mathcal{C}$-normal form of a positive n-braid β with respect to a normal finite Thurston type ordering $<$ by $W(\beta;<)$, or simply $W(\beta)$, and its code by $\mathcal{C}(\beta;<)$ or $\mathcal{C}(\beta;<)$.

Recall that from lemma 1, for two conjugate left invariant total orderings $<, <'$ of B_n, we can always choose a positive conjugating element α. In this mind, finally we can define the $\mathcal{C}$-normal form for arbitrary finite Thurston type ordering.

Definition 2 (The $\mathcal{C}$-normal form of positive braids)**.** Let $<$ be a finite Thurston type ordering on B_n which is conjugate to a normal finite Thurston type ordering $<_N$. Let P be a positive word representative

of a positive conjugating element between $<$ and $<_N$. Fix such $<_N$ and P.

For a positive n-braid β, the $\mathcal{C}$-normal form of β with respect to the ordering $<$ is a braid word $W(\beta;<) = W(\beta \cdot P;<_N) \cdot P^{-1}$. The code of β, denoted by $\mathcal{C}(\beta;<)$, is defined by $\mathcal{C}(\beta;<) = \mathcal{C}(\beta \cdot P;<_N)$.

From definition, the set of all codes $Code(n,<)$ with respect to the ordering $<$ is a subset of $Code(n,<_D)$ The lexicographical ordering $<_{left}$ and $<_{right}$ on the set $Code(n,<)$ are defined to be merely the restriction of the ordering $<_{left}$ and $<_{right}$ on the set $Code(n,<_N)$. Notice that for a general finite Thurston type ordering, the $\mathcal{C}$-normal form of a positive braid is no longer a positive braid word. Moreover, we need to fix a normal finite Thurston type ordering which is conjugate to $<_N$ and a positive word representative of a conjugating element P. Hence for a general finite Thurston type ordering, the $\mathcal{C}$-normal form are not canonically defined.

3.2. **Examples.** Before proceeding to the proof theorem 1, we give some examples of codes and the $\mathcal{C}$-normal forms, which give a better understanding of the $\mathcal{C}$-normal form description.

Example 3 (The Dehornoy orderings on 3-braids)**.** First of all, we consider the first non-trivial case, the Dehornoy ordering on 3-braids. For a positive 3-braid word W, as in the definition of codes, we decompose the word W into subwords.

$$W = A_m A_{m-1} \cdots A_0 A_{-1}$$

where $A_i, A_0, A_{-1} \in \langle \sigma_2 \rangle$ for even i and $A_i \in B_2^+ = \langle \sigma_1 \rangle$ for odd i. Therefore the above decomposition can be written explicitly by

$$W = \cdots \sigma_1^{m_{2i-1}} \sigma_2^{m_{2i}} \cdots \sigma_1^{m_1} \sigma_2^{m_0} \sigma_2^{m_{-1}}.$$

Thus the code of W is given by a sequence of integers

$$\mathcal{C}(W;<_D) = (\cdots, m_i, m_{i-1}, \cdots, m_0, m_{-1}).$$

Now we give more concrete examples. Let $W = \sigma_1|\sigma_2|\sigma_1||$. Then code of W for such a decomposition ($A_{-1} = A_0 = \varepsilon$, $A_1 = \sigma_1$, $A_2 = \sigma_2$, $A_3 = \sigma_1$), the code of W is given by $\mathcal{C}(W;<_D) = (1,1,1,0,0)$. This is not the $\mathcal{C}$-normal form. The $\mathcal{C}$-normal form of braid W is $\sigma_2|\sigma_1||\sigma_2$ and its code is $(1,1,0,1)$.

Example 4 (The Dehornoy ordering of 4-braids)**.** Next we proceed to more complicated example, the Dehornoy ordering $<_D$ on B_4. As in the example 3, decompose a positive 4-braid word W as

$$W = A_m A_{m-1} \cdots A_1 A_0 A_{-1}$$

where $A_{-1}, A_0, A_i \in \langle \sigma_2, \sigma_3 \rangle$ for even i and $A_i \in \langle \sigma_1, \sigma_2 \rangle$ for odd i.

From definition, $m_1 = 2$, $M_1 = 4$ and $m_2 = 3$, $M_2 = 4$. Thus, the decomposition of A_{-1} is written by $A_{-1} = X_1 X_2$ where $X_1 \in \langle \sigma_2, \sigma_3 \rangle$ and $X_2 \in \langle \sigma_3 \rangle$. The restriction of the Dehornoy ordering $<_D$ to $\langle \sigma_2, \sigma_3 \rangle$ is again the Dehornoy ordering $<_D$ of B_3, using the identification of the shift map. Since we have already described code of the Dehornoy ordering of B_3^+, we can explicitly write $\mathcal{C}(D^m(A_m); <_D)$, $\mathcal{C}(A_0; <_{res})$ and $\mathcal{C}(X_i; <_i)$.

Now we give a concrete example. Let W be a positive 4-braid word with decomposition

$$W = \sigma_1 | \sigma_3 \sigma_2 \sigma_3 | \sigma_2^2 \sigma_1^2 || \sigma_3.$$

That is, $A_3 = \sigma_1$, $A_2 = \sigma_3 \sigma_2 \sigma_3$, $A_1 = \sigma_2^2 \sigma_1^2$, $A_0 = \varepsilon$, and $A_{-1} = \sigma_3$. For a sake of simplicity, we choose the subword decomposition of each A_i so that they has the maximal code with respect to $<_{right}$ among all subword decompositions of A_i. From example 3, we can compute the code of each A_i.

$$\begin{cases} \mathcal{C}(D^2(A_3); <_D) = \mathcal{C}(\sigma_1 ||; <_D) = (1, 0, 0) \\ \mathcal{C}(D^1(A_2); <_D) = \mathcal{C}(D(\sigma_3 \sigma_2 \sigma_3); <_D) = \mathcal{C}(\sigma_2 | \sigma_1 || \sigma_2; <_D) = (1, 1, 0, 1) \\ \mathcal{C}(D^0(A_1); <_D) = \mathcal{C}(\sigma_2^2 | \sigma_1^2 ||; <_D) = (2, 2, 0, 0) \\ \mathcal{C}(A_0; <_{res}) = \mathcal{C}(Sh(\varepsilon); <_D) = \mathcal{C}(\varepsilon; <_D) = (0) \\ \mathcal{C}(A_{-1}; <) = \mathcal{C}(Sh(\sigma_3); <_D) = \mathcal{C}(\sigma_2 |; <_D) = (1, 0) \end{cases}$$

Consequently, the code of W is given by

$$C(W) = ((1, 0, 0), (1, 1, 0, 1), (2, 2, 0, 0), (0), (1, 0)).$$

The word W is not the $\mathcal{C}$-normal form. In fact, W has another equivalent word representative $W' = \sigma_3^3 | \sigma_2^2 \sigma_1 || \sigma_2 \sigma_3^2 |$, whose code is given by

$$\mathcal{C}(W') = ((3, 0, 0), (2, 1, 0, 0), (0), ((1, 0, 0), (2))).$$

Example 5 (A normal finite Thurston type ordering which is not conjugate to the Dehornoy ordering)**.** Finally, we study the Thurston type ordering $<$ defined in example 2, which corresponds to the permutation $(2, 1, 3)$.

For a positive 4-braid word W, we decompose W into subwords

$$W = A_m A_{m-1} \cdots A_0 A_{-1}$$

where $A_i \in \langle \sigma_1, \sigma_2 \rangle$ if i is odd and $A_i \in \langle \sigma_2, \sigma_3 \rangle$ if i is even. Since $k = k(1) = 2$, $A_{-1} \in \langle \sigma_1, \sigma_3 \rangle$.

First observe that the restriction of the ordering $<_T$ to $\langle \sigma_2, \sigma_3 \rangle$ is the Dehornoy ordering of B_3, under the identification by the shift map Sh.

Next consider a decomposition of A_{-1}. Since $m_1 = 1$, $M_1 = 2$ and $m_2 = 3$, $M_2 = 4$, the decomposition is written by $A_{-1} = X_1 X_2$ where

$X_1 \in \langle \sigma_1 \rangle$ and $X_2 \in \langle \sigma_2 \rangle$, so the code of the word $A_{-1} = \sigma_1^p | \sigma_3^q$ is given by a pair of integers (p, q).

Now we give a concrete example. Let W be the positive 4-braid word which appears in example 4, with decomposition

$$W = \sigma_1 | \sigma_3 \sigma_2 \sigma_3 \sigma_2^2 | \sigma_1^2 \sigma_3.$$

That is, the decomposition is defined by $A_{-1} = \sigma_1^2 \sigma_3$, $A_0 = \sigma_3 \sigma_2 \sigma_3 \sigma_2^2$, and $A_1 = \sigma_1$.

As observed above, we can compute each $\mathcal{C}(A_i; <)$.

$$\begin{cases} \mathcal{C}(A_1; <_D) = \mathcal{C}(\sigma_1 || ; <_D) = (1, 0, 0) \\ \mathcal{C}(A_0; <_{res}) = \mathcal{C}(Sh(\sigma_3 | \sigma_2 | \sigma_3 | \sigma_2^2 ||); <_D) \\ \qquad\qquad = \mathcal{C}(\sigma_2 | \sigma_1 | \sigma_2 | \sigma_1^2 || ; <_D) = (1, 1, 1, 2, 0, 0) \\ \mathcal{C}(A_{-1}; <) = \mathcal{C}(\sigma_1^2 | \sigma_3) = (2, 1) \end{cases}$$

Summarizing, the code of W is given by

$$\mathcal{C}(W; <_T) = ((1, 0, 0), (1, 1, 1, 2, 0, 0), (2, 1).$$

This is not the $\mathcal{C}$-normal form. The $\mathcal{C}$-normal form of braid W with respect to the ordering $<_T$ is

$$W' = \sigma_1 | \sigma_3^3 \sigma_2 | \sigma_1^2 \sigma_3^2,$$

and its code is

$$\mathcal{C}(W') = ((1, 0, 0), (3, 1, 0, 0), (2, 2)).$$

As this example suggests, different orderings give complete different $\mathcal{C}$-normal forms.

4. proof of Theorem 1, 2

Now we begin to prove theorem 1.

First we introduce a partial ordering $\succ$, which is bi-invariant under the multiplication of B_n from both sides.

Definition 3. Let W, V be positive braid words. We define $W \succ V$ if W is obtained from V by inserting positive generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$.

From Property S, if $W \succ V$ then $W > V$ holds for any Thurston type ordering $>$. More strongly, if $W \succ V$ then $\alpha W \beta > \alpha V \beta$ holds for any braid α, β and any Thurston type ordering $>$. Thus, we can consider the relation $\succ$ as a bi-invariant part of Thurston type orderings.

Now we begin with studying properties of the $\mathcal{C}$-normal form.

Lemma 2. *Let $W = A_m A_{m-1} \cdots A_{-1}$ be a $\mathcal{C}$-normal form.*

(1) *If $m \geq 0$ and $k \neq 1$, then A_0 is non-empty and the last letter of A_0 is σ_k.*

(2) *For all* $0 < i \leq m$, *the last letter of* A_i *is* σ_1 *if* i *is odd and* σ_{n-1} *if* i *is even.*

Proof. We only prove (2) for odd i. The proof of (1) or (2) for even i are similar. First assume that A_i is an empty word. Let σ_j be the last letter of $A_m A_{m-1} \cdots A_{i+1}$. If $j \neq n-1$, then a decomposition defined by $\cdots |\sigma_j| A_{i-1} | A_{i-2} | \cdots$ has the larger code. If $j = n-1$, then a decomposition $\cdots |\sigma_{n-1} A_{i-1} | A_{i-2} | \cdots$ also defines larger code. Thus A_i must be non-empty. Next assume that the last letter of A_i is not σ_1. Then the last letter of A_i is σ_j with $2 \leq j \leq n-2$, so there exists another subword decomposition $W' = \cdots A_{i+1} | A'_i | \sigma_j A_{i-1} | A_{i-2} \cdots$ which has the strictly larger code, which is a contradiction. □

We remark that in $k = 1$ case, A_0 might be empty word. For example, the word $W = \sigma_1 ||$ is the $\mathcal{C}$-normal form with respect to the Dehornoy ordering $<_D$, which has empty A_0. Thus, we can write the $\mathcal{C}$-normal form as the following form.

$$\begin{aligned} W &= A_m A_{m-1} \cdots A_0 A_{-1}. \\ &= \cdots A'_{2j+1} \sigma_1 A'_{2j} \sigma_{n-1} \cdots A'_0 \sigma_k A_{-1} \end{aligned}$$

Thus, the codes of the $\mathcal{C}$-normal form must have the following form.

$$\mathcal{C}(W) = ((\cdots, c_m, \underbrace{0, \cdots 0, 0}_{*}), (\cdots, c_{m-1}, \underbrace{0, \cdots 0, 0}_{*}), \cdots)$$

where c_i must be non-zero. We call zeros $(*)$ in the code of the $\mathcal{C}$-normal form *trivial zeros*. To consider code of $\mathcal{C}$-normal forms, we do not need to consider trivial zeros.

We then next examine each word A'_i more precisely.

Lemma 3. *Let* W *be the* $\mathcal{C}$*-normal form of a positive braid and let* A_i, A'_i *as the above.*

(1) *If* $m > 1$, *then* $A'_0 \succ \sigma_2 \sigma_3 \cdots \sigma_{k-1}$.
(2) *For all* $0 < i < m$, $A'_i \succ \sigma_{n-2} \sigma_{n-1} \cdots \sigma_2$ *if* i *is odd and* $A'_i \succ \sigma_2 \sigma_3 \cdots \sigma_{n-2}$ *if* i *is even.*

Proof. We only prove (2) for odd i. The proof of (1) and (2) for even i are similar. Suppose $A'_i \not\succ \sigma_{n-2}$. Then $A'_i \in \langle \sigma_1, \sigma_2 \cdots \sigma_{n-3} \rangle$, so A'_i commute with σ_{n-1}. Since $i < m$, from lemma 2, the last letter of A_{i+1} is σ_{n-1}. Thus, we obtain another word decomposition

$$W = \cdots A'_{i+1} | A'_i \sigma_1 | \sigma_{n-1} A_{i-1} | A_{i-2} \cdots,$$

which has the larger code. This is a contradiction. Thus, $A'_i \succ \sigma_{n-2}$. Now write $A'_i = V \sigma_{n-2} V'$, where $V' \in \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-3} \rangle$. By the similar arguments , we obtain $V' \succ \sigma_{n-3}$. By iterating this argument, we conclude that $A'_i \succ \sigma_{n-2} \sigma_{n-1} \cdots \sigma_2$.

□

In some special cases, the above estimation can be improved.

Lemma 4. *Let $W = A_m A_{m-1} \cdots A_0 A_{-1}$ be the $\mathcal{C}$-normal form of a positive braid and $i > 0$.*

(1) *If $m > 0$ and the last letter of A'_1 is not σ_1, then*

$$A'_0 \succ (\sigma_3\sigma_2)(\sigma_4\sigma_3)\cdots(\sigma_{j+1}\sigma_j)\sigma_j\sigma_{j+1}\cdots\sigma_{k-1}$$

for some $j < k$.

(2) *If $i < m$ is odd and the last letter of A'_{i+1} is not σ_{n-1}, then*

$$A'_i \succ (\sigma_{n-3}\sigma_{n-2})(\sigma_{n-4}\sigma_{n-3})\cdots(\sigma_{j-1}\sigma_j)\sigma_j\sigma_{j-1}\cdots\sigma_2$$

for some $j > 2$.

(3) *If $i < m$ is even and the last letter of A'_{i+1} is not σ_1, then*

$$A'_i \succ (\sigma_3\sigma_2)(\sigma_4\sigma_3)\cdots(\sigma_{j+1}\sigma_j)\sigma_j\sigma_{j+1}\cdots\sigma_{n-2}$$

for some $j < n-2$.

Proof. We only prove (3). Other cases are similar. From lemma 3, we already know $A_i \succ \sigma_2\sigma_3\cdots\sigma_{n-2}$. Thus, we can write $A_i = V_1\sigma_2 V_2\sigma_3\cdots V_{n-3}\sigma_{n-2}V_{n-2}$ where $V_j \in \langle\sigma_{j+1}, \sigma_{j+2}, \cdots, \sigma_{n-2}\rangle$, possibly empty.

Let $A'_{i+1} = A''_{i+1}\sigma_j$. First observe that in this assumption, $j = 2$ because otherwise, the last letter of A'_{i+1} commute with σ_1, so we obtain another word decomposition

$$W' = \cdots|A''_{i+1}\sigma_1|\sigma_j A_i|A_{i-1}\cdots .$$

which has the larger code, so it is contradiction.

Thus, the word W has a form

$$W = \cdots A''_{i+1}\sigma_2\sigma_1|(V_1\sigma_2 V_2\cdots V_{n-3}\sigma_{n-2}V_{n-2}\sigma_{n-1}|A_i\cdots$$

If $V_1 \not\succ \sigma_3$, then V_1 do not contain σ_1, σ_2 so V_1 commute with σ_2. Then σ_1 commute each V_i for $i > 1$, we can change W as

$$\begin{aligned} W &= \cdots|A''_{i+1}\sigma_2\sigma_1\sigma_2 V_1\cdots V_{n-3}\sigma_{n-2}V_{n-2}\sigma_{n-1}|A_{i-1}|\cdots \\ &= \cdots|A''_{i+1}\sigma_1\sigma_2\sigma_1 V_1\cdots V_{n-3}\sigma_{n-2}V_{n-2}\sigma_{n-1}|A_{i-1}|\cdots \\ &= \cdots|A''_{i+1}\sigma_1|\sigma_2 V_1\cdots V_{n-3}\sigma_{n-2}V_{n-2}\sigma_{n-1}|\sigma_1 A_{i-1}|\cdots \end{aligned}$$

by using the braid relations. Then the last word has the strictly bigger code under the indicated decomposition, this is contradiction. Thus we obtain $V_1 \succ \sigma_3$. If $V_1 \succ \sigma_3\sigma_2$, we are done. Otherwise, by iterating the similar argument for each V_i, we obtain the desired result. □

Let $W = A_m A_{m-1} \cdots A_0 A_{-1}$ be the $\mathcal{C}$-normal form of a positive braid. Now we construct two new positive word $\overline{W}$ and $\underline{W}$ by the following way.

First we define the integer a_i. For odd $0 < i < m$, we define $a_i = 2$ if the last letter of A'_i is σ_1 and $a_i = 1$ otherwise. Similarly, for even i, we define $a_i = 2$ if the last letter of A'_i is σ_{n-1} and $a_i = 1$ otherwise. From this definition, $a_{i+1} = 1$ is equivalent to the assumption of lemma 4 is satisfied for i. According to the value of a_{i+1}, we define $\underline{A_i}$ by the following manner. If $a_1 = 1$, then we define

$$\underline{A_0} = (\sigma_3\sigma_2)(\sigma_4\sigma_3)\cdots(\sigma_{j+1}\sigma_j)\sigma_j\sigma_{j+1}\cdots\sigma_k.$$

Similarly, if $a_{i+1} = 1$ then we define

$$\underline{A_i} = (\sigma_3\sigma_2)(\sigma_4\sigma_3)\cdots(\sigma_{j+1}\sigma_j)\sigma_j\sigma_{j+1}\cdots\sigma_{n-1}^{a_i}$$

for even i and

$$\underline{A_i} = (\sigma_{n-3}\sigma_{n-2})(\sigma_{n-4}\sigma_{n-3})\cdots(\sigma_{j-1}\sigma_j)\sigma_j\sigma_{j-1}\cdots\sigma_1^{a_i}$$

for odd i. Otherwise, we simply define

$$\underline{A_0} = (\sigma_2\sigma_3\cdots\sigma_k)$$

and

$$\underline{A_i} = (\sigma_{n-2}\sigma_{n-3}\cdots\sigma_2\sigma_1^{a_i})$$

for odd $i > 0$ and

$$\underline{A_i} = (\sigma_2\sigma_3\cdots\sigma_{n-2}\sigma_{n-1}^{a_i})$$

for even $i > 0$.

From lemma 3 and lemma 4, in either case $A_i \succ \underline{A_i}$ holds for all i. Using the words $\underline{A_i}$, we define the braid word $\underline{W}$ by

$$\underline{W} = A_m \underline{A_{m-1}}\,\underline{A_{m-2}}\cdots\underline{A_0}A_{-1}.$$

Let $d_0 = (\sigma_2\sigma_3\cdots\sigma_{n-1})^{n-1}$, $d_1 = (\sigma_1\sigma_2\cdots\sigma_{n-2})^{n-1}$ and

The braid word $\overline{W}$ is defined by

$$\overline{W} = A_m \cdots d_{[m-1]}^p d_{[m-2]}^p \cdots d_{[0]}^p A_{-1}$$

where p is sufficiently large integer, and $[i] = 0$ if i is even and $[i] = 1$ if i is odd. By choosing a sufficiently large p, we can obtain $\overline{W}$ by inserting generator's $\{\sigma_i\}$ into the word W. Thus, $\overline{W} \succ W \succ \underline{W}$ holds. We remark that $\overline{W}$ is not a $\mathcal{C}$-normal form, whereas $\underline{W}$ is.

Our idea to prove theorem 1 is that, for two $\mathcal{C}$-normal forms W and V, we compare $\underline{W}$ and $\overline{V}$ instead of comparing V and W themselves. An idea to insert or annihilate the generators for a given braid word so that the action of the braid become simple and easy to compare had already appeared in the author's previous papers [I1],[I2]. In fact, the argument given below is a straightforward generalization and the

refinement of arguments given in [I1],[I2] used to estimate the Dehornoy floor.

Since most parts of the words $\overline{W}$ and $\underline{V}$ can be described explicitly, we can "draw" the image of curve diagram Γ under their braid actions at least for some initial parts, which is enough to compare $\overline{W}$ and $\underline{V}$. The action of the braids d_0 and d_1 on the punctured disc D_n is easy to describe. The braid d_0 corresponds to a full Dehn-twist along the circle enclosing punctures $\{p_1, p_2, \cdots, p_{n-1}\}$ and the braid d_1 corresponds to a full Dehn-twist along the circle enclosing puncture $\{p_2, p_3, \cdots, p_n\}$.

Similarly, it is also not hard to see actions of $\underline{A_i}, \underline{B_i}$ and $\underline{E}$, though an entire picture might become complex.

To treat curve diagrams more accurately, now we introduce the cutting sequence presentation of embedded arcs. This is a method of encoding an embedded curve into the sequence of signed integers and introduced in [FGRRW]. Let Σ be a curve diagram consists of the horizontal arcs which connects puncture points (see figure 3). For a properly embedded oriented arc C in D_n which transversely intersects Σ, we associate the sequence of signed integers as the following manner:

Let the i-th intersection point of C with Σ be q_i. We define $C(i) = +k$ (resp. $-k$) if q_i lies on Σ_k and the sign of intersection at q_i is positive (resp. negative). Since the complement of the curve diagram Σ is a disc, the isotopy class of C can be uniquely determined by the finite sequence of integers $(C(1), C(2), \cdots, C(m))$.

We call this sequence *cutting sequence presentation* of an arc C. See example below. In a cutting sequence presentation, an existence of bigon between C and Σ corresponds to an existence of an subsequence $(\pm i, \mp i)$, therefore C is tight to Σ if and only if the cutting sequence presentation of C contains no subsequences of form $(\pm i, \mp i)$. Removing a bigon corresponds to removing the subsequence $(\pm i, \mp i)$ from the cutting sequence presentation. We call the cutting sequence presentation is *tight* if the corresponding arc is tight to Σ.

Example 6. In the right figure of figure 3, the cutting sequence presentation of the arc is $(+3, -2, +1, -4)$. As is easily observed, this is the tight cutting sequence presentation.

The actions of braid groups on cutting sequence presentations are easy to describe. Let $\mathcal{S} = (C(1), C(2), \cdots)$ be a cutting sequence presentation of an arc C. Since the braid σ_{i-1} corresponds to a half Dehn-twist along Σ_i, the cutting sequence presentation of the arc $\sigma_{i-1}(C)$ is obtained from the original cutting sequence $\mathcal{S}$ by replacing a subsequence of form $(+i)$ by $(+(i-1), -i, +(i+1))$ and a subsequence of form $(-i)$ by $(-(i+1), +i, -(i-1))$. We remark that the obtained

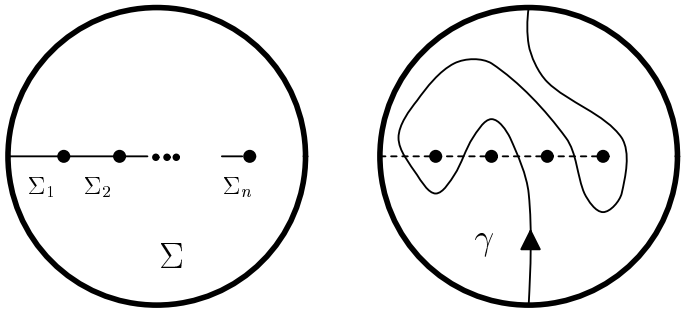


FIGURE 3. Cutting sequence presentation

cutting sequence presentation might be non-tight even if the original cutting sequence presentation $\mathcal{S}$ is tight. Therefore, sometimes the tight cutting sequence presentation of an arc might change dramatically. However, in some cases we can obtain some initial segments of tight cutting sequence presentation of arcs as the following lemma shows.

Lemma 5. *Let β be a positive n-braid and $(C(1), C(2), \cdots, C(k), C(k+1), \cdots)$ be the tight cutting sequence presentation of $\beta(\Gamma_1)$ and assume that $|C(j)| \neq i$ for all $j < k$.*

(1) *If $C(k) = +i$ and $C(k-1) \neq -(i-1)$, then the tight cutting sequence presentation of the braid $\sigma_{i-1}\beta$ is*

$$(C(1), C(2), \cdots, C(k-1), +(i-1), -i, \cdots).$$

(2) *If $C(k) = -i$ and $C(k-1) \neq +(i-1)$, then the tight cutting sequence presentation of the braid $\sigma_{i-1}\beta$ is*

$$(C(1), C(2), \cdots, C(k-1), -(i+1), +i, \cdots).$$

Proof. We only prove (1). The proof of (2) is similar. From a direct computation, the cutting sequence presentation of $\sigma_i\beta(\Gamma_1)$ is given by

$$(C(1), C(2), \cdots, C(k-1), +(i-1), -i, +i+1, \cdots).$$

Since original cutting sequence is tight, this sequence do not contain a subsequence of form $(\pm a, \pm b, \mp b, \mp a)$. This means, under process of making sequence tight, term $C(j)$ disappear if and only if $C(j \pm 1) = -C(j)$. From the assumption of $C(k-1) \neq -(i-1)$, the term $+(i-1)$ does not disappear, so the tight cutting sequence presentation of $\sigma_i\beta(\Gamma_i)$ is given by

$$(C(1), C(2), \cdots, C(k-1), +(i-1), -i, \cdots).$$

□

Iterated use of this lemma, we can calculate initial segments of the tight cutting sequence presentations of the following speicial braids.

Calculation 1. Let $W = A_m A_{m-1} \cdots A_{-1}$ be the $\mathcal{C}$-normal form of a positive n-braid β and let

$$\overline{W} = A_m d^p_{[m-1]} d^p_{[m-2]} \cdots d^p_{[0]} A_{-1}$$

and

$$\underline{W} = A_m \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1}.$$

- For odd m, the tight cutting sequence presentation of the arc
$$[\sigma_1 d^p_{[m-1]} d^p_{[m-2]} \cdots d^p_{[0]} A_{-1}](\Gamma_1)$$
is
$$(\underbrace{+1, +1, \cdots, +1}_{(m+1)/2}, -2, +3, \cdots).$$
- For even m, the tight cutting sequence presentation of the arc
$$[\sigma_{n-1} d^p_{[m-1]} d^p_{[m-2]} \cdots d^p_{[0]} A_{-1}](\Gamma_1)$$
is
$$(\underbrace{+1, +1, \cdots, +1}_{m/2}, +n, -(n-1), \cdots).$$
- For odd m, the tight cutting sequence presentation of the arc
$$[\sigma_1^{a_m} \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1}](\Gamma_1)$$
is
$$(\underbrace{+1, +1, \cdots, +1}_{(m+1)/2}, -2, +3, \cdots).$$
- For even m, the tight cutting sequence presentation of the arc
$$[\sigma_{n-1}^{a_m} \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1}](\Gamma_1)$$
is
$$(\underbrace{+1, +1, \cdots, +1}_{m/2}, +n, -(n-1), \cdots).$$

Now we are ready to prove theorem 1.

Proof of Theorem 1. First we remark that from definition of the $\mathcal{C}$-normal form and codes, we only need to prove theorem for normal finite Thurston type orderings.

We prove the theorem by the induction of n. The case $n = 2$ is trivial. Assume that the assertion is proved for all normal finite Thurston type orderings of B_i for all $i < n$.

Let $<$ be a normal finite Thurston type ordering on B_n and Γ be the normal curve diagram defining the ordering $<$. Let $\alpha, \beta \in B_n^+$ and assume that $C(\alpha;<) <_{left} C(\beta;<)$. Let $W = A_m A_{m-1} \cdots A_{-1}$ be the $\mathcal{C}$-normal form of β and $V = C_l C_{l-1} \cdots C_{-1}$ be the $\mathcal{C}$-normal form of α. We prove $\alpha < \beta$ by showing $\overline{V} < \underline{W}$.

Since the ordering $<$ is left-invariant, we can always assume that the initial letters of $W(\alpha;<)$ and $W(\beta;<)$ are different, by annihilating the initial common letters. Let

$$\underline{W} = \underline{W(\beta;<)} = A_m \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1}$$

and

$$\overline{V} = \overline{W(\alpha;<)} = C_l d_{[l-1]}^p d_{[l-2]}^p \cdots d_{[0]}^p A_{-1}.$$

Since $\mathcal{C}(\alpha;<) <_{left} \mathcal{C}(\beta;<)$, $l \leq m$. If $m = 0$ or -1 then $\alpha < \beta$ follows from an inductive hypothesis. If $l \neq m$, then from calculation 1, we conclude $\underline{W}(\Gamma_1)$ moves more left of $\overline{V}(\Gamma_1)$, so $\alpha < \overline{V} < \underline{W} < \beta$.

Thus we assume $l = m$ and $m > 0$. We only show m is even case. m is odd case is similar. From the calculation 1, the tight cutting sequence presentation of the arc

$$[\sigma_1^{a_m} \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1}](\Gamma_1)$$

and

$$[\sigma_1 d_{[m-1]}^p d_{[m-2]}^p \cdots d_{[0]}^p A_{-1}](\Gamma_1)$$

have the same initial segment,

$$(\underbrace{+1, +1, \cdots, +1}_{(m-1)/2}, \underbrace{+1, -2, +3}_{(*)}).$$

Let $W' = A_m \sigma_1^{-a_m+1}$ and $V' = C_m$. From the definition of $\mathcal{C}$-normal form, W' and V' is also the $\mathcal{C}$-normal form with respect to the Dehornoy ordering on B_{n-1}. Let Π be the normal curve diagram which defines the Dehornoy ordering $<_D$ as shown in the example 1. Since $\mathcal{C}(V;<) <_{left} \mathcal{C}(W;<)$, $\mathcal{C}(V';<_D) <_{left} \mathcal{C}(W';<_D)$ holds. Therefore from an inductive hypothesis, the arc $W'(\Pi_1)$ moves more left than $V'(\Pi_1)$. Especially, $\underline{W'}(\Pi_1)$ moves more left than $\overline{V'}(\Pi_1)$. Observe that the tight cutting sequence presentation of arc $\sigma_1(\Pi_1)$ is $(+1, -2, +3)$, which is identical with the subsequence $(*)$ above. Thus, we can apply the results in calculation 1 for the actions of $\underline{W'}$ and $\overline{V'}$ for the subsequence $(*)$. Therefore, the arc

$$\underline{W'} \sigma_1^{-1} (\sigma_1^{a_m} \underline{A_{m-1}}\, \underline{A_{m-2}} \cdots \underline{A_0} A_{-1})(\Gamma_1)$$

has the initial segment

$$\begin{aligned} & \underline{W'}\sigma_1^{-1}(\underbrace{+1,+1,\cdots,+1}_{(m/2)-1},\underbrace{+1,-2,+3}_{(*)}) \\ = \ & (\underbrace{+1,+1,\cdots}_{(m/2)-1},\underline{W'}(\Pi_1)) \end{aligned}$$

where $\underline{W'}(\Pi_1)$ is a initial segment of the tight cutting sequence presentation of arc $\underline{W'}(\Pi_1)$. Similarly, the arc

$$\overline{V'}\sigma_1^{-1}(\sigma_1 d^p_{[m-1]} d^p_{[m-2]} \cdots d^p_{[0]} A_{-1})(\Gamma_1)$$

has the initial segment

$$\begin{aligned} & \overline{V'}\sigma_1^{-1}(\underbrace{+1,+1,\cdots,+1}_{(m/2)-1},\underbrace{+1,-2,+3}_{(*)}) \\ = \ & (\underbrace{+1,+1,\cdots}_{(m/2)-1},\overline{V'}(\Pi_1)) \end{aligned}$$

where $\overline{V'}(\Pi_1)$ is a initial segment of the tight cutting sequence presentation of arc $\overline{V'}(\Pi_1)$.

Since $\underline{W'}(\Pi_1)$ moves more left than $\overline{V'}(\Pi_1)$, we conclude that

$$\alpha < \overline{V} < \overline{V'} d^p_{[m-1]} \cdots < \underline{W'}\sigma_1^{a_m - 1}\underline{A_{m-1}} \cdots < \underline{W} < \beta.$$

□

Next we proceed to a computation of order-types of the well-ordered set $(B_n^+, <)$ for arbitrary finite Thurston type orderings. First we observe following simple lemma.

Lemma 6. *Let $<$, $<'$ be two conjugate left-invariant total orderings on B_n, whose restriction on positive braid monoid B_n^+ defines a well-ordering. Then the order type of $(B_n^+, <)$ and $(B_n^+, <')$ are the same.*

Proof. Let α be a conjugating element between $<$ and $<'$. From lemma 1, we can choose α as a positive braid. From property S, the map $\Phi : (B_n^+, <') \to (B_n^+, <)$ defined by $\Phi(\beta) = \beta\alpha$ is an order-preserving injection, so the order-type of $(B_n^+, <')$ is smaller than that of $(B_n^+, <)$. By interchange the role of $<$ and $<'$, we also obtain that the order type of $(B_n^+, <)$ is smaller than that of $(B_n^+, <')$. Therefore they coincide. □

Thus, it is sufficient to compute the order-type of $(B_n^+, <)$ for a normal finite Thurston type ordering $<$ to determine the order-type of $(B_n^+, <')$ for general normal finite Thurston type ordering $<'$.

Proof of theorem 2. We prove the theorem by induction on n. It is trivial that statements holds for $n = 2$ because the well-ordered set $(B_2^+, <_D)$ is order-isomorphic to the set of natural numbers with the canonical ordering $(\mathbb{N}, <)$. Let $<$ be a normal finite Thurston type orderings on B_n and let $Ncode(n, <)$ be the set of all codes of the $\mathcal{C}$-normal forms with respect to $<$. Theorem 1 asserts that $(Ncode(n, <), <_{left})$ is order-isomorphic to $(B_n^+, <)$. A direct computation shows that the order-type of well-ordered set $(Code(n, <), <_{left})$ is $\omega^{\omega^{n-2}}$. Since $(Ncode(n, <), <_{left})$ is ordered subset of $(Code(n, <), <_{left})$, we obtain order type of $(Ncode(n, <), <_{left})$ is at most $\omega^{\omega^{n-2}}$.

To show converse inequality, we consider the set of even codes

$$Code_{even}(n, <) = \left\{ \mathcal{C} \in Code(n, <) \;\middle|\; \begin{array}{l} \mathcal{C} \text{ contains only} \\ \text{positive even integers} \\ \text{when neglecting trivial zeros} \end{array} \right\}$$

That is, the code $\mathcal{C}$ belongs to the set $Code_{even}(n, <)$ if each entry of $\mathcal{C}$ is positive even integer except trivial zeros. From an inductive hypothesis, a direct computation shows that the order type of $(Code_{even}(n, <), <_{left})$ is $\omega^{\omega^{n-2}}$. On the other hand, Tits conjecture, proved in [CP], asserts the sub group of B_n generated by the square of generators, $(\sigma_1^2, \sigma_2^2, \cdots, \sigma_{n-1}^2)$ has only trivial commutative relation $\sigma_i^2\sigma_j^2 = \sigma_j^2\sigma_i^2$ for $|i - j| > 1$, we conclude that every braid corresponds to element of $Code_{even}(n, <)$ is a subset of $NCode(n, <)$. So the order type of $(Ncode(n, <), <_{left})$ is at least $\omega^{\omega^{n-2}}$. Therefore the order type of $(Ncode(n, <), <_{left})$ is $\omega^{\omega^{n-2}}$. □

5. Relationships between Dehornoy's Φ-normal form

In this section we study the relationships between the $\mathcal{C}$-normal form and Dehornoy's Φ-normal form or Burckel's normal form. This gives an algebraic description of the $\mathcal{C}$-normal forms and the computational complexities.

5.1. Φ-normal form of braid groups.

The Φ-normal form is a normal form of positive braids defined by Dehornoy in [D3]. First we review the definition of the Φ-normal form of the positive braid monoid B_n^+. The construction of such a normal form is valid for much wider class of monoids, called locally Garside monoid. Since we are interested in the braid groups, we only describes the positive braid monoid case.

As well known, the positive braid monoid B_n^+ defines Garside structure of B_n. Here we do not describe what a Garside structure is. Only we need is an existence and uniqueness of the maximal right divisor.

For a subset $I \subset \{1, 2, \cdots, n-1\}$, let B_I^+ be the submonoid of B_n^+ generated by $\{\sigma_i \mid i \in I\}$.

For each $\beta \in B_n^+$, there exists the unique maximal right divisor of β which belongs to B_I^+. That is, there exists the unique element $\beta' \in B_I$ with the following property:

- $\beta\beta'^{-1} \in B_n^+$
- For $\beta'' \in B_I^+$, if $\beta\beta''^{-1} \in B_n^+$ then $\beta'\beta''^{-1} \in B_I^+$.

We denote the maximal right divisor of β which belongs to B_I^+ by $\beta \wedge B_I^+$.

Let $I = \{2, 3, \cdots, n-1\}$ and $J = \{1, 2, \cdots, n-2\}$. For a positive n-braid β, the (B_I^+, B_J^+)-decomposition (the alternate decomposition) of β is a factorization of β by

$$\beta = \beta_m \beta_{m-1} \cdots \beta_0$$

where β_i is defined by the inductive formula

$$\begin{cases} \beta_0 &= \beta \wedge B_I^+ \\ \beta_i &= (\beta \cdot \beta_0^{-1}\beta_1^{-1} \cdots \beta_{i-1}^{-1}) \wedge B_J^+ \ i : odd \\ &= (\beta \cdot \beta_0^{-1}\beta_1^{-1} \cdots \beta_{i-1}^{-1}) \wedge B_I^+ \ i : even \end{cases} .$$

Since the length of β is finite, the above factorization must stop in finite time.

For even $i > 0$, we identify B_I^+ with B_{n-1}^+ by the map D. Similarly, we identify B_J with B_{n-1}^+ by the identity map for odd $i > 0$. For $i = 0$, we use another identification. we identify B_I^+ with B_{n-1}^+ by the shift map Sh. Then by using these identification, we can perform the above alternate decompositions for each β_i. By iterating the alternate decomposition until each B_I^+ is generated by only one generator, we obtain the unique positive word representative of β. We call this unique positive word representative the Φ-*normal form*.

Example 7. Let $\beta = \sigma_1\sigma_3\sigma_2\sigma_3\sigma_2^2\sigma_1^2\sigma_3$ be a positive 4-braid. Then, the alternate decomposition of β is given by

$$\beta = \beta_2\beta_1\beta_0 = (\sigma_3^3)(\sigma_2^2\sigma_1)(\sigma_2\sigma_3^2).$$

Now we iterate the alternate decomposition for each β_i. Using identification $D(\beta_2) = D(\sigma_3^3) = \sigma_1^3 \in B_3^+$, the alternate decomposition of β_2 is given by $\beta_2 = (\sigma_3^3)$. Similarly, the alternate decomposition of $\beta_1 = \sigma_2^2\sigma_1$ is given by $\beta_1 = (\sigma_2^2)(\sigma_1)$. Finally, by using identification $Sh(\beta_0) = Sh(\sigma_2\sigma_3^2) = \sigma_1\sigma_2^2$, the alternate decomposition of β_0 is given by $\beta_0 = (\sigma_2)(\sigma_3^2)$. Summarizing, we finally obtain the Φ-normal form of β,

$$\beta = ((\sigma_3^3))((\sigma_2^2)(\sigma_1))((\sigma_2)(\sigma_3^2)) = \sigma_3^3\sigma_2^2\sigma_1\sigma_2\sigma_3^2.$$

Notice that this Φ-normal form coincide with the $\mathcal{C}$-normal form given in example 4.

As this example suggests, though definition of $\mathcal{C}$-normal form and Dehornoy's Φ-normal form, which defined algebraically, seems to be different, but they coincides.

Propotition 2. *Let $<_D$ be the Dehornoy ordering of the braid group B_n. Then for each element β in B_n, the $\mathcal{C}$-normal form $W(\beta;<_D)$ of β with respect to the ordering $<_D$ coincide with Dehornoy's Φ-normal form.*

Proof. Let β in B_n^+ and $W = A_m A_{m-1} \cdots A_{-1}$ be the $\mathcal{C}$-normal form of β, and $\beta = \beta_k \beta_{k-1} \cdots \beta_0$ be the alternate decomposition of β. From the definition of the $\mathcal{C}$-normal form, A_0 is always an empty word. We show that $A_{-1} = \beta_0$. That is, $A_{-1} = W \wedge \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle$. Since $A_{-1} \in \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle$, $\beta_0 = F A_{-1}$ for some $F \in \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle$. If F is not trivial braid, then the decomposition $W = \cdots | F A_{-1}$ gives the strictly larger code, so it leads a contradiction. So $A_{-1} = \beta_0$

By the similar argument, we obtain $\beta_i = A_i$ for $i > 0$. Iteration of this arguments shows $\mathcal{C}$-normal form and Φ-normal form coincide. □

Remark 1. The code defined in this paper corresponds to *the associated exponent sequence* in Dehornoy's paper [D3].

Since in [D3], Dehornoy shows Dehornoy's Φ-normal form coincide with Burckel's normal form, we obtain another description of Burckel's normal form.

Corollary 2. *The $\mathcal{C}$-normal form with respect to the Dehornoy ordering is identical with Burckel's normal form.*

5.2. **Tail-twisted Φ-normal form and $\mathcal{C}$-normal form.** Now we generalize proposition 2 for general normal finite Thurston type orderings. To describe the $\mathcal{C}$-normal form for general normal finite Thurston type orderings in view of Dehornor's Φ-normal form, we introduce a slightly generalized notion of Φ-normal form, the *Tail-twisted Φ-normal form.*

Let $\mathbf{k} = \{k(1), k(2), \cdots, k(n-1)\}$ be a permutation of $\{1, 2, \cdots, n-1\}$. For $\beta \in B_n^+$, the $\mathbf{k}$-tail twisted Φ-normal form is a positive word representative of β defined by the following inductive way.

In the case $n = 2$, we simply define the $\mathbf{k}$-tail normal form of a positive braid σ_1^p by the word σ_1^p. Assume that we have already constructed the $\mathbf{k}$-tail twisted normal form for all B_i^+ and the permutation $\mathbf{k} \in S_i$ for all $i < n$. Here S_i is the permutation group of degree i. Then we

define the $\mathbf{k}$-tail twisted normal form of positive n-braids β for $\mathbf{k} \in S_n$ as follows.

First we define the permutations $\mathbf{k}_i$ for $i = 0, 1, \cdots, n-2$. The permutation $\mathbf{k}_0$ of $\{1, 2, \cdots, n-2\}$ is defined by

$$\mathbf{k}_0 = \{k(j_1) - 1, k(j_2) - 1, \cdots, k(j_{n-2}) - 1 \mid j_i < j_{i+1},\ k(j_i) > 1\}.$$

Let m_j and M_j as in the definition of code (See section 3.1), and let $I_j = \{m_j, m_j + 1, \cdots, M_j - 1\}$. The permutation $\mathbf{k}_j$ of $\{1, 2, \cdots, M_j - m_j\}$is defined by

$$\mathbf{k}_j = \{k(i_1), k(i_2), \cdots, k(i_{M_j - m_j}) \mid j < i_q < i_{q+1},\ m_j \leq k(i_q) \leq M_j\}.$$

Let

$$\begin{cases} \beta_{Tail} = \beta \wedge \langle \sigma_1, \sigma_2, \cdots, \sigma_{k(1)-1}, \sigma_{k(1)+1}, \cdots, \sigma_{n-1} \rangle \\ \beta_{T0} = (\beta \beta_{Tail}^{-1}) \wedge \langle \sigma_2, \sigma_3, \cdots, \sigma_{n-1} \rangle \\ \beta_{Main} = \beta \beta_0^{-1} \beta_{Tail}^{-1} \end{cases}$$

We decompose β_{Tail} by $\beta_{Tail} = \beta_{T1} \beta_{T2} \cdots \beta_{Tn-2}$ where

$$\begin{cases} \beta_{Tn-2} &= \beta_{Tail} \wedge B^+_{I_{n-2}} \\ \beta_{Tn-j} &= (\beta_{Tail} \cdot \beta_{Tn-2}^{-1} \beta_{Tn-3}^{-1} \cdots \beta_{Tn-j+1}^{-1}) \wedge B^+_{I_{n-j}} \end{cases}.$$

By some powers of shift map, each B^+_{Ij} is identified with $B^+_{M_j - m_j + 1}$. Using this identification, we denote the Φ-normal form of the braid β_{Main} by $N(\beta_{Main})$ and the $\mathbf{k}_j$-tail twisted Φ-normal form of the braid β_{Tj} by $\widetilde{N}(\beta_{Tj}; \mathbf{k}_j)$. From an inductive assumption, we have already defined these tail-twisted Φ-normal forms. The $\mathbf{k}$-tail twisted Φ-normal form of a positive n-braid β is a positive word representative $\widetilde{N}(\beta; \mathbf{k})$ defined by

$$\widetilde{N}(\beta; \mathbf{k}) = N(\beta_{Main}) \widetilde{N}(\beta_{T0}; \mathbf{k}_0) \cdots \widetilde{N}(\beta_{Tn-2}; \mathbf{k}_{n-2}).$$

We remark that for the trivial permutation $\mathbf{k} = \{1, 2, \cdots, n-1\}$, the $\mathbf{k}$-tail twisted Φ-normal form is indeed the usual (Dehornoy's) Φ-normal forms. Thus, This definition can be seen as a generalization of Dehornoy's Φ-normal form. Using the tail-twisted Φ-normal form, proposition 2 is generalized as the following forms.

Propotition 3. *Let $<$ be a normal finite Thurston type ordering of B_n which corresponds to the permutation $\mathbf{k} = \{k(1), k(2), \cdots, k(n-1)\}$. Then for each $\beta \in B_n^+$, the $\mathbf{k}$-tail twisted Φ-normal form $\widetilde{N}(\beta; \mathbf{k})$ coincide with the $\mathcal{C}$-normal form $W(\beta; <)$.*

Proof. The proof is almost the same as the proof of proposition 2, so we only describe the correspondence between terms of both normal forms. Let β be a positive n-braid. Let $W = A_m \cdots A_{-1}$ be the $\mathcal{C}$-normal form

of β, and denote the decomposition of A_{-1} by $A_{-1} = X_1 X_2 \cdots X_{n-2}$. Similarly, let $\beta = \beta_{Main}\beta_{T0}\beta_{Tail} = \beta_{Main}\beta_{T0}\beta_{T1}\cdots\beta_{Tn-2}$ be a decomposition of the braid β as in the definition of $\mathbf{k}$-tail twisted Φ-normal forms, and $\beta_{Main} = \beta_p\beta_{p-1}\cdots\beta_0$ be the alternate decomposition of β_{Main}. As is easily observed, the last term β_0 is always empty word, so we omit it. By the same arguments of proof of the proposition 2, we obtain a correspondence $\beta_{Tail} = A_{-1}$, $\beta_{T0} = A_0$, and $\beta_i = A_i$, and $\beta_{Ti} = X_i$. Thus, an inductive argument shows these two normal forms are indeed identical. □

Example 8. The definition of the tail-twisted Φ-normal form is a bit complex, so we give an concrete example of the tail-twisted normal form in a simple case. Let $\mathbf{k} = \{2, 1, 3\}$. The normal finite Thurston type ordering corresponding to the permutation $\mathbf{k}$ is the ordering given in example 5.

Consider a positive 4-braid $\beta = \sigma_1\sigma_3\sigma_2\sigma_3\sigma_2^2\sigma_1^2\sigma_3$ which appears in example 5.

By a computation,

$$\beta_{Tail} = \beta \wedge \langle \sigma_1, \sigma_3 \rangle = \sigma_1^2\sigma_3^2$$

So $\beta_{T1} = \beta_1^2$ and $\beta_{T2} = \beta_3^2$ and

$$\beta_{T0} = \beta\beta_{Tail}^{-1} \wedge \langle \sigma_2, \sigma_3 \rangle = \sigma_3^3\sigma_2.$$

Thus,

$$\beta_{Main} = \beta\beta_{T0}^{-1}\beta_{Tail}^{-1} = \sigma_1.$$

Since for this permutation $\mathbf{k}$, all of permutation $\mathbf{k}_0$, $\mathbf{k}_1$ and $\mathbf{k}_2$ become trivial permutations, so all of tail-twisted ϕ-normal forms with respect to these permutations are merely the non-twisted (that is, the ordinary) Φ-normal form. Thus,

$$\begin{cases} \widetilde{N}(\beta_{T1}; \mathbf{k}_1) = \sigma_1^2, \ \widetilde{N}(\beta_{T2}; \mathbf{k}_2) = \sigma_3^2 \\ \widetilde{N}(\beta_0; \mathbf{k}_0) = \sigma_3^3\sigma_2 \\ N(\beta_{Main}) = \sigma_1 \end{cases}$$

Combing above results, we conclude that the $\mathbf{k}$-tail twisted normal form of β is

$$\widetilde{N}(\beta; \mathbf{k}) = \sigma_1\sigma_3^3\sigma_2\sigma_1^2\sigma_3^2.$$

which is of course identical with the $\mathcal{C}$-normal form of β given in the example 5.

These results, combined with Dehornoy's result about the Φ-normal forms, leads to a computational complexity for computing the $\mathcal{C}$-normal forms and comparing a normal finite Thurston-type orderings for arbitrary (that is, necessarily non-positive) n-braids. This is interesting

in following sense: although in [RW] a computational complexity of finite Thurston-type orderings are given, it relies on Mosher's automatic structure of mapping class groups so the relationships between a usual word representative and a computational complexity is not explicit. Our formulation leads to a computational complexity through usual word representatives, so it relies on Thurston's automatic structure of the braid groups ([E]).

Theorem 3. *Let $<$ be an arbitrary finite Thurston type ordering of the braid group B_n.*

1. *The $\mathcal{C}$-Normal form $W(\beta;<)$ of a positive braid $\beta \in B_n^+$ is computed by the time $O(l^2 n \log n)$, where l is the word length of β.*
2. *For a braid (not necessarily positive) $\beta \in B_n$, whether $\beta > 1$ holds or not can be decided by the time $O(l^2 n^3 \log n)$ where l is the word length of β.*

Proof. For Dehornoy's Φ-normal form, these results are proved in [D3]. Since the $\mathcal{C}$-normal form with respect to normal Thurston-type orderings is identical with the tail-twisted Φ-normal forms, and a computation of twisted tail parts (that is, a computation of the parts β_0, β_{Ti} and tail-twisted normal forms $\widetilde{N}(\beta_{Ti};\mathbf{k}_i)$) are merely computations of maximal right divisors and the tail-twisted Φ-normal forms with fewer n, so computation of these parts does not affect the total computational complexity. Therefore the theorem also holds for normal Thurston type orderings. For general finite Thurston type ordering, computation of $\mathcal{C}$-normal form for word length l braid is merely computation of $\mathcal{C}$-normal form with respect to normal finite type of a braid with the word length $l + p$, where p is the word length of a positive conjugating elements. Since p takes a constant value, therefore the conclusion holds. □

The $\mathcal{C}$-normal form constructions and arguments in this paper can be seen as a geometric background of the Φ-normal form and Burckel's normal form. Therefore our $\mathcal{C}$-normal form brings some information about the questions raised by Dehornoy in [D3]. Especially our construction gives an answer of the question 6.5 in [D3], which is in short, "Give an direct proof of the fact Φ-normal form gives combinatorial description of the Dehornoy ordering $<_D$ on B_n^+." Here the word "direct" means without using Burckel's result and proof about Dehornoy orderings. This is because our proof requires neither Burckel's normal form nor its tree construction and uses only "Geometric" definition of the Dehornoy ordering given in [FGRRW] or [SW] and property S, which is easily confirmed from the geometric definition, thus our proof

is independent of Burckel's results. Of course, above question is still open when we restrict to use only the usual "Algebraic" definition of the Dehornoy ordering and algebraic method.

References

[Be] D.Bessis, *The dual braid monoid,* Ann. Sci. Ecole. Noem. Sup. **36**, No.5 (2003), 647-683.

[Bi] J.Birman, *Braids, Links, and Mapping Class Groups,* Annals of Math. Studies **82**, Princeton Univ. Press (1975).

[BKL] J.Birman, K.Ko, and S.Lee, *A New approach to the word problem in the braid groups,* Adv. Math. **139** (1998), 322-353.

[B1] S.Burckel, *The well-ordering on positive braids,* J. of pure and appl. Alg, **120**, No.1 (1997),1-17.

[B2] S.Burckel, *Computation of the ordinal of braids,* Order, **16**,(1999),291-304.

[CP] J.Crisp, L.Paris, *The solution to a conjecture of Tits on the sub group generated by the squares of the generators of an Artin group,* Invent. Math, **145**,(2001),19-36.

[D1] P.Dehornoy, *Braids and left distributive operations,* Trans. of AMS, **345**, No.1 (1994), 115-151.

[D2] P.Dehornoy, *Still another approach to the braid ordering,* Pacific Journal of Mathematics, **232**, No.1 (2007), 139-176.

[D3] P.Dehornoy, *Alternating normal forms in braid monoids and locally Garside monoids,* J. Pure Appl. Algebra, **212**, No.11 (2008), 2416-2439.

[DDRW1] P.Dehornoy, I.Dynnkov, D.Rolfsen and B.Wiest, *Why are the braids orderable ?,* Panoramas et Synthéses **14**, Soc. Math. France. 2002.

[DDRW2] P.Dehornoy, I.Dynnkov, D.Rolfsen and B.Wiest, *Ordering Braids,* Mathematical Surveys and Monographs **148**, Amer. Math. Soc. 2008.

[E] D.B.A. Epstein, J.Cannon, D.Holt, S.Levy, M. Paterson and W. Thurston, *Word processing in Groups,* Jones and Bartlett Publishers, Boston, MA. 1992.

[F1] J.Fromentin, *The cycling normal form on dual braid monoids,* arXiv:math.GR/0712.3836.

[F2] J.Fromentin, *A well-ordering of dual braid monoids,* Comptes Rendus Mathematique, **346**,(2008), 729-734.

[F3] J.Fromentin, *Every braid admits a short sigma-definite representative* arXiv:0811.3902v1, **346**,(2008), 729-734.

[FGRRW] R.Fenn, M.Greene, D.Rolfsen, C.Rourke and B.Wiest, *Ordering the braid groups,* Pacific J. of Math, **191**, No.1 (1999), 49-73.

[I1] T.Ito, *Braid ordering and the geometry of closed braid,* e-print, arXiv:0805.1447v2

[I2] T.Ito, *Braid ordering and knot genus,* e-print, arXiv:0805.2042v2

[I3] T.Ito, *Finite Thurston type orderings on the dual braid monoids,* e-print, arXiv:0902.0833v1

[RW] C.Rourke, B.Wiest, *Order automatic mapping class groups* Pacific J. Math, **194**, No.1 (2000), 209-227.

[SW] H.Short, B.Wiest, *Ordering of mapping class groups after Thurston,* L'Enseignement Mathématique, **46**,(2000), 279-312.

Graduate School of Mathematical Sciences, the University of Tokyo, Japan
E-mail address: tetitoh@ms.u-tokyo.ac.jp