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Some Examples of Blackadar and Kirchberg's MF Algebras

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Abstract: In the paper, we provide some examples of MF algebras by considering minimal or maximal tensor products of MF algebras and crossed products of MF algebras by finite groups or an integer group. We also present some examples of C*-algebras, whose BDF extension semigroup is not a group. These examples include, for example, $C_r^*(F_n) \otimes_{max} C^*(F_n)$, $C_r^*(F_n) \otimes_{min} C^*(F_n)$, $C_r^*(H_1 * H_2)$ with $2 \leq |H_1| < \infty$ and $3 \leq |H_2| < \infty$ where $|H_1|$, $|H_2|$ are the orders of thr groups H_1, or H_2 respectively, and several others.

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1. Introduction

The notion of MF algebra was introduced by Blackadar and Kirchberg in [**3**]. A separable C*-algebra is an MF algebra if and only if it can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers n_k, $k = 1, 2, \ldots$. It follows from the definition that a separable quasidiagonal C*-algebra is an MF algebra.

The class of MF algebras is closely connected to Brown, Douglas and Fillmore's extension theory (see [**7**]). In [**14**], Haagerup and Thorbjørnsen solved a long standing open problem by showing that $Ext(C_r^*(F_n))$ is not a group for $n \geq 2$, where $C_r^*(F_n)$ is the reduced C*-algebra of the free group F_n and $Ext(C_r^*(F_n))$ is the BDF's extension semigroup. Their result follows from a combination of Voiculescu's argument in [**25**] and their remarkable work on proving that $C_r^*(F_n)$ is an MF algebra. Basing on Haagerup and Thorbjørnsen's approach to BDF's extension semigroup, one would be interested in finding other natural examples of C*-algebras whose BDF extension semigroup is not a group besides the ones in [**1**], [**14**] and [**27**]. This is one of motivations of our investigation on MF algebras. In [**18**], we considered the full free products of MF algebras and showed that $Ext(C_r^*(F_n) *_{\mathbb{C}} \mathcal{A})$ is not a group when $\mathcal{A}$ is any MF algebra and $n \geq 2$ is any positive integer. In this paper, we consider tensor products of MF algebras and crossed products of MF algebras by finite groups or an integer group. In fact, we are able to show that, for example, both $Ext(C_r^*(F_n) \otimes_{max} C^*(F_n))$ and $Ext(C_r^*(F_n) \otimes_{min} C^*(F_n))$ are not groups (see Corollary 3.2, Corollary 3.3, Corollary 4.1 and Example 3.1).

Blackadar and Kirchberg's MF algebra is also closely connected to Voiculescu's topological free entropy dimension. In [**26**], for a family of self-adjoint elements $x_1, \ldots, x_n$ in a unital C*-algebra $\mathcal{A}$, Voiculescu introduced the notion of topological free entropy dimension of $x_1, \ldots, x_n$.

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In the definition of topological free entropy dimension, it requires that Voiculescu's norm microstate space of $x_1, \ldots, x_n$ is "eventually" nonempty, which is equivalent to say that the C*-subalgebra generated by $x_1, \ldots, x_n$ in $\mathcal{A}$ should be an MF algebra. Thus it is important to determine which C*-algebra belongs to the class of MF algebra, where Voiculescu's topological free entropy dimension is well defined (More discussion on topological free entropy dimension can be found in [**16**], [**17**], [**18**]).

More connections between MF algebra and the classification of C*-algebras can also be found in [**3**], [**4**], [**5**].

Now we outline the main results obtained in the paper. First, we study the minimal and maximal tensor products of MF algebras and obtain the following results:

Theorem 3.1: Suppose $C^*(F_N)$ is the full C*-algebra of the free group F_N (N is a positive integer or ∞) and $\mathcal{B}$ is a unital separable MF algebra. Then $C^*(F_N) \otimes_{max} \mathcal{B}$ is an MF algebra.

Proposition 3.1 & 3.2: Suppose $\mathcal{A}$ and $\mathcal{B}$ are separable MF algebras. If $\mathcal{A}$ is either exact or quasidiagonal, then $\mathcal{A} \otimes_{min} \mathcal{B}$ is an MF algebra.

We also consider the crossed product of an MF algebra by finite groups or an integer group and obtain an analogue of Pimsner and Voiculescu's result [**21**] in the context of MF algebra.

Theorem 4.2: Suppose that $\mathcal{A}$ is a finitely generated unital MF algebra and α is a homomorphism from $\mathbb{Z}$ into $Aut(\mathcal{A})$ such that there is a sequence of integers $0 \leq n_1 < n_2 < \cdots$ satisfying $\lim_{j\to\infty} \|\alpha(n_j)a - a\| = 0$ for any $a \in \mathcal{A}$. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is an MF algebra.

Combining with Haagerup and Thorbjørnsen's result that $C_r^*(F_n)$ is an MF algebra, we obtain more examples of C*-algebras whose BDF extension semigroup is not a group.

Corollary 3.2: Suppose $n, m \geq 2$ are positive integers and $\mathcal{B}$ is a unital separable MF algebra. Then $(C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}) \otimes_{max} C^*(F_m)$ is an MF algebra and $Ext((C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}) \otimes_{max} C^*(F_m))$ is not a group.

Corollary 3.3: Suppose that $\mathcal{B}$ is a unital separable MF algebras. Then $Ext(C_r^*(F_n) \otimes_{min} \mathcal{B})$ is not a group for $n \geq 2$.

Corollary 4.2: Assume $n \geq 1, m \geq 1$ are positive integers and G is a finite group with $|G| \geq 2$, where $|G|$ is the order of group G. Then $Ext(C_r(F_n * (G \times F_m)))$ is not a group.

Corollary 4.3: Assume that $n \geq 2$ is a positive integer and $H_1, \ldots, H_n$ is a family of finite groups. Let

$$H = H_1 * H_2 * \cdots * H_n.$$

Moreover, if there are $1 \leq i \neq j \leq n$ such that

$$|H_i| \geq 2 \qquad and \qquad |H_j| \geq 3,$$

where $|H|$ denotes the order of the group H, then $Ext(C_r^*(H))$ is not a group.

Corollary 4.4: Let $SL_2(\mathbb{Z})$ to be the spacial linear group of 2×2 matrices with integer entries. Then $C_r^*(SL_2(\mathbb{Z}))$ is an MF algebra and $Ext(C_r^*(SL_2(\mathbb{Z})))$ is not a group.

Corollary 4.5: Let $C_r^*(F_2)$ be the reduced C*-algebra of the free group F_2. Let u_1, u_2 be the canonical unitary generators of $C_r^*(F_2)$ and $0 < \theta < 1$ be a positive number. Let α be a homomorphism from $\mathbb{Z}$ into $Aut(C_r^*(F_2))$ induced by the following mapping: $\forall\ n \in \mathbb{Z}$,

$$\alpha(n)(u_1) = e^{2n\pi\theta\cdot i}u_1 \qquad and \qquad \alpha(n)(u_2) = e^{2n\pi\theta\cdot i}u_2.$$

Then $C_r^*(F_2) \rtimes_\alpha \mathbb{Z}$ is an MF algebra and $Ext(C_r^*(F_2) \rtimes_\alpha \mathbb{Z})$ is not a group.

We hope our investigation on MF algebra will be helpful in the study of Blackadar and Kirchberg's MF algebras, BDF's extension semigroup and Voiculescu's topological free entropy theory.

The organization of the paper is as follows. In section 2, we introduce some notation and preliminaries. In section 3, we study tensor products of MF algebras. In section 4, we consider crossed product of an MF algebra by the actions of a finite group or an integer group.

2. Notation and Preliminaries

Suppose $\mathcal{H}$ is a separable complex Hilbert space and $B(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. Suppose $\{x, x_k\}_{k=1}^\infty$ is a family of elements in $B(\mathcal{H})$. We say $x_k \to x$ in $*$-SOT ($*$-strong operator topology) if and only if $x_k \to x$ in SOT and $x_k^* \to x^*$ in SOT.

For any sequence of C* algebras $\mathcal{A}_k \subseteq B(\mathcal{H}_k)$, $k = 1, 2, \ldots$, we introduce the direct product of $\mathcal{A}_k$ as follows.

$$\prod_k \mathcal{A}_k = \left\{ \langle x_k\rangle_{k=1}^\infty \mid \sup_k \|x_k\| < \infty, \quad where \quad x_k \in \mathcal{A}_k \right\},$$

where the norm of any element $\langle x_k\rangle_{k=1}^\infty$ in $\prod_k \mathcal{A}_k$ is defined by

$$\|\langle x_k\rangle_{k=1}^\infty\| = \sup_k \|x_k\|.$$

Let

$$\sum_k \mathcal{A}_k = \left\{ \langle x_k\rangle_{k=1}^\infty \in \prod_k \mathcal{A}_k \mid \limsup_{k\to\infty} \|x_k\| = 0, \quad where \quad x_k \in \mathcal{A}_k \right\}.$$

It is easy to see that $\sum_k \mathcal{A}_k$ is a closed two-sided idea of the C*-algebra $\prod_k \mathcal{A}_k$. Thus $\prod_k \mathcal{A}_k / \sum_k \mathcal{A}_k$ is also a C*-algebra. Denote by π the canonical quotient mapping from $\prod_k \mathcal{A}_k$ onto $\prod_k \mathcal{A}_k / \sum_k \mathcal{A}_k$, and denote $\pi(\langle x_k\rangle_{k=1}^\infty)$ by $[\langle x_k\rangle_{k=1}^\infty]$ for any element $\langle x_k\rangle_{k=1}^\infty$ in $\prod_k \mathcal{A}_k$.

Suppose $\mathcal{A}$ is a C*-algebra on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^\infty = \oplus_{\mathbb{N}}\mathcal{H}$, and for any $x \in \mathcal{A}$, let x^∞ be the element $\langle x, x, x, \ldots\rangle = \oplus_{\mathbb{N}} x$ in $\prod_k \mathcal{A}^{(k)} \subseteq B(\mathcal{H}^\infty)$, where $\mathcal{A}^{(k)}$ is the k-th copy of $\mathcal{A}$. Let $\mathcal{A}^\infty = \{x^\infty \in \prod_k \mathcal{A}^{(k)} \mid x \in \mathcal{A}\}$ be a C*-subalgebra of $\prod_k \mathcal{A}^{(k)}$.

For each positive integer n, denote the $n \times n$ matrix algebra over the complex numbers by $\mathcal{M}_n(\mathbb{C})$ and the group of unitary matrices in $\mathcal{M}_n(\mathbb{C})$ by $\mathcal{U}_n(\mathbb{C})$.

Recall the definition of residually finite dimensional C*-algebras as follows.

Definition 2.1. *A separable C*-algebra $\mathcal{A}$ is residually finite dimensional if there is an embedding from $\mathcal{A}$ into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})$ for some positive integers $\{n_k\}_{k=1}^\infty$.*

Recall the definition of quasidiagonal C*-algebras as follows.

Definition 2.2. *A set of elements $\{a_1, \ldots, a_n\} \subseteq B(\mathcal{H})$ is quasidiagonal if there is an increasing sequence of finite-rank projections $\{p_i\}_{i=1}^{\infty}$ on H tending strongly to the identity such that $\|a_j p_i - p_i a_j\| \to 0$ as $i \to \infty$ for any $1 \leq j \leq n$. A separable C*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is quasidiagonal if there is an increasing sequence of finite-rank projections $\{p_i\}_{i=1}^{\infty}$ on H tending strongly to the identity such that $\|xp_i - p_i x\| \to 0$ as $i \to \infty$ for any $x \in \mathcal{A}$. An abstract separable C*-algebra $\mathcal{A}$ is quasidiagonal if there is a faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ such that $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$ is quasidiagonal.*

Let $\mathbb{C}\langle X_1, \ldots, X_n\rangle$ be the unital noncommutative polynomials in the indeterminates $X_1, \ldots, X_n, X_1^*, \ldots, X_n^*$. The following lemma is well known.

Lemma 2.1. *Suppose $\mathcal{A} \subseteq B(\mathcal{H})$ is a unital separable quasidiagonal C*-algebra and $x_1, \ldots, x_n$ are elements in $\mathcal{A}$. For any $\epsilon > 0$, a finite subset $\{P_1, \ldots, P_r\}$ of $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, and a finite subset $\{\xi_1 \ldots, \xi_s\}$ of $\mathcal{H}$, there is a finite rank projection p in $B(\mathcal{H})$ such that:*

(i) $\|P_i(p\ x_1 p, \ldots, p\ x_n p)\xi_j - P_i(x_1, \ldots, x_n)\xi_j\| < \epsilon$; *for all* $1 \leq i \leq r$ *and* $1 \leq j \leq s$.
(ii) $\big|\|P_i(p\ x_1 p, \ldots, p\ x_n p)\|_{B(p\mathcal{H})} - \|P_i(x_1, \ldots, x_n)\|_{\mathcal{A}}\big| < \epsilon$, *for all* $1 \leq i \leq r$.

The following well known result of Voiculescu plays a crucial role in the study of quasidiagonal C*-algebras.

Lemma 2.2. *Suppose $\mathcal{A}$ is a separable quasidiagonl C*-algebra and π is an essentially faithful $*$-representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$, i.e. $\pi(\mathcal{A}) \cap \mathcal{K} = \{0\}$ where $\mathcal{K}$ is the set of compact operators on $\mathcal{A}$. Then $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$ is quasidiagonal.*

The notion of MF algebra was introduced by Blackadar and Kirchberg in [**3**].

Definition 2.3. *A separable C*-algebra $\mathcal{A}$ is an MF algebra if there is an embedding from $\mathcal{A}$ into $\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for some positive integers $\{n_k\}_{k=1}^{\infty}$.*

Remark 2.1. *The following implication is obvious from the definitions: If a C*-algebra $\mathcal{A}$ is residually finite dimensional then it is a quasidiagonal C*-algebra. If $\mathcal{A}$ is a quasidiagonal C*-algebra, then it is an MF algebra.*

The following lemma gives equivalent definitions of an MF algebra.

Lemma 2.3. *Suppose that $\mathcal{A}$ is a unital C*-algebra generated by a family of elements $x_1, \ldots, x_n$ in $\mathcal{A}$. Then the following are equivalent:*

(1) *$\mathcal{A}$ is an MF algebra;*
(2) *For any $\epsilon > 0$ and any finite family $\{P_1, \ldots, P_r\}$ in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, there are a positive integers k and family of matrices $\{A_1, \ldots, A_n\}$ in $\mathcal{M}_k(\mathbb{C})$, such that*

$$\max_{1 \leq j \leq r} \Big| \|P_j(A_1^{(k)}, \ldots, A_n^{(k)})\| - \|P_j(x_1, \ldots, x_n)\| \Big| < \epsilon.$$

(3) *Suppose $\pi : \mathcal{A} \to B(\mathcal{H})$ is a faithful $*$-representation of $\mathcal{A}$ on an infinite dimensional separable complex Hilbert space $\mathcal{H}$. Then there are a sequence of positive integers $\{m_k\}_{k=1}^{\infty}$, families of self-adjoint matrices $\{A_1^{(k)}, \ldots, A_n^{(k)}\}$ in $\mathcal{M}_{m_k}(\mathbb{C})$ for $k = 1, 2, \ldots$, and unitary operators $U_k : \mathcal{H} \to (\mathbb{C}^{m_k})^{\infty}$ for $k = 1, 2, \ldots$, such that*

(a) $\lim_{k\to\infty}\|P(A_1^{(k)},\dots,A_n^{(k)})\| = \|P(x_1,\dots,x_n)\|,\ \forall\ P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$;
(b) $U_k^*\ (A_i^{(k)})^\infty\ U_k \to \pi(x_i)$ *in* $*-SOT$ *as* $k\to\infty$, *for* $1\le i\le n$, *where* $(A_i^{(k)})^\infty = A_i^{(k)}\oplus A_i^{(k)}\oplus A_i^{(k)}\cdots \in B((\mathbb{C}^n)^\infty)$.

(4) *Suppose* $\pi:\mathcal{A}\to B(\mathcal{H})$ *is a faithful* $*$*-representation of* $\mathcal{A}$ *on an infinite dimensional separable complex Hilbert space* $\mathcal{H}$. *Then there is a family of elements* $\{a_1^{(k)},\dots,a_n^{(k)}\}_{k=1}^\infty\subseteq B(\mathcal{H})$ *such that*
(a) *For each* $k\ge 1$, $\{a_1^{(k)},\dots,a_n^{(k)}\}\subseteq B(\mathcal{H})$ *is quasidiagonal;*
(b) $\lim_{k\to\infty}\|P(a_1^{(k)},\dots,a_n^{(k)})\| = \|P(x_1,\dots,x_n)\|,\ \forall\ P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$;
(c) $\forall\ 1\le i\le n,\ \ a_i^{(k)}\to\pi(x_i),\ in\ *-SOT\ as\ k\to\infty$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was proved in [**18**]. (3) $\Rightarrow$ (4) is immediate. We only need to show (4) $\Rightarrow$ (2). For any $\epsilon>0$ and $\{P_1,\dots,P_r\}$ in $\mathbb{C}\langle X_1,\dots,X_n\rangle$, by condition (b), there is some $m\ge 1$ such that

$$\max_{1\le j\le r}\left|\|P_j(a_1^{(m)},\dots,a_n^{(m)})\| - \|P_j(x_1,\dots,x_n)\|\right| < \epsilon/2;$$

and $a_1^{(m)},\dots,a_n^{(m)}\subseteq B(\mathcal{H})$ is quasidiagonal. Thus from Lemma 2.1 it follows that there is a finite rank projection p such that

$$\max_{1\le j\le r}\left|\|P_j(pa_1^{(m)}p,\dots,pa_n^{(m)}p)\|_{B(p\mathcal{H})} - \|P_j(a_1^{(m)},\dots,a_n^{(m)})\|\right| < \epsilon/2.$$

Hence

$$\max_{1\le j\le r}\left|\|P_j(pa_1^{(m)}p,\dots,pa_n^{(m)}p)\|_{B(p\mathcal{H})} - \|P_j(x_1,\dots,x_n)\|\right| < \epsilon.$$

Let $k = dim(B(p\mathcal{H}))$. Note that $B(p\mathcal{H})\simeq\mathcal{M}_k(\mathbb{C})$. Let $A_i = pa_i^{(m)}p$ for $1\le i\le n$. Then we have the desired result. □

The following result can be found in [**8**] (See also [**24**],[**14**]) which connects MF algebras to BDF's extension semigroups.

Lemma 2.4. *If* $\mathcal{A}$ *is a unital separable MF algebra and* $Ext(\mathcal{A})$ *is a group, then* $\mathcal{A}$ *is quasidiagonal.*

Proof. For the purpose of completeness, we sketch its proof here. By Voiculescu's characterization of quasidiagonality, to show that $\mathcal{A}$ is quasidiagonal, it suffices to show that for any $a_1,\dots,a_n$ in $\mathcal{A}$ and any $\epsilon>0$, there is a unital completely positive map $\psi:\mathcal{A}\to\mathcal{M}_k(\mathbb{C})$ for some positive integer k such that (1) $\|\psi(a_i)\|\ge\|a\|-\epsilon$; and (2) $\|\psi(a_ia_j)-\psi(a_i)\psi(a_j)\|\le\epsilon$.

Note that $\mathcal{A}$ is an MF algebra. There is a unital embedding $\rho:\mathcal{A}\to\prod_k\mathcal{M}_{n_k}(\mathbb{C})/\sum_k\mathcal{M}_{n_k}(\mathbb{C})$ for some positive integers $\{n_k\}_{k=1}^\infty$. Let $\mathcal{H}=\mathbb{C}^{n_1}\oplus\mathbb{C}^{n_2}\oplus\cdots$ be a Hilbert space and p_k the projection from $\mathcal{H}$ onto the subspace $0\oplus\cdots\oplus 0\oplus\mathbb{C}^{n_k}\oplus 0\oplus\cdots$. Let $\mathcal{K}(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$ and π the quotient map from $B(\mathcal{H})$ onto $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Then

$$\rho:\mathcal{A}\to\prod_k\mathcal{M}_{n_k}(\mathbb{C})/\sum_k\mathcal{M}_{n_k}(\mathbb{C})\hookrightarrow B(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Let $[\langle A_i^{(k)}\rangle_{k=1}^{\infty}] = \rho(a_i) \in \prod_k \mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$, and $x_i = \langle A_i^{(k)}\rangle_{k=1}^{\infty} \in \prod_k \mathcal{M}_{n_k}(\mathbb{C}) \hookrightarrow B(\mathcal{H})$ for $1 \leq i \leq n$. Thus (i) $p_k x_i = p_k x_i p_k$. By Proposition 3.3.5 in [**3**], we can assume that (ii) $\|a_i\| = \lim_k \|p_k x_i p_k\| = \lim_k \|A_i^{(k)}\|$; and (iii) $\|a_i a_j\| = \lim_k \|p_k x_i p_k x_j p_k\| = \lim_k \|A_i^{(k)} A_j^{(k)}\|$ for $1 \leq i, j \leq n$.

Since $Ext(\mathcal{A})$ is a group, by [**2**], there is a unital completely positive map $\phi : \mathcal{A} \to B(\mathcal{H})$ such that $\rho = \pi \circ \phi$. It follows that $\pi(\phi(a_i)) = \pi(x_i)$ and $\pi(\phi(a_i a_j)) = \pi(x_i x_j)$ for $1 \leq i, j \leq n$. Then $\phi(a_i) - x_i$ and $\phi(a_i a_j) - x_i x_j$ are in $\mathcal{K}(\mathcal{H})$, whence (iv) $\lim_k \|p_k x_i p_k - p_k \phi(a_i) p_k\| = 0$ and (v) $\lim_k \|p_k x_i x_j p_k - p_k \phi(a_i a_j) p_k\| = 0$

Define

$$\psi_k : \mathcal{A} \to B(p_k\mathcal{H}) \simeq \mathcal{M}_{n_k}(\mathbb{C}) \qquad by \qquad \psi_k(a) = p_k \phi(a) p_k, \quad \forall\ a \in \mathcal{A}.$$

Then each ψ_k is a unital completely positive map from $\mathcal{A}$ to $\mathcal{M}_{n_k}(\mathbb{C})$. When k is large enough, by (ii) and (iv), we know that ψ_k satisfies (1). By (i), (iv) and (v), ψ_k satisfies (2). Hence $\mathcal{A}$ is quasidiagonal. □

We also need the following result by Rosenberg [**22**].

Lemma 2.5. *Suppose that G is a countable discrete group. If $C_r^*(G)$ is quasidiagonal, then G is an amenable group.*

3. Tensor products of MF Algebras

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two C*-algebras. Denote the minimal tensor product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} \otimes_{min} \mathcal{B}$ and the maximal tensor product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} \otimes_{max} \mathcal{B}$.

Suppose that F_n is the free group on n many generators with $n \in \mathbb{N}$. Let $C_r^*(F_n)$ be the reduced C*-algebra associated with F_n and $C^*(F_n)$ be the full C*-algebra of the group F_n.

3.1. Maximal Tensor Product. In this subsection, we consider the question whether the maximal tensor product of $C^*(F_n)$ and an MF algebra is again an MF algebra. Before we present our main result, we will introduce a few lemmas first.

Lemma 3.1. *Suppose $\mathcal{H}$ is a Hilbert space. For any $\delta > 0$, there is a polynomial $P \in C\langle X\rangle$ with $P(0) = 0$ such that, if x in $B(\mathcal{H})$ with $0 < x < I$, then $\|\sqrt{x} - P(x)\| \leq \delta$.*

Proof. Consider $C[0, 1]$. For any $\delta > 0$, there is a polynomial $P \in C\langle X\rangle$ with $P(0) = 0$ such that

$$\max_{0 \leq t \leq 1} |\sqrt{t} - P(t)| \leq \delta.$$

Now it follows from the functional calculus that, if x in $B(\mathcal{H})$ with $0 < x < I$, then

$$\|\sqrt{x} - P(x)\| = \sup_{t \in \sigma(x)} |\sqrt{t} - P(t)| \leq \max_{0 \leq t \leq 1} |\sqrt{t} - P(t)| \leq \delta.$$

□

Lemma 3.2. *Suppose that $\mathcal{H}$ is a Hilbert space. For any $\delta > 0$, there is a function*

$$f_\delta : (0, \infty) \to (0, \infty)$$

such that the following hold:

(1) $\lim_{t\to 0} f_\delta(t) \leq \delta$;

(2) *Suppose (i) p and q are mutually orthogonal, equivalent, finite rank, projections in $B(\mathcal{H})$ with a partial isometry $w \in B(\mathcal{H})$ satisfying $w^*w = p, ww^* = q$. (ii) Suppose $\{u_1, \ldots, u_n, v_1, \ldots v_m\}$ are unitary elements in $B(\mathcal{H})$ such that*

$$u_i v_j = v_i u_j, \qquad \text{and} \qquad \|pv_j - v_j p\| \leq t, \quad \forall\ 1 \leq i \leq n, 1 \leq j \leq m.$$

Let

$$U_i = pu_ip + \sqrt{p - pu_ipu_i^*p}\ w^* + w\ \sqrt{p - pu_i^*pu_ip} - wpu_i^*pw^*, \quad \forall\ 1 \leq i \leq n$$
$$V_j = pv_jp + wv_jw^*, \qquad \forall\ 1 \leq j \leq m.$$

Then the following are true:

(a) *$U_1, \ldots, U_n$ are uniatry elements in $B((p+q)\mathcal{H})$;*

(b) *$\forall\ 1 \leq i \leq n$, $1 \leq j \leq m$,*

$$\|U_iV_j - V_jU_i\| \leq f_\delta(t).$$

Proof. Take a positive number t. It is trivial to verify that $U_1, \ldots, U_n$ are unitary elements in $B((p+q)\mathcal{H})$.

Note $u_iv_j = v_ju_i$ and $\|pv_j - v_jp\| \leq t$. We know that

$$\|pv_jp \cdot pu_i^*pu_ip - pu_i^*pu_ip \cdot pv_jp\| \leq 3t. \tag{3.1}$$

Similarly,

$$\|pv_jp \cdot pu_ipu_i^*p - pu_ipu_i^*p \cdot pv_jp\| \leq 3t \tag{3.2}$$

For such $\delta > 0$, by Lemma 3.1, there is a polynomial $P \in \mathbb{C}\langle X\rangle$ such that

$$\max_{0\leq\theta\leq 1} |\sqrt{\theta} - P(\theta)| \leq \delta/4.$$

This implies

$$\|\sqrt{p - pu_i^*pu_ip} - P(p - pu_i^*pu_ip)\| \leq \delta/4, \qquad \|\sqrt{p - pu_ipu_i^*p} - P(p - pu_i^*pu_ip)\| \leq \delta/4.$$

For such polynomial P, by (3.1) and (3.2), there is some positive number $D_\delta > 0$ (only depending on P, not on t) so that

$$\|pv_jpP(p - pu_i^*pu_ip) - P(p - pu_i^*pu_ip)pv_jp\| \leq D_\delta \cdot t,$$

and

$$\|pv_jpP(p - pu_ipu_i^*p) - P(p - pu_ipu_i^*p)pv_jp\| \leq D_\delta \cdot t.$$

Then

$$\|pv_jp\sqrt{p - pu_i^*pu_ip} - \sqrt{p - pu_i^*pu_ip}\ pv_jp\| \leq D_\delta \cdot t + \delta/2, \tag{3.3}$$

and

$$\|pv_jp\sqrt{p-pu_ipu_i^*p}-\sqrt{p-pu_ipu_i^*p}\,pv_jp\|\le D_\delta\cdot t+\delta/2. \tag{3.4}$$

Since

$$V_j=pv_jp+wv_jw^*,\qquad \forall\ 1\le j\le m,$$

from (3.3) and (3.4) we have

$$\begin{aligned}\|U_iV_j-V_jU_i\|&=\|pu_ipv_jp+\sqrt{p-pu_ipu_i^*p}\ v_j\ w^*+w\ \sqrt{p-pu_i^*pu_ip}\ pv_jp-wpu_i^*pv_jw^*\\&\quad-(pv_jpu_ip+pv_jp\sqrt{p-pu_ipu_i^*p}\ w^*+wv_j\ \sqrt{p-pu_i^*pu_ip}-wv_jpu_i^*pw^*)\|\\&\le 4t+2tD_\delta+\delta.\end{aligned} \tag{3.5}$$

Let $f_\delta(t)=4t+2tD_\delta+\delta$ and we have desired result. □

Lemma 3.3. *Suppose that $\mathcal{A}$ is a unital MF algebra generated by a family of unitary elements $x_1,\dots,x_n$, and $\mathcal{B}$ is a unital C^* algebra generated by $y_1,\dots,y_m$. For any $\epsilon>0$, any $\{P_1,\dots,P_r\}\subseteq\mathbb{C}\langle X_1,\dots,X_n,Y_1,\dots,Y_n\rangle$ and $\{Q_1,\dots,Q_t\}\subseteq\mathbb{C}\langle X_1,\dots,X_n\rangle$, there is a quasidiagonal C^*-algebra $\mathcal{A}_1$ with a family of unitary generators $\{z_1,\dots,z_n\}$ so that the following hold:*

(1) *For any $1\le j\le r$,*

$$\begin{aligned}&\|P_j(x_1\otimes 1,\dots,x_n\otimes 1,1\otimes y_1,\dots,1\otimes y_m)\|_{\mathcal{A}\otimes_{max}\mathcal{B}}\\&\qquad\le\|P_j(z_1\otimes 1,\dots,z_n\otimes 1,1\otimes y_1,\dots,1\otimes y_m)\|_{\mathcal{A}_1\otimes_{max}\mathcal{B}}.\end{aligned}$$

(2) *For any $1\le j\le t$,*

$$|\|Q_j(z_1,\dots,z_n)\|_{\mathcal{A}_1}-\|Q_j(x_1,\dots,x_n)\|_{\mathcal{A}}|\le\epsilon.$$

Proof. Note that $\mathcal{A}$ is an MF algebra and $x_1,\dots,x_n$ are unitary elements in $\mathcal{A}$. Thus there are a sequence of positive integers $\{t_l\}_{l=1}^\infty$ and families of unitary matrices $\{B_1^{(l)},\dots,B_n^{(l)}\}$ in $\mathcal{M}_{t_l}(\mathbb{C})$ for $l=1,2,\dots$, such that

$$\lim_{l\to\infty}\|Q(B_1^{(l)},\dots,B_n^{(l)})\|=\|Q(x_1,\dots,x_n)\|,\ \forall\ Q\in\mathbb{C}\langle X_1,\dots,X_n\rangle.$$

For any positive integers N_1 and $1\le i\le n$, we define

$$D(N_1;i)=B_i^{(N_1)}\oplus B_i^{(N_1+1)}\oplus B_i^{(N_1+2)}\oplus\cdots$$

It is not hard to see that, for any $Q\in\mathbb{C}\langle X_1,\dots,X_n\rangle$,

$$\|Q(D(N_1;1),\dots,D(N_1;n))\|=\sup_{k\ge N_1}\|Q(B_1^{(k)},\dots,B_n^{(k)})\|\ge\|Q(x_1,\dots,x_n)\| \tag{3.6}$$

$$\lim_{N_1\to\infty}\|Q(D(N_1;1),\cdots,D(N_1;n))\|=\|Q(x_1,\dots,x_n)\| \tag{3.7}$$

Therefore, for the chosen set $\{Q_1,\dots Q_t\}$, by (3.7) there exists some N_1 such that

$$|\|Q_j(D(N_1;1),\dots,D(N_1;n))\|-\|Q_j(x_1,\dots,x_n)\||\le\epsilon,\quad for\ \ 1\le j\le t. \tag{3.8}$$

Let $z_i = D(N_1; i)$ for $1 \leq i \leq n$ and $\mathcal{A}_1$ be the unital C*-algebra generated by $z_1, \ldots, z_n$. Hence, by (3.8)

$$|\|Q_j(z_1, \ldots, z_n)\| - \|Q_j(x_1, \ldots, x_n)\|| \leq \epsilon, \quad for \ \ 1 \leq j \leq t. \tag{3.9}$$

It is trivial to see that $z_1, \ldots, z_n$ are unitary elements in $\mathcal{A}_1$. Moreover by inequality (3.6), there is a $*$-homomorphism

$$\rho_1 : \mathcal{A}_1 \to \mathcal{A},$$

such that $\rho_1(z_i) = x_i$ for $1 \leq i \leq n$. By the property of the maximal tensor product, it induces a $*$-homomorphism

$$\rho_1 \otimes_{max} id : \ \mathcal{A}_1 \otimes_{max} \mathcal{B} \to \mathcal{A} \otimes_{max} \mathcal{B}.$$

In follows that, for any $1 \leq j \leq r$,

$$\begin{aligned} &\|P_j(x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m)\|_{\mathcal{A} \otimes_{max} \mathcal{B}} \\ &\qquad \leq \|P_j(z_1 \otimes 1, \ldots, z_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m)\|_{\mathcal{A}_1 \otimes_{max} \mathcal{B}}. \end{aligned} \tag{3.10}$$

Combining with (3.9) and (3.10), we have completed the proof of the Lemma.

□

Lemma 3.4. *Suppose $C^*(F_n)$ is the full C*-algebra of the free group F_n and $u_1, \ldots, u_n$, are the canonical generators of $C^*(F_n)$. Suppose $\mathcal{B}$ is a unital quasidiagonal C*-algebra generated by unitary elements $z_1, \ldots, z_m$. Suppose ρ is a faithful representation of $C^*(F_n) \otimes_{max} \mathcal{B}$ on a separable Hilbert space $\mathcal{H}$.*

For any $\epsilon > 0$, any $\{P_1, \ldots, P_r\}$ of $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$, and any $\{Q_1, \ldots, Q_t\}$ in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, there are a positive integer k_1 and $k_1 \times k_1$ matrices

$$\{U_1, \ldots, U_n\} \subseteq \mathcal{U}_{k_1}(\mathbb{C}) \qquad \{V_1, \ldots, V_m\} \subseteq \mathcal{M}_{k_1}(\mathbb{C})$$

such that (1)

$$\|U_i V_j - V_j U_i\| \leq \epsilon, \qquad \forall \ \ 1 \leq i \leq n, \ 1 \leq j \leq m;$$

(2), $\forall\ 1 \leq j \leq t$,

$$|\|Q_j(V_1, \ldots, V_m)\| - \|Q_j(z_1, \ldots, z_m)\|| \leq \epsilon;$$

and (3), $\forall\ 1 \leq i \leq r$,

$$\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes z_1, \ldots, 1 \otimes z_m)\|_{C^*(F_n) \otimes \mathcal{B}} \leq \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| + \epsilon.$$

Proof. Since ρ is a faithful representation of $C^*(F_n) \otimes_{max} \mathcal{B}$, without loss of generality we can further assume that ρ is an essentially faithful representation. Then $\rho(1 \otimes \mathcal{B}) \subseteq B(\mathcal{H})$ is quasidiagonal. It follows that there is a sequence of finite rank projections p_k, $k = 1, 2, \ldots$, on $\mathcal{H}$ such that (i) $1 \leq j \leq m$,

$$\|p_k \rho(1 \otimes z_j) - \rho(1 \otimes z_j) p_k\| \to 0;$$

and (ii) $\forall\ 1 \leq j \leq t$,

$$\|Q_j(p_k \rho(1 \otimes z_1) p_k, \ldots, p_k \rho(1 \otimes z_m) p_k)\| \to \|Q_j(\rho(1 \otimes z_1), \ldots, \rho(1 \otimes z_m))\|. \tag{3.11}$$

Let

$$t_k = \max_{1\le j\le m}\{\|p_k\rho(1\otimes z_j) - \rho(1\otimes z_j)p_k\|\}.$$

Then t_k tends to 0. Let q_k be a projection in $B(\mathcal{H})$ so that p_k, q_k are mutually orthogonal, equivalent projections, and let w_k be the partial isometry so that $w_k^*w_k = p_k, w_kw_k^* = q_k$. Let

$$\begin{aligned} U_i^{(k)} &= p_k\rho(u_i\otimes 1)p_k + \sqrt{p_k - p_k\rho(u_i\otimes 1)p_k\rho(u_i\otimes 1)^*p_k}\ w_k^* \\ &\quad + w_k\ \sqrt{p_k - p_k\rho(u_i\otimes 1)^*p_k\rho(u_i\otimes 1)p_k} - w_kp_k\rho(u_i\otimes 1)^*p_kw_k^*, \quad \forall\ 1\le i\le n \end{aligned} \tag{3.12}$$

$$V_j^{(k)} = p_k\rho(1\otimes z_j)p_k + w_k\rho(1\otimes z_j)w_k^*, \qquad \forall\ 1\le j\le m \tag{3.13}$$

to be elements in $B((p_k+q_k)\mathcal{H})$. For $\delta = \frac{\epsilon}{4} > 0$, let

$$f_\delta : (0,\infty)\to(0,\infty)$$

be the function as in Lemma 3.2. Then

(a) $\lim_{t\to 0} f_\delta(t) \le \epsilon/4$ and
(b) $U_i^{(k)}$ is a unitary element in $B((p_k+q_k)\mathcal{H})$; and
(c) $\forall\ 1\le i\le n,\ 1\le j\le m$

$$\|U_i^{(k)}V_j^{(k)} - V_j^{(k)}U_i^{(k)}\| \le f_\delta(t_k). \tag{3.14}$$

We observe that $U_i^{(k)}, V_j^{(k)}$, as k goes to infinity, converges in strong operator topology to

$$\begin{pmatrix}\rho(u_i\otimes 1) & 0\\ 0 & -\rho(u_i\otimes 1)^*\end{pmatrix} \qquad and \qquad \begin{pmatrix}\rho(1\otimes z_j) & 0\\ 0 & \rho(1\otimes z_j)\end{pmatrix}.$$

It follows that, for any $1\le i\le r$,

$$\begin{aligned} &\liminf_{k\to\infty}\|P_i(U_1^{(k)},\dots,U_n^{(k)},V_1^{(k)},\dots,V_m^{(k)}\| \\ &\qquad \ge \|P_i(\rho(u_1\otimes 1),\dots,\rho(u_n\otimes 1),\rho(1\otimes z_1),\dots,\rho(1\otimes z_m))\| \\ &\qquad = \|P_i(u_1\otimes 1,\dots,u_n\otimes 1,1\otimes z_1,\dots,1\otimes z_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} \end{aligned} \tag{3.15}$$

On the other hand,

$$\begin{aligned} &\lim_k\Big|\|Q_j(V_1^{(k)},\dots,V_n^{(k)})\| - \|Q_j(z_1,\dots,z_m)\|_{\mathcal{B}}\Big| \\ &\quad = \lim_k|\|Q_j(p_k\rho(1\otimes z_1)p_k,\dots,p_k\rho(1\otimes z_m)p_k)\| - \|Q_j(z_1,\dots,z_m)\|_{\mathcal{B}}| \\ &\quad = |\|Q_j(\rho(1\otimes z_1),\dots,\rho(1\otimes z_m))\| - \|Q_j(z_1,\dots,z_m)\|_{\mathcal{B}}| \qquad \text{(By (3.11))} \\ &\quad = \Big|\|Q_j(1\otimes z_1,\dots,1\otimes z_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} - \|Q_j(z_1,\dots,z_m)\|_{\mathcal{B}}\Big| \\ &\quad = 0. \end{aligned} \tag{3.16}$$

By (3.14), (3.15), and (3.16), we know when k is large enough, there are a positive integer $k_1 = dim((p_{k_1}+q_{k_1})\mathcal{H})$ and $k_1\times k_1$ matrices

$$\{U_1,\dots,U_n\}\subseteq\mathcal{U}_{k_1}(\mathbb{C}), \qquad \{V_1,\dots,V_m\}\subseteq\mathcal{M}_{k_1}(\mathbb{C})$$

such that (i)

$$\|U_iV_j - V_jU_i\| \leq \epsilon, \qquad \forall\ 1 \leq i \leq n,\ 1 \leq j \leq m;$$

(ii) $\forall\ 1 \leq j \leq t$,

$$|\|Q_j(V_1, \ldots, V_m)\| - \|Q_j(z_1, \ldots, z_m)\|_{\mathcal{B}}| \leq \epsilon;$$

and (iii), $\forall\ 1 \leq i \leq r$,

$$\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes z_1, \ldots, 1 \otimes z_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} \leq \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| + \epsilon.$$

□

Recall that $C^*(F_n)$ is the full C*-algebra of the free group F_n and $u_1, \ldots, u_n$ is a family of canonical generators of $C^*(F_n)$. We assume that $\mathcal{B}$ is an MF algebra with a family of unitary generators $v_1, \ldots, v_m$. From now on till the end of this subsection, we will assume that $\{P_r\}_{r=1}^{\infty}$, or $\{Q_t\}_{t=1}^{\infty}$ respectively, is the collection of all polynomials in $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$, or $\mathbb{C}\langle Y_1, \ldots, Y_m\rangle$ respectively, with the rational coefficients.

Lemma 3.5. *For any $r_0 \geq 1$, there is some $r_1 \geq 1$ so that the following hold: for any positive integer $k \geq 1$, if*

$$\{U_1, \ldots, U_n\} \subseteq \mathcal{U}_k(\mathbb{C}) \qquad \{V_1, \ldots, V_m\} \in \mathcal{M}_k(\mathbb{C}),$$

is a family of $k \times k$ matrices satisfying,

$$\|U_iV_j - V_jU_i\| \leq \frac{1}{r_1}, \qquad \forall\ 1 \leq i \leq n,\ 1 \leq j \leq m;$$

and

$$|\|Q_j(V_1, \ldots, V_m)\| - \|Q_j(v_1, \ldots, v_m)\|_{\mathcal{B}}| \leq \frac{1}{r_1}, \qquad \forall\ 1 \leq j \leq r_1,$$

then, $\forall\ 1 \leq i \leq r_0$

$$\|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| \leq \|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes v_1, \ldots, \otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} + \frac{1}{r_0}.$$

Proof. We will prove the lemma by using contradiction. Suppose the result of the lemma does not hold. There are some positive integer $r_0 > 0$, a sequence of positive integers $\{k_r\}_{r=1}^{\infty}$, and sequences of matrices

$$\{U_1^{(r)}, \ldots, U_n^{(r)}\} \subseteq \mathcal{U}_{k_r}(\mathbb{C}) \qquad \{V_1^{(r)}, \ldots, V_m^{(r)}\} \in \mathcal{M}_{k_r}(\mathbb{C}),$$

satisfying

(i)

$$\|U_i^{(r)}V_j^{(r)} - V_j^{(r)}U_i^{(r)}\| \leq \frac{1}{r}, \qquad \forall\ 1 \leq i \leq n,\ 1 \leq j \leq m;$$

(ii) and

$$\left|\|Q_j(V_1^{(r)}, \ldots, V_m^{(r)})\| - \|Q_j(v_1, \ldots, v_m)\|_{\mathcal{B}}\right| \leq \frac{1}{r}, \qquad \forall\ 1 \leq j \leq r;$$

(iii) but

$$\max_{1\leq i\leq r_0}\{\|P_i(U_1^{(r)},\ldots,U_n^{(r)},V_1^{(r)},\ldots,V_m^{(r)})\|-\|P_i(u_1\otimes 1,\ldots,u_n\otimes 1,1\otimes v_1,\ldots,\otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}}\}>\frac{1}{r_0}. \tag{3.17}$$

Consider the C*-algebra $\prod_r \mathcal{M}_{k_r}(\mathbb{C})/\sum_r \mathcal{M}_{k_r}(\mathbb{C})$. Let

$$x_i=[\langle U_i^{(r)}\rangle_{r=1}^{\infty}], \qquad \forall\ 1\leq i\leq n$$
$$y_j=[\langle V_j^{(r)}\rangle_{r=1}^{\infty}], \qquad \forall\ 1\leq j\leq m$$

be the elements in $\prod_r \mathcal{M}_{k_r}(\mathbb{C})/\sum_r \mathcal{M}_{k_r}(\mathbb{C})$. It is not hard to check the following

(iv) $x_1,\ldots,x_n$ are unitary element;
(v) $x_iy_j=y_jx_i$ for all $1\leq i\leq n$, $1\leq j\leq m$;
(vi) for all $j\geq 1$,

$$\|Q_j(y_1,\ldots,y_m)\|_{\prod_r \mathcal{M}_{k_r}(\mathbb{C})/\sum_r \mathcal{M}_{k_r}(\mathbb{C})}=\|Q_j(v_1,\ldots,v_m)\|_{\mathcal{B}}$$

By the universal property of the C*-algebra $C^*(F_n)\otimes_{max}\mathcal{B}$, we know there is $*$-homomorphism

$$\psi: C^*(F_n)\otimes_{max}\mathcal{B}\to C^*(x_1,\ldots,x_n,y_1,\ldots,y_m)$$

such that

$$\psi(u_i\otimes 1)=x_i, \qquad \psi(1\otimes v_j)=y_j, \qquad \forall\ 1\leq i\leq n, 1\leq j\leq m,$$

where $C^*(x_1,\ldots,x_n,y_1,\ldots,y_m)$ is the unital C*-algebra generated by $x_1,\ldots,x_n,y_1,\ldots,y_m$ in $\prod_r \mathcal{M}_{k_r}(\mathbb{C})/\sum_r \mathcal{M}_{k_r}(\mathbb{C})$. Hence, for all $i\geq 1$,

$$\limsup_r \|P_i(U_1^{(r)},\ldots,U_n^{(r)},V_1^{(r)},\ldots,V_m^{(r)})\|\leq\|P_i(u_1\otimes 1,\ldots,u_n\otimes 1,1\otimes v_1,\ldots,\otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}}.$$

This contradicts with the inequality (3.29). The proof of the lemma is completed. □

Now we are able to show our main result in this section.

Theorem 3.1. *Suppose n is a positive integer and $\mathcal{B}$ is a finitely generated unital MF algebra. Then $C^*(F_n)\otimes_{max}\mathcal{B}$ is an MF algebra.*

Proof. Suppose that $u_1,\ldots,u_n$ are the canonical unitary generators of $C^*(F_n)$ and $v_1,\ldots,v_m$ are unitary generators of $\mathcal{B}$. Let $r_0\geq 1$, $0<\epsilon<1/r_0$, $\{P_i\}_{i=1}^{r_0}$ be a finite subset of $\mathbb{C}\langle X_1,\ldots,X_n, Y_1,\ldots,Y_m\rangle$ and $\{Q_j\}_{j=1}^{t}$ a subset of $\mathbb{C}\langle Y_1,\ldots,Y_m\rangle$. By Lemma 3.3, there is a quasidiagonal C*-algebra $\mathcal{B}_1$ with a family of unitary generators $\{z_1,\ldots,z_n\}$ so that the following hold:

(i) For any $1\leq i\leq r_0$,

$$\|P_i(u_1\otimes 1,\ldots,u_n\otimes 1,1\otimes v_1,\ldots,1\otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}}$$
$$\leq\|P_i(u_1\otimes 1,\ldots,u_n\otimes 1,1\otimes z_1,\ldots,1\otimes z_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}_1}.$$

(ii) For any $1\leq j\leq t$,

$$|\|Q_j(v_1,\ldots,v_m)\|_{\mathcal{B}}-\|Q_j(z_1,\ldots,z_m)\|_{\mathcal{B}_1}|\leq\epsilon/2.$$

Let ρ be a faithful $*$-representation $\rho : C^*(F_n) \otimes_{max} \mathcal{B}_1 \to B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. By lemma 3.4, there are a positive integer k and $k \times k$ matrices

$$\{U_1, \ldots, U_n\} \subseteq \mathcal{U}_k(\mathbb{C}) \qquad \{V_1, \ldots, V_m\} \subseteq \mathcal{M}_k(\mathbb{C})$$

such that

(iii)

$$\|U_iV_j - V_jU_i\| \leq \epsilon, \qquad \forall \ 1 \leq i \leq n, \ 1 \leq j \leq m; \tag{3.18}$$

(iv) $\forall\ 1 \leq j \leq t$,

$$|\|Q_j(V_1, \ldots, V_m)\| - \|Q_j(z_1, \ldots, z_m)\|_{\mathcal{B}_1}| \leq \epsilon/2;$$

(v) $\forall\ 1 \leq i \leq r_0$,

$$\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes z_1, \ldots, 1 \otimes z_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}_1} \leq \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| + \epsilon/2.$$

By (ii) and (iv) we know that $\forall\ 1 \leq j \leq t$,

$$|\|Q_j(V_1, \ldots, V_m)\| - \|Q_j(v_1, \ldots, v_m)\|_{\mathcal{B}}| \leq \epsilon; \tag{3.19}$$

By (i) and (v), we know that $\forall\ 1 \leq i \leq r_0$,

$$\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes v_1, \ldots, 1 \otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} \leq \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| + \epsilon. \tag{3.20}$$

By Lemma 3.5 and inequalities (3.18), (3.19)we know that, when ϵ is small enough and t is large enough, for $1 \leq i \leq r_0$

$$\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes v_1, \ldots, 1 \otimes v_m)\|_{C^*(F_n)\otimes_{max}\mathcal{B}} \geq \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\| - 1/r_0, \tag{3.21}$$

Combining with (3.20), we obtain

$$\big|\|P_i(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes v_1, \ldots, 1 \otimes v_m)\|_{C^*(F_m)\otimes_{max}\mathcal{B}} - \|P_i(U_1, \ldots, U_n, V_1, \ldots, V_m)\|\big| \leq 1/r_0. \tag{3.22}$$

By (3.18) and (3.22), we know that, for any $r_0 \in \mathbb{N}$, there are k and matrices

$$\{U_1, \ldots, U_n\} \subseteq \mathcal{U}_k(\mathbb{C}) \qquad \{V_1, \ldots, V_m\} \subseteq \mathcal{M}_k(\mathbb{C})$$

satisfying

$$\|U_iV_j - V_jU_i\| \leq 1/r_0, \qquad \forall \ 1 \leq i \leq n, \ 1 \leq j \leq m;$$

and $\forall\ 1 \leq i \leq r_0$,

$$\big|\|P_j(u_1 \otimes 1, \ldots, u_n \otimes 1, 1 \otimes v_1, \ldots, 1 \otimes v_m)\|_{C^*(F_m)\otimes_{max}\mathcal{B}} - \|P_j(U_1, \ldots, U_n, V_1, \ldots, V_m)\|\big| \leq 1/r_0.$$

By Lemma 2.3, $C^*(F_n) \otimes_{max} \mathcal{B}$ is an MF algebra. □

Remark 3.1. *The argument in the proof of Theorem 3.1 also applies to the case when $\mathcal{B}$ is countably generated. Thus the preceding result can be generalized as follows: Suppose that n be a positive integer and $\mathcal{B}$ is a unital separable MF algebra. Then $C^*(F_n) \otimes_{max} \mathcal{B}$ is an MF algebra.*

The following result by Brown and Ozawa (see Proposition 7.4.5 of [**9**]) follows directly from the preceding theorem.

Corollary 3.1. *Suppose $n, m \geq 2$ are positive integers. Then $C^*(F_n) \otimes_{max} C^*(F_m)$ is an MF algebra, i.e. there is an embedding*

$$\rho : C^*(F_n) \otimes_{max} C^*(F_m) \to \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$$

for some positive integers $\{n_k\}_{k=1}^{\infty}$.

Note Haagerup and Thorbjørnsen proved that $C_r^*(F_n)$ is an MF algebra. By Lemma 2.4 and 2.5 and Theorem 3.1 and Corollary 5.3 in [**18**], we have the following result.

Corollary 3.2. *Suppose $n, m \geq 2$ are positive integers and $\mathcal{B}$ is a separable MF algebra. By Corollary 5.3 in [**18**], $C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}$ is an MF algbebra, where $C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}$ is the unital full free product of $C_r^*(F_n)$ and $\mathcal{B}$. Then $(C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}) \otimes_{max} C^*(F_m)$ is an MF algebra and $Ext((C_r^*(F_n) *_{\mathbb{C}} \mathcal{B}) \otimes_{max} C^*(F_m))$ is not a group by Lemma 2.4 and Lemma 2.5.*

3.2. Minimal Tensor Product. The following result might have been known to experts. For the purpose of completeness, we also include it here.

Proposition 3.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two separable MF algebras. If $\mathcal{A}$ is exact, then $\mathcal{A} \otimes_{min} \mathcal{B}$ is also an MF algebra.*

Proof. Since $\mathcal{B}$ is an MF algebra, there is an embedding

$$\rho : \mathcal{B} \to \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$$

for a family of positive integers $\{n_k\}_{k=1}^{\infty}$. This embedding ρ induces another embedding

$$id \otimes_{min} \rho : \mathcal{A} \otimes_{min} \mathcal{B} \to \mathcal{A} \otimes_{min} \left(\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \right).$$

Note $\mathcal{A}$ is an exact C*-algebra. We have

$$\mathcal{A} \otimes_{min} \left(\prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \right) \simeq \left(\mathcal{A} \otimes_{min} \prod_k \mathcal{M}_{n_k}(\mathbb{C}) \right) / \left(\mathcal{A} \otimes_{min} \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \right).$$

Since $\mathcal{A}$ is an MF algebra, by Corollary 3.4.3 in [**3**], any separable C*-subalgebra of

$$\left(\mathcal{A} \otimes_{min} \prod_k \mathcal{M}_{n_k}(\mathbb{C}) \right) / \left(\mathcal{A} \otimes_{min} \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \right)$$

is an MF algebra. Therefore, $\mathcal{A} \otimes_{min} \mathcal{B}$ is also an MF algebra. □

Note Haagerup and Thorbjørnsen proved that $C_r^*(F_n)$ is an MF algebra. By Lemma 2.4 and 2.5 and preceding proposition, we have the following result.

Corollary 3.3. *Suppose that $\mathcal{B}$ is a unital separable MF algebra. Then $C_r^*(F_n) \otimes_{min} \mathcal{B}$ is an MF algebra for any $n \geq 2$. Moreover, $Ext(C_r^*(F_n) \otimes_{min} \mathcal{B})$ is not a group by Lemma 2.4 and Lemma 2.5.*

When both two C* algebras was not exact, we have the following result.

Proposition 3.2. *Suppose $\mathcal{A}$ is a separable quasidiagonal C*-algebra and $\mathcal{B}$ a separable MF algebras. Then $\mathcal{A} \otimes_{min} \mathcal{B}$ is an MF algebra.*

Proof. We need only to show that if $\mathcal{A}_1$, and $\mathcal{B}_1$ respectively, is a finite generated C*-subalgebra of $\mathcal{A}$, and $\mathcal{B}$ respectively, then $\mathcal{A}_1 \otimes_{min} \mathcal{B}_1$ is MF algebra. Suppose that $x_1, \ldots, x_n$, and $y_1, \ldots, y_m$ respectively, is a family of generators of $\mathcal{A}_1$, and $\mathcal{B}_1$ respectively. By Lemma 2.3, to show $\mathcal{A}_1 \otimes_{min} \mathcal{B}_1$ is MF algebra, it suffice to show the following: *for any $\epsilon > 0$ and any finite subset $\{P_1, \ldots, P_r\}$ of noncommutative $*$-polynomials $\mathbb{C}\langle X_1, \ldots, X_n, Y_1, \ldots, Y_m\rangle$, there are a positive integer k and matrices*

$$\{A_1, \ldots, A_n, B_1, \ldots, B_m\} \subseteq \mathcal{M}_k(\mathbb{C})$$

such that

$$\max_{1\leq j\leq r} |\|P_j(A_1, \ldots, A_n, B_1, \ldots, B_m)\| - \|P_j(x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_m)\|_{\mathcal{A}\otimes_{min}\mathcal{B}}| \leq \epsilon.$$

Let ρ be a faithful $*$-representation of $\mathcal{B}$ on a separable Hilbert space $\mathcal{H}_1$. Let π be an essentially faithful $*$-representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}_2$. By Lemma 2.2, we know that $\pi(\mathcal{A}) \subseteq B(\mathcal{H}_2)$ is quasidiagonal. Then there is a sequence of finite rank projections q_s, $s = 1, 2, \ldots$, in $B(\mathcal{H}_2)$ such that (i) $q_s \to I$ in SOT and (ii) $\|q_s\pi(x) - \pi(x)q_s\| \to 0$ for any $x \in \mathcal{A}$. Let $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_1$ be a Hilbert space. By (i), we know that

$$\begin{aligned}\langle q_s\pi(x_1)q_s \otimes 1, \ldots, q_s\pi(x_n)q_s \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m)\rangle \to^{*-SOT}& \\ \langle \pi(x_1) \otimes 1, \ldots, \pi(x_n) \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m)\rangle.&\end{aligned}$$

Thus, for $1 \leq j \leq r$,

$$\begin{aligned}&\|P_j(\pi(x_1) \otimes 1, \ldots, \pi(x_n) \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m))\| \\ &\quad \leq \liminf_{s\to\infty} \|P_j(q_s\pi(x_1)q_s \otimes 1, \ldots, q_s\pi(x_n)q_s \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m))\|. \qquad (3.23)\end{aligned}$$

By (ii), we know that, for $1 \leq j \leq r$,

$$\begin{aligned}\lim_{s\to\infty} &\|P_j(q_s\pi(x_1)q_s \otimes 1, \ldots, q_s\pi(x_n)q_s \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m)) \\ &-(q_s \otimes 1)\left(P_j(\pi(x_1) \otimes 1, \ldots, \pi(x_n) \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m))\right)(q_s \otimes 1)\| = 0. \qquad (3.24)\end{aligned}$$

Therefore, by (3.23) and (3.24), there is a positive integer t such that

$$\begin{aligned}\max_{1\leq j\leq r} |\|P_j(q_t\pi(x_1)q_t \otimes 1, \ldots, q_t\pi(x_n)q_t \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m))\| & \\ - \|P_j(\pi(x_1) \otimes 1, \ldots, \pi(x_n) \otimes 1, 1 \otimes \rho(y_1), \ldots, 1 \otimes \rho(y_m))\|| \leq \epsilon/2. \qquad (3.25)&\end{aligned}$$

Moreover, we know

$$\{q_t\pi(x_1)q_t\otimes 1,\ldots,q_t\pi(x_n)q_t\otimes 1, 1\otimes\rho(y_1),\ldots,1\otimes\rho(y_m)\}\subseteq B(q_t\mathcal{H}_2)\otimes\rho(\mathcal{B}).$$

Because $\mathcal{B}$ is an MF algebra, $B(q_t\mathcal{H}_2)\otimes\rho(\mathcal{B})$ is an MF algebra. Thus there are a positive integer k and

$$\{A_1,\ldots,A_n,B_1,\ldots,B_m\}\subseteq\mathcal{M}_k(\mathbb{C})$$

such that

$$\max_{1\leq j\leq r}|\|P_j(A_1,\ldots,A_n,B_1,\ldots,B_m)\| - \|P_j(q_t\pi(x_1)q_t\otimes 1,\ldots,q_t\pi(x_n)q_t\otimes 1,1\otimes\rho(y_1),\ldots,1\otimes\rho(y_m))\|\,|\leq\epsilon/2,$$

Combining with (3.25), it implies

$$\max_{1\leq j\leq r}|\|P_j(A_1,\ldots,A_n,B_1,\ldots,B_m)\| - \|P_j(x_1\otimes 1,\ldots,x_n\otimes 1,1\otimes y_1,\ldots,1\otimes y_m)\|_{\mathcal{A}\otimes_{min}\mathcal{B}}\,|\leq\epsilon/2.$$

This completes the proof of the proposition.

□

Example 3.1. *Note $C^*(F_n)$ is a residually finite dimensional C*-algebra, thus a quasidiagonal C*-algebra. Therefore we have the following result: Suppose $\mathcal{B}$ is a unital separable MF algebra. By Corollary 5.3 in* [**18**]*, $C_r^*(F_n)*_{\mathbb{C}}\mathcal{B}$ is an MF algbebra, where $C_r^*(F_n)*_{\mathbb{C}}\mathcal{B}$ is the unital full free product of $C_r^*(F_n)$ and $\mathcal{B}$. Then by Proposition 3.1, $C^*(F_n)\otimes_{min}(C_r^*(F_n)*_{\mathbb{C}}\mathcal{B})$ is an MF algebra. Thus, $Ext(C^*(F_n)\otimes_{min}(C_r^*(F_n)*_{\mathbb{C}}\mathcal{B}))$ is not a group because of Lemma 2.4.*

4. Crossed Products

In this section, we assume that $\mathcal{A}$ is an MF algebra and G is a discrete countable amenable group such that there is a homomorphism $\alpha: G\to Aut(\mathcal{A})$. We will consider the question when the reduced, or full, crossed product of $\mathcal{A}$ by G is an MF algebra. This is not always the case if we are not going to add extra conditions on the actions of G on $\mathcal{A}$. Note Cuntz algebra is stably isometric to the crossed product of AF algebras by $\mathbb{Z}$. Obviously Cuntz algebra is not an MF algebra but AF algebra is.

Because G is an amenable group, the reduced crossed product of $\mathcal{A}$ by G coincides with the full crossed product of $\mathcal{A}$ by G, which we will denote by $\mathcal{A}\rtimes_\alpha G$.

4.1. Finite group action. When G is a finite group, we have the following result.

Theorem 4.1. *Suppose $\mathcal{A}$ is an MF algebra and G is a finite group such that there is a homomorphism $\alpha: G\to Aut(\mathcal{A})$. Then $\mathcal{A}\rtimes_\alpha G$ is an MF algebra.*

Proof. Since G is a finite group, by Green's result in [**13**], we know that

$$\mathcal{A}\rtimes_\alpha G\hookrightarrow\mathcal{A}\otimes_{min}C(G)\rtimes_{\alpha\otimes\gamma}G\simeq\mathcal{A}\otimes_{min}\mathcal{K},$$

where $C(G)$ is the C*-algebra consisting all (continuous) functions on G, γ is the action induced by left translation by G on $C(G)$, and $\mathcal{K}$ is the set of compact operators on $l^2(G)$. By the fact the $\mathcal{A}$ is an MF algebra and results in [**3**] or Proposition 3.1, $\mathcal{A}\otimes\mathcal{K}$ is an MF algebra. Hence $\mathcal{A}\rtimes_\alpha G$ is an MF algebra. □

Corollary 4.1. *If G is a finite group with $|G| \geq 2$, where $|G|$ is the order of group G, then $C_r^*(\mathbb{Z} * G)$ is an MF algebra and $Ext(C_r^*(\mathbb{Z} * G))$ is not a group.*

PROOF. Since a finite group is a subgroup of a finite permutation group, we need only to prove the result when G is a finite permutation group S_n. If $\alpha : S_n \to Aut(F_n)$ is a mapping defined by

$$\alpha(\sigma)(g_i) = g_{\sigma(i)},$$

where $g_1, \ldots, g_n$ are the canonical generators of the free group F_n, then α induces an action G on $C_r^*(F_n)$ satisfying

$$C_r^*(F_n) \rtimes_\alpha S_n \simeq C_r^*(\mathbb{Z} * S_n).$$

Since Haagerup and Thorbjørnsen proved that $C_r^*(F_n)$ is an MF algebra, it follows immediately from last proposition that $C_r^*(\mathbb{Z} * S_n)$ is an MF algebra. Moreover since $\mathbb{Z} * S_n$ is not an amenable group for $n \geq 2$, $C_r^*(\mathbb{Z} * S_n)$ is not a quasidiagonal C*-algebra for $n \geq 2$ by Lemma 2.5, whence $Ext(C_r^*(\mathbb{Z} * S_n))$ is not a group by Lemma 2.4. So, for any finite group G with $|G| \geq 2$, $C_r^*(\mathbb{Z} * G)$ is an MF algebra and $Ext(C_r^*(\mathbb{Z} * G))$ is not a group.

□

Remark 4.1. *Assume $n \geq 1, m \geq 0$ are nonnegative integers and $r = n^2 + m$. Let F_r be the free group on r many generators, $g_1, \ldots, g_{n^2}, g_{n^2+1}, \cdots, g_r$. Let S_n be the permutation groups on the integers $\{1, \ldots, n\}$.*

Let α be the homomorphism from S_n into $Aut(F_r)$ defined the following mapping: $\forall\ \sigma \in S_n$

$$\alpha(\sigma)(g_i) = g_{\sigma(i-kn)} \qquad for \quad 0 \leq k \leq n-1 \ \ and \ \ 1 + kn \leq i \leq (k+1)n$$
$$\alpha(\sigma)(g_j) = g_j \qquad for \quad n+1 \leq j \leq r.$$

Then we observe that

$$F_r \rtimes_\alpha S_n \simeq F_n * (S_n \times F_m).$$

By Theorem 4.1 and the same arguments as in Corollary 4.1, we have the following result.

Corollary 4.2. *Assume $n \geq 1, m \geq 0$ are nonnegative integers and G is a nontrivial finite group. Then $C_r(F_n * (G \times F_m))$ is an MF algebra and $Ext(C_r(F_n * (G \times F_m)))$ is not a group by Lemma 2.4 and Lemma 2.5.*

Remark 4.2. *Assume that $n \geq 2$ is a positive integer and $H_1, \ldots, H_n$ is a family of finite groups. Let*

$$H = H_1 * H_2 * \cdots * H_n$$

be the free product of the groups $H_1, \ldots, H_n$. Then there is a finite group G satisfying

$$H \subseteq \mathbb{Z} * G.$$

Thus $C_r^*(H)$ is a C-subalgebra of $C_r^*(\mathbb{Z} * G)$. It follows from the Corollary 4.1 that $C_r^*(H)$ is an MF algebra. Hence we have the following result.*

Corollary 4.3. *Assume that $n \geq 2$ is a positive integer and $H_1, \ldots, H_n$ is a family of finite groups. Let*

$$H = H_1 * H_2 * \cdots * H_n.$$

then $C_r^*(H)$ is an MF algebra. Moreover, if there are $1 \leq i \neq j \leq n$ such that

$$|H_i| \geq 2 \qquad \text{and} \qquad |H_j| \geq 3,$$

where $|H|$ denotes the order of the group H, then $Ext(C_r^*(H))$ is not a group.

Proof. By the explanation in the Remark 4.2, we know that $C_r^*(H)$ is an MF algebra. if there are $1 \leq i \neq j \leq n$ such that $|H_i| \geq 2$ and $|H_j| \geq 3$, then H is not an amenable group. It follows that $C_r^*(H)$ is not a quasidiagonal C*-algebra, whence $Ext(C_r^*(H))$ is not a group. □

Example 4.1. *For any positive integers $p \geq 2, q \geq 3$, $Ext(C_r^*(\mathbb{Z}_p * \mathbb{Z}_q))$ is not a group.*

4.2. Subgroup of finite index. We have the following result.

Proposition 4.1. *Assume that G is a discrete countable group and H is a subgroup of G with a finite index. If $C_r^*(H)$ is an MF algebra, then $C_r^*(G)$ is also an MF algebra.*

Proof. Let $l^2(G)$ be the Hilbert space with an orthonormal basis $\{e_g\}_{g\in G}$. Recall the left regular representation of the group G is the unitary representation $\lambda : G \to \mathcal{U}(l^2(G))$ given by $\lambda(g)(e_h) = e_{gh}$, $g \in G$. Then $C_r^*(G)$ is the C*-algebra generated by $\{\lambda(g)\}_{g\in G}$ in $B(l^2(G))$ and the C*-subalgebra $\mathcal{B}$ generated by $\{\lambda(g)\}_{g\in H}$ in $C_r^*(G)$ is $*$-isomorphic to $C_r^*(H)$.

Note $[G : H] < \infty$. There are elements $e = g_1, g_2, \ldots, g_n$ in G such that $G = \cup_i g_i H$ and $g_i H \cap g_j H = \emptyset$ if $i \neq j$. Let $l^2(H)$ be the closed subspace spanned by $\{e_h\}_{h\in H}$ in $l^2(G)$ and E be the projection from $l^2(G)$ onto $l^2(H)$. Then for each $1 \leq i \leq n$, $\lambda(g_i)E\lambda(g_i^{-1})$ is the projection from $l^2(G)$ onto the closed subspace $l^2(g_i H)$. Moreover

$$\sum_{i=1}^{n} \lambda(g_i)E\lambda(g_i^{-1}) = I.$$

Note that $E\mathcal{B}E \simeq C_r^*(H)$ and $\{\lambda(g_i)E\lambda(g_j^{-1})\}_{1\leq i,j\leq n}$ consists a system of matrix units of $\mathcal{M}_n(\mathbb{C})$. Let $\mathcal{A}$ be the C*-algebra generated by elements $\{\lambda(g_i)E\mathcal{B}E\lambda(g_j^{-1})\}_{1\leq i,j\leq n}$ in $B(l^2(G))$. Then we observe that

$$\mathcal{A} \simeq C_r^*(H) \otimes \mathcal{M}_n(\mathbb{C}).$$

On the other hand, for any $g \in G$, there is a permutation σ_g on the integers $\{1, 2, \ldots, n\}$ such that

$$gg_iH = g_{\sigma_g(i)}H, \qquad \forall\ 1 \leq i \leq n, \qquad i.e. \qquad g_{\sigma_g(i)}^{-1} g g_i \in H, \qquad \forall\ 1 \leq i \leq n.$$

Thus there is some $h_i \in H$ such that

$$g_{\sigma_g(i)}^{-1} g g_i = h_i, \qquad i.e. \qquad \lambda(g_{\sigma_g(i)}^{-1})\lambda(g)\lambda(g_i) = \lambda(h_i).$$

Moreover,

$$E\lambda(g_j^{-1})\lambda(g)\lambda(g_i)E = 0, \qquad if \quad j \neq \sigma_g(i).$$

Therefore

$$\begin{aligned}\lambda(g) &= \left(\sum_{j=1}^{n}\lambda(g_j)E\lambda(g_j^{-1})\right)\lambda(g)\left(\sum_{i=1}^{n}\lambda(g_i)E\lambda(g_i^{-1})\right)\\ &= \sum_{i,j=1}^{n}\lambda(g_j)E\lambda(g_j^{-1})\lambda(g)\lambda(g_i)E\lambda(g_i^{-1})\\ &= \sum_{i=1}^{n}\lambda(g_{\sigma_g(i)})E\lambda(g_{\sigma_g(i)}^{-1})\lambda(g)\lambda(g_i)E\lambda(g_i^{-1})\\ &= \sum_{i=1}^{n}\lambda(g_{\sigma_g(i)})E\lambda(h_i)E\lambda(g_i^{-1}) \in \mathcal{A}.\end{aligned}$$

It follows that $C_r^*(G) \subseteq \mathcal{A}$. Apparently, $\mathcal{A}$ is an MF algebra, because $C_r^*(H)$ is an MF algebra. Hence, being a subalgebra of an MF algebra, $C_r^*(G)$ is also an MF algebra. □

Corollary 4.4. *Let $SL_2(\mathbb{Z})$ to be the spacial linear group of 2×2 matrices with integer entries. Then $C_r^*(SL_2(\mathbb{Z}))$ is an MF algebra and $Ext(C_r^*(SL_2(\mathbb{Z})))$ is not a group.*

Proof. Note $SL_2(\mathbb{Z})$ has a subgroup, which is isomorphic to F_2, with an index 12. The result of the corollary follows immediately from the preceding proposition and Haagerup and Thorbjørnsen's result that $C_r^*(F_2)$ is an MF algebra. □

4.3. Integer group action. In [**21**], Pimsner and Voiculescu proved the following result: *Suppose $\mathcal{A}$ is a unital sepaprable C*-algebra and $\alpha : \mathbb{Z} \to Aut(\mathcal{A})$ is a homomorphism, such that there exists a sequence of integers $0 \le n_1 < n_2 < \cdots$, so that*

$$\lim_{j\to\infty}\|\alpha(n_j)a - a\| = 0$$

for all $a \in \mathcal{A}$. Assume that $\mathcal{A}$ is quasidiagonal. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is also quasidiagonal. In this subsection, we will prove an analogue of Pimsner and Voiculescu's result in the context of MF algebras.

Let $l^2(\mathbb{Z})$ be the Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. Recall the left regular representation of the group $\mathbb{Z}$ is the unitary representation $\lambda : \mathbb{Z} \to \mathcal{U}(l^2(\mathbb{Z}))$ given by $\lambda(n)(e_{n_1}) = e_{n+n_1}$, $n_1 \in \mathbb{Z}$. Then $C_r^*(\mathbb{Z})$ is the C*-algebra generated by $\{\lambda(n)\}_{n\in\mathbb{Z}}$ in $B(l^2(\mathbb{Z}))$. Let $u = \lambda(1)$ be the canonical generator of $C_r^*(\mathbb{Z})$, a bilateral shift operator.

Suppose that $\mathcal{A}$ is a unital MF algebra generated by a family of self-adjoint elements

$$x_1, \ldots, x_m \in \mathcal{A}. \tag{4.1}$$

Suppose α is a homomorphism from $\mathbb{Z}$ to $Aut(\mathcal{A})$ such that there exists a sequence of integers $0 \le n_1 < n_2 < \cdots$, satisfying

$$\lim_{j\to\infty}\|\alpha(n_j)a - a\| = 0 \qquad \text{for all } a \in \mathcal{A}.$$

Without loss of generality, we assume that

$$\|\alpha(n_j)x_i - x_i\| < \frac{1}{j}\ , \qquad \forall\ j \geq 1,\ 1 \leq i \leq m. \tag{4.2}$$

Assume $\mathcal{A}$ acts on a Hilbert space $\mathcal{H}$. Define $\sigma : \mathcal{A} \to B(\mathcal{H} \otimes l^2(\mathbb{Z}))$ to be

$$\sigma(a)(\xi \otimes e_n) = (\alpha(-n)a)\xi \otimes e_n, \qquad \forall\ \ n \in \mathbb{Z}.$$

Then the C*-algebra generated by $\{\sigma(a)\}_{a\in\mathcal{A}} \cup \{I_{\mathcal{H}} \otimes \lambda(n)\}_{n\in\mathbb{Z}}$ in $B(\mathcal{H} \otimes l^2(\mathbb{Z}))$ is called reduced crossed product $\mathcal{A} \rtimes_\alpha \mathbb{Z}$.

Following the notation from [**21**], we let for $j = 1, 2, \ldots$

$$f_{k,j} = \cos\frac{k\pi}{2n_j} \cdot e_k + \sin\frac{k\pi}{2n_j} \cdot e_{k-n_j} \qquad \text{for } 0 \leq k \leq n_j \tag{4.3}$$

be a unit vector in $l^2(\mathbb{Z})$, and $q_{k,j}$ be the rank one projection from $l^2(\mathbb{Z})$ onto $f_{k,j}$. Let

$$q_j = q_{1,j} + \ldots + q_{n_j,j}.$$

Then by [**21**]

$$q_j \to I_{l^2(\mathbb{Z})} \text{ in SOT} \quad \text{and} \quad \lim_{j\to\infty} \|uq_j - q_j u\|_{B(l^2(\mathbb{Z}))} = 0. \tag{4.4}$$

Let $y_i = \alpha(-1)x_i \in \mathcal{A}$, for $1 \leq i \leq m$. We have the following result.

Lemma 4.1. *Let $s, t, r, N_1, \ldots, N_r$ be positive integers. For any $\epsilon > 0$, $\{G_i\}_{i=1}^s$ in $\mathbb{C}\langle Z\rangle$, $\{H_i\}_{i=1}^t$ in $\mathbb{C}\langle X_1, \ldots, X_m, Y_1, \ldots, Y_m\rangle$ and $\{\{P_j^{(i)}\}_{j=-N_i}^{N_i}\}_{i=1}^r$ in $\mathbb{C}\langle X_1, \ldots, X_m\rangle$, there are a positive integer k and $k \times k$ matrices*

$$A_1, \ldots, A_m, B_1, \ldots, B_m, U \in \mathcal{M}_k(\mathbb{C})$$

such that (i)

$$\|U^*A_i - B_iU^*\| \leq \epsilon, \quad 1 \leq i \leq m;$$

(ii)

$$\big|\|G_i(U)\| - \|G_i(u)\|_{C_r^*(\mathbb{Z})}\big| \leq \epsilon, \quad 1 \leq i \leq s;$$

(iii)

$$|\|H_i(A_1, \ldots, A_m, B_1, \ldots, B_m)\| - \|H_i(x_1, \ldots, x_m, y_1, \ldots, y_m)\|_{\mathcal{A}}| \leq \epsilon, \quad 1 \leq i \leq t;$$

and (iv) $\forall\ \ 1 \leq i \leq r$

$$\|\sum_{j=-N_i}^{N_i} P_j^{(i)}(\sigma(x_1), \ldots, \sigma(x_m))(1\otimes u)^j\| - \epsilon \leq \|\sum_{j=-N_i}^{0} P_j^{(i)}(A_1, \ldots, A_m)(U^*)^{-j} + \sum_{j=1}^{N_1} P_j^{(i)}(A_1, \ldots, A_m)U^j\|.$$

Proof. Without loss of generality, we assume that each $\|x_i\| \leq 1$ for $1 \leq i \leq m$. Let

$$x_{k,i} = \alpha(-k)x_i, \quad y_{k,i} = \alpha(-k)y_i \qquad \text{for } 1 \leq i \leq m,\ k \in \mathbb{Z}. \tag{4.5}$$

By (4.2), we have

$$\|x_{k+n_j,i} - x_{k,i}\| < \frac{1}{j} \qquad \text{and} \qquad \|y_{k+n_j,i} - y_{k,i}\| < \frac{1}{j} \qquad \text{for } \ 1 \leq i \leq m,\ \text{and } k, j \geq 1.$$

For the purpose of simplicity, we will assume that $r = 1$. The general case when $r \geq 1$ follows from the similar argument. Thus we let

$$P(X_1, \ldots, X_m, Z) = \sum_{i=-N_1}^{0} P_i^{(1)}(X_1, \ldots, X_m)(Z^*)^{-i} + \sum_{i=1}^{N_1} P_i^{(1)}(X_1, \ldots, X_m)Z^i \tag{4.6}$$

be in $\mathbb{C}\langle X_1, \ldots, X_m, Z\rangle$ and

$$M = \sum_{i=-N_1}^{N_1} \|P_i^{(1)}(x_1, \ldots, x_m)\|_{\mathcal{A}}.$$

For any $\epsilon > 0$, there are a positive integer $L \in \mathbb{N}$ and a unit vector

$$\eta = \sum_{l=-L}^{L} \xi_l \otimes e_l \tag{4.7}$$

in $\mathcal{H} \otimes l^2(\mathbb{Z})$, where each vector ξ_l is in $\mathcal{H}$, such that

$$\|P(\sigma(x_1), \ldots, \sigma(x_m), 1 \otimes u)\|_{B(\mathcal{H}\otimes l^2(\mathbb{Z}))} - 2\epsilon \leq \|P(\sigma(x_1), \ldots, \sigma(x_m), 1 \otimes u)\eta\|. \tag{4.8}$$

For such ϵ and η, by (4.3) and (4.4), there is a positive integer $j > 8/\epsilon$ and a finite rank projection

$$q_j = q_{1,j} + \ldots + q_{n_j,j}$$

on $l^2(\mathbb{Z})$ such that

$$|\cos\frac{(n_j-1)\pi}{2n_j}| + \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L}(M+1)\|e_{l+i} - \cos\frac{(l+i)\pi}{2n_j}f_{l+i,j}\| \leq \frac{\epsilon}{8}; \tag{4.9}$$

$$\|(1 \otimes q_j)\eta - \eta\| \leq \frac{\epsilon}{16(M+1)(N_1)^2}; \tag{4.10}$$

$$\|uq_j - q_j u\|_{B(l^2(\mathbb{Z}))} \leq \frac{\epsilon}{16(M+1)(N_1)^2}; \tag{4.11}$$

$$\big|\|G_i(q_j u q_j)\|_{B(l^2(\mathbb{Z}))} - \|G_i(u)\|_{B(l^2(\mathbb{Z}))}\big| \leq \frac{\epsilon}{8}, \qquad 1 \leq i \leq s, \tag{4.12}$$

where M, N_1, L are introduced as above.

Since $\mathcal{A}$ is an MF algebra with a family of elements $\{x_i, y_i, x_{k,i}, y_{k,i}\}_{1\leq i\leq m, |k|\leq n_j+1}$ in $\mathcal{A}$, for such ϵ, j, by Lemma 2.3 there exists a family of quasidiagonal operators

$$\{a_i, b_i, a_{k,i}, b_{k,i}\}_{1\leq i\leq m, |k|\leq n_j+1} \subseteq B(\mathcal{H})$$

satisfying, by (4:b) of Lemma 2.3,

$$\max_{1\le i\le t,|k|\le n_j} |\|H_i(a_{k,1},\dots,a_{k,m},b_{k,1},\dots,b_{k,m})\| - \|H_i(x_{k,1},\dots,x_{k,m},y_{k,1},\dots,y_{k,m})\|| \le \frac{\epsilon}{8} \quad (4.13)$$

$$\max_{1\le i\le m,|k|\le n_j} \|a_{k+1,i} - b_{k,i}\| \le \frac{\epsilon}{8} \quad (4.14)$$

$$\max_{1\le i\le m,|k|\le n_j} \{\|a_{k+n_j,i} - a_{k,i}\|, \|b_{k+n_j,i} - b_{k,i}\|\} < \frac{\epsilon}{8} \quad (4.15)$$

$$\max_{1\le i\le N_1,|k|\le n_j} |\|P_i^{(1)}(x_1,\dots,x_m)\| - \|P_i^{(1)}(a_{k,1},\dots,a_{k,m})\|| \le \epsilon/8; \quad (4.16)$$

and by (4:c) of Lemma 2.3,

$$\sum_{|k|\le n_j+1} \sum_{-N_1\le i\le N_1} \sum_{1\le l\le L} \|P_i^{(1)}(a_{k,1},\dots,a_{k,m})\xi_l - P_i^{(1)}(x_{k,1},\dots,x_{k,m})\xi_l\| \le \epsilon/8. \quad (4.17)$$

For such ϵ, j, we let

$$\begin{aligned}
\tilde{A}_i &= \sum_{0\le k\le n_j-1} a_{k,j}\otimes q_{k,j}, \qquad 1\le i\le m\\
\tilde{B}_i &= \sum_{0\le k\le n_j-1} b_{k+1,j}\otimes q_{k,j}, \qquad 1\le i\le m\\
\tilde{U} &= 1\otimes q_j u q_j
\end{aligned}$$

By the choice of η in (4.7) and the fact $u=\lambda(1)$, we know that

$$\begin{aligned}
P(\sigma(x_1),\dots,\sigma(x_m),1\otimes u)\eta &= \left(\sum_{i=-N_1}^{N_1} P_i^{(1)}(\sigma(x_1),\dots,\sigma(x_m))(1\otimes u)^i\right)\left(\sum_{l=-L}^{L}\xi_l\otimes e_l\right)\\
&= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L}\left(P_i^{(1)}(\alpha(-i-l)x_1,\dots,\alpha(-i-l)x_m)\xi_l\right)\otimes e_{l+i}\\
&= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L}\left(P_i^{(1)}(x_{i+l,1},\dots,x_{i+l,m})\xi_l\right)\otimes e_{l+i} \quad (4.18)
\end{aligned}$$

Note $\{q_{k,j}\}_{k=1}^{n_j}$ is a family of mutually orthogonal projections. By inequality (4.16), we know

$$\|P_i^{(1)}(\tilde{A}_1,\dots,\tilde{A}_m)\| = \sup_{0\le k\le n_j-1}\|P_i^{(1)}(a_{k,1},\dots,a_{k,m})\| \le \sup_{0\le k\le n_j-1}\|P_i^{(1)}(x_{k,1},\dots,x_{k,m})\|+1 \le M+1 \quad (4.19)$$

for all $-N_1 \leq i \leq N_1$. By inequality (4.11) and (4.19), we know that

$$\begin{aligned}
&\|P(\tilde{A}_1,\ldots,\tilde{A}_m,\tilde{U}) - P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)(1\otimes q_j)\| \\
&= \| \left(\sum_{i=-N_1}^{0} (P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(1\otimes \tilde{U}^*)^{-i} + \sum_{i=1}^{N_1} (P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(1\otimes \tilde{U})^i \right) \\
&\quad - \left(\sum_{i=-N_1}^{0} (P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(1\otimes u^*)^{-i} + \sum_{i=1}^{N_1} (P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(1\otimes u)^i \right) (1\otimes q_j)\| \\
&\leq \sum_{i=-N_1}^{0} \|P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)\|\|(1\otimes q_j u^* q_j)^{-i} - (1\otimes u^*)^{-i}(1\otimes q_j)\| \\
&\quad + \sum_{i=1}^{N_1} \|P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)\|\|(1\otimes q_j u q_j)^{i} - (1\otimes u)^{i}(1\otimes q_j)\| \quad \leq \quad \frac{\epsilon}{8}.
\end{aligned} \tag{4.20}$$

And, by (4.10) and (4.19),

$$\begin{aligned}
\|P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)(1\otimes q_j)\eta - P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)\eta\| & \\
&\leq \|P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)\|\|(1\otimes q_j)\eta - \eta\| \\
&\leq \sum_{i=-N_1}^{N_1} \|P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)\|\|(1\otimes q_j)\eta - \eta\| \\
&\leq \frac{\epsilon}{8}.
\end{aligned} \tag{4.21}$$

Moreover, by the definitions of $\tilde{A}_1,\ldots,\tilde{A}_m$, we have

$$\begin{aligned}
P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)\eta &= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L} P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(1\otimes u)^i(\xi_l\otimes e_l) \\
&= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L} P_i^{(1)}(\tilde{A}_1,\ldots,\tilde{A}_m)(\xi_l\otimes e_{l+i}) \\
&= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L} (P_i^{(1)}(a_{l+i,1},\ldots,a_{l+i,m})\xi_l)\otimes (q_{l+i,j}e_{l+i}) \\
&= \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L} (P_i^{(1)}(a_{l+i,1},\ldots,a_{l+i,m})\xi_l)\otimes e_{l+i} \\
&\quad - \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L} (P_i^{(1)}(a_{l+i,1},\ldots,a_{l+i,m})\xi_l)\otimes (e_{l+i} - \cos\frac{(l+i)\pi}{2n_j} f_{l+i,j})
\end{aligned} \tag{4.22}$$

By inequalities (4.20), (4.21), (4.18), (4.22), (4.17) and (4.9) we know,

$$\begin{aligned}
&\|P(\tilde{A}_1,\ldots,\tilde{A}_m,\tilde{U})\eta - P(\sigma(x_1),\ldots,\sigma(x_m),1\otimes u)\eta\| \\
&\quad\le \|P(\tilde{A}_1,\ldots,\tilde{A}_m,\tilde{U})\eta - P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)(1\otimes q_j)\eta\| \\
&\qquad + \|P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)(1\otimes q_j)\eta - P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)\eta\| \\
&\qquad + \|P(\tilde{A}_1,\ldots,\tilde{A}_m,1\otimes u)\eta - P(\sigma(x_1),\ldots,\sigma(x_m),1\otimes u)\eta\| \\
&\quad\le \frac{\epsilon}{8}+\frac{\epsilon}{8}+\sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L}\|P_i^{(1)}(a_{l+i,1},\ldots,a_{l+i,m})\xi_l - P_i^{(1)}(x_{l+i,1},\ldots,x_{l+i,m})\xi_l\| \\
&\qquad\quad + \sum_{i=-N_1}^{N_1}\sum_{l=-L}^{L}\|P_i^{(1)}(a_{l+i,1},\ldots,a_{l+i,m})\|\|e_{l+i}-\cos\frac{(l+i)\pi}{2n_j}f_{l+i,j}\| \\
&\quad\le \frac{\epsilon}{2}. \qquad (4.23)
\end{aligned}$$

Note that $\{a_i, b_i, a_{k,i}, b_{k,i}\}_{1\le i\le m,|k|\le n_j+1} \subseteq B(\mathcal{H})$ is a family of quasidiagonal operators. There is a finite rank projection p on $\mathcal{H}$ such that, by Lemma 2.1 part (ii),

$$\max_{1\le i\le t}|\|H_i(a_{k,1},\ldots,a_{k,m},b_{k,1},\ldots,b_{k,m})\| - \|H_i(pa_{k,1}p,\ldots,pa_mp,pb_{k,1}p,\ldots,b_{k,m})\|| \le \frac{\epsilon}{8}; \quad (4.24)$$

and, by Lemma 2.1 part (i),

$$\|P(A_1,\ldots,A_m,U)\eta - P(\tilde{A}_1,\ldots,\tilde{A}_m,\tilde{U})\eta\| \le \frac{\epsilon}{2}, \qquad (4.25)$$

where

$$\begin{aligned}
A_i &= p\tilde{A}_ip = \sum_{0\le k\le n_j-1} pa_{k,j}p\otimes q_{k,j}, \qquad 1\le i\le m \\
B_i &= p\tilde{B}_ip = \sum_{0\le k\le n_j-1} pb_{k+1,j}p\otimes q_{k,j}, \qquad 1\le i\le m \\
U &= p\tilde{U}p = p\otimes q_juq_j
\end{aligned}$$

are the elements in $B((p\otimes q_j)(\mathcal{H}\otimes l^2(\mathbb{Z})))$. By (4.8), (4.23) and (4.25), we have

$$\|P(A_1,\ldots,A_m,U)\| \ge \|P(\sigma(x_1),\ldots,\sigma(x_m),1\otimes u)\|_{B(\mathcal{H}\otimes l^2(\mathbb{Z}))} - 4\epsilon. \qquad (4.26)$$

Since $\{q_{k,j}\}_{k=1}^{n_j}$ is a family of mutually orthogonal projections, we have

$$\|H_i(A_1,\ldots,A_m,B_1,\ldots,B_m)\| = \max_{1\le k\le n_j}\{\|H_i(pa_{k,1}p,\ldots,pa_{k,m}p,pb_{k,1}p,\ldots,pb_{k,m}p)\|\}. \quad (4.27)$$

Now it follows from (4.13), (4.24) and (4.27) that for $1\le i\le t$

$$|\|H_i(A_1,\ldots,A_m,B_1,\ldots,B_m)\| - \|H_i(x_1,\ldots,x_m,y_1,\ldots,y_m)\|_{\mathcal{A}}| \le \epsilon. \qquad (4.28)$$

By (4.12), we know that for $1\le i\le s$

$$|\|G_i(U)\| - \|G_i(u)\|| \le \epsilon. \qquad (4.29)$$

Let E_n be the rank one projection from $l^2(\mathbb{Z})$ onto the vector e_n. For any unit vector $h \in \mathcal{H}$

$$\begin{aligned}
&\left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) h \otimes f_{k,j} \\
&\quad = \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (h \otimes (\cos \frac{k\pi}{2n_j} \cdot e_k + \sin \frac{k\pi}{2n_j} \cdot e_{k-n_j})) \\
&\quad = (\cos \frac{k\pi}{2n_j} \cdot a_{k,i}h) \otimes e_k + (\sin \frac{k\pi}{2n_j} \cdot a_{k-n_j,i}) \otimes e_{k-n_j} \\
&\quad = (\cos \frac{k\pi}{2n_j} \cdot a_{k,i}h) \otimes e_k + (\sin \frac{k\pi}{2n_j} \cdot a_{k,i}) \otimes e_{k-n_j} + (\sin \frac{k\pi}{2n_j} \cdot (a_{k-n_j,i} - a_{k,i})) \otimes e_{k-n_j} \\
&\quad = (a_{k,i} \otimes q_{k,j})(h \otimes f_{k,j}) + (\sin \frac{k\pi}{2n_j} \cdot (a_{k-n_j,i} - a_{k,i})) \otimes e_{k-n_j}; \qquad \forall\ 1 \le i \le m.
\end{aligned}$$

By (4.15), we have that

$$\left\|\left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (1 \otimes q_{k,j}) - (a_{k,i} \otimes q_{k,j})\right\| \le \epsilon/8; \qquad \forall\ 1 \le i \le m.$$

Together with the fact that $\{q_{k,j}\}_{k=1}^{n_j}$ is a family of mutually orthogonal projections, we have

$$\begin{aligned}
&\left\|\left(\sum_{0\le k\le n_k-1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right)\right) (1 \otimes q_j) - \sum_{0\le k\le n_k-1} (a_{k,i} \otimes q_{k,j})\right\| \\
&\qquad = \left\|\sum_{0\le k\le n_k-1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (1 \otimes q_{k,j}) - \sum_{0\le k\le n_k-1} (a_{k,i} \otimes q_{k,j})\right\| \\
&\qquad = \left\|\sum_{0\le k\le n_k-1} \left(\left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (1 \otimes q_{k,j}) - (a_{k,i} \otimes q_{k,j})\right)\right\| \\
&\qquad \le \max_{0\le k\le n_k-1} \left\|\left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (1 \otimes q_{k,j}) - (a_{k,i} \otimes q_{k,j})\right\| \\
&\qquad \le \epsilon/8; \qquad \forall\ 1 \le i \le m.
\end{aligned} \tag{4.30}$$

Similarly,

$$\left\|\left(\sum_{0\le k\le n_k-1} \left(b_{k,i} \otimes E_k + b_{k-n_j,i} \otimes E_{k-n_j}\right)\right) (1 \otimes q_j) - \sum_{0\le k\le n_k-1} (b_{k,i} \otimes q_{k,j})\right\| \le \epsilon/8. \tag{4.31}$$

On the other hand, we observe that

$$(I \otimes u^*) \sum_{0\le k\le n_j-1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) = \sum_{0\le k\le n_j-1} \left(a_{k,i} \otimes E_{k-1} + a_{k-n_j,i} \otimes E_{k-n_j-1}\right) (I \otimes u^*) \tag{4.32}$$

and by (4.9), for $1 \le i \le m$,

$$(a_{-n_j,i} \otimes E_{-n_j-1})(1 \otimes q_j) = 0; \qquad \|(b_{n_j-1,i} \otimes E_{n_j-1})(1 \otimes q_j)\| \le |\cos \frac{(n_j-1)\pi}{2n_j}| \le \epsilon/8. \tag{4.33}$$

It follows that for $1 \leq i \leq m$

$$\begin{aligned}
T_i = (I \otimes u^*) & \sum_{0 \leq k \leq n_j - 1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) (I \otimes q_j) \\
& - \left(\sum_{0 \leq k \leq n_j - 1} \left(b_{k,i} \otimes E_k + b_{k-n_j,i} \otimes E_{k-n_j}\right) \right) (I \otimes q_j)(I \otimes u^*) \\
= & \sum_{0 \leq k \leq n_j - 1} \left(a_{k,i} \otimes E_{k-1} + a_{k-n_j,i} \otimes E_{k-1-n_j}\right) (I \otimes u^*)(I \otimes q_j) \qquad \text{(By (4.32))} \\
& - \left(\sum_{0 \leq k \leq n_j - 1} \left(b_{k,i} \otimes E_k + b_{k-n_j,i} \otimes E_{k-n_j}\right) \right) (I \otimes q_j u^*) \\
= & \sum_{0 \leq k \leq n_j - 1} \left(a_{k,i} \otimes E_{k-1} + a_{k-n_j,i} \otimes E_{k-1-n_j}\right) ((I \otimes q_j u^*) - I \otimes (q_j u^* - u^* q_j)) \\
& - \left(\sum_{0 \leq k \leq n_j - 1} \left(b_{k,i} \otimes E_k + b_{k-n_j,i} \otimes E_{k-n_j}\right) \right) (I \otimes q_j u^*) \\
= & - \left(\sum_{0 \leq k \leq n_j - 1} \left(a_{k,i} \otimes E_{k-1} + a_{k-n_j,i} \otimes E_{k-1-n_j}\right) \right) (I \otimes (q_j u^* - u^* q_j)) \\
& + \left(\sum_{-n_j \leq k \leq n_k - 2} (a_{k+1,i} - b_{k,i}) \otimes E_k \right) (I \otimes q_j u^*) \\
& + \left(a_{-n_j,i} \otimes E_{-n_j-1} - b_{n_i-1,i} \otimes E_{n_j-1}\right) (I \otimes q_j u^*)
\end{aligned}$$

By (4.11), (4.14) and (4.33), we have that

$$\|T_i\| \leq \frac{\epsilon}{2}. \tag{4.34}$$

Moreover,

$$\begin{aligned}
U^* A_i &= (p \otimes q_j u^* q_j)\Big(\sum_{0 \leq k \leq n_k - 1} p a_{k,j} p \otimes q_{k,j} \Big) \\
&= (p \otimes q_j)(I \otimes u^*) \left(\sum_{0 \leq k \leq n_k - 1} (a_{k,i} \otimes q_{k,j}) - \sum_{0 \leq k \leq n_k - 1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) \right. \\
&\qquad \left. + \sum_{0 \leq k \leq n_k - 1} \left(a_{k,i} \otimes E_k + a_{k-n_j,i} \otimes E_{k-n_j}\right) \right) (p \otimes q_j)
\end{aligned}$$

And

$$B_iU^* = (\sum_{0\le k\le n_k-1} pb_{k,j}p\otimes q_{k,j})(p\otimes q_ju^*q_j)$$
$$= (p\otimes q_j)\left(\sum_{0\le k\le n_k-1}(b_{k,i}\otimes q_{k,j}) - \sum_{0\le k\le n_k-1}\left(b_{k,i}\otimes E_k + b_{k-n_j}\otimes E_{k-n_j}\right)\right.$$
$$\left. + \sum_{0\le k\le n_k-1}\left(b_{k,i}\otimes E_k + b_{k-n_j}\otimes E_{k-n_j}\right)\right)(1\otimes q_j)(1\times u^*)(p\otimes q_j)$$

By (4.30), (4.31), (4.34), we have

$$\|U^*A_i - B_iU^*\| \le \epsilon. \tag{4.35}$$

Let $k = rank(p\otimes q_j)$. Then $B((p\otimes q_j)(\mathcal{H}\otimes l^2(\mathbb{Z}))) \simeq \mathcal{M}_k(\mathbb{C})$. By (4.35), (4.29), (4.28) and (4.26), we have completed the proof of the lemma. □

From now on, we let $\{H_r\}_{r=1}^\infty$ (respectively, $\{G_r\}_{r=1}^\infty$) be the collection of all polynomials in $\mathbb{C}\langle X_1,\dots,X_n,Y_1,\dots,Y_m\rangle$ (or $\mathbb{C}\langle Z\rangle$ respectively) with rational coefficients.

Lemma 4.2. *For any $r_0\in\mathbb{N}$ and $P_1,\dots,P_{r_0}$ in $\mathbb{C}\langle X_1,\dots,X_n,Z\rangle$, there is some $r_1>0$ such that the following is true: for any $k\in\mathbb{N}$ and any*

$$A_1,\dots,A_m,B_1,\dots,B_m,U\in\mathcal{M}_k(\mathbb{C})$$

satisfying (i)

$$\|U^*A_i - B_iU^*\|_{\mathcal{M}_k(\mathbb{C})} \le \frac{1}{r_1},\quad 1\le i\le m;$$

and (ii)

$$\left|\|G_i(U)\|_{\mathcal{M}_k(\mathbb{C})} - \|G_i(u)\|_{C_r^*(\mathbb{Z})}\right| \le \frac{1}{r_1},\quad 1\le i\le r_1;$$

and (iii)

$$\left|\|H_i(A_1,\dots,A_m,B_1,\dots,B_m)\|_{\mathcal{M}_k(\mathbb{C})} - \|H_i(x_1,\dots,x_m,y_1,\dots,y_m)\|_{\mathcal{A}}\right| \le \frac{1}{r_1},\quad 1\le i\le r_1,$$

we have

$$\|P_j(A_1,\dots,A_m,U)\|_{\mathcal{M}_k(\mathbb{C})} - \|P_j(x_1,\dots,x_m,u)\|_{\mathcal{A}\rtimes_\alpha\mathbb{Z}} \le \frac{1}{r_0},\quad 1\le j\le r_0.$$

Proof. Assume the result of the lemma is not true. Thus there are some $r_0>0$, a sequence of positive integers $\{k_l\}_{l=1}^\infty$, and matrices

$$A_1^{(l)},\dots,A_m^{(l)},B_1^{(l)},\dots,B_m^{(l)},U^{(l)}\in\mathcal{M}_{k_l}(\mathbb{C})$$

satisfying

(a)

$$\|(U^{(l)})^*A_i^{(l)} - B_i^{(l)}(U^{(l)})^*\|_{\mathcal{M}_{k_l}(\mathbb{C})} \le \frac{1}{l},\quad 1\le i\le m;$$

(b)

$$\left|\|G_i(U^{(l)})\|_{\mathcal{M}_{k_l}(\mathbb{C})} - \|G_i(u)\|_{C_r^*(\mathbb{Z})}\right| \leq \frac{1}{l}, \quad 1 \leq i \leq l;$$

(c)

$$\left|\|H_i(A_1^{(l)}, \ldots, A_m^{(l)}, B_1^{(l)}, \ldots, B_m^{(l)})\|_{\mathcal{M}_{k_l}(\mathbb{C})} - \|H_i(x_1, \ldots, x_m, y_1, \ldots, y_m)\|_{\mathcal{A}}\right| \leq \frac{1}{l}, \quad 1 \leq i \leq l.$$

(d) but

$$\max_{1\leq j\leq r_0} \{\|P_j(A_1, \ldots, A_m, U)\|_{\mathcal{M}_{k_l}(\mathbb{C})} - \|P_j(x_1, \ldots, x_m, u)\|_{\mathcal{A}\rtimes_\alpha\mathbb{Z}}\} > \frac{1}{r_0}.$$

Consider the unital C*-algebra $\prod_l \mathcal{M}_{k_l}(\mathbb{C})/\sum_l \mathcal{M}_{k_l}(\mathbb{C})$. Let

$$a_i = [\langle A_i^{(l)}\rangle_{l=1}^\infty], \quad b_i = [\langle B_i^{(l)}\rangle_{l=1}^\infty], \quad w = [\langle U^{(l)}\rangle_{l=1}^\infty], \quad 1 \leq i \leq m$$

be the elements in $\prod_l \mathcal{M}_{k_l}(\mathbb{C})/\sum_l \mathcal{M}_{k_l}(\mathbb{C})$ and $C^*(a_1, \ldots, a_m, b_1, \ldots, b_m, w)$ be the C*-algebra generated by $a_1, \ldots, a_m, b_1, \ldots, b_m, w$ in $\prod_l \mathcal{M}_{k_l}(\mathbb{C})/\sum_l \mathcal{M}_{k_l}(\mathbb{C})$. By (b) and (c), there exist embedding

$$\rho_1 : \mathcal{A} \to C^*(a_1, \ldots, a_m, b_1, \ldots, b_m, w) \quad \textit{and} \quad \rho_2 : C_r^*(\mathbb{Z}) \to C^*(a_1, \ldots, a_m, b_1, \ldots, b_m, w)$$

such that

$$\rho_1(x_i) = a_i, \ \rho_1(y_i) = b_i, \quad \rho_2(u) = w.$$

By (a), we know that

$$w^* a_i w = b_i.$$

Note that $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is also the full crossed product of $\mathcal{A}$ by $\mathbb{Z}$. From the universal property of the full crossed product, it follows that there is a $*$-homomorphism

$$\rho : \mathcal{A} \rtimes_\alpha \mathbb{Z} \to C^*(a_1, \ldots, a_m, b_1, \ldots, b_m, w)$$

such that

$$\rho(x_i) = a_i, \ \rho(y_i) = b_i, \quad \rho(u) = w.$$

It follows that for all $1 \leq j \leq r_0$

$$\begin{aligned}\|P_j(x_1, \ldots, x_m, u)\|_{\mathcal{A}\rtimes_\alpha\mathbb{Z}} &\geq \|P_j(a_1, \ldots, a_m, w)\|_{\prod_l \mathcal{M}_{k_l}(\mathbb{C})/\sum_l \mathcal{M}_{k_l}(\mathbb{C})} \\ &= \limsup_l \|P_j(A_1, \ldots, A_m, U)\|_{\mathcal{M}_{k_l}(\mathbb{C})}.\end{aligned}$$

This contradicts with the assumption (d). Thus the proof of the lemma is completed. □

Combining Lemma 4.1 and Lemma 4.2, we have the following conclusion.

Theorem 4.2. *Suppose that $\mathcal{A}$ is a finitely generated unital MF algebra and α is a homomorphism from $\mathbb{Z}$ into $Aut(\mathcal{A})$ such that there is a sequence of integers $0 \leq n_1 < n_2 < \cdots$ satisfying*

$$\lim_{j\to\infty} \|\alpha(n_j)a - a\| = 0$$

for any $a \in \mathcal{A}$. Then $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is an MF algebra.

PROOF. Assume that $\mathcal{A}$ is generated by a family of self-adjoint elements $x_1, \ldots, x_m$ and u is the canonical generator of $C_r(\mathbb{Z})$. Without loss of generality, we assume that $x_1, \ldots, x_m, u$ generate $\mathcal{A} \rtimes_\alpha \mathbb{Z}$. Let r_0 be a positive integer.

Assume that $\epsilon < 1/r_0$ is a positive number. Assume that $s, t, N_1, \ldots, N_{r_0}$ are positive integers and $\{G_i\}_{i=1}^s \subseteq \mathbb{C}\langle Z\rangle$, $\{H_i\}_{i=1}^t \subseteq \mathbb{C}\langle X_1, \ldots, X_m, Y_1, \ldots, Y_m\rangle$ and $\{\{P_j^{(i)}\}_{j=-N_i}^{N_i}\}_{i=1}^{r_0} \subseteq \mathbb{C}\langle X_1, \ldots, X_m\rangle$.

For $1 \leq i \leq r_0$, we let

$$P_i(X_1, \ldots, X_m, Z) = \sum_{j=-N_1}^{0} P_j^{(i)}(X_1, \ldots, X_m)(Z^*)^{-j} + \sum_{i=1}^{N_1} P_j^{(i)}(X_1, \ldots, X_m)Z^j$$

By Lemma 4.1, there are a positive integer k and $k \times k$ matrices

$$A_1, \ldots, A_m, B_1, \ldots, B_m, U \in \mathcal{M}_k(\mathbb{C})$$

such that (i)

$$\|U^* A_i - B_i U^*\| \leq \epsilon \leq 1/r_0, \quad 1 \leq i \leq m; \tag{4.36}$$

(ii)

$$\big|\|G_i(U)\| - \|G_i(u)\|_{C_r^*(\mathbb{Z})}\big| \leq \epsilon, \quad 1 \leq i \leq s;$$

(iii)

$$|\|H_i(A_1, \ldots, A_m, B_1, \ldots, B_m)\| - \|H_i(x_1, \ldots, x_m, y_1, \ldots, y_m)\|_{\mathcal{A}}| \leq \epsilon, \quad 1 \leq i \leq t;$$

and (iv) $\forall \ \ 1 \leq i \leq r_0$

$$\|P_i(x_1, \ldots, x_m, u)\|_{\mathcal{A} \rtimes_\alpha \mathbb{Z}} - \epsilon \leq \|P_i(A_1, \ldots, A_m, U)\|.$$

By Lemma 4.2, we know when ϵ is small enough and s, t are large enough,

$$\|P_i(A_1, \ldots, A_m, U)\| \leq \|P_i(x_1, \ldots, x_m, u)\|_{\mathcal{A} \rtimes_\alpha \mathbb{Z}} + 1/r_0, \ \ \forall \ \ 1 \leq i \leq r_0.$$

Combining with (iv), we have

$$|\|P_i(A_1, \ldots, A_m, U)\| - \|P_i(x_1, \ldots, x_m, u)\|_{\mathcal{A} \rtimes_\alpha \mathbb{Z}}| \leq 1/r_0, \ \ \forall \ \ 1 \leq i \leq r_0. \tag{4.37}$$

By (4.36), (4.37) and Lemma 2.3, we know that $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is an MF algebra □

The following corollary follows directly from the preceding theorem.

Corollary 4.5. *Let $C_r^*(F_2)$ be the reduced C*-algebra of the free group F_2. Let u_1, u_2 be the canonical unitary generators of $C_r^*(F_2)$ and $0 < \theta < 1$ be a positive number. Let α be a homomorphism from $\mathbb{Z}$ into $Aut(C_r^*(F_2))$ induced by the following mapping: $\forall\ n \in \mathbb{Z}$,*

$$\alpha(n)(u_1) = e^{2n\pi\theta \cdot i} u_1 \qquad \text{and} \qquad \alpha(n)(u_2) = e^{2n\pi\theta \cdot i} u_2.$$

Then $C_r^*(F_2) \rtimes_\alpha \mathbb{Z}$ is an MF algebra and $Ext(C_r^*(F_2) \rtimes_\alpha \mathbb{Z})$ is not a group.

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