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On the fundamental group of II_1 factors and equivalence relations arising from group actions

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Abstract

Given a countable group G, we consider the sets $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$, of subgroups $H \subset \mathbb{R}$ for which there exists a free ergodic probability measure preserving action $G \curvearrowright X$ such that the fundamental group of the associated II_1 factor $\mathrm{L}^\infty(X) \rtimes G$, respectively orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$, equals $\exp(H)$. We prove that if $G = \Gamma^{*\infty} * \mathbb{Z}$, with $\Gamma \neq 1$, then $\mathcal{S}_{\rm factor}(G)$ and $\mathcal{S}_{\rm equiv}(G)$ contain $\mathbb{R}$ itself, all of its countable subgroups, as well as uncountable subgroups of any Hausdorff dimension $\alpha \in (0,1)$. We then prove that if $G = \Gamma * \Lambda$, with Γ, Λ finitely generated ICC groups, one of which has property (T), then $\mathcal{S}_{\rm factor}(G) = \mathcal{S}_{\rm equiv}(G) = \{\{0\}\}$. We also show that there exist II_1 factors of the form $M = L^\infty(X) \rtimes \mathbb{F}_\infty$ such that the fundamental group of M is $\mathbb{R}_+$, but $M \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))$ admits no continuous trace scaling action of $\mathbb{R}_+$.

1 Introduction

We continue in this paper our investigation of fundamental groups of II_1 factors $L^\infty(X) \rtimes G$ and orbit equivalence relations $\mathcal{R}(G \curvearrowright X)$, arising from free ergodic measure preserving (m.p.) actions of groups on probability spaces, $G \curvearrowright (X, \mu)$, initiated in [29]. In [29] we concentrated on the case $G = \mathbb{F}_\infty$, showing the existence of "many" orbit inequivalent actions of $\mathbb{F}_\infty$ having prescribed fundamental group and prescribed outer automorphism group. Also, we considered the sets $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$, of subgroups $H \subset \mathbb{R}$ for which there exists a free ergodic m.p. action $G \curvearrowright X$ such that $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes G) = \exp(H)$, respectively $\mathcal{F}(\mathcal{R}(G \curvearrowright X)) = \exp(H)$, and exhibited a large family $\mathcal{S}$ of subgroups of $\mathbb{R}$, containing $\mathbb{R}$ itself, all of its countable subgroups, as well as uncountable subgroups of any Hausdorff dimension $\alpha \in (0,1)$, such that $\mathcal{S}$ is contained in both $\mathcal{S}_{\rm factor}(\mathbb{F}_\infty)$ and $\mathcal{S}_{\rm equiv}(\mathbb{F}_\infty)$. (Note however that an abstract characterization of the sets $\mathcal{S}_{\rm factor}(\mathbb{F}_\infty)$, $\mathcal{S}_{\rm equiv}(\mathbb{F}_\infty)$ remained elusive, though it was noted that any subgroup in either set, as in fact any subgroup in $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$ for an arbitrary G, must be a Borel set and Polishable.)

Our purpose now is to estimate (or even completely calculate) the invariants $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$ for other classes of groups G. We will target two types of problems: on the one hand, detecting classes of groups G for which $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$ are "large", containing for instance all countable groups and groups of arbitrary Hausdorff dimension (like in the case case $G = \mathbb{F}_\infty$); on the other hand, detecting classes of groups G for which $\mathcal{S}_{\rm factor}(G)$, $\mathcal{S}_{\rm equiv}(G)$ are "small" (e.g. countable, or even trivial).

Thus, our first result enlarges considerably the class of groups G for which we can show that the above set $\mathcal{S}$ is contained in both $\mathcal{S}_{\rm factor}(G)$ and $\mathcal{S}_{\rm equiv}(G)$. In fact, for this larger class of groups G we show that $\mathcal{S}_{\rm factor}(G) \cap \mathcal{S}_{\rm equiv}(G)$ contains an even larger set of subgroups, denoted $\mathcal{S}_{\rm centr}$ and

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consisting of all $H \subset \mathbb{R}$ for which there exists a free ergodic action of an amenable group Λ on an infinite measure space (Y, ν), such that the set of scalars $t > 0$ that can appear as scaling constants of non-singular automorphisms θ of (Y, ν) commuting with $\Lambda \curvearrowright Y$, equals $\exp H$.

Theorem 1.1. *Let Γ be a non-trivial group and Σ an amenable group. Then,*

$$\mathcal{S}_{\rm centr} \subset \mathcal{S}_{\rm factor}(\Gamma^{*\infty} * \Sigma) \qquad \textit{and} \qquad \mathcal{S}_{\rm centr} \subset \mathcal{S}_{\rm equiv}(\Gamma^{*\infty} * \Sigma)\ .$$

We mention that in Section 7, we show that if an equivalence relation $\mathcal{R}$ on a probability space (X, μ) contains a property (T) group acting ergodically on X, then $\mathcal{F}(\mathcal{R})$ follows countable. Thus, if a group Γ appearing in the above statement contains an infinite subgroup Λ with the property (T) and if $G = \Gamma^{*\infty} * \Sigma \curvearrowright X$ is a free ergodic p.m.p. action such that $\mathcal{R}_G$ has fundamental group equal to an uncountable group in $\mathcal{S}_{\rm centr}$, then the restriction of $G \curvearrowright X$ to Λ cannot be ergodic.

Note that all groups of the form $G = \Gamma^{*\infty} * \Sigma$, covered by the above theorem, have infinite first ℓ^2-Betti number, $\beta_1^{(2)}(G) = \infty$. On the other hand, by Gaboriau's results ([11]), any free ergodic p.m.p. action $G \curvearrowright X$ of a group G with $\beta_n^{(2)}(G) \neq 0, \infty$, for some n, gives rise to an equivalence relation $\mathcal{R}_G$ with trivial fundamental group, $\mathcal{F}(\mathcal{R}_G) = \{1\}$. In other words, $\mathcal{S}_{\rm equiv}(G) = \{\{0\}\}$. While it is still an open question whether the corresponding II_1 factors $M = \mathrm{L}^\infty(X) \rtimes G$ satisfy $\mathcal{F}(M) = \{1\}$ as well (i.e. $\mathcal{S}_{\rm factor}(G) = \{\{0\}\}$), our next result provides a large class of groups G for which this is indeed the case.

Theorem 1.2. *Let Γ and Λ be infinite, finitely generated groups. Assume that Γ is ICC and that one of the following conditions holds.*

a) $\Gamma = \Gamma_1 \times \Gamma_2$, with Γ_1 non-trivial and Γ_2 non-amenable,

b) Γ admits a non virtually abelian, normal subgroup Γ_1 with the relative property (T).

Then, $\mathcal{S}_{\rm factor}(\Gamma * \Lambda) = \mathcal{S}_{\rm equiv}(\Gamma * \Lambda) = \{\{0\}\}$.

When viewed from the perspective of Connes' discrete decomposition of type III_λ factors with $0 < \lambda < 1$ ([5]) and respectively Connes-Takesaki continuous decomposition of type III_1 factors ([9]), the above result provides a large class of groups G with the property that no II_1 factor M arising from an arbitrary free ergodic p.m.p. action of G can appear in the decomposition of a type III factor (i.e., as Connes puts it, no such M can appear as the "shadow" of a type III factor).

Our next result provides a class of group measure space II_1 factors $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ which do have fundamental group equal to $\mathbb{R}_+$ yet cannot appear in the continuous decomposition of a type III_1 factor. The problem of whether such II_1 factors exist was posed over the years by several people, including Connes, Takesaki, and more recently Shlyakhtenko. The fact that there are even factors of the form $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ satisfying this property should be contrasted with the fact that the II_∞ factor associated with $L(\mathbb{F}_\infty)$ does admit a trace scaling action of $\mathbb{R}_+$, by [32].

Theorem 1.3. *There exist free ergodic p.m.p. actions $\mathbb{F}_\infty \curvearrowright (X, \mu)$ such that the II_1 factor $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ has fundamental group $\mathcal{F}(M) = \mathbb{R}_+$, but nevertheless the II_∞ factor $M \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))$ admits no trace scaling action of $\mathbb{R}_+$.*

While II_1 factors $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ arising from free ergodic p.m.p. actions $\Gamma \curvearrowright X$ of ICC property (T) groups always have countable fundamental group (cf. [23], [20]), it was not known whether there exist cases when $\mathcal{F}(M) \neq \{1\}$. Our next result gives the first such examples. It also provides

the first concrete examples of free ergodic p.m.p. actions $\Gamma \curvearrowright X$ with the associated II_1 factors M having fundamental group $\neq \{1\}, \mathbb{R}_+$. Indeed, the actions in Theorem 1.1 above and in [29] are shown to exist by using a Baire-category argument, at some point, while in 1.4 below they are concrete Γ-actions, obtained as diagonal products of Bernoulli and profinite actions.

Theorem 1.4. *Let $\mathcal{F} \subset \mathbb{Q}_+$ be a subgroup generated by a subset of the prime numbers. Let $\Gamma = \mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$. Then, Γ admits a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ such that the fundamental group of $\mathrm{L}^\infty(X) \rtimes \Gamma$ and of $\mathcal{R}(\Gamma \curvearrowright X)$ equals $\mathcal{F}$.*

We in fact believe that any subgroup of $\mathbb{Q}_+$ can be realized as the fundamental group of a factor or equivalence relation arising from a free ergodic p.m.p. action of $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$. The question of whether there exist free ergodic p.m.p. actions of property (T) groups $\Gamma \curvearrowright X$ such that $\mathcal{F}(\mathcal{R}_\Gamma)$ contains irrational numbers, remains open.

2 Preliminaries

The *fundamental group* $\mathcal{F}(M)$ of a II_1 factor M, introduced in [19], is defined as the following subgroup of $\mathbb{R}_+$.

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid p, q \text{ are non-zero projections in } M \text{ such that } pMp \cong qMq\} .$$

We call II_1 equivalence relation on a standard probability space (X, μ) every ergodic probability measure preserving (p.m.p.) measurable equivalence relation with countable equivalence classes. The fundamental group $\mathcal{F}(\mathcal{R})$ of a II_1 equivalence relation $\mathcal{R}$ is defined as

$$\mathcal{F}(\mathcal{R}) = \{\mu(Y)/\mu(Z) \mid \mathcal{R}|_Y \cong \mathcal{R}|_Z\} .$$

Whenever $\Gamma \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, we denote by $\mathcal{R}(\Gamma \curvearrowright X)$ the associated II_1 orbit equivalence (OE) relation and by $\mathrm{L}^\infty(X) \rtimes \Gamma$ the associated group measure space II_1 factor [19].

Definition 2.1. A free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is called *rigid* if the corresponding inclusion $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is rigid in the sense of [27, Proposition 4.1].

Some sets of subgroups of $\mathbb{R}$ and ergodic measures

Given a countable group Γ, we are interested in

$$\begin{aligned} \mathcal{S}_{\mathrm{factor}}(\Gamma) := \{H \subset \mathbb{R} \mid & \text{ there exists a free ergodic p.m.p. action } \Gamma \curvearrowright (X, \mu) \\ & \text{such that } \mathcal{F}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \exp(H)\} , \\ \mathcal{S}_{\mathrm{equiv}}(\Gamma) := \{H \subset \mathbb{R} \mid & \text{ there exists a free ergodic p.m.p. action } \Gamma \curvearrowright (X, \mu) \\ & \text{such that } \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \exp(H)\} . \end{aligned}$$

In [29, Theorem 5.3 and formula (2.2)], we have shown that both $\mathcal{S}_{\mathrm{factor}}(\mathbb{F}_\infty)$ and $\mathcal{S}_{\mathrm{equiv}}(\mathbb{F}_\infty)$ contain $\mathcal{S}_{\mathrm{centr}}$ defined as

$$\begin{aligned} \mathcal{S}_{\mathrm{centr}} := \{H \subset \mathbb{R} \mid & \text{ there exists } \Lambda \curvearrowright (Y, \eta) \text{ free ergodic m.p. action,} \\ & \text{with } \Lambda \text{ amenable and } \mathrm{mod}(\mathrm{Centr}_\Lambda(Y)) = \exp(H) \} . \end{aligned}$$

Following [1, Section 4], we call *ergodic measure on* $\mathbb{R}$ any σ-finite measure ν on the Borel sets of $\mathbb{R}$ having the following properties, where we denote $\lambda_x(y) = x + y$.

- For all $x \in \mathbb{R}$, either $\nu \circ \lambda_x = \nu$ or $\nu \circ \lambda_x \perp \nu$.
- There exists a countable subgroup $Q \subset \mathbb{R}$ such that $\nu \circ \lambda_x = \nu$ for all $x \in Q$ and such that every Q-invariant Borel function on $\mathbb{R}$ is ν-almost everywhere constant.

For every ergodic measure ν on $\mathbb{R}$, one defines

$$H_\nu := \{x \in \mathbb{R} \mid \nu \circ \lambda_x = \nu\} .$$

The groups H_ν can have arbitrary Hausdorff dimension and all belong to $\mathcal{S}_{\mathrm{centr}}$, see [29, Section 2 and proof of Theorem 5.3] for a detailed exposition.

Intertwining by bimodules and the notation $A \prec_M B$

In Sections 3 and 4, we use the method of intertwining by bimodules, introduced by the first author in [27]. Let (M, τ) be a von Neumann algebra with faithful normal tracial state τ. We use the notation $M^n = \mathrm{M}_n(\mathbb{C}) \otimes M$. When $A, B \subset M^n$ are possibly non-unital embeddings, we write $A \prec_M B$ if there exists a non-zero partial isometry $v \in 1_A(\mathrm{M}_{n,m}(\mathbb{C}) \otimes M)$ and a, possibly non-unital, normal *-homomorphism $\rho : A \to B^m$ satisfying $av = v\rho(a)$ for all $a \in A$. Several equivalent formulations of this property can be given, see [24, Theorem 2.1] (see also [35, Theorem C.3]).

Suppose that A and B are Cartan subalgebras of the II_1 factor M. Let $A_0 \subset A$ be a von Neumann subalgebra such that $A_0' \cap M = A$. By [27, Theorem A.1], $A_0 \prec_M B$ if and only if there exists a unitary $u \in M$ such that $uAu^* = B$.

3 Groups G for which $\mathcal{S}_{\mathrm{factor}}(G)$ contains uncountable groups

The following theorem, whose proof is given at the end of the section, provides a large family of groups G such that $\mathcal{S}_{\mathrm{factor}}(G)$ and $\mathcal{S}_{\mathrm{equiv}}(G)$ is large, in the sense that both contain $\mathcal{S}_{\mathrm{centr}}$. The groups G are infinite free product groups and should be opposed to the groups G treated in Theorem 4.1, for which $\mathcal{S}_{\mathrm{factor}}(G)$ is trivial (cf. Remark 4.2).

Theorem 3.1. *Let Γ, Σ be infinite groups with Σ being amenable. Then,*

$$\mathcal{S}_{\mathrm{centr}} \subset \mathcal{S}_{\mathrm{factor}}(\Gamma^{*\infty} * \Sigma) \qquad \textit{and} \qquad \mathcal{S}_{\mathrm{centr}} \subset \mathcal{S}_{\mathrm{equiv}}(\Gamma^{*\infty} * \Sigma) .$$

Let G be a countable group with subgroup Γ. Suppose that $G \curvearrowright (X, \mu)$ is a free p.m.p. action such that the restriction to Γ is ergodic. Slightly changing notations compared to [29, Section 2], denote by $\mathrm{Emb}(\Gamma, G)$ the set of non-singular partial automorphisms ϕ of (X, μ) satisfying $\phi(g \cdot x) \in G \cdot \phi(x)$ for all $g \in \Gamma$ and almost all $x \in X$ with $x, g \cdot x \in D(\phi)$. Denote by $[[G]]$ the full pseudo-group of the OE relation $\mathcal{R}(G \curvearrowright X)$, i.e. the set of a partial automorphisms ϕ of (X, μ) satisfying $\phi(x) \in G \cdot x$ for almost all $x \in D(\phi)$.

The following lemma generalizes [29, Theorem 4.1].

Lemma 3.2. *Let Γ be an infinite group, Λ an arbitrary group, both acting freely and p.m.p. on (X, μ). There exists a free p.m.p. action $\Gamma^{*\infty} * \Lambda \overset{\alpha}{\curvearrowright} (X, \mu)$ with the following properties.*

- *The restriction of α to $\Gamma^{*\infty}$ is ergodic and rigid (in the sense of Definition 2.1).*

- $\mathrm{Emb}(\Gamma^{*\infty}, \Gamma^{*\infty} * \Lambda) = [[\Gamma^{*\infty} * \Lambda]]$.
- *The restriction of α to any of the copies of Γ, resp. to Λ, is conjugate with the originally given action.*

Proof. Denote the given actions by $\Gamma \overset{\beta}{\curvearrowright} (X, \mu)$ and $\Lambda \overset{\rho}{\curvearrowright} (X, \mu)$. We introduce the following notations:

$$\Gamma^{*\infty} = \overset{\infty}{\underset{n=-1}{*}} G_n \quad \text{with all } \; G_n \cong \Gamma \; ,$$

$$\Gamma_n := \overset{n}{\underset{k=-1}{*}} G_k \; ,$$

$$\Gamma_E := G_{-1} * G_0 * \underset{n \in E}{*} G_n \quad \text{whenever } \; E \subset \mathbb{N} \; .$$

By [12, Theorem 1.2], take a free ergodic p.m.p. action $\Gamma_0 \overset{\alpha_0}{\curvearrowright} (X, \mu)$ such that α_0 is a rigid action and such that the restrictions of α_0 to G_{-1} and G_0 are conjugate with the action β. By [34, Category Lemma] and [15, Lemma A.1], extend α_0 to a free action of $\Gamma_0 * \Lambda$ on (X, μ), still denoted by α_0, whose restriction to Λ is conjugate with the action ρ.

Extend the action α_0 inductively to free actions $\Gamma_n * \Lambda \overset{\alpha_n}{\curvearrowright} (X, \mu)$ following the procedure in [29, Section 3] and such that the restriction of α_n to $G_k \subset \Gamma_n$ is conjugate with β for all $k \leq n$. We end up with the free action $\Gamma^{*\infty} * \Lambda \overset{\alpha_\infty}{\curvearrowright} (X, \mu)$. For every infinite subset $E \subset \mathbb{N}$, we denote by α_E the restriction of α_∞ to $\Gamma_E * \Lambda$. Following the proof of [29, Theorem 4.1], there exists an infinite subset $E \subset \mathbb{N}$ such that $\mathrm{Emb}(\Gamma_E, \Gamma_E * \Lambda) = [[\Gamma_E * \Lambda]]$. Since $\Gamma_E \cong \Gamma^{*\infty}$, the lemma is proved. □

Remark 3.3. Using the methods of [12, Section 2.3], Lemma 3.2 can be shown for $\Gamma_1 * \Gamma_2$ instead of the infinite free product $\Gamma^{*\infty}$, for arbitrary infinite groups Γ_1, Γ_2 with given free p.m.p. actions on (X, μ). Such a generalization does not provide a refinement for Theorem 3.1 though, since the proof of Theorem 3.1 involves taking once more an infinite free product.

Theorem 3.4. *Let $\Gamma_1 * \Gamma_2 \overset{\alpha}{\curvearrowright} (X, \mu)$ be a free p.m.p. action. Let $\Gamma_2 \curvearrowright (Y, \eta)$ be a free ergodic action preserving the infinite standard measure η. Consider the action $\Gamma_1 * \Gamma_2 \curvearrowright X \times Y$ given by*

$$g \cdot (x, y) = (g \cdot x, y) \;\; \forall g \in \Gamma_1 \;\; , \;\; h \cdot (x, y) = (h \cdot x, h \cdot y) \;\; \forall h \in \Gamma_2 \; . \tag{3.1}$$

Make the following assumptions.

- *The restriction of α to Γ_1 is ergodic and rigid.*
- *We have $\mathrm{Emb}(\Gamma_1, \Gamma_1 * \Gamma_2) = [[\Gamma_1 * \Gamma_2]]$.*
- *Γ_2 is amenable.*

Then, the following holds.

1. *The map*

$$\Theta : \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma_2) \to \mathrm{Aut}(\mathcal{R}(\Gamma_1 * \Gamma_2 \curvearrowright X \times Y)) : \Delta \mapsto \Theta_\Delta \;\; \textit{where} \;\; \Theta_\Delta(x, y) = (x, \Delta(y))$$

induces an onto isomorphism between $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma_2)$ and $\mathrm{Out}(\mathcal{R}(\Gamma_1 * \Gamma_2 \curvearrowright X \times Y))$.

2. *Every automorphism of the II_∞ factor $\mathrm{L}^\infty(X \times Y) \rtimes (\Gamma_1 * \Gamma_2)$ preserves the Cartan subalgebra $\mathrm{L}^\infty(X \times Y)$ up to unitary conjugacy.*

Proof. The proof of point 1 is identical to [29, Lemma 5.1]. It remains to prove point 2. Write $A = \mathrm{L}^\infty(X)$ and $B = \mathrm{L}^\infty(Y)$. Let θ be an automorphism of $N := (A \overline{\otimes} B) \rtimes (\Gamma_1 * \Gamma_2)$. Take a projection $p \in A \overline{\otimes} B$ of finite trace and put $q = \theta(p)$. After unitary conjugacy, we may assume that $q \in A \overline{\otimes} B$. By [27, Theorem A.1], it is sufficient to prove that $\theta(Ap) \underset{qNq}{\prec} (A \overline{\otimes} B)q$.

Since $\theta(Ap) \subset qNq$ is rigid, [15, Theorem 5.1] implies that

$$\theta(Ap) \underset{qNq}{\prec} q\big((A \overline{\otimes} B) \rtimes \Gamma_i\big)q \quad \text{for some } \ i = 1, 2.$$

Since $\theta(Ap)$ is quasi-regular in qNq, [15, Theorem 1.1] implies that $\theta(Ap) \underset{qNq}{\prec} (A \overline{\otimes} B)q$. □

Lemma 3.5. *Let $\Gamma * \Lambda \curvearrowright (X, \mu)$ be a free p.m.p. action with the restriction to Γ being ergodic. Let $\Lambda \curvearrowright (Y, \eta)$ be a free ergodic action preserving the infinite standard measure η. Assume that Λ is amenable. Consider $\Gamma * \Lambda \curvearrowright X \times Y$ as in (3.1). Let $Z \subset Y$ be a subset of finite measure and define the II_1 equivalence relation $\mathcal{R}$ as the restriction of $\mathcal{R}(\Gamma * \Lambda \curvearrowright X \times Y)$ to Z.*

Whenever Σ is an infinite amenable group, there exists a free ergodic p.m.p. action $\Gamma^{*\infty} * \Sigma \curvearrowright X \times Z$ such that $\mathcal{R} = \mathcal{R}(\Gamma^{*\infty} * \Sigma \curvearrowright X \times Z)$.

Proof. Denote by $\mathcal{R}_1$ the equivalence relation given by the restriction of $\mathcal{R}(\Lambda \curvearrowright X \times Y)$ to $X \times Z$. Note that $\mathcal{R}_1$ need not be ergodic. Since Λ is amenable and almost every equivalence class of $\mathcal{R}_1$ is infinite, the results in [6] and [21] allow to take a free p.m.p. action $\Sigma \curvearrowright X \times Z$ whose OE relation is precisely $\mathcal{R}_1$.

Since the action of Λ on (Y, η) is ergodic, take $\phi_n \in [[\Lambda]]$ with $\mathrm{dom}(\phi_n) = X \times Z$ and $\mathrm{range}(\phi_n) = X \times Z_n$, where Z_n, $n \in \mathbb{N}$, forms a partition of Y (up to measure zero). Since the action of Γ leaves every $X \times Z_n$ globally invariant, we can view $\phi_n^{-1} \Gamma \phi_n$ as a group of automorphisms of $X \times Z$. It is now an exercise to check that $\mathcal{R}$ is freely generated by the OE relations of $\phi_n^{-1} \Gamma \phi_n$, $n \in \mathbb{N}$, together with $\Sigma \curvearrowright X \times Z$. This provides us with the required free action of $\Gamma^{*\infty} * \Sigma \curvearrowright X \times Z$. □

Proof of Theorem 3.1. Let Γ, Σ be infinite groups with Σ being amenable. Let Λ be an infinite amenable group and $\Lambda \curvearrowright (Y, \eta)$ a free ergodic action preserving the infinite standard measure η. Set $\mathcal{F} = \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda))$. We have to prove the existence of a free ergodic p.m.p. action of $\Gamma^{*\infty} * \Sigma$ such that the associated equivalence relation and II_1 factor have fundamental group $\mathcal{F}$.

Choose any free p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X, \mu)$. Take a free p.m.p. action $\Gamma^{*\infty} * \Lambda \curvearrowright (X, \mu)$ satisfying the conclusions of Lemma 3.2. Define $\Gamma^{*\infty} * \Lambda \curvearrowright X \times Y$ by (3.1), with $\Gamma_1 = \Gamma^{*\infty}$ and $\Gamma_2 = \Lambda$. Define the II_1 equivalence relation $\mathcal{R}$ by restricting $\mathcal{R}(\Gamma^{*\infty} * \Lambda \curvearrowright X \times Y)$ to a subset $X \times Z$ of finite measure. By Lemma 3.5, we can take a free ergodic p.m.p. action $\Gamma^{*\infty} * \Sigma \curvearrowright X \times Z$ whose OE relation equals $\mathcal{R}$. Then, Theorem 3.4 ensures us that the associated II_1 factor and equivalence relation have both fundamental group equal to $\mathcal{F}$. □

4 Groups G for which $\mathcal{S}_{\mathrm{factor}}(G)$ is trivial

Combining results from [2, 11, 15], we prove that for the following groups G, $\mathcal{S}_{\mathrm{factor}}(G)$ is trivial.

Theorem 4.1. *Let Γ and Λ be infinite, finitely generated groups. Assume that Γ is ICC and that one of the following conditions holds.*

a) $\Gamma = \Gamma_1 \times \Gamma_2$ is a non-trivial direct product with Γ_2 being non-amenable,

b) Γ admits a non virtually abelian, normal subgroup Γ_1 with the relative property (T).

Then, $\mathcal{S}_{\text{factor}}(\Gamma * \Lambda) = \mathcal{S}_{\text{equiv}}(\Gamma * \Lambda) = \{\{0\}\}$.

Remark 4.2. Observe that Theorem 4.1 implies that, in general, Theorem 3.1 is false if we only take a finite free product.

Proof of Theorem 4.1. Let $\Gamma * \Lambda \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Note first that by [11, Propriétés 1.5], we have $0 < \beta_1^{(2)}(\Gamma * \Lambda) < \infty$. Hence, by [11, Corollaire 5.7], the fundamental group of the OE relation $\mathcal{R}(\Gamma * \Lambda \curvearrowright X)$ is trivial.

Write $A = \mathrm{L}^\infty(X)$, $M_1 = A \rtimes \Gamma$, $M_2 = A \rtimes \Lambda$. Finally, set $M = A \rtimes (\Gamma * \Lambda) = M_1 *_A M_2$. Suppose that $p \in A$ is a projection and $\theta : M \to pMp$ a *-isomorphism. It remains to prove that $\theta(A)$ and pA are unitarily conjugate, since this implies that $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}(\Gamma * \Lambda \curvearrowright X))$.

Under assumption a), we invoke [2, Theorem 4.2] and under assumption b), we invoke [15, Theorem 5.1] and conclude in both cases that $\theta(L(\Gamma_1)) \underset{M}{\prec} M_i$ for some $i = 1, 2$. Take a projection $q \in M_i^n$, a non-zero partial isometry $v \in p(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M)q$ and a unital *-homomorphism $\rho : L(\Gamma_1) \to qM_i^n q$ satisfying $\theta(a)v = v\rho(a)$ for all $a \in L(\Gamma_1)$. In both cases a) and b), the group Γ_1 is not virtually abelian. Hence, $\rho(L(\Gamma_1)) \underset{M_i}{\not\prec} A$. By [15, Theorem 1.1], the normalizer of $\rho(L(\Gamma_1))$ inside qM^nq is contained in qM_i^nq. Since v^*v commutes with $\rho(L(\Gamma_1))$, we may first of all assume that $q = v^*v$. Next, it follows that $v^*\theta(L(\Gamma))v \subset qM_i^nq$. Hence, $\theta(L(\Gamma)) \underset{M}{\prec} M_i$.

Repeating the previous paragraph, we may assume that $\rho : L(\Gamma) \to qM_i^nq$, $\theta(a)v = v\rho(a)$ for all $a \in L(\Gamma)$ and $v^*v = q$. Since Γ is an ICC group, we get that

$$M \cap L(\Gamma)' = M_1 \cap L(\Gamma)' = A^\Gamma \ .$$

So, $vv^* \in \theta(A^\Gamma)$. It follows that $v^*\theta(A)v$ is a Cartan subalgebra of qM^nq. Moreover, for all $g \in \Gamma$, the unitary $v^*\theta(u_g)v = \rho(u_g)$ belongs to qM_i^nq and normalizes $v^*\theta(A)v$. Then, [15, Theorem 1.8] implies that there exists $w \in qM^n$ such that $ww^* = q$, $w^*w \in A^n$ and $w^*v^*\theta(A)vw = w^*wA^n$. It follows that $\theta(A)$ and pA are unitarily conjugate. □

5 Fundamental groups and trace scaling actions of $\mathbb{R}_+$

When M is a II_1 factor, the fundamental group $\mathcal{F}(M)$ equals the image of the homomorphism

$$\mathrm{mod} : \mathrm{Aut}(M \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))) \to \mathbb{R}_+ : \mathrm{Tr} \circ \alpha = \mathrm{mod}\, \alpha \ \mathrm{Tr} \ .$$

So, when $\mathcal{F}(M) = \mathbb{R}_+$, the homomorphism mod is surjective. As we show in the following theorem, this does not imply the existence of a splitting, i.e. a strongly continuous action of $\mathbb{R}_+$ on $M \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))$ scaling the trace.

Theorem 5.1. *There exist free ergodic p.m.p. actions $\mathbb{F}_\infty \curvearrowright (X, \mu)$ such that for $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$, we have $\mathcal{F}(M) = \mathbb{R}_+$, but nevertheless the II_∞ factor $M \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))$ admits no trace scaling action of $\mathbb{R}_+$.*

In the course of the proof of Theorem 5.1, we will also obtain the following result.

Theorem 5.2. *There exist II_1 factors M_1 and M_2 such that $\mathcal{F}(M_1) \neq \mathbb{R}_+ \neq \mathcal{F}(M_2)$, but nevertheless $\mathcal{F}(M_1 \overline{\otimes} M_2) = \mathbb{R}_+$.*

The proof of Theorem 5.1 will make use of the following lemma. Recall that the automorphism group $\mathrm{Aut}(N)$ of a von Neumann algebra with separable predual is a Polish group under the topology making the maps $\mathrm{Aut}(N) \to N_* : \alpha \mapsto \omega \circ \alpha$ continuous for all $\omega \in N_*$.

Let $\Gamma \curvearrowright (Y, \eta)$ be a free ergodic action preserving the (possibly infinite) measure η. Set $A = \mathrm{L}^\infty(Y)$ and $N = \mathrm{L}^\infty(Y) \rtimes \Gamma$.

One defines $\mathrm{Z}^1(\Gamma \curvearrowright Y)$ to be the group of S^1-valued cocycles for $\Gamma \curvearrowright Y$, i.e. the group of measurable maps $\omega : \Gamma \times Y \to S^1$ satisfying $\omega(gh, y) = \omega(g, h \cdot y)\omega(h, y)$ for all $g, h \in \Gamma$ and almost all $y \in Y$. Two cocycles are identified if they are equal almost everywhere. Note that every $\omega \in \mathrm{Z}^1(\Gamma \curvearrowright Y)$ defines an automorphism θ_ω of N satisfying $\theta_\omega(u_g a) = u_g \omega(g, \cdot) a$ for all $g \in \Gamma$, $a \in A$, where we view $\omega(g, \cdot)$ as a unitary in A. As such, $\mathrm{Z}^1(\Gamma \curvearrowright Y)$ equals the Polish group $\{\theta \in \mathrm{Aut}\, N \mid \theta(a) = a \ \text{ for all } \ a \in A\}$.

Every $\Delta \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma)$ defines an automorphism θ_Δ given by $\theta_\Delta(u_g a) = u_g(a \circ \Delta^{-1})$ for all $g \in \Gamma$, $a \in A$. Note that $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma)$ is a Polish group, being a closed subgroup of $\mathrm{Aut}\, Y = \mathrm{Aut}\, A$. The homomorphism $\Delta \mapsto \theta_\Delta$ is continuous.

Lemma 5.3. *Let Γ be a countable group acting freely and ergodically on the standard infinite measure space (Y, η), preserving the measure η. Set $N = \mathrm{Aut}(\mathrm{L}^\infty(Y) \rtimes \Gamma)$. Assume that $\mathrm{Aut}\, N$ is generated by*

$$\{\mathrm{Ad}\, u \mid u \in \mathcal{U}(N)\} \ , \quad \{\theta_\omega \mid \omega \in \mathrm{Z}^1(\Gamma \curvearrowright Y)\} \ \text{ and } \ \{\theta_\Delta \mid \Delta \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma)\} \, .$$

Assume that $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma) \cap [\Gamma] = \{\mathrm{id}\}$. Then, the subgroup H of $\mathrm{Aut}\, N$ generated by $\mathrm{Inn}(N)$ and $\mathrm{Z}^1(\Gamma \curvearrowright Y)$ is closed and normal in $\mathrm{Aut}\, N$. Moreover, the homomorphism

$$H \rtimes \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma) \to \mathrm{Aut}\, N : (\rho, \Delta) \mapsto \rho\theta_\Delta$$

is an isomorphism and homeomorphism of Polish groups.

Proof. Write Z^1 instead of $\mathrm{Z}^1(\Gamma \curvearrowright Y)$. The Polish group Z^1 acts continuously on the Polish group $\mathcal{U}(N)/S^1$ by $\omega \cdot u = \theta_\omega(u)$. We form the semi-direct product Polish group $G_1 := \mathcal{U}(N)/S^1 \rtimes \mathrm{Z}^1$. Define the continuous homomorphism $\partial : \mathcal{U}(A)/S^1 \to \mathrm{Z}^1$ given by $(\partial u)(g, x) = u(g \cdot x)u^*(x)$. It follows that

$$G_0 := \{(u^*, \partial u) \mid u \in \mathcal{U}(A)/S^1\}$$

is a closed subgroup of G_1, contained in the center of G_1. Denote by G the quotient Polish group G_1/G_0. By construction, the natural homomorphism

$$\pi : G \rtimes \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Gamma) \to \mathrm{Aut}\, N$$

is bijective and continuous, between two Polish groups. Hence, π is a homeomorphism. Since $\pi(G) = H$, we are done. □

Lemma 5.4. *There exist ergodic measures ν, ν' on $\mathbb{R}$ such that $H_\nu \neq \mathbb{R} \neq H_{\nu'}$ and $H_\nu + H_{\nu'} = \mathbb{R}$.*

Proof. As explained in [29, Section 2], an ergodic measure ν on $\mathbb{R}$ can be associated to any pair $(a_n), (b_n)$ of sequences in $\mathbb{N}$ satisfying $\sum_{n=1}^{\infty} b_n^{-1} < \infty$ and $b_n < a_n/2$ for all n, in such a way that

$$H_\nu = \Big\{ x \in \mathbb{R} \;\Big|\; \sum_{n=1}^{\infty} \frac{a_n}{b_n} \|a_1 \cdots a_{n-1} x\| < \infty \Big\}$$

where $\|x\|$ denotes the distance of $x \in \mathbb{R}$ to $\mathbb{Z} \subset \mathbb{R}$. Take $a_n = 2^{2n+2}$, $b_n = 2^{2n}$ and associate with it the ergodic measure ν. Take $a'_n = 2^{2n+1}$, $b'_n = 2^{2n-1}$ and associate with it the ergodic measure ν'. First of all,

$$\sum_{n=1}^{\infty} \frac{b_n}{a_1 \cdots a_n} \notin H_\nu \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{b'_n}{a'_1 \cdots a'_n} \notin H_{\nu'}$$

proving that $H_\nu \neq \mathbb{R} \neq H_{\nu'}$.

Let now $x \in \mathbb{R}$ and write

$$x = x_0 + \sum_{n=1}^{\infty} \frac{x_n}{a_1 \cdots a_n} \quad \text{with} \quad x_n \in \{0, \ldots, a_n - 1\} \; .$$

Write for every $n \in \mathbb{N}$, $x_n = y_n + \sqrt{a_n} z_n$ with $y_n, z_n \in \{0, \ldots, \sqrt{a_n}\}$. Define

$$y = x_0 + \sum_{n=1}^{\infty} \frac{y_n}{a_1 \cdots a_n} \quad \text{and} \quad z = \sum_{n=1}^{\infty} \frac{2 z_n}{a'_1 \cdots a'_n} \; .$$

One checks that $y \in H_\nu$, $z \in H_{\nu'}$ and $x = y + z$. So, $H_\nu + H_{\nu'} = \mathbb{R}$. □

Proof of Theorem 5.1. Let Λ be an amenable group and $\Lambda \curvearrowright (Y, \eta)$ a free ergodic action preserving the infinite measure η. Set $G = \Gamma^{*\infty} * \Lambda$ and let $G \curvearrowright (X, \mu)$ be a free p.m.p. action satisfying the conclusions of Lemma 3.2. Define $G \curvearrowright X \times Y$ as in (3.1), where $\Gamma_1 = \Gamma^{*\infty}$ and $\Gamma_2 = \Lambda$. Set $N = \mathrm{L}^\infty(X \times Y) \rtimes G$.

We claim that the II_∞ factor N admits a trace scaling action of $\mathbb{R}_+$ if and only if the homomorphism

$$\mathrm{mod} : \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda) \to \mathbb{R}_+$$

is onto and admits a continuous splitting. If mod is onto and splits continuously, N obviously admits a trace scaling action of $\mathbb{R}_+$. Suppose conversely that (α_t) is a trace scaling action of $\mathbb{R}_+$ on N. Theorem 3.4 tells us that the assumptions of Lemma 5.3 are fulfilled. So, the homomorphism $\mathbb{R}_+ \to \mathrm{Aut}\, N : t \mapsto \alpha_t$ translates to a continuous homomorphism $\mathbb{R}_+ \to H \rtimes \mathrm{Centr}_{\mathrm{Aut}\, Y}(G)$. Since the automorphisms of the subgroup H are trace preserving, we find a continuous splitting $\mathbb{R}_+ \to \mathrm{Centr}_{\mathrm{Aut}\, Y}(G)$, proving the claim.

By Lemma 5.4, we can take ergodic measures ν_1, ν_2 on $\mathbb{R}$ such that $H_{\nu_1} \neq \mathbb{R} \neq H_{\nu_2}$, while $H_{\nu_1} + H_{\nu_2} = \mathbb{R}$. By formula (2.2) in [29], we can take amenable groups Λ_1, Λ_2 and free ergodic infinite measure preserving actions $\Lambda_i \curvearrowright (Y_i, \eta_i)$ such that $\mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}\, Y_i}(\Lambda_i) = \exp(H_{\nu_i})$. Since the homomorphism mod is continuous, we equip $\exp(H_{\nu_i})$ with the (Polish) quotient topology. In this way, the H_{ν_i} become Polish groups and the embedding $H_{\nu_i} \hookrightarrow \mathbb{R}$ continuous.

Construct as above the II_∞ factor N, starting from the direct product action $\Lambda_1 \times \Lambda_2 \curvearrowright Y_1 \times Y_2$. It remains to prove that the homomorphism

$$\mathrm{mod} : \mathrm{Centr}_{\mathrm{Aut}(Y_1 \times Y_2)}(\Lambda_1 \times \Lambda_2) \to \mathbb{R}_+$$

admits no continuous splitting. Assume that it does. Since the left-hand side of the previous formula equals $\mathrm{Centr}_{\mathrm{Aut}\, Y_1}(\Lambda_1) \times \mathrm{Centr}_{\mathrm{Aut}\, Y_2}(\Lambda_2)$, the homomorphism

$$H_{\nu_1} \times H_{\nu_2} \to \mathbb{R} : (x, y) \mapsto x + y$$

admits a continuous splitting. We then find continuous homomorphisms $\theta_i : \mathbb{R} \to H_{\nu_i}$ such that $x = \theta_1(x) + \theta_2(x)$ for all $x \in \mathbb{R}$. Since the embedding $H_{\nu_i} \hookrightarrow \mathbb{R}$ is continuous, there exist $\lambda_i \in \mathbb{R}$ such that $\theta_i(x) = \lambda_i x$ for all $x \in \mathbb{R}$. But, $H_{\nu_i} \neq \mathbb{R}$, forcing $\lambda_i = 0$ for $i = 1, 2$, a contradiction. □

Proof of Theorem 5.2. By Lemma 5.4, we can take ergodic measures ν_1, ν_2 on $\mathbb{R}$ such that $H_{\nu_1} \neq \mathbb{R} \neq H_{\nu_2}$, while $H_{\nu_1} + H_{\nu_2} = \mathbb{R}$. By formula (2.2) in [29], $H_{\nu_i} \in \mathcal{S}_{\mathrm{centr}}$, so that by Theorem 3.1, we can take II_1 factors M_i with $\mathcal{F}(M_i) = \exp(H_{\nu_i})$. Then, the fundamental group of $M_1 \overline{\otimes} M_2$ contains $\exp(H_{\nu_1} + H_{\nu_2})$ and hence equals $\mathbb{R}_+$. □

6 Property (T) groups with rational fundamental groups

When Γ is an ICC property (T) group, all groups in $\mathcal{S}_{\mathrm{factor}}(\Gamma)$ are countable (see [13, Proof of Theorem 1.7]). Nevertheless, $\mathcal{S}_{\mathrm{factor}}(\Gamma)$ can be non-trivial, as shown by the following theorem.

It is unclear though whether there exists a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ of a property (T) group Γ such that $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) \not\subset \mathbb{Q}_+$.

Theorem 6.1. *Let $\mathcal{F} \subset \mathbb{Q}_+$ be a subgroup generated by a subset of the prime numbers. Let $\Gamma = \mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$. Then, Γ admits a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ such that the fundamental group of $\mathrm{L}^\infty(X) \rtimes \Gamma$ and of $\mathcal{R}(\Gamma \curvearrowright X)$ equals $\mathcal{F}$.*

We prove Theorem 6.1 as a consequence of the following more general result.

Theorem 6.2. *Let Γ be a group having a normal, non virtually abelian subgroup Σ with the relative property (T) and with Γ/Σ being finitely generated. Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a decreasing sequence of finite index subgroups such that the action $\Gamma \curvearrowright (X, \mu) := \varprojlim \Gamma/\Gamma_n$ is essentially free. Consider the diagonal product action $\Gamma \curvearrowright X \times [0,1]^\Gamma$ of $\Gamma \curvearrowright X$ and the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$.*

Then, the fundamental groups of the associated II_1 factor and II_1 equivalence relation are both equal to

$$\left\{ \frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} \;\middle|\; \begin{array}{l} \Gamma_n \subset \Lambda_1 \cap \Lambda_2 \text{ for large enough } n, \text{ and} \\ \qquad \Lambda_1 \curvearrowright \varprojlim \Lambda_1/\Gamma_n \text{ is conjugate with } \Lambda_2 \curvearrowright \varprojlim \Lambda_2/\Gamma_n \end{array} \right\}. \tag{6.1}$$

Conjugacy of two profinite actions can be expressed in purely group-theoretic terms, see e.g. [14, Proposition 1.8].

Before proving Theorem 6.2, we introduce some terminology and an auxiliary result. Recall that a 1-*cocycle* $\omega : \Gamma \times X \to \Lambda$ for an action $\Gamma \curvearrowright (X, \mu)$ with values in a countable group Λ, is a measurable map satisfying

$$\omega(gh, x) = \omega(g, h \cdot x)\, \omega(h, x) \quad \text{for all } g, h \in \Gamma \text{ and almost all } x \in X .$$

The 1-cocycles $\omega, \omega' : \Gamma \times X \to \Lambda$ are called *cohomologous* if there exists a measurable map $\varphi : X \to \Lambda$ satisfying $\omega'(g, x) = \varphi(g \cdot x)\omega(g, x)\varphi(x)^{-1}$ almost everywhere. We identify homomorphisms from Γ to Λ with 1-cocyles ω that are independent of the x-variable.

Definition 6.3. Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. We say that a 1-cocycle $\omega : \Gamma \times X \to \Lambda$ *virtually untwists* if there exists

- a finite index subgroup $\Gamma_0 < \Gamma$ and a quotient map $\pi : \Gamma \to \Gamma/\Gamma_0$ satisfying $\pi(g \cdot x) = g\pi(x)$ almost everywhere,
- a 1-cocycle $\omega' : \Gamma \times \Gamma/\Gamma_0 \to \Lambda$ for the action $\Gamma \curvearrowright \Gamma/\Gamma_0$,

such that ω is cohomologous with the 1 cocycle $(g, x) \mapsto \omega'(g, \pi(x))$.

We call $\Gamma \curvearrowright (X, \mu)$ *virtually cocycle superrigid* (with countable target groups) if every 1-cocycle with values in a countable group Λ, virtually untwists.

A *stable orbit equivalence* between free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X', \mu')$ is a map $\Delta : X \to X'$ satisfying the following properties.

- For almost every $x \in X$, we have $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.
- There exists a partition $X = \bigsqcup_n X_n$ of X into measurable subsets $X_n \subset X$ and there exist measurable subsets $X'_n \subset X'$ such that for every $n \in \mathbb{N}$, the restriction of Δ to X_n is a non-singular isomorphism between X_n and X'_n.

By ergodicity, every of these non-singular isomorphisms $\Delta|_{X_n} : X_n \to X'_n$ is measure scaling, with the scaling being independent of n. This scaling factor is called the *compression constant* of the stable OE Δ and denoted by $c(\Delta)$.

The *Zimmer* 1*-cocycle* $\omega : \Gamma \times X \to \Lambda$ associated with the stable OE Δ is defined by

$$\Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x) \quad \text{almost everywhere.}$$

Two stable OEs $\Delta_1, \Delta_2 : X \to X'$ are called *similar* if $\Delta_1(x) \in \Lambda \cdot \Delta_2(x)$ for almost all $x \in X$. Note that similar stable OEs give rise to cohomologous 1-cocycles.

Whenever $X_0 \subset X$ and $X'_0 \subset X'$ are non-negligible measurable subsets and $\Delta_0 : X_0 \to X'_0$ is a non-singular isomorphism satisfying $\Delta_0(\Gamma \cdot x \cap X_0) = \Lambda \cdot \Delta_0(x) \cap X'_0$ for almost all $x \in X_0$, ergodicity allows to choose a measurable map $p : X \to X_0$ with $p(x) \in \Gamma \cdot x$ for almost all $x \in X$ and then, $\Delta := \Delta_0 \circ p$ defines a stable OE. Another choice of p gives rise to a similar stable OE. It follows that

$$\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \{c(\Delta) \mid \Delta \text{ is a stable OE between } \Gamma \curvearrowright X \text{ and } \Gamma \curvearrowright X\} .$$

Let $\Gamma \curvearrowright (X, \mu)$. We say that the action $\Gamma \curvearrowright X$ is *induced* from $\Gamma_1 \curvearrowright X_1$ if X_1 is a non-neglible measurable subset of X and $\Gamma_1 < \Gamma$ is a finite index subgroup such that $g \cdot X_1 = X_1$ for all $g \in \Gamma_1$ and $\mu(g \cdot X_1 \cap X_1) = 0$ if $g \in \Gamma - \Gamma_1$. Obviously, in this situation $\Gamma \curvearrowright X$ is stably orbit equivalent with $\Gamma_1 \curvearrowright X_1$ with compression constant $[\Gamma : \Gamma_1]^{-1}$.

The following provides one more instance of a general principle going back to [37, Proposition 4.2.11]. For other versions of this, see [26, Proposition 5.11] and [35, Lemma 4.7].

Proposition 6.4. *Let $\Delta : X \to X'$ be a stable OE between the free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (X', \mu')$. If the associated Zimmer 1-cocycle virtually untwists (see Definition 6.3), there exist finite index subgroups $\Gamma_1 < \Gamma$, $\Lambda_1 < \Lambda$, non-negligible measurable subsets $X_1 \subset X$, $X'_1 \subset X'$ and a finite normal subgroup $H \lhd \Gamma_1$ such that*

1. $\Gamma \curvearrowright X$ *is induced from* $\Gamma_1 \curvearrowright X_1$,

2. $\Lambda \curvearrowright Y$ *is induced from* $\Lambda_1 \curvearrowright Y_1$,

3. *the actions* $\Gamma_1/H \curvearrowright X_1/H$ *and* $\Lambda_1 \curvearrowright Y_1$ *are conjugate,*

and such that the stable OE Δ *is similar to the composition of the canonical stable OEs given by 1, 3 and 2. In particular, the compression constant of* Δ *equals*

$$c(\Delta) = \frac{[\Lambda : \Lambda_1]}{[\Gamma : \Gamma_1]|H|} \; .$$

Proof. Let $\Delta(g \cdot x) = \omega(g,x) \cdot \Delta(x)$ almost everywhere. By our assumption, take a finite index subgroup Γ_1, a quotient map $\pi : X \to \Gamma/\Gamma_1$ and a 1-cocycle $\omega' : \Gamma \times \Gamma/\Gamma_1 \to \Lambda$ such that $\pi(g \cdot x) = g\pi(x)$ almost everywhere and such that ω is cohomologous with the 1-cocycle $(g,x) \mapsto \omega'(g,\pi(x))$. Define $X_1 = \pi^{-1}(e\Gamma_1)$. By construction, $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$.

Denote by Δ_1 the restriction of Δ to X_1. Then, Δ_1 is a stable OE between $\Gamma_1 \curvearrowright X_1$ and $\Lambda \curvearrowright Y$. By construction, the 1-cocycle associated with Δ_1 is cohomologous to a homomorphism from Γ_1 to Λ. The conclusion of the proposition now follows from [35, Lemma 4.7]. $\square$

In order to show the equality of the fundamental groups of the II_1 factor and the II_1 equivalence relation associated with the Γ-actions defined in Theorem 6.2, we need the following result about automatic preservation of Cartan subalgebras.

Proposition 6.5. *Let* Γ *be a countable group having a normal, non virtually abelian subgroup* Σ *with the relative property (T). Let* $\Gamma \curvearrowright (X,\mu)$ *be a free ergodic p.m.p. action and assume that this action admits a free and profinite quotient: there exists a free profinite p.m.p. action* $\Gamma \curvearrowright (X_1,\mu_1)$ *and a quotient map* $\pi : X \to X_1$ *satisfying* $\pi(g \cdot x) = g \cdot \pi(x)$ *almost everywhere.*

Let (Y_0,η_0) *be a non-trivial standard probability space and set* $(Y,\eta) = (Y_0,\eta_0)^\Gamma$. *Consider the diagonal action* $\Gamma \curvearrowright (X \times Y, \mu \times \eta)$. *Set* $M = \mathrm{L}^\infty(X \times Y) \rtimes \Gamma$.

Then, every isomorphism $\theta : M \to pMp$ *preserves, up to unitary conjugacy, the natural Cartan subalgebras of* M, pMp.

Proof. Set $A = \mathrm{L}^\infty(X)$ and $B = \mathrm{L}^\infty(Y)$. Let $\theta : M \to pMp$ be an isomorphism. Denote $A_1 = \mathrm{L}^\infty(X_1)$ and view A_1 as a globally Γ-invariant von Neumann subalgebra of A.

Almost literally repeating [24, Theorem 4.1] (see also [35, Lemma 6.1]), we find that $\theta(L(\Sigma)) \underset{M}{\prec} A \rtimes \Gamma$. Take a projection $q \in (A \rtimes \Gamma)^n$, a non-zero partial isometry $v \in p(\mathrm{M}_{1,n}(\mathbb{C}) \otimes M)q$ and a unital *-homomorphism $\alpha : L(\Sigma) \to q(A \rtimes \Gamma)^n q$ such that $\theta(a)v = v\alpha(a)$ for all $a \in L(\Sigma)$.

Since Σ is normal in Γ and since $\Sigma \curvearrowright X_1$ is profinite, the quasi-normalizer of $L(\Sigma)$ inside M contains $A_1 \rtimes \Gamma$. Since Σ is non virtually abelian, $L(\Sigma)$ cannot be embedded in an amplification of A. So, by [35, Proposition D.5], $v^*\theta(A_1 \rtimes \Gamma)v \subset (A \rtimes \Gamma)^n$. It follows in particular that $\theta(A_1) \underset{M}{\prec} A \rtimes \Gamma$. We claim that in fact $\theta(A_1) \underset{M}{\prec} A$. Indeed, if this would not be the case, applying once more [35, Proposition D.5] (and using the regularity of $A_1 \subset (A \overline{\otimes} B) \rtimes \Gamma$) would yield $M \underset{M}{\prec} A \rtimes \Gamma$, a contradiction. This proves the claim.

Since $\Gamma \curvearrowright X_1$ is free, we have $A_1' \cap M = A \overline{\otimes} B$. Hence, the proposition follows from [27, Theorem A.1]. $\square$

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. Put $(X,\mu) = \varprojlim \Gamma/\Gamma_n$ as in the formulation of the theorem. We assume $\Gamma \curvearrowright X$ to be (essentially) free. Let $Y = [0,1]^\Gamma$ and denote by η the infinite product of the Lebesgue measure on $[0,1]$. We consider the diagonal action $\Gamma \curvearrowright X \times Y$.

By Proposition 6.5, we have

$$\mathcal{F}(\mathrm{L}^\infty(X \times Y) \rtimes \Gamma) = \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X \times Y)) \; .$$

Whenever $\Lambda_1 < \Gamma$ is a subgroup containing Γ_n for large enough n, the action $\Gamma \curvearrowright (X,\mu)$ is induced from the action $\Lambda_1 \curvearrowright X_1 := \varprojlim \Lambda_1/\Gamma_n$ and hence, $\Gamma \curvearrowright X \times Y$ is induced from $\Gamma_1 \curvearrowright X_1 \times Y$. Since $\Gamma_1 \curvearrowright [0,1]^\Gamma$ and $\Gamma_1 \curvearrowright [0,1]^{\Gamma_1}$ are isomorphic actions, it follows that the set defined by (6.1) is part of the fundamental group $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X \times Y))$.

A combination of [26, Theorem 0.1] and [14, Theorem B] yields that the diagonal action $\Gamma \curvearrowright X \times Y$ is virtually cocycle superrigid in the sense of Definition 6.3. So, we can apply Proposition 6.4.

Let $\Delta : X \times Y \to X \times Y$ be a stable OE between $\Gamma \curvearrowright X \times Y$ and itself. We have to prove that $c(\Delta)$ belongs to the set defined in (6.1). Proposition 6.4 provides us with finite index subgroups G_1, G_2 of Γ, non-negligible measurable subsets $Z_1, Z_2 \subset X \times Y$ and a finite normal subgroup H of G_1 such that $\Gamma \curvearrowright X \times Y$ is induced from $G_i \curvearrowright Z_i$ and such that $G_1/H \curvearrowright Z_1/H$ is conjugate with $G_2 \curvearrowright Z_2$, say through the isomorphism $\Delta : Z_1/H \to Z_2$ and the group isomorphism $\delta : G_1 \to G_2$. Finally, $c(\Delta) = \frac{[\Gamma:G_2]}{[\Gamma:G_1]|H|}$.

Since the Bernoulli action $G_i \curvearrowright Y$ is mixing, we have $Z_i = X_i \times Y$, with $\Gamma \curvearrowright X$ being induced from $G_i \curvearrowright X_i$. Moreover, still because the Bernoulli action is mixing, $\Delta(x,y) = (\Delta_0(x), \ldots)$, where $\Delta_0 : X_1/H \to X_2$ is an isomorphism conjugating the actions $G_1/H \curvearrowright X_1/H$ and $G_2 \curvearrowright X_2$ through the group isomorphism $\delta : G_1/H \to G_2$.

Denote by $\pi_n : X \to \Gamma/\Gamma_n$ the natural quotient map. By [14, Lemma 4.1], we find $k \in \mathbb{N}$ and $g \in \Gamma$ such that $g\Gamma_k g^{-1} \subset G_1$ and $X_1 = \pi_k^{-1}(G_1 g \Gamma_k)$. Moreover, since $\Gamma \curvearrowright X$ is free, we can take k large enough and assume that $H \cap \Gamma_k = \{e\}$. Replacing G_1 by $g^{-1}G_1 g$ and X_1 by $g^{-1} \cdot X_1$, we may assume that $g = e$. Note that $G_1/H \curvearrowright X_1/H = \varprojlim G_1/(\Gamma_n H)$ is induced from $(\Gamma_k H)/H \curvearrowright \varprojlim (\Gamma_k H)/(\Gamma_n H)$ and that the latter is conjugate with $\Gamma_k \curvearrowright \varprojlim \Gamma_k/\Gamma_n$, because $\Gamma_k \cap H = \{e\}$.

It follows that $\Gamma \curvearrowright X$ is induced from $\delta((\Gamma_k H)/H) \curvearrowright \Delta_0(\varprojlim (\Gamma_k H)/(\Gamma_n H))$. Applying as above [14, Lemma 4.11], we find $h \in \Gamma$ such that, after replacing δ by $g \mapsto h\delta(g)h^{-1}$ and Δ_0 by $x \mapsto h \cdot \Delta_0(x)$, we have $\Gamma_n \subset \Lambda_2 := \delta((\Gamma_k H)/H)$ for n large enough and

$$\Delta_0(\varprojlim (\Gamma_k H)/(\Gamma_n H)) = \varprojlim \Lambda_2/\Gamma_n \; .$$

Denoting $\Lambda_1 = \Gamma_k$, we have constructed finite index subgroups $\Lambda_1, \Lambda_2 \subset \Gamma$ such that $\Gamma_n \subset \Lambda_1 \cap \Lambda_2$ for n large enough and such that the actions $\Lambda_i \curvearrowright \varprojlim \Lambda_i/\Gamma_n$ are conjugate for $i = 1,2$. Tracing back the construction, we also have

$$c(\Delta) = \frac{[\Gamma : \Lambda_2]}{[\Gamma : \Lambda_1]}$$

concluding the proof of the theorem. ∎

Proof of Theorem 6.1. Let $\mathcal{F}$ be a subgroup of $\mathbb{Q}_+$ generated by a non-empty subset $\mathcal{P}$ of the prime numbers. The case $\mathcal{F} = \{1\}$ will be discussed at the end of the proof. Denote by R the subring of $\mathbb{Q}$ generated by $\mathcal{P}^{-1}$. Note that $R^* = \mathcal{F} \cup (-\mathcal{F})$. Set $G = R^n \rtimes \mathrm{GL}(n,R)$ and $\Gamma = \mathbb{Z}^n \rtimes \mathrm{SL}(n,\mathbb{Z})$. Let $G = \{g_1, g_2, \cdots\}$ and define the finite index subgroups $\Gamma_k < \Gamma$ as $\Gamma_k = \Gamma \cap \bigcap_{i=1}^k g_i \Gamma g_i^{-1}$. Define the profinite action $\Gamma \curvearrowright (X,\mu) := \varprojlim \Gamma/\Gamma_k$.

We first argue why $\Gamma \curvearrowright (X,\mu)$ is essentially free. Let $p \in \mathcal{P}$ and take $k_1 < k_2 < k_3 < \cdots$ such that $(0, p^l 1) \in \{g_1, \ldots, g_{k_l}\}$. One checks that $\Gamma_{k_l} \subset G_l := p^l\mathbb{Z}^n \rtimes \mathrm{SL}(n,\mathbb{Z})$ and hence, it suffices to prove freeness of $\Gamma \curvearrowright \varprojlim \Gamma/G_l$. The latter has been shown in [14, discussion before Corollary 5.8].

Consider the diagonal action $\Gamma \curvearrowright X \times [0,1]^\Gamma$. Denote by $\mathcal{F}$ the set defined in (6.1). By Theorem 6.2, we have to prove that $\mathcal{F} = R_+^*$. It is more convenient to write $X = \varprojlim \Gamma/\Gamma_F$, where F runs through the finite subsets of G and $\Gamma_F := \Gamma \cap \bigcap_{g\in F} g\Gamma g^{-1}$. Whenever $g \in G$, the action $\Gamma \curvearrowright X$ is induced from

$$\Gamma \cap g\Gamma g^{-1} \curvearrowright \varprojlim_{F, g\in F} \frac{\Gamma \cap g\Gamma g^{-1}}{\Gamma_F}$$

and is induced from

$$g^{-1}\Gamma g \cap \Gamma \curvearrowright \varprojlim_{F, g^{-1}\in F} \frac{g^{-1}\Gamma g \cap \Gamma}{\Gamma_F} \; .$$

Both actions are conjugate by construction. If $g = (x, A)$, one checks that

$$\frac{[\Gamma : \Gamma \cap g\Gamma g^{-1}]}{[\Gamma : g^{-1}\Gamma g \cap \Gamma]} = |\det A| \; .$$

It follows that $R_+^* \subset \mathcal{F}$.

Conversely, we claim that whenever $\Lambda_1, \Lambda_2 < \Gamma$ are isomorphic finite index subgroups of Γ such that $\Gamma_k \subset \Lambda_1 \cap \Lambda_2$ for some k, then $[\Gamma : \Lambda_1]/[\Gamma : \Lambda_2] \in R_+^*$. Once this claim is proven, we get the required equality $\mathcal{F} = \mathbb{R}_+^*$. Let $\delta : \Lambda_1 \to \Lambda_2$ be an isomorphism and $\Gamma_k \subset \Lambda_1 \cap \Lambda_2$. We have $\delta(\Lambda_1 \cap \mathbb{Z}^n) = \Lambda_2 \cap \mathbb{Z}^n$. An elementary argument for this fact can be given by repeating the beginning of the proof of [30, Proposition 7.1]. Denoting by $\pi : \Gamma \to \mathrm{SL}(n,\mathbb{Z})$ the quotient map, $\pi(\Lambda_1)$ and $\pi(\Lambda_2)$ are isomorphic finite index subgroups of $\mathrm{SL}(n,\mathbb{Z})$. Using [14, Lemma 5.2], it follows that $[\mathrm{SL}(n,\mathbb{Z}) : \pi(\Lambda_1)] = [\mathrm{SL}(n,\mathbb{Z}) : \pi(\Lambda_2)]$. Hence, we get

$$\frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} = \frac{[\mathrm{SL}(n,\mathbb{Z}) : \pi(\Lambda_1)]\; [\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_1]}{[\mathrm{SL}(n,\mathbb{Z}) : \pi(\Lambda_2)]\; [\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_2]} = \frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_1]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_2]} \; .$$

Being finite index subgroups of $\mathbb{Z}^n$, we have $\Lambda_i \cap \mathbb{Z}^n = B_i\mathbb{Z}^n$ for some $B_i \in \mathrm{M}_n(\mathbb{Z})$ with $\det B_i \neq 0$, $i = 1, 2$. It follows that there exists $A \in \mathrm{GL}(n,\mathbb{Q})$ such that $\delta(x, 1) = (Ax, 1)$ for all $(x, 1) \in \Lambda_1 \cap \mathbb{Z}^n$. Hence,

$$\frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_1]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap \Lambda_2]} = \frac{[\mathbb{Z}^n : \mathbb{Z}^n \cap A^{-1}\mathbb{Z}^n]}{[\mathbb{Z}^n : \mathbb{Z}^n \cap A\mathbb{Z}^n]} \; .$$

Since $\Gamma_k \subset \Lambda_1 \cap \Lambda_2 \subset \mathbb{Z}^n \cap A\mathbb{Z}^n \cap A^{-1}\mathbb{Z}^n$, we find $\alpha \in R^* \cap (\mathbb{N} - \{0\})$ such that $\alpha\mathbb{Z}^n \subset \mathbb{Z}^n \cap A\mathbb{Z}^n \cap A^{-1}\mathbb{Z}^n$ for $i = 1, 2$. We conclude that $A \in \mathrm{GL}(n,R)$ and finally,

$$\frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} = |\det A|^{-1} \in R_+^* \; .$$

To conclude the proof of the theorem, we need to construct a free ergodic p.m.p. action $\Gamma \curvearrowright (X,\mu)$ such that the associated II_1 factor has trivial fundamental group. By [25, Corollary 0.2], the

Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$ has this property. Other examples can be given as follows. Let $p_1, p_2, \ldots$ be an enumeration of the prime numbers and set $\Gamma_k = p_1 \cdots p_k \mathbb{Z}^n \rtimes \mathrm{SL}(n,\mathbb{Z})$. By Theorem 6.2 and [14, Corollary 5.8], the diagonal product of the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$ and the profinite action $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_k$, provides a crossed product II_1 factor with trivial fundamental group. □

7 Property (T) and countability of the fundamental group

In his celebrated "rigidity paper" [3], Connes showed that II_1 factors arising from ICC groups with the property (T) of Kazhdan have countable fundamental group. Using the same ideas, it was later shown that for a separable II_1 factor M to have countable $\mathcal{F}(M)$, it is in fact sufficient that M contains a subfactor with the property (T) in the sense of [8] and having trivial relative commutant, $N' \cap M = \mathbb{C}$ (cf. Theorem 4.5.1 in [23]; see also [20] for a more general statement). It was also shown that if M is a separable II_1 factor, then the family of subfactors $N_i \subset M, i \in I$, having property (T) and trivial relative commutant, is countable modulo conjugacy by unitaries in M (cf. Theorem 4.6.1 in [23]; see also [22] for a related result). In this section, we prove some analogous results for II_1 equivalence relations.

To state them, we need some notations. Thus, if $\mathcal{R}$ is a II_1 equivalence relation on the standard probability space (X,μ), then we denote by $[\mathcal{R}]$ the *full group* of the equivalence relation $\mathcal{R}$, consisting of all non-singular isomorphisms $\phi : X \to X$ satisfying $(x,\phi(x)) \in \mathcal{R}$ for almost all $x \in X$. The *full pseudogroup* of $\mathcal{R}$ is denoted by $[[\mathcal{R}]]$ and consists of all non-singular partial automorphisms ϕ between measurable subsets $D(\phi), R(\phi) \subset X$, satisfying $(x,\phi(x)) \in \mathcal{R}$ for almost all $x \in D(\phi)$. Note that, since $\mathcal{R}$ is II_1, every $\phi \in [[\mathcal{R}]]$ is measure preserving. If $\Gamma \subset [[\mathcal{R}]]$ is a subgroup, we denote by $s(\Gamma) \subset X$ its *support*, i.e. the subset $Y \subset X$ with the property that $R(g) = D(g) = Y$, $\forall g \in \Gamma$. Two such subgroups $\Gamma, \Lambda \subset [[cR]]$ are *conjugate* by an element in $[[\mathcal{R}]]$ if there exists $\phi \in [[\mathcal{R}]]$ such that $D(\phi) = s(\Gamma)$, $R(\phi) = s(\Lambda)$ and $\phi\Gamma = \Lambda\phi$.

Theorem 7.1. *Let $\mathcal{R}$ be a II_1 equivalence relation on the probability space (X,μ).*

1. *If $[\mathcal{R}]$ contains a property (T) group Γ implementing an ergodic action on (X,μ), then $\mathcal{F}(\mathcal{R})$ is countable. More generally, if $[\mathcal{R}]$ contains a countable group Γ having a subgroup $H \subset \Gamma$ with the relative property (T) implementing an ergodic action on (X,μ), then $\mathcal{F}(\mathcal{R})$ is countable.*

2. *Let $\mathcal{T}$ be the set of property (T) subgroups $\Gamma \subset [[\mathcal{R}]]$ acting ergodically on $s(\Gamma)$. Then $\mathcal{T}$ is countable, modulo conjugacy by elements in $[[\mathcal{R}]]$.*

Note that the ergodicity assumption of the action of Γ on (X,μ) in Part 1 of the above statement is crucial. Indeed, Theorem 3.1 provides examples of free ergodic p.m.p. actions $G \curvearrowright (X,\mu)$ such that $\mathcal{R}(G \curvearrowright X)$ has uncountable fundamental group, but nevertheless G contains a subgroup having property (T) (which, a fortiori, acts non-ergodically on (X,μ)). In turn, the existence of a property (T) subgroup of $[\mathcal{R}]$ acting freely and ergodically on X, does not insure that the II_1 factor $L(\mathcal{R})$ has countable fundamental group. Indeed, by [7] there exist free ergodic p.m.p. actions of groups of the form $G = \Gamma \times \Sigma$, with Γ having property (T) and acting by Bernoulli shifts (thus ergodically), such that $M = \mathrm{L}^\infty(X) \rtimes G$ splits off the hyperfinite II_1 factor, and thus $\mathcal{F}(M) = \mathbb{R}_+$. In fact, as pointed out in [26], more than being countable, the fundamental group of $\mathcal{R}(G \curvearrowright X)$ is trivial.

Note also that in the case $\mathcal{R}$ comes from a free ergodic action of a property (T) group, $\Gamma \curvearrowright (X,\mu)$, Part 1 of Theorem 7.1 was already shown in [13, Corollary 1.8], in the case Γ is ICC, and in [14, Theorem 5.9], in the general case. We will use the above result in [31], to prove that the II_1

equivalence $\mathcal{R}$ obtained by restricting the II_∞ equivalence relation implemented by $\mathrm{SL}(n,\mathbb{Z}) \curvearrowright \mathbb{R}^n$, $n \geq 4$, to a subset of measure 1, has property (T) in the sense of Zimmer, yet cannot be implemented by an action (even non-free) of a property (T) group because $\mathcal{F}(\mathcal{R}) = \mathbb{R}_+$.

We will prove Theorem 7.1 by contradiction, using the property (T) of the subgroups and a "separability" argument, in the spirit of [23]. For more on this strategy of proofs, which grew out of Connes' rigidity paper [3], we send the interested reader to Section 4 in [28]. As a result of this argument, we obtain two copies $\Gamma_1, \Gamma_2 \subset [[\mathcal{R}]]$ of the same property (T) group, which are uniformly close one to the other. This in turn gives rise to a non-zero intertwiner $\phi \in [[\mathcal{R}]]$ between Γ_1, Γ_2. But if the Γ_i-actions are assumed ergodic, this forces $\mu(s(\Gamma_1)) = \mu(s(\Gamma_2))$ and the conjugacy of Γ_1, Γ_2.

The existence of an intertwiner between uniformly close subgroups in $[[\mathcal{R}]]$ is the subject of the next lemma. Recall that $[[\mathcal{R}]]$ has a natural metric space structure, inherited from the Hilbert-norm $\|\cdot\|_2$ of the underlying II_1 factor $L(\mathcal{R})$ associated with $\mathcal{R}$, by viewing every $\phi \in [[\mathcal{R}]]$ as a partial isometry in $L(\mathcal{R})$ and using the $\|\cdot\|_2$-norm on the latter. The metric can be concretely written as

$$d(\phi,\psi)^2 = \mu\big(D(\phi) \,\triangle\, D(\psi)\big) + 2\mu\big(\{x \in D(\phi) \cap D(\psi) \mid \phi(x) \neq \psi(x)\}\big) \;,$$

where $\triangle$ denotes the symmetric difference of two sets. We will also need the natural σ-finite measure $\mu^{(1)}$ on $\mathcal{R} \subset X \times Y$, defined by the formula

$$\mu^{(1)}(\mathcal{U}) = \int_X \#\{y \mid (x,y) \in \mathcal{U}\}\, d\mu(x) = \int_X \#\{x \mid (x,y) \in \mathcal{U}\}\, d\mu(y)$$

for all measurable subsets $\mathcal{U} \subset \mathcal{R}$.

Lemma 7.2. *Let $\mathcal{R}$ be a II_1 equivalence relation on the standard probability space (X,μ). Suppose that Γ is a countable group, $X_0, Y_0 \subset X$ and let*

$$\alpha : \Gamma \to [\mathcal{R}|_{X_0}] \quad \textit{and} \quad \beta : \Gamma \to [\mathcal{R}|_{Y_0}]$$

be group morphisms satisfying $d(\alpha_g, \beta_g) \leq 1/5$ for all $g \in \Gamma$ and $\mu(X_0), \mu(Y_0) \geq 3/4$. Then, there exist non-negligible measurable subsets $X_1 \subset X_0$, $Y_1 \subset Y_0$ and $\phi \in [[\mathcal{R}]]$ with $D(\phi) = X_1$, $R(\phi) = Y_1$ such that

$$\begin{aligned} &X_1 \text{ is globally } (\alpha_g)_{g\in\Gamma}\text{-invariant} \;,\; Y_1 \text{ is globally } (\beta_g)_{g\in\Gamma}\text{-invariant, and} \\ &\phi(\alpha_g(x)) = \beta_g(\phi(x)) \text{ for almost all } x \in D(\phi) \;. \end{aligned}$$

Proof. Denote by Tr the normal faithful semi-finite trace on $\mathrm{L}^\infty(\mathcal{R})$ given by integration along $\mu^{(1)}$. Let $p \in \mathrm{L}^\infty(\mathcal{R})$ be the projection onto $\mathcal{R} \cap X_0 \times Y_0$ and $e \leq p$ the projection onto $\{(z,z) \mid z \in X_0 \cap Y_0\}$. Set $B = \mathrm{L}^\infty(\mathcal{R})p$. The group Γ acts by automorphisms ρ_g of B given by

$$(\rho_g F)(x,y) = F(\alpha_{g^{-1}}(x), \beta_{g^{-1}}(y)) \quad \text{for almost all } (x,y) \in \mathcal{R} \cap X \times Y \;.$$

Since $\|\rho_g(e) - e\|_{2,\mathrm{Tr}}^2 = 2\mu\big(\{z \in X_0 \cap Y_0 \mid \alpha_{g^{-1}}(z) \neq \beta_{g^{-1}}(z)\}\big)$, we get

$$\|\rho_g(e) - e\|_{2,\mathrm{Tr}} \leq \frac{1}{5} \quad \text{for all } g \in \Gamma \;.$$

Define $a \in B^+$ as the unique element of minimal $\|\cdot\|_{2,\mathrm{Tr}}$ in the weakly closed convex hull $\overline{\mathrm{conv}}\{\rho_g(e) \mid g \in \Gamma\}$. It follows that $\|a - e\|_{2,\mathrm{Tr}} \leq 1/5$ and that $\rho_g(a) = a$ for all $g \in \Gamma$. Note that $0 \leq a \leq 1$.

Defining f as the spectral projection $f = \chi_{[1/2,1]}(a)$, we find that $\|f - e\|_{2,\mathrm{Tr}} \leq 2/5$ and $\rho_g(f) = f$ for all $g \in \Gamma$. We write $f = \chi_W$, where $W \subset \mathcal{R} \cap X_0 \times Y_0$ is globally $(\rho_g)_{g\in\Gamma}$-invariant and satisfies

$$\mu^{(1)}\big(W \;\triangle\; \{(z,z) \mid z \in X_0 \cap Y_0\}\big) \leq \frac{4}{25}\,. \tag{7.1}$$

Denote ${}_xW := \{y \in Y_0 \mid (x,y) \in W\}$ and $W_y := \{x \in X_0 \mid (x,y) \in W\}$. Define

$$W_0 := \{(x,y) \in W \mid {}_xW \text{ and } W_y \text{ are singletons }\}\,.$$

Then, W_0 is still globally $(\rho_g)_{g\in\Gamma}$-invariant. Since $\mu(X_0), \mu(Y_0) \geq 3/4$ and $\mu(X_0 \triangle Y_0) \leq 1/25$, we have $\mu(X_0 \cap Y_0) \geq 3/4 - 1/25$. By (7.1), the set of $x \in X_0$ such that ${}_xW$ is a singleton, then has measure at least $3/4 - 1/25 - 4/25$. The same holds for the set of $y \in Y_0$ such that W_y is a singleton. So, W_0 has measure at least $1/10$. By construction, W_0 is the graph of a partial automorphism $\phi \in [[\mathcal{R}]]$ satisfying all the conclusions of the lemma. □

Proof of Theorem 7.1. Let us first prove Part 1 of the theorem. By the relative property (T) of $H \subset \Gamma$, there exist $F \subset \Gamma$ and $0 < \varepsilon < 1/4$ such that whenever $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ on a Hilbert space $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ a unit vector satisfying $\|\pi(g)\xi_0 - \xi_0\| \leq \varepsilon$ for all $g \in F$, then $\|\pi(h)\xi_0 - \xi_0\| \leq 1/8$ for all $h \in H$.

Choose for every $t \in (0,1)$ a measurable subset $Y_t \subset X$ with $\mu(Y_t) = t$ and such that $Y_s \subset Y_t$ if $s \leq t$. Assume that the fundamental group of $\mathcal{R}$ is uncountable. For every $t \in \mathcal{F}(\mathcal{R}) \cap (3/4, 1)$, choose an isomorphism $\Delta_t : X \to Y_t$ between $\mathcal{R}$ and $\mathcal{R}|_{Y_t}$. Note that Δ_t scales the measure μ by t. Define $\alpha_g^t = \Delta_t \circ \alpha_g \circ \Delta_t^{-1}$. Note that $\alpha_g^t \in [[\mathcal{R}]]$ with $D(\alpha_g^t) = R(\alpha_g^t) = Y_t$. Since $\mathcal{F}(\mathcal{R}) \cap (3/4,1)$ is uncountable, separability of the metric space $([[\mathcal{R}]], d)$ yields $s, t \in \mathcal{F}(\mathcal{R}) \cap (3/4,1)$ with $s < t$ and $d(\alpha_g^s, \alpha_g^t) \leq \varepsilon/2$ for all $g \in F$.

Define the Hilbert space $\mathcal{H} = \mathrm{L}^2(\mathcal{R} \cap Y_s \times Y_t, \mu^{(1)})$ and the unitary representation

$$\pi : \Gamma \to \mathcal{U}(\mathcal{H}) : (\pi(g)\xi)(x,y) = \xi(\alpha_{g^{-1}}^s(x), \alpha_{g^{-1}}^t(y))\,.$$

Set $\Delta_s := \{(y,y) \mid y \in Y_s\}$ and $\xi_0 := s^{-1/2}\chi_{\Delta_s}$. Then, ξ_0 is a unit vector in $\mathcal{H}$ and, for all $g \in F$,

$$\|\pi(g)\xi_0 - \xi_0\|^2 = 2s^{-1}\mu\big(\{y \in Y_s \mid \alpha_{g^{-1}}^s(y) \neq \alpha_{g^{-1}}^t(y)\}\big) \leq s^{-1} d(\alpha_g^s, \alpha_g^t)^2 \leq \varepsilon^2\,.$$

It follows that $\|\pi(h)\xi_0 - \xi_0\| \leq 1/8$ for all $h \in H$. So, for all $h \in H$, we have

$$2\mu\big(\{y \in Y_s \mid \alpha_h^s(y) \neq \alpha_h^t(y)\}\big) \leq \frac{s}{64} \leq \frac{1}{64}\,.$$

Since also, given $g \in F$,

$$\mu(Y_t \setminus Y_s) \leq d(\alpha_g^s, \alpha_g^t)^2 \leq \frac{\varepsilon^2}{4} \leq \frac{1}{64}\,,$$

it follows that

$$d(\alpha_h^s, \alpha_h^t)^2 \leq \mu(Y_t \setminus Y_s) + \frac{1}{64} < \frac{1}{25}$$

for all $h \in H$. Since $(\alpha_h)_{h\in H}$ implements an ergodic action on (X,μ), the same holds for $(\alpha_h^s)_{h\in H}$, $(\alpha_h^t)_{h\in H}$ and so, Lemma 7.2 provides an element $\phi \in [[\mathcal{R}]]$ with $D(\phi) = Y_s$ and $R(\phi) = Y_t$. Since ϕ is a measure preserving isomorphism between Y_s and Y_t and $\mu(Y_s) = s < t = \mu(Y_t)$, we reached a contradiction.

To prove Part 2 of Theorem 7.1, assume by contradiction that there exist uncountably many subgroups $\{\Gamma_i \mid i \in I\}$ in $[[\mathcal{R}]]$ which have property (T) and are non-conjugate in $[[\mathcal{R}]]$. We continue to use the measurable subsets $Y_t \subset X$ with $\mu(Y_t) = t$ and $Y_s \subset Y_t$ whenever $s \leq t$. By the ergodicity of $\mathcal{R}$, we may assume that for every $i \in I$, the support of Γ_i is one of the Y_s. Note that by [33] we may assume all Γ_i's are representations α_i of the same property (T) group Γ. Finally, we may assume that there exists $t \in (0,1)$ such that $\mu(s(\Gamma_i)) \in (3t/4, t)$ for all $i \in I$. So, replacing $\mathcal{R}$ by $\mathcal{R}|_{Y_t}$, we may assume that $\mu(s(\Gamma_i)) \in (3/4, 1)$ for all $i \in I$.

By the property (T) of Γ, there exist $F \subset \Gamma$ and $0 < \varepsilon < 1/4$ such that whenever $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ on a Hilbert space $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ a unit vector satisfying $\|\pi(g)\xi_0 - \xi_0\| \leq \varepsilon$ for all $g \in F$, then $\|\pi(g)\xi_0 - \xi_0\| \leq 1/8$ for all $g \in \Gamma$.

Now, by the separability of $([[\mathcal{R}]], d)$, there exist $i \neq j$ such that $d(\alpha_i(g), \alpha_j(g)) \leq \varepsilon/2$, $\forall g \in F$. Let $Y_i \subset X$, $Y_j \subset X$ be the support of Γ_i resp. Γ_j and assume $Y_i \subset Y_j$. We define $\mathcal{H}$, $\xi_0 \in \mathcal{H}$, $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ as before, but replacing α_g^s by $\alpha_i(g)$, α_g^t by $\alpha_j(g)$, Y_s by Y_i and Y_t by Y_j. The same estimates then show that ξ_0 is a unit vector satisfying $\|\pi_g(\xi_0) - \xi_0\| \leq \varepsilon$, $\forall g \in F$. Thus, $\|\pi(g)\xi_0 - \xi_0\| \leq 1/8$ for all $g \in \Gamma$. As before, this translates into $d(\alpha_i(g), \alpha_j(g)) \leq 1/4$, $\forall g \in \Gamma$. By Lemma 7.2, this implies Γ_i, Γ_j are conjugate by an element in $[[\mathcal{R}]]$, contradicting our initial assumption. □

Part 2 of Theorem 7.1 readily implies that the functor $\Gamma \mapsto \mathcal{R}_\Gamma$, from free ergodic p.m.p. actions of property (T) groups with morphisms given by conjugacy, to the associated equivalence relations with morphisms given by orbital isomorphism, is "countable to one". In other words, there are at most countably many non-conjugate free ergodic p.m.p. actions in each OE class of a free ergodic p.m.p. action of a property (T) group. In fact, even more is true: any free ergodic p.m.p. action of a property (T) group follows "orbit equivalent superrigid, modulo countable classes", in a sense made precise below.

Corollary 7.3. *Let $\Gamma \curvearrowright X$ be a free ergodic p.m.p. action of a property (T) group. Let $\Gamma_i \curvearrowright X_i$, $i \in I$, be a family of free ergodic p.m.p. actions such that $\mathcal{R}_\Gamma \simeq \mathcal{R}_{\Gamma_i}^{t_i}$, for some $t_i \geq 0$. Then the family I is countable, modulo conjugacy of actions.*

Proof. We may assume that $t_i \geq 1/c$ for all $i \in I$ and some $c > 0$. Setting $\mathcal{R} = (\mathcal{R}_\Gamma)^c$, we can view all Γ_i as subgroups of $[[\mathcal{R}]]$, with the action of Γ_i on $s(\Gamma_i) \subset X$ being conjugate to $\Gamma_i \curvearrowright X_i$. By [10, Corollary 1.4], all Γ_i have property (T). So, by Part 2 of Theorem 7.1, the family I is countable modulo conjugacy of actions. □

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